

Geometry of right exact 'spaces'.

Alexander Rosenberg

Introduction

There are two parts of noncommutative algebraic geometry – pseudo-geometry and geometry. Pseudo-geometry stretches our language identifying (different types of) categories, or algebras, or sheaves of sets on certain (pre)sites with 'spaces'; and geometry tries to associate with these 'spaces' some *geometric spaces* which are topological spaces with additional structures. In the commutative case, the additional structure is a *structure sheaf* – a sheaf of rings whose stalks at points are local rings.

The main conceptual achievement of the geometric part of non-commutative algebraic geometry was the discovery (in the middle of eighties) of the spectrum $\mathbf{Spec}(-)$ of 'spaces' represented by abelian categories. Thanks to remarkable properties of this spectrum, a 'space' X represented by a svelte abelian category C_X is naturally realized as a stack of categories on the topological space $\mathbf{Spec}(X)$ whose fibers at points are "local categories", which is a non-commutative analog of a locally ringed topological space [R5]. If C_X is the category of quasi-coherent sheaves on a quasi-compact quasi-separated commutative scheme, then the spectrum $\mathbf{Spec}(X)$ is homeomorphic to the underlying space of the scheme and the scheme itself can be reconstructed uniquely up to isomorphism from the geometric realization of the 'space' X (hence from the category C_X). This spectrum is also used for the reconstruction of non-quasi-compact commutative schemes [R4].

On the other hand, the key notions of noncommutative algebraic geometry, like schemes, smooth morphisms etc. are naturally defined in the context of arbitrary categories, without any abelian or even additivity hypothesis [R2], [R3], [KR2]. Several spectra of ('spaces' represented by) arbitrary categories were found some time ago [R8]. But, none of them turns into the spectrum $\mathbf{Spec}(X)$ when C_X is an arbitrary abelian category.

In this work, we extend the spectrum $\mathbf{Spec}(X)$ to 'spaces' represented by svelte *right exact* categories with *weak equivalences*. We call them *right exact 'spaces' with weak equivalences*, or, simply, *right exact 'spaces'*. By definition, a right exact category with weak equivalences is a triple $(C_X, \mathfrak{E}_X, \mathcal{W}_X)$, where C_X is a category, \mathfrak{E}_X is a class of *strict epimorphisms* which forms a pretopology on C_X and \mathcal{W}_X is a subpretopology of \mathfrak{E}_X . In other words, the class \mathfrak{E}_X and its subclass \mathcal{W}_X contain all isomorphisms of C_X and are closed under composition and pull-backs. Every exact category (in particular, any abelian category) is identified with a right exact category with trivial (that is consisting only of isomorphisms) class of weak equivalences, whose deflations are admissible epimorphisms. Right exact categories with the trivial class of weak equivalences came into life as a (half of the) base for a version of homological algebra developed in [R9] (and outlined in [R11]).

One of the motivations behind choosing right exact 'spaces' as a setting for spectral theory comes from fact that they form a natural (although not the most general) domain for K-theory. Grothendieck introduced K-theory for studying cycles on commutative schemes. Having K-groups of right exact 'spaces' already defined [R9], [R11], the next question is

what are cycles in this case. Other motivations are of a more pragmatical nature. Abelian categories are too restrictive even for commutative algebraic geometry. Already extending the spectral picture to 'spaces' represented by exact categories (which are a special case of right exact categories) considerably increases the area of potential applications, because, for instance, the category of Banach vector spaces has a natural exact structure. This exact structure is the *finest right exact structure* which exists on any category.

Some important fragments of spectral theory of 'spaces' represented by abelian categories [R5] served as a guide for this work. This spectral theory is based on the notions of *topologizing*, *thick* and *Serre* subcategories and their basic properties. A starting observation was that a subcategory of an abelian category can be replaced by the class of epimorphisms whose kernels are objects of this subcategory. So that the idea was to describe classes of epimorphisms corresponding to topologizing, thick and Serre subcategories, and then use this description for right exact categories. The realization of this program turned out to be way more subtle than it looked in the beginning.

The paper is organized as follows. We start, in Section 1, with preliminaries (borrowed from [R11]) on kernels of arrows in categories with initial objects and then continue with right exact categories with weak equivalences. An important for this work new notion which appears here is that of *stable* class of deflations. In Section 2, we introduce topologizing, thick and Serre systems of deflations and establish their main properties which in abelian case turn into the known properties of respectively topologizing, thick and Serre subcategories [R, Ch.3]. In Section 3, we define the spectra of a right exact 'space' $(X, \bar{\mathfrak{C}}_X)$ with weak equivalences related with topologizing, thick and Serre systems of deflations. In particular, we define the spectrum $\mathbf{Spec}_t(X, \bar{\mathfrak{C}}_X)$ which, in the case of abelian category C_X with the standard exact structure, is naturally isomorphic to the spectrum $\mathbf{Spec}(X)$.

In Section 4, we sketch an alternative version of spectral theory based on the notions of *semitopologizing* and *strongly closed* (– a replacement of Serre) systems. This theory requires less conditions on the right exact categories and, therefore, is much more universal. In general, the spectra of 'spaces' differ from those defined in Section 3. Both spectral theories coincide in the abelian case. In Section 5, we introduce *strongly 'exact'* functor and, in particular, *strongly 'exact'* localizations, and establish their basic properties. In Section 6, we study functorial properties of the spectra with respect to strongly 'exact' localizations. We establish the so called *locality* theorems for the spectrum $\mathbf{Spec}_t(X, \bar{\mathfrak{C}}_X)$ and its semitopological analog $\mathbf{Spec}_{st}(X, \bar{\mathfrak{C}}_X)$, which is one of the most important properties of these spectra. In Section 7, the main notions and facts of the work are specialized for right exact categories with initial objects. In particular, we obtain a spectral theory of 'spaces' represented by exact categories and, in the abelian case, recover main constructions and facts of [R5]. We conclude with a couple of examples of illustrative nature.

Appendix 1 contains some useful complementary facts about kernels of morphisms.

In Appendix 2, we remind, for the reader convenience, the main notions of spectral theory in the abelian case. It might be useful to look into this appendix prior to the reading the main bulk of this work, that is before starting Section 2.

This text was written at the Max Planck Institut für Mathematik in Bonn in 2008. I am grateful to the Institute for hospitality and excellent working conditions.

1. Preliminaries.

1. Kernels of arrows.

Let C_X be a category with an initial object, x . For a morphism $M \xrightarrow{f} N$ we define the *kernel of f* as the upper horizontal arrow in a cartesian square

$$\begin{array}{ccc} \text{Ker}(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ f' \downarrow & \text{cart} & \downarrow f \\ x & \longrightarrow & N \end{array}$$

when the latter exists.

Cokernels of morphisms are defined dually, via a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{c}(f)} & \text{Cok}(f) \\ f \uparrow & \text{cocart} & \uparrow f' \\ M & \longrightarrow & y \end{array}$$

where y is a final object of C_X .

If C_X is a pointed category (i.e. its initial objects are final), then the notion of the kernel is equivalent to the usual one: the diagram $\text{Ker}(f) \xrightarrow{\mathfrak{k}(f)} M \xrightarrow[f]{f} N$ is exact.

Dually, the cokernel of f makes the diagram $M \xrightarrow[f]{f} N \xrightarrow[\mathfrak{c}(f)]{} \text{Cok}(f)$ exact.

