On short graded algebras

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Introduction.

Let (A, m, k) be a local Cohen-Macaulay ring of dimension d. We denote by e the multiplicity of A, by N its embedding dimension and by $h := N - d$ the codimension of A. The Hilbert function of A is the numerical function defined by $H_A(n) := dim_k(m^n/m^{n+1})$ and the Poincare series is the series $P_A(z) := \sum_{n\geq 0} H_A(n)z^n$. By the theorem of Hilbert-Serre there exists a polynomial $f(z) \in \mathbf{Z}[z]$ such that $f(1) = e$ and $P_A(z) = f(z)/(1-z)^d$. From this it follows that there exists a polynomial $h_A(x) \in \mathbf{Q}[x]$ such that $H_A(n) = h_A(n)$ for all $n \gg 0$. This polynomial is called the Hilbert polynomial of A. If we denote by $s = s(A) := deg(f(z))$ and by $i = i(A) := max\{n \in \mathbf{Z} | H_A(n) \neq h_A(n)\} + 1$, then it is well known that $i = s - d + 1$ (see [EV]). Also we denote by $t = t(A)$ the initial degree of A, which is by definition $t = t(A) := min\{j | H_A(j) \neq \binom{N+j-1}{j}\}$. It is clear from the definition that $t \geq 2$. In [RV] we proved that $e \geq \binom{h+t-1}{h}$. Also in the same paper we proved that if $e = \binom{h+t-1}{h}$ then $gr_m(A) := \oplus(m^n/m^{n+1})$ is a Cohen-Macaulay graded ring and

$$P_A(z) = \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i/(1-z)^d.$$

If $e = \binom{h+t-1}{h} + 1$ then $gr_m(A)$ needs not to be Cohen-Macaulay (see [S]) but if the Cohen-Macaulay type $\tau(A)$ verifies $\tau(A) < \binom{h+t-2}{t-1}$ then again $gr_m(A)$ is Cohen-Macaulay and

$$P_A(z) = \left(\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^t \right) /(1-z)^d.$$

(see [RV]). On the other hand if we consider a set X of e distinct points in the projective space $\mathbf{P}^h$ and we let $A = k[X_0, \ldots, X_h]/I$ be the coordinate ring of X, then A is a graded

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Cohen-Macaulay ring of dimension 1. Hence the Hilbert function of A is strictly increasing up to the degree of X, which is e. Many authors (see [GO1],[G],[GO2],[GM],[GGR], [B],[Br1],[Br2],[BK],[L1],[L2],[R],[TV]) have studied the notion of points in "generic" position. This means by definition that

$$H_A(n) = min\left(e, \binom{h+n}{n}\right).$$

It is easy to prove that almost every set of e points in $\mathbf{P}^h$ are in generic position, in the sense that the points in generic position in $\mathbf{P}^h$ form a dense open set U of $\mathbf{P}^h \times \mathbf{P}^h \times \cdots \times \mathbf{P}^h$ (e times). Now it is clear that if X is a set of points in generic position in $\mathbf{P}^h$ then

$$P_A(z) = \left(\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + cz^t\right) /(1-z)$$

where t is defined to be the integer such that $\binom{h+t-1}{h} \leq e < \binom{h+t}{h}$.

Thus we are led to consider graded algebras $A = k[X_0, \ldots, X_r]/I$ over an infinite field k which are Cohen-Macaulay and whose Poincare series is given by

$$P_A(z) = \left(\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + cz^t\right) /(1-z)^d$$

where d is the Krull dimension of A, t is an integer ≥ 2, and c is an integer $0 \leq c < \binom{h+t-1}{t}$.

We call such an algebra a **Short Graded Algebra**.

It is easy to see that short graded algebras are the Cohen-Macaulay graded algebras A such that H_A^{1-d} is maximal according to the definition given by Orecchia in [O] . Also extremal Cohen-Macaulay graded algebras in the sense of Schenzel (see [Sc]) are short graded algebras with $c = 0$.

Generalities on short graded algebras.