1.1. Lemma. *Let C_X be a category with an initial object x .*

- (a) *Let a morphism $M \xrightarrow{f} N$ of C_X have a kernel. The canonical morphism $\text{Ker}(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism, if the unique arrow $x \xrightarrow{i_N} N$ is a monomorphism.*
 (b) *If $M \xrightarrow{f} N$ is a monomorphism, then $x \xrightarrow{i_M} M$ is the kernel of f .*

Proof. The pull-backs of monomorphisms are monomorphisms. ■

1.2. Corollary. *Let C_X be a category with an initial object x . The following conditions are equivalent:*

- (a) *If $M \xrightarrow{f} N$ has a kernel, then the canonical arrow $\text{Ker}(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism.*
 (b) *The unique arrow $x \xrightarrow{i_M} M$ is a monomorphism for any $M \in \text{Ob}C_X$.*

Proof. (a) \Rightarrow (b), because, by 1.1(b), the unique morphism $x \xrightarrow{i_M} M$ is the kernel of the identical morphism $M \rightarrow M$. The implication (b) \Rightarrow (a) follows from 1.1(a). ■

1.3. Note. The converse assertion is not true in general: a morphism might have a trivial kernel without being a monomorphism. It is easy to produce an example in the category of pointed sets.

1.4. Examples.

1.4.1. Kernels of morphisms of unital k -algebras. Let C_X be the category Alg_k of associative unital k -algebras. The category C_X has an initial object – the k -algebra k . For any k -algebra morphism $A \xrightarrow{\varphi} B$, we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \mathfrak{k}(\varphi) \uparrow & & \uparrow \\ k \oplus K(\varphi) & \xrightarrow{\epsilon(\varphi)} & k \end{array}$$

where $K(\varphi)$ denote the kernel of the morphism φ in the category of non-unital k -algebras and the morphism $\mathfrak{k}(\varphi)$ is determined by the inclusion $K(\varphi) \rightarrow A$ and the k -algebra structure $k \rightarrow A$. This square is cartesian. In fact, if

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \gamma \uparrow & & \uparrow \\ C & \xrightarrow{\psi} & k \end{array}$$

is a commutative square of k -algebra morphisms, then C is an augmented algebra: $C = k \oplus K(\psi)$. Since the restriction of $\varphi \circ \gamma$ to $K(\psi)$ is zero, it factors uniquely through $K(\varphi)$. Therefore, there is a unique k -algebra morphism $C = k \oplus K(\psi) \xrightarrow{\beta} Ker(\varphi) = k \oplus K(\varphi)$ such that $\gamma = \mathfrak{k}(\varphi) \circ \beta$ and $\psi = \epsilon(\varphi) \circ \beta$.

This shows that each (unital) k -algebra morphism $A \xrightarrow{\varphi} B$ has a canonical kernel $Ker(\varphi)$ equal to the augmented k -algebra corresponding to the ideal $K(\varphi)$.

It follows from the description of the kernel $Ker(\varphi) \xrightarrow{\mathfrak{k}(\varphi)} A$ that it is a monomorphism iff the k -algebra structure $k \rightarrow A$ is a monomorphism.

Notice that cokernels of morphisms are not defined in Alg_k , because this category does not have final objects.

1.4.2. Kernels and cokernels of maps of sets. Since the only initial object of the category $Sets$ is the empty set \emptyset and there are no morphisms from a non-empty set to \emptyset , the kernel of any map $X \rightarrow Y$ is $\emptyset \rightarrow X$. The cokernel of a map $X \xrightarrow{f} Y$ is the projection $Y \xrightarrow{\mathfrak{c}(f)} Y/f(X)$, where $Y/f(X)$ is the set obtained from Y by the contraction of $f(X)$ into a point. So that $\mathfrak{c}(f)$ is an isomorphism iff either $X = \emptyset$, or $f(X)$ is a one-point set.

1.4.3. Presheaves of sets. Let C_X be a svelte category and C_X^\wedge the category of non-trivial presheaves of sets on C_X (that is we exclude the *trivial* presheaf which assigns to every object of C_X the empty set). The category C_X^\wedge has a final object which is the constant presheaf with values in a one-element set. If C_X has a final object, y , then $\hat{y} = C_X(-, y)$ is a final object of the category C_X^\wedge . Since C_X^\wedge has small colimits, it has cokernels of arbitrary morphisms which are computed object-wise, that is using 1.4.2.

If the category C_X has an initial object, x , then the presheaf $\hat{x} = C_X(-, x)$ is an initial object of the category C_X^\wedge . In this case, the category C_X^\wedge has kernels of all its morphisms (because C_X^\wedge has limits) and the Yoneda functor $C_X \xrightarrow{h} C_X^\wedge$ preserves kernels.

Notice that the initial object of C_X^\wedge is not isomorphic to its final object unless the category C_X is pointed, i.e. initial objects of C_X are its final objects.

1.5. Some properties of kernels. See Appendix 1.

1.6. A construction. For a class of arrows \mathcal{S} of a category C_X , we denote by \mathcal{S}^\wedge the class of all arrows \mathfrak{s} of C_X such that some pull-back of \mathfrak{s} belongs to \mathcal{S} .

1.6.1. Proposition. Fix a category C_X .

(a) $\bigcup_{i \in \mathcal{J}} \mathcal{T}_i^\wedge = \left(\bigcup_{i \in \mathcal{J}} \mathcal{T}_i \right)^\wedge$ for any set $\{\mathcal{T}_i \mid i \in \mathcal{J}\}$ of classes of arrows of C_X .

(b) $\mathcal{S} \subseteq \mathcal{S}^\wedge$ and $\mathcal{S}^\wedge = (\mathcal{S}^\wedge)^\wedge$ for any class of arrows \mathcal{S} of the category C_X .

(c) Suppose that the category C_X is filtered in the sense that any diagram of the form $L \longrightarrow M \longleftarrow N$ in the category C_X can be completed to a commutative square

$$\begin{array}{ccc} \tilde{L} & \longrightarrow & L \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \end{array}$$

(for instance, C_X is a category with fibred products, or C_X has initial objects).

(i) Let \mathcal{S} be a class of arrows of C_X stable under arbitrary pull-backs. Then the class \mathcal{S}^\wedge is stable under pull-backs.

(ii) Suppose that \mathcal{S} is stable under pull-backs and satisfies the following condition:

(#) If in the commutative diagram

$$\begin{array}{ccccc} \tilde{\mathcal{L}} & \xrightarrow{\tilde{\mathfrak{s}}} & \mathcal{K} & & \\ \text{j}' \downarrow & \text{cart} & \downarrow \text{j} & & \\ \mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{K} \end{array}$$

with cartesian square $\mathfrak{t} \circ \text{j} = \text{id}_{\mathcal{K}}$ and morphisms \mathfrak{t} and $\tilde{\mathfrak{s}}$ belong to \mathcal{S} , then $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$.

Then the class \mathcal{S}^\wedge is multiplicative (that is closed under composition).

Proof. (a)&(b). The equality $\bigcup_{i \in \mathcal{J}} \mathcal{T}_i^\wedge = \left(\bigcup_{i \in \mathcal{J}} \mathcal{T}_i \right)^\wedge$, the inclusion $\mathcal{S} \subseteq \mathcal{S}^\wedge$ and the equality $\mathcal{S}^\wedge = (\mathcal{S}^\wedge)^\wedge$ are obvious.

(i) Let $L \xrightarrow{\mathfrak{s}} M$ be a morphism of \mathcal{S}^\wedge and

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\tilde{\mathfrak{s}}} & \tilde{M} \\ \text{f}' \downarrow & \text{cart} & \downarrow \text{f} \\ L & \xrightarrow{\mathfrak{s}} & M \end{array}$$

a cartesian square. Since $\mathfrak{s} \in \mathcal{S}^\wedge$, there exists a cartesian square

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\mathfrak{t}} & \mathfrak{M} \\ \phi' \downarrow & \text{cart} & \downarrow \phi \\ L & \xrightarrow{\mathfrak{s}} & M \end{array}$$

with $\mathfrak{t} \in \mathcal{S}$. By condition, there exists a commutative square

$$\begin{array}{ccc} \widetilde{\mathfrak{M}} & \xrightarrow{\bar{\phi}} & \widetilde{M} \\ f'' \downarrow & & \downarrow f \\ \mathfrak{M} & \xrightarrow{\phi} & M \end{array}$$

Set $\varphi = f \circ \bar{\phi} = \phi \circ f'$ and consider the cartesian square

$$\begin{array}{ccc} \widetilde{\mathfrak{L}} & \xrightarrow{\gamma} & \widetilde{\mathfrak{M}} \\ \varphi' \downarrow & \text{cart} & \downarrow \varphi \\ L & \xrightarrow{\mathfrak{s}} & M \end{array} \quad (1)$$

The equalities $f \circ \bar{\phi} = \varphi = \phi \circ f'$ imply two decompositions of the square (1),

$$\begin{array}{ccc} \widetilde{\mathfrak{L}} & \xrightarrow{\gamma} & \widetilde{\mathfrak{M}} \\ \bar{\phi}' \downarrow & \text{cart} & \downarrow \bar{\phi} \\ \widetilde{L} & \xrightarrow{\bar{\mathfrak{s}}} & \widetilde{M} \\ f' \downarrow & \text{cart} & \downarrow f \\ L & \xrightarrow{\mathfrak{s}} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{\mathfrak{L}} & \xrightarrow{\gamma} & \widetilde{\mathfrak{M}} \\ \bar{f} \downarrow & \text{cart} & \downarrow f'' \\ \mathfrak{L} & \xrightarrow{\mathfrak{t}} & \mathfrak{M} \\ \bar{\phi} \downarrow & \text{cart} & \downarrow \phi \\ L & \xrightarrow{\mathfrak{s}} & M \end{array} \quad (2)$$

Since the class \mathcal{S} is stable under pull-backs and $\mathfrak{L} \xrightarrow{\mathfrak{t}} \mathfrak{M}$ belongs to \mathcal{S} , it follows from the upper cartesian square of the right diagram (2) that the morphism $\widetilde{\mathfrak{L}} \xrightarrow{\gamma} \widetilde{\mathfrak{M}}$ belongs to \mathcal{S} . The upper cartesian square of the left diagram (2) shows that the pull-back $\widetilde{L} \xrightarrow{\bar{\mathfrak{s}}} \widetilde{M}$ of the morphism $L \xrightarrow{\mathfrak{s}} M$ belongs to the class \mathcal{S}^\wedge .

(ii) Let $L \xrightarrow{\mathfrak{s}} M$ and $M \xrightarrow{\mathfrak{t}} N$ be morphisms of \mathcal{S}^\wedge ; i.e. there exist cartesian squares

$$\begin{array}{ccc} \widetilde{L} & \xrightarrow{\bar{\mathfrak{s}}} & \widetilde{M} \\ f' \downarrow & \text{cart} & \downarrow f \\ L & \xrightarrow{\mathfrak{s}} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{\bar{\mathfrak{t}}} & \mathcal{N} \\ g' \downarrow & \text{cart} & \downarrow g \\ M & \xrightarrow{\mathfrak{t}} & N \end{array}$$

whose upper horizontal arrows belong to \mathcal{S} . By hypothesis, there exists a commutative square

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\bar{f}} & \mathcal{M} \\ g'' \downarrow & & \downarrow g' \\ \widetilde{M} & \xrightarrow{f} & M \end{array}$$

which gives rise to a pair of diagrams

$$\begin{array}{ccc}
 \widetilde{\mathfrak{M}} & \xrightarrow{\widetilde{s}} & \mathfrak{M} \\
 \bar{\phi}' \downarrow & \text{cart} & \downarrow g'' \\
 \widetilde{L} & \xrightarrow{\widetilde{s}} & \widetilde{M} \\
 f' \downarrow & \text{cart} & \downarrow f \\
 L & \xrightarrow{s} & M
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \mathfrak{L} & \xrightarrow{\gamma} & \mathfrak{M} \\
 \varphi' \downarrow & \text{cart} & \downarrow \varphi \\
 \mathcal{M} & \xrightarrow{\bar{t}} & \mathcal{N} \\
 g' \downarrow & \text{cart} & \downarrow g \\
 M & \xrightarrow{t} & N
 \end{array}
 \quad (3)$$

with cartesian squares, where $\varphi = \bar{t} \circ \bar{f}$. The latter equality implies the existence of a unique arrow $\mathfrak{M} \xrightarrow{j} \widetilde{\mathfrak{M}}$ such that $\gamma \circ j = id_{\mathfrak{M}}$.

Notice that in the diagram

$$\begin{array}{ccccc}
 \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{u}} & \mathfrak{M} & & \\
 \bar{j} \downarrow & \text{cart} & \downarrow j & & \\
 \widetilde{\mathfrak{L}} & \xrightarrow{u} & \mathfrak{L} & \xrightarrow{\gamma} & \mathfrak{M} \\
 \bar{\phi} \downarrow & \text{cart} & \downarrow \phi & \text{cart} & \downarrow \psi = g \circ \varphi \\
 L & \xrightarrow{s} & M & \xrightarrow{t} & N
 \end{array}
 \quad (4)$$

built of cartesian squares, we can take $\widetilde{\mathcal{M}} = \widetilde{\mathfrak{M}}$ and $\widetilde{u} = \widetilde{s}$.

In fact, it follows from the equality $\phi \circ j = f \circ g''$, universal property of cartesian squares (and the fact that composition of cartesian squares is a cartesian square) that the cartesian square

$$\begin{array}{ccc}
 \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{u}} & \mathfrak{M} \\
 \bar{\phi} \circ \bar{j} \downarrow & \text{cart} & \downarrow \phi \circ j \\
 L & \xrightarrow{s} & M
 \end{array}$$

is isomorphic to the cartesian square

$$\begin{array}{ccc}
 \widetilde{\mathfrak{M}} & \xrightarrow{\widetilde{s}} & \mathfrak{M} \\
 f' \circ \bar{\phi}' \downarrow & \text{cart} & \downarrow f \circ g'' \\
 L & \xrightarrow{s} & M
 \end{array}$$

In particular, $\mathcal{M} \xrightarrow{\widetilde{u}} \mathfrak{M}$ is an arrow of \mathcal{S} . Applying the condition (#) to the subdiagram

$$\begin{array}{ccccc}
 \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{u}} & \mathfrak{M} & & \\
 \bar{j} \downarrow & \text{cart} & \downarrow j & & \\
 \widetilde{\mathfrak{L}} & \xrightarrow{u} & \mathfrak{L} & \xrightarrow{\gamma} & \mathfrak{M}
 \end{array}$$

of the diagram (3), we obtain that the composition $\tilde{\mathcal{L}} \xrightarrow{\gamma_{\text{ou}}} \mathfrak{M}$ belongs to \mathcal{S} . Since the square

$$\begin{array}{ccc} \tilde{\mathcal{L}} & \xrightarrow{\gamma_{\text{ou}}} & \mathfrak{M} \\ \bar{\phi} \downarrow & \text{cart} & \downarrow \psi \\ L & \xrightarrow{\text{tos}} & N \end{array}$$

(derived from the lower two squares of (3)) is cartesian, this means that $\text{t} \circ \mathfrak{s} \in \mathcal{S}^{\bar{\wedge}}$. ■

1.6.2. Remarks. (a) It follows that a class of arrows \mathcal{S} satisfying the condition (#) of 1.6.1 is multiplicative, that is closed under composition.