Let $A = k[X_0, \ldots, X_r]/I$ be a short graded algebra with Poincare series

$$P_A(z) = \left(\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + cz^t\right) /(1-z)^d.$$

The multiplicity of A is denoted by $e = e(A)$. We have $e = \binom{h+t-1}{h} + c$. Also we have $i = i(A) = t - d + 1$. Since k is an infinite field, we can find d linear forms $L_1, \dots, L_d$ in $R = k[X_0, \dots, X_r]$ such that if $J = (L_1, \dots, L_d)$, the graded algebra $B = A/JA$ is of dimension 0, codimension h and has $e(A) = e(B)$. If we denote by - reduction modulo J, we get $B = \bar{R}/\bar{I}$ and we call B an artinian reduction of A. It is clear that B is a short graded algebra with

$$P_B(z) = \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + cz^t.$$

It follows that $s(B) = s(A) = t(B) = t(A) = t$. Now let

$$\bar{\mathbf{F}} : 0 \to \bar{F}_h \to \cdots \to \bar{F}_1 \to \bar{R} \to B \to 0$$

be a minimal graded free resolution of B with $\bar{F}_i = \oplus_{j=1}^{\beta_i} \bar{R}(-d_{ij})$. The positive integers β_i are called the Betti numbers of B; the integers d_{ij} are called the shifting in the resolution of B and , along with the β_i, are unique. Since $t(B) = t$ we have $t \leq d_{1j}$ for every j. Further it is well known that we have a graded isomorphism $Tor_h^{\bar{R}}(B, k) \simeq (0 : B_1)(-h)$, hence we get $d_{hj} \leq s + h$ for every j. The following lemma is possibly well known, but we insert here a proof for the sake of completeness.

Let

$$\bar{\mathbf{F}} : 0 \to \bar{F}_h \to \bar{F}_{h-1} \to \cdots \to \bar{F}_0 \to M \to 0$$

be a minimal graded free resolution of the graded R-module M, with $F_i = \oplus_{j=1}^{\beta_i} R(-d_{ij})$.

LEMMA 1.1. *If $i > 0$, for every j there exists q such that $d_{i-1,q} < d_{ij}$. If $i < h$, for every j there exists p such that $d_{ij} < d_{i+1,p}$.*

PROOF: It is clear that d_{ij} is the degree of the element of F_{i-1} which is the j-th column of the matrix Δ_i representing the map of free modules $F_i \to F_{i-1}$. Hence we get for every $q = 1, \dots, \beta_{i-1}$

$$\delta_q + d_{i-1,q} = d_{ij}$$

where $\delta_1, \dots, \delta_{\beta_{i-1}}$ are the degree of the elements of this column vector. Now if for some j we have $d_{ij} = d_{i-1,q}$ for every q, then Δ_i would have a column of zeros, a contradiction to the minimality of the resolution. The other result follows in the same way, by using the fact that the transpose of Δ_i cannot have a column of zeros since it is a matrix in the minimal graded free resolution of $Ext_R^h(M, R)$.

Using this lemma we get that in the resolution $\bar{\mathbf{F}}$ of B we have

$$\bar{F}_i = \bar{R}(-t-i)^{b_i} \oplus \bar{R}(-t-i+1)^{a_i}$$

for every $i \geq 1$. Now it is well known that the graded free resolution of A as an R-module has the same Betti numbers and shifting as the resolution of B as an $\bar{R}$-module. Hence a graded free resolution of A can be written as

$$0 \to R(-t-h)^{b_h} \oplus R(-t-h+1)^{a_h} \to \cdots \to R(-t-i)^{b_i} \oplus R(-t-i+1)^{a_i} \to \cdots \to R \to A \to 0$$

for some integers $a_i, b_i \geq 0$. By the particular Hilbert function of A we get $a_1 = \binom{h+t-1}{t} - c$ and $b_h = c$.

A detailed proof of these observations can be found in [L2].

We close this section by remarking that for a short graded algebra the Betti numbers β_i determine all the resolution. This can be easily seen by using the fact that in each degree $n > t$ we have

$$dim(\bar{R}_n) + \sum_{i=1}^{h} (-1)^i \left[a_i dim(\bar{R}(-t-i+1)_n) + b_i dim(\bar{R}(-t-i)_n) \right] = 0.$$

Pure and linear resolution.

Recall that given a graded free resolution

$$\mathbf{F} : 0 \to F_h \to \cdots \to F_1 \to R \to A \to 0$$

of the graded algebra A with $F_i = \oplus_{j=1}^{\beta_i} R(-d_{ij})$ we say that the resolution is pure of type $(d_1, \ldots, d_h)$ if for every $i = 1, \ldots, h$ we have $d_{ij} = d_i$ for every j. If the resolution is pure of type $(t, t+m, t+2m, \ldots, t+(h-1)m)$, we shall say that it is pure of type (t, m). A pure resolution of type $(t, 1)$ is just called a t-linear resolution (see [W],[HK])

In this section we investigate what short graded algebras have pure or linear resolution.