(b) It follows from the argument of 1.6.1(ii) that it suffices to require in the condition (#) that the object \mathcal{K} runs through a cofinal class of objects \mathfrak{K} . The word *cofinal* means that for any $M \in \text{Ob}C_X$, there is an arrow $\mathcal{K} \rightarrow M$ with $\mathcal{K} \in \mathfrak{K}$.

Thus, if C_X is a category with an initial object, \mathfrak{x} , then the condition (#) of 1.6.1 can be replaced by the following condition:

(#') If a morphism $\mathcal{L} \xrightarrow{\mathfrak{s}} \mathcal{M}$ is such that canonical arrow $\text{Ker}(\mathfrak{s}) \rightarrow \mathfrak{x}$ belongs to \mathcal{S} and $\mathcal{M} \xrightarrow{\text{t}} \mathfrak{x}$ is from \mathcal{S} , then the composition $\mathcal{M} \xrightarrow{\text{tos}} \mathfrak{x}$ is a morphism of \mathcal{S} .

Notice that if \mathcal{S} is a class of arrows of C_X stable under pull-backs along morphisms from initial objects (in particular, morphisms of \mathcal{S} have kernels), then the class $\mathcal{S}^{\bar{\wedge}}$ consists of all arrows $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{L}$ such that the canonical morphism $\text{Ker}(\mathfrak{s}) \rightarrow \mathfrak{x}$ (– the pull-back of \mathfrak{s} along the unique arrow $\mathfrak{x} \rightarrow \mathcal{L}$) belongs to \mathcal{S} .

(c) Let \mathfrak{K} be a cofinal class of objects of the category C_X . Suppose that a functor $C_X \xrightarrow{F} C_Y$ is such that pull-backs of retracts $\mathcal{K} \rightarrow \mathcal{M}$ from

$$\Sigma_F \stackrel{\text{def}}{=} \{\mathfrak{s} \in \text{Hom}C_X \mid F(\mathfrak{s}) \in \text{Iso}(C_Y)\}$$

with $\mathcal{K} \in \mathfrak{K}$ along arrows from $\Sigma_F^{\bar{\wedge}}$ belong to Σ_F . Then the system Σ_F satisfies the condition (#).

In fact, if the condition above holds and

$$\begin{array}{ccccc} \tilde{\mathcal{L}} & \xrightarrow{\tilde{\mathfrak{s}}} & \mathcal{K} & & \\ j' \downarrow & \text{cart} & \downarrow j & & \\ \mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} & \xrightarrow{\text{t}} & \mathcal{K} \end{array}$$

is a commutative diagram with cartesian square such that $\text{t} \circ j = \text{id}_{\mathcal{K}}$ and morphisms t and $\tilde{\mathfrak{s}}$ belong to Σ_F , then j is a retract from Σ_F and, therefore, both vertical arrows and upper horizontal arrow of the diagram

$$\begin{array}{ccc} F(\tilde{\mathcal{L}}) & \xrightarrow{F(\tilde{\mathfrak{s}})} & F(\mathcal{K}) \\ F(j') \downarrow & & \downarrow F(j) \\ \mathcal{L} & \xrightarrow{F(\mathfrak{s})} & F(\mathcal{M}) \end{array}$$

are invertible. Therefore $F(\mathfrak{s})$ is invertible, i.e. $\mathfrak{s} \in \Sigma_F$. In particular, $\mathfrak{t} \circ \mathfrak{s} \in \Sigma_F$.

(c') Suppose that the category C_X has an initial object \mathfrak{x} and a functor $C_X \xrightarrow{F} C_Y$ preserves pull-backs along the morphisms from \mathfrak{x} (for instance, C_Y has initial objects too and F preserves kernels of arrows). Then the system Σ_F satisfies the condition ($\#'$) above.

If the categories C_X , C_Y and the functor F are additive and all morphisms of C_X have kernels, then

$$[F \text{ preserves pull-backs of retracts}] \Leftrightarrow [F \text{ preserves kernels}] \Leftrightarrow [F \text{ is left exact}].$$

1.6.3. Morphisms with a trivial kernel. Let $Iso(C_X)$ denote the class of all isomorphisms of a category C_X . We call elements of $Iso(C_X)^\wedge$ *morphisms with trivial kernel*. It follows from the observation 1.6.2(b) that if C_X is a category with initial objects, then $Iso(C_X)^\wedge = \{\mathfrak{s} \in Hom C_X \mid Ker(\mathfrak{s}) \text{ is an initial object}\}$. If the category C_X is additive, then the class $Iso(C_X)^\wedge$ coincides with the class of all monomorphisms of the category C_X . There are many non-additive categories having this property. One of them is the category Alg_k of unital associative k -algebras (see 1.4.1).

1.7. Proposition. *Suppose that C_X is a filtered category, i.e. any diagram of the form $L \rightarrow M \leftarrow N$ in the category C_X can be completed to a commutative square (say, C_X has initial objects). Then*

$$\left(\bigcap_{i \in J} \mathcal{S}_i\right)^\wedge = \bigcap_{i \in J} \mathcal{S}_i^\wedge$$

for any finite set $\{\mathcal{S}_i \mid i \in J\}$ of classes of arrows of C_X which are stable under pull-backs.

Proof. Evidently, $\left(\bigcap_{i \in J} \mathcal{S}_i\right)^\wedge \subseteq \bigcap_{i \in J} \mathcal{S}_i^\wedge$ for any set $\{\mathcal{S}_i \mid i \in J\}$ of classes of arrows of the category C_X . The claim is that the inverse inclusion holds when J is finite and each \mathcal{S}_i is stable under pull-backs.

In fact, let $J = \{1, 2, \dots, n\}$, and let $\mathfrak{s} \in \bigcap_{i \in J} \mathcal{S}_i^\wedge$. Then a pull-back, \mathfrak{s}_1 , of \mathfrak{s} belongs to \mathcal{S}_1 . Since, by 1.6.1, each of the classes \mathcal{S}_i^\wedge is closed under pull-backs. So that \mathfrak{s}_1 is an element of $\mathcal{S}_1 \cap \left(\bigcap_{2 \leq i \leq n} \mathcal{S}_i^\wedge\right)$. By a standard induction argument, $\left(\bigcap_{2 \leq i \leq n} \mathcal{S}_i\right)^\wedge = \bigcap_{i \in J} \mathcal{S}_i^\wedge$.

Therefore, there is a pull-back, $\tilde{\mathfrak{s}}_1$ of \mathfrak{s}_1 which belongs to $\bigcap_{2 \leq i \leq n} \mathcal{S}_i$. By hypothesis, \mathcal{S}_1

is stable under pull-backs, in particular, $\tilde{\mathfrak{s}}_1 \in \bigcap_{1 \leq i \leq n} \mathcal{S}_i$. Since $\tilde{\mathfrak{s}}_1$ is a pull-back of \mathfrak{s} , this

proves the desired inverse inclusion $\left(\bigcap_{i \in J} \mathcal{S}_i\right)^\wedge \supseteq \bigcap_{i \in J} \mathcal{S}_i^\wedge$. ■

1.8. The dual construction. For a class \mathcal{S} of arrows of a category C_X , we denote by \mathcal{S}^\vee the class of all arrows \mathfrak{s} of C_X such that some push-forward of \mathfrak{s} belongs to \mathcal{S} . The dual versions of the facts above are left to the reader.

We shall call the arrows of $Iso(C_X)^\vee$ *morphisms with trivial cokernel*. If the category C_X is additive, $Iso(C_X)^\vee$ coincides with the class of all epimorphisms of C_X (see 1.6.3). In this case, the intersection $Iso(C_X)^\wedge \cap Iso(C_X)^\vee$ consists of all bimorphisms of C_X .

1.9. Right exact 'spaces' with weak equivalences.

Right exact categories and 'spaces' (they represent) were introduced in [R9]. Here we need a slightly more flexible structure – right exact categories with *weak equivalences*.

1.9.1. Right exact categories and 'exact' functors. A *right exact category* is a pair (C_X, \mathfrak{E}_X) , where C_X is a category and \mathfrak{E}_X is a Grothendieck pretopology on C_X whose covers are strict epimorphisms called (after P. Gabriel) *deflations*.

1.9.2. Right exact categories and 'spaces' with weak equivalences. We call this way triples $(C_X, \mathfrak{E}_X, \mathcal{W}_X)$, where (C_X, \mathfrak{E}_X) is a right exact category and (C_X, \mathcal{W}_X) is an exact subcategory of (C_X, \mathfrak{E}_X) (i.e. \mathcal{W}_X is a subpretopology of \mathfrak{E}_X). We call arrows of \mathcal{W}_X *weak equivalences*. For convenience, we denote the pair $(\mathfrak{E}_X, \mathcal{W}_X)$ by $\bar{\mathfrak{E}}_X$ and write $(C_X, \bar{\mathfrak{E}}_X)$ instead of $(C_X, \mathfrak{E}_X, \mathcal{W}_X)$. An 'exact' functor from $(C_X, \bar{\mathfrak{E}}_X)$ to $(C_Y, \bar{\mathfrak{E}}_Y)$ is an 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{F} (C_Y, \mathfrak{E}_Y)$ such that $F(\mathcal{W}_X) \subseteq \mathcal{W}_Y$.

1.9.3. Examples. Fix a right exact category (C_X, \mathfrak{E}_X) .

(a) The smallest class of weak equivalences is $\mathcal{W}_X = Iso(C_X)$ – the class of all isomorphisms of the category C_X .

(b) An example essential for this work is

$$\mathcal{W}_X = \mathfrak{E}_X^{\otimes} \stackrel{\text{def}}{=} Iso(C_X)^{\bar{\wedge}} \cap \mathfrak{E}_X.$$

In other words, the class \mathcal{W}_X consists of deflations with trivial kernels (see 1.6.3). If the category C_X is additive, then, as it is observed in 1.6.3, the class $Iso(C_X)^{\bar{\wedge}}$ consists of all monomorphisms of C_X . Since deflations are strict epimorphisms, it follows that $\mathfrak{E}_X^{\otimes} = Iso(C_X)$; i.e. weak equivalences are isomorphisms in this case. There are many natural examples of non-additive categories having this property.

(c) One of them is the category Alg_k of unital associative k -algebras with strict epimorphisms as deflations. In fact, a k -algebra morphism has a trivial kernel iff its kernel as a k -module morphism is trivial (see 1.4.1). So that algebra morphisms with trivial kernels are monomorphisms.

1.10. Stable classes of deflations. Fix a right exact category (C_X, \mathfrak{E}_X) . We call a class of deflations \mathcal{S} of (C_X, \mathfrak{E}_X) *stable* if it is closed under pull-backs and $\mathcal{S} = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$.

1.10.1. Proposition. *Let (C_X, \mathfrak{E}_X) be a right exact category such that the category C_X is filtered, i.e. any pair of arrows $L \rightarrow M \leftarrow N$ can be completed to a commutative square (for instance, C_X has initial objects, or it has fiber products). Then,*

(a) *For any class \mathcal{S} of arrows of the category C_X which is closed under pull-backs, the intersection $\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$ is a stable class.*

(b) *The union and the intersection of any set of stable classes are stable classes.*

Proof. (a) It follows from 1.7 and the equality $\mathcal{S}^{\bar{\wedge}} = (\mathcal{S}^{\bar{\wedge}})^{\bar{\wedge}}$ (see 1.6.1) that

$$\mathfrak{E}_X \cap (\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}})^{\bar{\wedge}} = \mathfrak{E}_X \cap (\mathfrak{E}_X^{\bar{\wedge}} \cap (\mathcal{S}^{\bar{\wedge}})^{\bar{\wedge}}) = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}.$$

Since the category C_X is filtered, by 1.6.1, the class $\mathcal{S}^{\bar{\wedge}}$ inherits from \mathcal{S} the stability under pull-backs. Therefore, the intersection $\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$ is stable under pull-backs too.

(b1) By 1.6.1(a), $(\bigcup_{i \in J} \mathcal{S}_i)^\wedge = \bigcup_{i \in J} \mathcal{S}_i^\wedge$ for any $\{\mathcal{S}_i \mid i \in J\}$ of classes of arrows of the category C_X . Therefore, if all classes \mathcal{S}_i are stable, then

$$\bigcup_{i \in J} \mathcal{S}_i = \bigcup_{i \in J} (\mathfrak{E}_X \cap \mathcal{S}_i^\wedge) = \mathfrak{E}_X \cap (\bigcup_{i \in J} \mathcal{S}_i)^\wedge.$$

(b2) For any set $\{\mathcal{S}_i \mid i \in J\}$ of classes of arrows of the category C_X , there is an obvious inclusion $(\bigcap_{i \in J} \mathcal{S}_i)^\wedge \subseteq \bigcap_{i \in J} \mathcal{S}_i^\wedge$. If $\{\mathcal{S}_i \mid i \in J\}$ is a family of stable classes of deflations, then the inclusion above implies that

$$\bigcap_{i \in J} \mathcal{S}_i \subseteq \mathfrak{E}_X \cap (\bigcap_{i \in J} \mathcal{S}_i)^\wedge \subseteq \mathfrak{E}_X \cap (\bigcap_{i \in J} \mathcal{S}_i^\wedge) = \bigcap_{i \in J} (\mathfrak{E}_X \cap \mathcal{S}_i^\wedge) = \bigcap_{i \in J} \mathcal{S}_i.$$

In particular, $\bigcap_{i \in J} \mathcal{S}_i = \mathfrak{E}_X \cap (\bigcap_{i \in J} \mathcal{S}_i)^\wedge$, which means, by definition, that the intersection

$\bigcap_{i \in J} \mathcal{S}_i$ is a stable class. ■

1.10.2. Proposition. *Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ be a right exact category with a class of weak equivalences containing all deflations with trivial kernels and a left divisible class of deflations: $\mathfrak{s} \circ \mathfrak{t} \in \mathfrak{E}_X \ni \mathfrak{t}$ implies that $\mathfrak{s} \in \mathfrak{E}_X$. Then every class \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ which is closed under pull-backs and push-forwards (we assume that arbitrary push-forwards of arrows of \mathcal{S} exist) and coincides with $\mathcal{W}_X \circ \mathcal{S}$ is stable.*

Proof. Let $\mathcal{L} \xrightarrow{u} \mathcal{M}$ be an arrow of $\mathcal{S}^\wedge \cap \mathfrak{E}_X$; that is there exists a cartesian square