The first proposition deals with the case of a linear resolution.

PROPOSITION 2.1. *Let A be a Cohen-Macaulay graded algebra. The following conditions are equivalent*

a) A is short and has a t-linear resolution.

b) A is short with $c = 0$.

c) $e = \binom{h+t-1}{h}$ and $t = indeg(A)$

d) I is generated by $\binom{h+t-1}{t}$ forms of degree t

PROOF: The conditions b), c) and d) are equivalent by theorem 3.3 in [RV]. If A is short and $c = 0$ then $b_h = 0$. By lemma 1.1 this implies $b_i = 0$ for every $i = 1, \ldots, h$ and the resolution is linear. If the resolution is linear then $b_h = 0$, hence $c = 0$.

The case of a pure resolution of type (t, m) is considered in the next proposition which extends Theorem 2 in [Br1].

PROPOSITION 2.2. *Let A be a short graded algebra. A has a pure resolution of type (t, m) if and only if one of the following occurs*

a) $e = \binom{h+t-1}{h}$

or

b) $h = 2$, $e = \binom{t+1}{2} + \frac{t}{2}$ where t is even and I is generated by forms of degree t.

PROOF: If the resolution is linear a) holds by the above proposition. If the resolution is pure of type (t, m) with $m \geq 2$, we get $d_h = t + (h-1)m \leq t + h$, hence $(h-1)m \leq h$. This implies $m = 1$ or $m = h = 2$. In the first case a) holds by the above proposition, while in the latter case we get a resolution

$$0 \rightarrow R(-t-2)^{a-1} \rightarrow R(-t)^{a} \rightarrow R \rightarrow A \rightarrow 0$$

From this it follows easily that t is even, $e = \binom{t+1}{2} + \frac{t}{2}$ and I is generated by forms of degree t.

Conversely if a) holds the conclusion follows by the above proposition, while if b) holds we get a resolution

$$0 \rightarrow R(-t-2)^{b_2} \oplus R(-t-1)^{a_2} \rightarrow R(-t)^{a_1} \rightarrow R \rightarrow A \rightarrow 0$$

It follows that $b_2 + a_2 = a_1 - 1$ where $a_1 = t + 1 - c$ and $b_2 = c = \frac{t}{2}$. Hence $a_2 = t + 1 - \frac{t}{2} - 1 - \frac{t}{2} = 0$

The next result says that a short graded algebra has a pure resolution if and only if it has some special Betti numbers. It extends Theorem 3 in [Br1] (see also [L1]).

Proposition 2.3. *Let A be a short graded algebra with $\binom{h+t-1}{h} < e < \binom{h+t}{h}$. A has a pure resolution if and only if there exists an integer p such that $1 \leq p \leq h-1$ and*

$$\beta_i = \begin{cases} \binom{t+i-2}{i-1}\binom{h+t}{h-i+1}\frac{p-i+1}{t+p}, & \text{for i=1,...,p} \\ \binom{t+i-1}{i}\binom{h+t}{h-i}\frac{i-p}{t+p}, & \text{for i=p+1,...,h.} \end{cases}$$

Proof: If A is short and has a pure resolution of type $(d_1, \ldots, d_h)$, then $d_1 = t$ and $d_h = t+h$, otherwise if $d_h = t+h-1$ then the resolution would be linear and by Proposition 2.1 $e = \binom{h+t-1}{h}$. Hence there exists an integer p, $1 \leq p \leq h-1$ such that

$$d_i = \begin{cases} t+i-1, & \text{for i=1, \ldots ,p} \\ t+i, & \text{for i=p+1, \ldots ,h.} \end{cases}$$

Now, by a result of Herzog and Kuhl (see [HK]), if the graded algebra A has a pure resolution of type $(d_1, \ldots, d_h)$ then $\beta_i = \left|\Pi_{j \neq i} \frac{d_j}{d_j - d_i}\right|$. In our case the conclusion follows by an easy computation. Conversely, we have seen at the end of section 1 that for a short graded algebra the Betti numbers determine all the resolution. Now it is easy to prove that the particular Betti numbers of the proposition determine a pure resolution.

For example let us consider the case $h=3$, $t=3$, $p=2$. We get $\beta_1 = 8$, $\beta_2 = 9$,$\beta_3 = 2$, hence we have a resolution

$$0 \to R(-6)^{b_3} \oplus R(-5)^{a_3} \to R(-5)^{b_2} \oplus R(-4)^{a_2} \to R(-4)^{b_1} \oplus R(-3)^{a_1} \to R \to A \to 0$$

with $a_1 = 10 - c$, hence $b_1 = c - 2$. Now $b_3 = c \leq \beta_3 = 2$, hence $c = 2$, $a_1 = 8$, $b_1 = 0$, $b_3 = 2$, $a_3 = 0$. Further we have

$$dim(\bar{R}_4) + a_2 = b_1 + a_1 dim(\bar{R}_1).$$

Since $b_1 = 0$, $dim(\bar{R}_4) = \binom{3+4-1}{4} = 15$, $dim(\bar{R}_1) = 3$ we get $a_2 = 9$, hence $b_2 = 0$ and the resolution is pure of type $(3, 4, 6)$.

We finally remark that if A is a short graded algebra with a pure resolution, then for the same p as in the above prposition, we get $e = \frac{t\binom{h+t}{h}}{t+p}$ (see [HM]).

A particular case of pure resolution is considered in the last result of this section.

THEOREM 2.4. *Let $A = R/I$ be a graded algebra which is Cohen-Macaulay. Then the following conditions are equivalent:*

a) A is Gorenstein and short.

b) A has a pure resolution and $e = h + 2$.

c) The resolution of A is

$$0 \rightarrow R(-h-2)^{\beta_h} \rightarrow R(-h)^{\beta_{h-1}} \rightarrow \cdots \rightarrow R(-2)^{\beta_1} \rightarrow R \rightarrow A \rightarrow 0$$

PROOF: If A is Gorenstein the Hilbert function of its artinian reduction is symmetric, hence we get $c = 1$, $e = h + 2$ and $t = 2$. This proves that A is an extremal Gorenstein algebra according to the definition given by Schenzel in [Sc]. But extremal Gorenstein algebras have a pure resolution of type $(2, 3, \ldots, h, h + 2)$ as proved in the same paper [Sc]. Hence a) implies b) and c). Let now prove that b) implies c). It is clear that $P_A(z) = (1 + hz + z^2)/(1 - z)^d$, hence $c = 1$ and $b_h = c = 1$. Since the resolution is pure we get $\beta_h = 1$ and A is Gorenstein. Finally we prove that c) implies a). By the formula of Herzog and Kuhl we get

$$\beta_h = \left|\Pi_{j<h} \frac{d_j}{d_j - h - 2}\right| = \left|\frac{2}{-h} \frac{3}{-h+1} \cdots \frac{h}{-2}\right| = \frac{h!}{h!} = 1$$

hence A is Gorenstein. Further I is generated by forms of degree 2 and we get

$$\beta_1 = a_1 = \left|\Pi_{j>1} \frac{d_j}{d_j - 2}\right| = \left|\frac{3}{1}\frac{4}{2} \cdots \frac{h}{h-2} \frac{h+2}{h}\right| = \frac{h!(h+2)}{2(h-2)!h} = \binom{h+1}{2} - 1.$$

The conclusion follows by using theorem 3.10 in [RV].

Right almost linear resolution

Let A be a graded algebra with graded free resolution

$$\mathbf{F} : 0 \rightarrow F_h \rightarrow F_{h-1} \rightarrow \cdots \rightarrow F_1 \rightarrow R \rightarrow A \rightarrow 0$$

where $F_i = \oplus_{j=1}^{\beta_i} R(-d_{ij})$. Following [L1] we say that $\mathbf{F}$ is right almost linear if it is linear except possibly at F_1. In [L1] Lorenzini proved that the coordinate ring of a set of points in $\mathbf{P}^h$ has a right almost linear resolution in some particular cases. All these results are consequence of the following theorem which proves that a suitable condition on the defining ideal of a short graded algebra forces the resolution to be right almost linear with special Betti numbers.

We recall that for a short graded algebra $A = R/I$, N denotes the embedding dimension of A. Hence we may assume $A = R/I$ where R is a polynomial ring of dimension N. As before we let $B = \bar{R}/\bar{I}$ be an artinian reduction of A. (see section 1).