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathfrak{M} \\ \mathfrak{f}' \downarrow & \text{cart} & \downarrow \mathfrak{f} \\ \mathcal{L} & \xrightarrow{u} & \mathcal{M} \end{array} \quad (5)$$

whose upper horizontal arrow belongs to \mathcal{S} . Taking a push-forward of \mathfrak{s} along the morphism \mathfrak{f}' , we obtain a decomposition of this diagram into a commutative diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathfrak{M} & & \\ \mathfrak{f}' \downarrow & (\text{co})\text{cart} & \downarrow \tilde{\mathfrak{f}}_1 & & \\ \mathcal{L} & \xrightarrow{u_1} & \tilde{\mathcal{M}} & \xrightarrow{u_2} & \mathcal{M} \end{array} \quad (6)$$

with a cocartesian square and the morphism u_2 uniquely determined by the equalities $u_2 \circ u_1 = u$, $u_2 \circ \tilde{\mathfrak{f}}_1 = \mathfrak{f}$. Notice that the square in the diagram (6) is also cartesian, because the square (5) is cartesian. By hypothesis, the arrow $\mathcal{L} \xrightarrow{u_1} \tilde{\mathcal{M}}$ belongs to \mathcal{S} and the arrow u_2 is a deflation. Taking a pull-back of the deflation u_2 along the morphism $\mathfrak{M} \xrightarrow{\mathfrak{f}} \mathcal{M}$, we obtain a further decomposition of the diagram (5) into the diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\mathfrak{s}_1} & \tilde{\mathfrak{M}} & \xrightarrow{\mathfrak{s}_2} & \mathfrak{M} \\ \mathfrak{f}' \downarrow & (\text{co})\text{cart} & \downarrow \tilde{\mathfrak{f}}_1 & \text{cart} & \downarrow \mathfrak{f} \\ \mathcal{L} & \xrightarrow{u_1} & \tilde{\mathcal{M}} & \xrightarrow{u_2} & \mathcal{M} \end{array} \quad (7)$$

whose both squares are cartesian and $\mathfrak{s} = \mathfrak{s}_2 \circ \mathfrak{s}_1$. A morphism $\mathcal{K} \xrightarrow{\mathfrak{v}} \widetilde{\mathfrak{M}}$ gives rise to the diagram

$$\begin{array}{ccccc}
\mathcal{K}(\mathfrak{s}) & \xrightarrow{\lambda_{\mathfrak{s}}} & \mathcal{K} & & \\
id \downarrow \wr & \text{cart} & \downarrow \mathfrak{k}' & & \\
\mathcal{K}(\mathfrak{s}) & \xrightarrow{\mathfrak{t}} & \mathcal{K}(\mathfrak{s}_2) & \xrightarrow{\lambda_{\mathfrak{s}_2}} & \mathcal{K} \\
\mathfrak{k}_1 \downarrow & \text{cart} & \downarrow \mathfrak{k} & \text{cart} & \downarrow \mathfrak{s}_2 \circ \mathfrak{v} \\
\mathcal{L} & \xrightarrow{\mathfrak{s}_1} & \widetilde{\mathfrak{M}} & \xrightarrow{\mathfrak{s}_2} & \mathfrak{M} \\
f' \downarrow & (\text{co})\text{cart} & \downarrow \widetilde{f}_1 & \text{cart} & \downarrow f \\
\mathcal{L} & \xrightarrow{u_1} & \widetilde{\mathcal{M}} & \xrightarrow{u_2} & \mathcal{M}
\end{array} \tag{8}$$

built of cartesian squares. Here the arrow $\mathcal{K} \xrightarrow{\mathfrak{k}'} \mathcal{K}(\mathfrak{s}_2)$ is uniquely determined by the equalities $\lambda_{\mathfrak{s}_2} \circ \mathfrak{k}' = id_{\mathcal{K}}$, $\mathfrak{k} \circ \mathfrak{k}' = \mathfrak{v}$. The upper cartesian square (with the identical left vertical arrow) is due to the fact that the square in the diagram (6) is cartesian. All horizontal arrows of the diagram (8) are deflations, because u_1 and u_2 are deflations and each of the remaining arrows is a pull-back of either u_1 , or u_2 . In particular, $\mathcal{K}(\mathfrak{s}) \xrightarrow{\mathfrak{t}} \mathcal{K}(\mathfrak{s}_2)$ is a deflation. The fact that $\mathfrak{t} = \mathfrak{k}' \circ \lambda_{\mathfrak{s}}$ is a strict epimorphism implies that \mathfrak{k}' is a strict epimorphism. On the other hand, \mathfrak{k}' is a monomorphism, due to the equality $\lambda_{\mathfrak{s}_2} \circ \mathfrak{k}' = id_{\mathcal{K}}$; hence \mathfrak{k}' is an isomorphism. Therefore, $\lambda_{\mathfrak{s}_2}$ is an isomorphism. The latter means that the arrow $\widetilde{\mathcal{M}} \xrightarrow{u_2} \mathcal{M}$ belongs to $\mathfrak{E}_X^{\circledast} = Iso(C_X)^{\wedge} \cap \mathfrak{E}_X$, i.e. it is a deflation with a trivial kernel. By hypothesis all deflations with a trivial kernel are weak equivalences. Thus, our arbitrary element u of $\mathcal{S}^{\wedge} \cap \mathfrak{E}_X$ is the composition $u_2 \circ u_1$, where $u_1 \in \mathcal{S}$ and $u_2 \in \mathcal{W}_X$. Since $\mathcal{W}_X \circ \mathcal{S} = \mathcal{S}$, the arrow u belongs to \mathcal{S} . ■

1.10.3. Right exact categories with stable classes of weak equivalences. We are particularly interested in the right exact categories (C_X, \mathfrak{E}_X) with a *stable* class weak equivalences \mathcal{W}_X , that is $\mathcal{W}_X = \mathcal{W}_X^{\wedge} \cap \mathfrak{E}_X$. Since any class of weak equivalences contains all isomorphisms of the category C_X , the smallest stable class coincides with the class $\mathfrak{E}_X^{\circledast} = Iso(C_X)^{\wedge} \cap \mathfrak{E}_X$ of all deflations with trivial kernels.

1.11. Coimages of morphisms and deflations with trivial kernels.

1.11.1. Coimages of morphisms. Fix a category C_X with an initial object x . Let $M \xrightarrow{f} N$ be an arrow which has a kernel, i.e. we have a cartesian square

$$\begin{array}{ccc}
Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\
f' \downarrow & \text{cart} & \downarrow f \\
x & \xrightarrow{i_N} & N
\end{array}$$

with which one can associate a pair of arrows $Ker(f) \xrightarrow[0_f]{\mathfrak{k}(f)} M$, where 0_f is the composition of the projection f' and the unique morphism $x \xrightarrow{i_M} M$. Since $i_N = f \circ i_M$, the morphism

f equalizes the pair $Ker(f) \xrightarrow[0_f]{\mathfrak{k}(f)} M$. If the cokernel of this pair of arrows exists, it will be called the *coimage of f* and denoted by $Coim(f)$, or, loosely, $M/Ker(f)$.

Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. Then f is the composition of the canonical strict epimorphism $M \xrightarrow{p_f} Coim(f)$ and a uniquely defined morphism $Coim(f) \xrightarrow{j_f} N$.