THEOREM 3.1. *Let A be a short graded algebra such that $e = \binom{h+t}{h} - p$ for some positive integer p. If $dim_k(I_t R_1) = Np$ then the resolution of A is right almost linear of type*

$$0 \to R(-t-h)^{b_h} \to \cdots \to R(-t-2)^{b_2} \to R(-t-1)^{b_1} \oplus R(-t)^{a_1} \to R \to A \to 0$$

where $a_1 = p$, $b_1 = \binom{h+t}{h-1} - hp$, $b_i = \binom{h}{i}e - \binom{i+t-1}{i}\binom{h+t}{h-i}$ for every $i = 2, \ldots, h$.

PROOF: Since $e = \binom{h+t}{h} - p = \binom{h+t-1}{h} + \binom{h+t-1}{t} - p$ we get $c = \binom{h+t-1}{t} - p$, hence $a_1 = p$. This means $dim_k(I_t) = p$, and since $dim_k(I_t R_1) = Np$ we get $a_2 = 0$. By lemma 1.1 this implies $a_i = 0$ for every $i \geq 2$. Since in each degree $n > t$ we have

$$dim(\bar{R}_n) + \sum_{i=1}^{h} (-1)^i \left[a_i dim(\bar{R}(-t-i+1)_n) + b_i dim(\bar{R}(-t-i)_n) \right] = 0.$$

we get $dim(\bar{R}_{t+1}) - a_1 dim(\bar{R}_1) - b_1 = 0$, hence $b_1 = \binom{h+t}{t+1} - ph$. In the same way we get $dim(\bar{R}_{t+2}) - a_1 dim(\bar{R}_2) - b_1 dim(\bar{R}_1) + b_2 = 0$, from which, by easy computation, one gets $b_2 = \binom{h}{2}e - \binom{t+1}{2}\binom{h+t}{h-2}$. By induction we get the right value of the remaining b_i's.

We remark that we can apply the above results to the following cases:
a) $e = \binom{h+t}{h} - 1$ points in generic position in $\mathbf{P}^h$
b) $e = \binom{h+t}{h} - 2$ points in uniform position in $\mathbf{P}^h$.

In fact in case a) I_t is a vector space of dimension 1, hence it is clear that the condition of the theorem is fullfilled. As for the case b) we recall that a set of e points in $\mathbf{P}^h$ is said to be in uniform position if every subset is in generic position. Now case b) follows from the following lemma a stronger version of which has been proved by Geramita and Maroscia in [GM] by completely different methods. We insert here a proof since the original one is rather complicate.

As usual we denote by $A = k[X_0, \ldots, X_n]/I$ the coordinate ring of a set of points in $\mathbf{P}^h$ and by t the initial degree of A.

LEMMA 3.2. *If $P_1, \dots, P_e$ are points in uniform position in $\mathbf{P}^h$, the forms of degree t in I cannot have a common factor (if $dim(I_t) = 1$ and $I_t = kF$ this means that F is irreducible).*

PROOF: Let F be a common factor of all the forms in I_t with $deg(F) = d$, $1 \le d \le t-1$. Let $\wp_1, \dots, \wp_e$ be the prime ideals of the poits $P_1, \dots, P_e$ respectively. Since $d < t = indeg(A)$ we must have $F \in \wp_1 \cap \dots \cap \wp_n$, $F \notin \wp_{n+1} \cup \dots \cup \wp_e$ for some n, $1 \le n < e$. Let $K = \wp_1 \cap \dots \cap \wp_n$, $J = \wp_{n+1} \cap \dots \cap \wp_e$. It is clear that $I_t = FJ_{t-d}$, hence $dim(I_t) = dim(J_{t-d})$ and we get $H_{R/J}(t-d) = \binom{h+t-d}{h} - dim(I_t)$.Since $P_{n+1}, \dots, P_e$ are in generic position we have $H_{R/J}(t-d) = min\left\{e-n, \binom{h+t-d}{h}\right\}$, hence we get $e - n = \binom{h+t-d}{h} - dim(I_t) = \binom{h+t-d}{h} - \binom{h+t}{h} + H_{R/I}(t) \le \binom{h+t-d}{h} - \binom{h+t}{h} + e$. This implies $n \ge \binom{h+t}{h} - \binom{h+t-d}{h} \ge \binom{h+d}{h}$ where the last inequality follows by an easy combinatorial argument. Thus we get $H_{R/K}(d) = min\left\{n, \binom{h+d}{h}\right\} = \binom{h+d}{h}$, a contradiction to the fact that $F \in K$.

The Cohen-Macaulay type

In this section we study the Cohen-Macaulay type of some special classes of short graded algebras. The first theorem extends and simplifies analogous results given by Brown and Roberts (see [Br2] and [R]).

THEOREM 4.1. *Let A be a short graded algebra with $e = \binom{h+t}{h} - p$ for some positive integer p. Let J be the ideal generated by the forms of degree t in I. If $h(J) > p - h + 1$ then $\beta_h = \binom{h+t-1}{t} - p$*

PROOF: Since k is an infinite field, it is clear that given a maximal regular sequence of forms of degree t in I we may complete this to a maximal regular sequence in R with linear forms $L_1, \dots, L_d$ such that $A/(L_1, \dots, L_d)A = \bar{R}/\bar{I}$ is an artinian reduction of A. Hence $h(J)$ coincides with the height of the corresponding ideal generated by the forms of degree t in $\bar{I}$. Thus we may assume $A = k[X_1, \dots, X_h]/I$ with $dim(A) = 0$. We have $b_h = c = \binom{h+t-1}{t} - p$, hence we need only to prove that $a_h = 0$, or which is the same, that if F is a form of degree $t-1$ such that $FR_1 \subseteq I$, then $F = 0$. We have $dim(I_t) = p$, hence if $p < h$ the conclusion is clear. Let $p \ge h$ and F be a form of degree $t-1$ such that $FR_1 \subseteq I$. Then $FX_1, \dots, FX_h$ are linearly independent vectors in I_t, hence we can find vectors $G_1, \dots, G_{p-h} \in I_t$ such that $(FX_1, \dots, FX_h, G_1, \dots, G_{p-h})$ is a k-vector base of I_t. This means that $J \subseteq (F, G_1, \dots, G_{p-h})$, hence $h(J) \le p - h + 1$, a contradiction.

The case of e points in generic position in $\mathbf{P}^h$ with $e = \binom{h+t}{h} - p$ and $p \leq h-1$ is the main result in [R].

On the other hand if we have $e = \binom{h+t}{h} - h$ points in uniform position, by lemma 3.2 we get $h(J) \geq 2$ and we may apply the above theorem. This is the main result in [Br2].

Let now $A = R/I$ be a Cohen-Macaulay graded algebra with codimension h, multiplicity e and initial degree t. It is clear that $e \geq \binom{h+t-1}{h}$ and we have seen in proposition 2.1 that if $e = \binom{h+t-1}{h}$ then A is short and the resolution is t-linear. In the following proposition we study the case $e = \binom{h+t-1}{h} + 1$.

PROPOSITION 4.2. *Let A be a Cohen-Macaulay graded algebra with $e = \binom{h+t-1}{h} + 1$. Then we have:*

a) A is short with $c = 1$.

b) $\beta_h \leq \binom{h+t-2}{t-1}$.

c) The following condition are equivalent:

c1) $\beta_h < \binom{h+t-2}{t-1}$

c2) $b_1 = 0$

c3) $\beta_1 = \binom{h+t-1}{t} - 1$

d) The following conditions are equivalent:

d1) $\beta_h = \binom{h+t-2}{t-1}$

d2) $b_1 = 1$

d3) $\beta_1 = \binom{h+t-1}{t}$.

PROOF: By passing to an artinian reduction of A we may assume $dim(A) = 0$. Then it is clear that A is short with $c = 1$ and $b_h = dim(A_t) = 1$. Also $(0 : A_1)_{t-1} \neq A_{t-1}$ otherwise $A_t = 0$, hence

$$\beta_h = dim(0 : A_1)_t + dim(0 : A_1)_{t-1} < dim(A_t) + dim(A_{t-1}) = 1 + \binom{h+t-2}{t-1}.$$

This proves b). The equivalence in c) has been proved in [RV] theorem 3.10. As for d), since $\beta_1 = b_1 + a_1 = b_1 + \binom{h+t-1}{t} - 1$, we get $\beta_1 = \binom{h+t-1}{t}$ if and only if $b_1 = 1$. If $b_1 = 1$, then by b) and c) we get $\beta_h = \binom{h+t-2}{t-1}$. Finally if $\beta_h = \binom{h+t-2}{t-1}$, then by b) and c) we get $b_1 > 0$ and we need only to prove that $dim(R_{t+1}/R_1I_t) \leq 1$. Now $dim(A_t) = 1$ implies $R_t = I_t + kM$ for some monomial M of degree t. Hence we may assume $M = X_1N$ for some monomial N of degree $t-1$ and we get

$$R_{t+1} = R_1 I_t + R_1 M = R_1 I_t + X_1 R_1 N \subseteq R_1 I_t + X_1(I_t + kM) = R_1 I_t + kX_1 M$$

This gives the conclusion.