1.11.1.1. Lemma. *Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. There is a natural isomorphism $Ker(f) \xrightarrow{\sim} Ker(p_f)$ and the kernel of the morphism $Coim(f) \xrightarrow{j_f} N$ is trivial.*

Proof. The outer square of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & \longrightarrow & x \\ \mathfrak{k}(f) \downarrow & \text{cart} & \downarrow & & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \quad (1)$$

is cartesian. Therefore, its left square is cartesian which implies, by A.3, that $Ker(p_f)$ is isomorphic to $Ker(f')$. But, $Ker(f') \simeq Ker(f)$. ■

1.11.2. Right exact categories with coimages of deflations.

1.11.2.1. Proposition. *Let (C_X, \mathfrak{E}_X) be a right exact category and \mathfrak{E}_X^c the class of deflations which are isomorphic to their coimage. The class \mathfrak{E}_X^c is closed under composition and contains all isomorphisms of the category C_X .*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} Ker(\tilde{f}) & \xrightarrow{\mathfrak{k}(\tilde{f})} & Ker(gf) & \xrightarrow{\tilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ \wr \downarrow & & \mathfrak{k}(gf) \downarrow & \text{cart} & \downarrow \mathfrak{k}(g) & \text{cart} & \downarrow i_N \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & \xrightarrow{g} & N \\ & & \phi \downarrow & & & & \\ & & \mathcal{V} & & & & \end{array} \quad (2)$$

where f and g belong to \mathfrak{E}_X^c and the morphism $L \xrightarrow{\phi} \mathcal{V}$ equalizes the pair of arrows $Ker(gf) \xrightarrow[0_{gf}]{\mathfrak{k}(gf)} L$. It follows from the left square of the diagram (2) that ϕ equalizes the pair of arrows $Ker(f) \xrightarrow[0_f]{\mathfrak{k}(f)} L$. Since, by hypothesis, $L \xrightarrow{f} M$ is an equalizer of this pair of arrows, there is a unique morphism $M \xrightarrow{\gamma_\phi} \mathcal{V}$ such that $\phi = \gamma_\phi \circ f$. Since ϕ equalizes the pair $Ker(gf) \xrightarrow[0_{gf}]{\mathfrak{k}(gf)} L$, it follows from the commutativity of the central square of (2)

that γ_ϕ equalizes the composition of the morphism $Ker(gf) \xrightarrow{\tilde{f}} Ker(g)$ and the pair of arrows $Ker(g) \xrightarrow[0_g]{\mathfrak{e}(g)} M$. Since $Ker(gf) \xrightarrow{\tilde{f}} Ker(g)$ is a pull-back of a deflation, it is a deflation, in particular, it is a strict epimorphism. Therefore, the cokernel of this composition is the cokernel of the pair $Ker(g) \xrightarrow[0_g]{\mathfrak{e}(g)} M$. Since $M \xrightarrow{g} N$ is an equalizer of the latter pair, there exists a unique morphism $N \xrightarrow{\lambda_\phi} \mathcal{V}$ such that $\gamma_\phi = \lambda_\phi \circ g$.

Thus, $\phi = (\lambda_\phi \circ \gamma_\phi) \circ (g \circ f)$. Since $g \circ f$ is an epimorphism, a morphism ξ such that $\phi = \xi \circ (g \circ f)$ is unique. Therefore, $g \circ f$ is a cokernel of the pair $Ker(gf) \xrightarrow[0_{gf}]{\mathfrak{e}(gf)} L$. ■

1.11.2.2. Corollary. *Let (C_X, \mathfrak{E}_X) be a right exact category such that every its deflation has a coimage which is also a deflation and the system of deflations \mathfrak{E}_X is left divisible, i.e. if a composition $\mathfrak{t} \circ \gamma$ of two arrows is a deflation, then \mathfrak{t} is a deflation. Then $\mathfrak{E}_X = \mathfrak{E}_X^\circledast \circ \mathfrak{E}_X^\mathfrak{e}$ and for every deflation, this decomposition is defined uniquely up to isomorphism.*

Proof. It follows from 1.11.1.1, 1.11.2.1 and the imposed conditions that every deflation \mathfrak{e} is the composition $\tilde{\mathfrak{e}} \circ \mathfrak{t}$, where \mathfrak{t} coincides with its coimage and $\tilde{\mathfrak{e}}$ is a morphism with a trivial kernel. Since, by hypothesis, \mathfrak{E}_X is left divisible, $\tilde{\mathfrak{e}}$ is a deflation. ■

1.11.3. Proposition. *Let $M \xrightarrow{\mathfrak{s}} N$ be a deflation from $\mathfrak{E}_X^\mathfrak{e}$. Then any cartesian square*

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ \mathfrak{f}' \downarrow & & \downarrow \mathfrak{f} \\ \mathcal{M} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array}$$

is a cocartesian square.

Proof. In fact, let

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ \mathfrak{f}' \downarrow & & \downarrow \xi_2 \\ \mathcal{M} & \xrightarrow{\xi_1} & \mathcal{L} \end{array}$$

be a commutative square. It follows from the commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{t}) & \xrightarrow{\mathfrak{e}(\mathfrak{t})} & \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ \wr \downarrow & & \mathfrak{f}' \downarrow & & \downarrow \xi_2 \\ Ker(\mathfrak{s}) & \xrightarrow{\mathfrak{e}(\mathfrak{s})} & \mathcal{M} & \xrightarrow{\xi_1} & \mathcal{L} \end{array}$$

and the fact that the morphism \mathfrak{t} (hence $\xi_2 \circ \mathfrak{t}$) equalizes the pair $Ker(\mathfrak{t}) \xrightarrow[0_{\mathfrak{t}}]{\mathfrak{e}(\mathfrak{t})} \widetilde{\mathcal{M}}$, that $\mathcal{M} \xrightarrow{\xi_1} \mathcal{L}$ equalizes the pair $Ker(\mathfrak{s}) \xrightarrow[0_{\mathfrak{s}}]{\mathfrak{e}(\mathfrak{s})} \mathcal{M}$. Therefore, since \mathfrak{s} is the cokernel of this pair

of arrows $\xi_1 = \tilde{\xi}_1 \circ \mathfrak{s}$ for a uniquely determined morphism $\mathcal{L} \xrightarrow{\tilde{\xi}_1} \mathcal{L}$. So that we have:

$$\xi_2 \circ \mathfrak{t} = \xi_1 \circ \mathfrak{f}' = \tilde{\xi}_1 \circ \mathfrak{s} \circ \mathfrak{f}' = (\tilde{\xi}_1 \circ \mathfrak{f}) \circ \mathfrak{t}.$$

Since \mathfrak{t} is a deflation, in particular, an epimorphism, the equality $\xi_2 \circ \mathfrak{t} = (\tilde{\xi}_1 \circ \mathfrak{f}) \circ \mathfrak{t}$ implies that $\xi_2 = \tilde{\xi}_1 \circ \mathfrak{f}$. ■

1.11.3.1. Note. Suppose that the conditions of 1.11.2.2 hold. Let

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ \mathfrak{f}' \downarrow & \text{cart} & \downarrow \mathfrak{f} \\ \mathcal{M} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array} \quad (3)$$

be a cartesian square and $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{L}$ is a deflation. By 1.11.3, $\mathfrak{s} = \mathfrak{e}_s \circ \mathfrak{s}_c$, where $\mathfrak{s}_c \in \mathfrak{E}_X^c$ and $\mathfrak{e}_s \in \mathfrak{E}_X^{\otimes}$. To this decomposition, there corresponds a decomposition

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N}_c & \xrightarrow{\mathfrak{t}_c} & \mathcal{N} \\ \mathfrak{f}' \downarrow & \text{(co)cart} & \downarrow \mathfrak{f}_c & \text{cart} & \downarrow \mathfrak{f} \\ \mathcal{M} & \xrightarrow{\mathfrak{s}_c} & \mathcal{L}_c & \xrightarrow{\mathfrak{e}_s} & \mathcal{L} \end{array} \quad (4)$$

of the square (3) into two cartesian squares. Since the class \mathfrak{E}_X^{\otimes} of deflations with trivial kernel is stable under pull-backs, the horizontal arrows of the right square belong to \mathfrak{E}_X^{\otimes} , in particular, they are weak equivalences. By 1.11.3, the left square of (4) is both cartesian and cocartesian.

2. Topologizing, thick and Serre systems.

2.0. Assumptions. Fix a right exact category with weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$. Below is the list of assumptions which appear (not necessarily simultaneously) in different assertions of this work.