The above Proposition can be applied for example in the following situation.

COROLLARY 4.3. *Let A be a Cohen-Macaulay graded algebra with $e = \binom{h+t-1}{h} + 1$. Let J be the ideal generated by the forms of degree t in I. If $h(J) = h$ then $\beta_1 = \binom{h+t-1}{t} - 1$.*

PROOF: As in theorem 4.1 we may assume $dim(A) = 0$. We have $dim(I_t) = \binom{h+t-1}{t} - 1$. This implies $R_1 I_t = R_{t+1}$, a fact proved in [RV] theorem 3.10. Hence $b_1 = 0$ and we may apply the above proposition to get the conclusion.

We remark that, again by lemma 3.2, we may apply the above corollary to the case of $e = \binom{t+1}{2} + 1$ points in uniform position in $\mathbf{P}^2$.

The last result of this section gives the Cohen-Macaulay type of some special one-dimensional short graded algebras. This extends a result in [TV].

THEOREM 4.4. *Let A be a one dimensional short graded algebra with $t = 2$. If $I \subseteq (X_i X_j)_{1 \le i < j \le h+1}$ and $X_i X_j \notin I$ for every $i \neq j$, then $\beta_h = b_h = c$.*

PROOF: We need only to prove that $a_h = dim(Tor_h^R(A,k)_{h+1}) = 0$. The crucial point is that one can compute $Tor_i^R(A,k)$ via the Koszul resolution of $k = R/(X_1, \ldots, X_{h+1})$

$$0 \to \overset{h+1}{\Lambda} V \otimes R(-h-1) \xrightarrow{\delta_{h+1}} \overset{h}{\Lambda} V \otimes R(-h) \to \cdots \to \Lambda V \otimes R(-1) \xrightarrow{\delta_1} R \to k \to 0$$

where V is a k-vector space of dimension $h+1$. Hence, in order to prove $Tor_h^R(A,k)_{h+1} = 0$, we need only to prove that the Koszul-type complex

$$\overset{h+1}{\Lambda} V \otimes A(-h-1)_{h+1} \to \overset{h}{\Lambda} V \otimes A(-h)_{h+1} \to \overset{h-1}{\Lambda} V \otimes A(-h+1)_{h+1}$$

is exact in the middle term. We may write this complex in the following way

$$\overset{h+1}{\Lambda} V \otimes k \xrightarrow{f=\delta_{h+1}} \overset{h}{\Lambda} V \otimes R_1 \xrightarrow{g} \overset{h-1}{\Lambda} V \otimes A_2$$

Now let $\xi \in Ker(g)$; this means that $\delta_h(\xi) \in \overset{h+1}{\Lambda} V \otimes I_2$ and we need to prove that $\xi \in Im(f) = Im(\delta_{h+1}) = Ker(\delta_h)$. This is equivalent to prove that if $\alpha \in \overset{h-1}{\Lambda} V \otimes I_2$ and $\alpha \in Im(\delta_h) = Ker(\delta_{h-1})$, then $\alpha = 0$. Let $e_1, \dots, e_{h+1}$ be a k-vector base of V and $\varepsilon_{ij} = e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_{h+1}$ be the corresponding vector base of $\overset{h-1}{\Lambda} V$. Then we can write $\alpha = \sum_{1 \le i < j \le h+1} \varepsilon_{ij} \otimes F_{ij}$ with $F_{ij} \in I_2$ and $\delta_{h-1}(\alpha) = 0$. This implies $F_{ij} = \lambda_{ij} X_i X_j$, otherwise if for example $F_{ij} = X_t X_s + \dots$ with $t \ne i, j$ then in $\delta_{h-1}(\alpha)$ we have a term

$$\pm e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge \hat{e}_t \wedge \dots \wedge e_{h+1} \otimes X_t^2 X_s$$

which cannot cancel out since every quadratic form in I_2 does not contain any pure square. This implies that $F_{ij} = 0$ and the conclusion follows.

COROLLARY 4.5.. *Let A be a one-dimensional short graded algebra with $e = h + 2$. If $I \subseteq (X_i X_j)_{1 \le i < j \le h+1}$ and $X_i X_j \notin I$ for every $i \ne j$, then A is Gorenstein.*

We remark that the conditions in the above theorem are verified for a set of $h + 1 < e < \binom{h+2}{2}$ points in generic position in $\mathbf{P}^h$ such that $h + 1$ of these points are not contained in an hyperplane. On the other hand it is easy to find a short graded algebra with $e = h + 2$ which is not Gorenstein.

Let $A = k[X, Y, Z]/(XZ, YZ, X^2Y - XY^2)$; then $h = 2$, $e = 4$, $I \subseteq (XY, XZ, YZ)$ but A is not Gorenstein since it is not a complete intersection.

A remark on a conjecture by Sally

Given a local Cohen-Macaulay ring (A, m) of dimension d, codimension h and multiplicity $e = h + 2$, the tangent cone $gr_m(A) = \oplus m^n/m^{n+1}$ is not necessarily Cohen-Macaulay. But Sally conjectured in [S] that in this case we always have $depth(gr_m(A)) \ge d - 1$. In the same paper she proves that if $d = 1$, then $H_A(n) \ge h + 1$, for every n, hence the Hilbert function of A does not decrease. This implies that $P_A(z) = \frac{1+hz+z^s}{1-z}$ for some $s \ge 2$. Hence we are led to consider graded algebra A, not necessarily Cohen-Macaulay, with Poincare series $P_A(z) = \left(\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^s\right) / (1 - z)^d$ for some integer $s \ge t$. This could be the right notion of short graded algebras in the non Cohen-Macaulay case.

Here we ask the following question. If (A,m) is a Cohen-Macaulay local ring of dimension d, codimension h and multiplicity $e = \binom{h+t-1}{h} + 1$ is it true that $P_A(z) = \left(\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^s\right)/(1-z)^d$ for some integer s ?

At the moment we are not able to answer this question, but in the case $t = 2$ we can show that this is equivalent to Sally's conjecture.

PROPOSITION 5.1. *Let (A,m) be a local Cohen-Macaulay ring of dimension d, codimension h and multiplicity $e = h+2$. The following conditions are equivalent.*

a) $depth(gr_m(A)) \geq d-1$.

b) $P_A(z) = \frac{1+hz+z^s}{(1-z)^d}$.

PROOF: By the result of Sally the conclusion holds in the case $d = 1$. Let $d \geq 2$ and $depth(gr_m(A)) \geq d-1$. We may assume that A/m is infinite and take $x_1, \dots, x_d$ a minimal reduction of m with x_i superficial for every i. The initial forms $x_1^*, \dots, x_d^*$ in $gr_m(A)_1$ are a system of parameters in $gr_m(A)$, hence we may assume $x_1^*, \dots, x_{d-1}^*$ form a regular sequence in $gr_m(A)$. This implies that if $B = A/(X_1, \dots, X_{d-1})$, then B is a 1-dimensional Cohen-Macaulay ring with the same codimension and multiplicity as A. Further we have $P_A(z) = P_B(z)/(1-z)^{d-1}$. By the result of Sally we get $P_B(z) = \frac{1+hz+z^s}{(1-z)}$ for some integer $s \geq 2$ and the conclusion follows. Conversely let us assume $P_A(z) = \frac{1+hz+z^s}{(1-z)^d}$ and let $B = A/(x_1, \dots, x_{d-1})$. As before B is a 1-dimensional Cohen-Macaulay ring with the same codimension and multiplicity as A. Since $d \geq 2$ we get $e_1(A) = e_1(B)$, where for a local ring S of dimension d and Poincare series $P_S(z) = \sum_{i=0}^{s} a_i z^i/(1-z)^d$, we define $e_1(S) = \sum_{j=1}^{s} j a_j$ (see [EV]). By the result of Sally we have $P_B(z) = \frac{1+hz+z^t}{1-z}$, hence $e_1(B) = h+t = e_1(A) = h+s$. This implies $s = t$ and $P_A(z) = P_B(z)/(1-z)^{d-1}$. Hence $x_1^*, \dots, x_{d-1}^*$ is a regular sequence in $gr_m(A)$ and the conclusion follows.

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