(a) The category C_X is filtered, i.e. every pair of arrows $L \longrightarrow M \longleftarrow N$ can be completed to a commutative square.

(b) The class of weak equivalences is stable, i.e. $\mathcal{W}_X = \mathcal{W}_X^{\bar{\lambda}} \cap \mathfrak{E}_X$, and has two more properties:

(b1) If $\mathfrak{s} \circ \mathfrak{t} \in \mathcal{W}_X$ and both \mathfrak{s} and \mathfrak{t} are deflations, then $\mathfrak{s} \in \mathcal{W}_X \ni \mathfrak{t}$.

(b2) The class \mathcal{W}_X is invariant under push-forwards along deflations; that is for any pair $L \xleftarrow{\mathfrak{e}} N \xrightarrow{\mathfrak{w}} M$ of deflations with $\mathfrak{w} \in \mathcal{W}_X$, there is a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{w}} & M \\ \mathfrak{e} \downarrow & \text{cocart} & \downarrow \mathfrak{e}' \\ L & \xrightarrow{\mathfrak{v}} & \widetilde{\mathcal{N}} \end{array}$$

with both horizontal arrows from \mathcal{W}_X .

The class \mathfrak{E}_X of deflations is *left divisible* in the sense that

(c) if $\mathfrak{t} \in \mathfrak{E}_X \ni \mathfrak{s} \circ \mathfrak{t}$, then $\mathfrak{s} \in \mathfrak{E}_X$,

and *weakly right divisible* in the following sense:

(d) if $\mathfrak{s} \in \mathfrak{E}_X \ni \mathfrak{s} \circ \mathfrak{t}$, then $\mathfrak{t} \in \mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X$.

(e) There is a multiplicative class \mathfrak{D}_X of arrows of the category C_X which includes all deflations and $\mathcal{W}_X^{\bar{\wedge}}$ and a map which assigns to every $\mathfrak{s} \in \mathfrak{D}_X$ a decomposition $\mathfrak{s} = \gamma_{\mathfrak{s}} \circ \mathfrak{e}_{\mathfrak{s}}$, where $\mathfrak{e}_{\mathfrak{s}}$ is a strict epimorphism, such that $\gamma_{\mathfrak{s}} \in \mathcal{W}_X^{\bar{\wedge}}$ and $\mathfrak{e}_{\mathfrak{s}} \in \mathfrak{E}_X$, whenever $\mathfrak{s} \in \mathfrak{E}_X \circ \mathcal{W}_X^{\bar{\wedge}}$.

A more detailed version of this condition is obtained by adding to (e) the following:

(e') There is a multiplicative subclass \mathcal{E}_X of the class of strict epimorphisms of the category C_X and a multiplicative subclass \mathfrak{W}_X of $\mathcal{W}_X^{\bar{\wedge}}$ such that $\mathfrak{e}_{\mathfrak{s}} \in \mathcal{E}_X$ and $\gamma_{\mathfrak{s}} \in \mathfrak{W}_X$ for all $\mathfrak{s} \in \mathfrak{D}_X$; and the arrows $\lambda \in \mathfrak{W}_X$ and $\mathfrak{t} \in \mathfrak{E}_X$ in the decomposition $\lambda \circ \mathfrak{t}$ are determined uniquely up to isomorphism. In particular, $\mathfrak{e}_{\mathfrak{s}}$ is isomorphic to \mathfrak{s} for all $\mathfrak{s} \in \mathcal{E}_X$ and $\gamma_{\mathfrak{t}} \simeq \mathfrak{t}$ for all $\mathfrak{t} \in \mathfrak{W}_X$.

2.0.1. Comments. (a) The condition (a) holds automatically if the category C_X has initial objects, or if it has fiber products.

(b) Every stable class \mathcal{W}_X of weak equivalences contains the class \mathfrak{E}_X^{\otimes} of deflations with trivial kernel. The class \mathfrak{E}_X^{\otimes} satisfies the condition (b1) and, if there exist push-forwards of deflations with trivial kernels along deflations with trivial kernels, it satisfies the condition (b2) as well.

(c) The largest class $\mathfrak{E}_X^{\mathfrak{s}\mathfrak{t}}$ of deflations of the category C_X (which consists of all strict epimorphisms such that their arbitrary pull-backs exist and are strict epimorphisms) satisfies the condition (c). This follows from two observations:

(i) if $\mathfrak{s} \circ \mathfrak{t}$ is a strict epimorphism, then \mathfrak{s} is a strict epimorphism;

(ii) if there exist arbitrary pull-backs of the composition $\mathfrak{s} \circ \mathfrak{t}$ and of the morphism \mathfrak{t} , then there are arbitrary pull-backs of the morphism \mathfrak{s} .

(d)&(e') Suppose that the class of all strict epimorphisms of the category C_X is stable under pull-backs and, for every morphism $\mathcal{M} \xrightarrow{f} \mathcal{N}$ of the category C_X , there exists a kernel pair $Ker_2(f) = \mathcal{M} \times_{\mathcal{N}} \mathcal{M} \xrightarrow[p_2]{p_1} \mathcal{M}$ and the cokernel $\mathcal{M} \xrightarrow{c_2(f)} Coim_2(f)$ of the pair (p_1, p_2) which we call *2-coimage* of the morphism f .

Suppose that the class $\mathcal{W}_X^{\bar{\wedge}}$ contains all monomorphisms.

(d) Then the largest class of deflations $\mathfrak{E}_X^{\mathfrak{s}\mathfrak{t}}$ (which coincides with the class of all strict epimorphisms) satisfies the conditions (d) by a trivial reason, because, under the conditions above, every morphism f is the composition $j(f) \circ c_2(f)$ of a strict epimorphism, $c_2(f)$, and a monomorphism; and, by hypothesis, $\mathcal{W}_X^{\bar{\wedge}}$ contains all monomorphisms.

(e1) By the same reason, the class $\mathfrak{E}_X^{\mathfrak{s}\mathfrak{t}}$ satisfies the condition (e') with $\mathcal{E}_X = \mathfrak{E}_X^{\mathfrak{s}\mathfrak{t}}$ and with \mathfrak{W}_X equal to the class of all monomorphisms of the category C_X .

Notice that $\mathcal{W}_X^{\bar{\wedge}}$ contains all monomorphisms automatically, if the category C_X has initial objects, because it contains all morphisms with a trivial kernel and monomorphisms have trivial kernels.

(d1) Actually, we need a weaker condition instead of 2.0(d) which is as follows. Let

$$\begin{array}{ccccc}
 L & \xrightarrow{\mathfrak{t}} & \mathcal{N} & \xrightarrow{\mathfrak{s}_1} & \mathcal{M} \\
 & & \mathfrak{s}_2 \downarrow & \text{cart} & \downarrow \mathfrak{s}'_2 \\
 & & \mathcal{L} & \xrightarrow{\xi_1} & \tilde{\mathcal{N}}
 \end{array}$$

be a diagram whose square is cartesian and formed by deflations and the compositions $\mathfrak{s}_2 \circ \mathfrak{t}$ and $\mathfrak{s}_2 \circ \mathfrak{t}$ are deflations. Then $\mathfrak{t} \in \mathcal{W}_X^\wedge \circ \mathfrak{E}_X$.

(e2) Suppose that the category C_X has initial objects and kernels and coimages of all morphisms. Then every morphism f is the composition $j_f \circ p_f$ of its coimage, p_f and a morphism j_f . The latter belongs to $\text{Iso}(C_X)^\wedge$, hence it belongs to \mathcal{W}_X^\wedge . Thus, if the coimage of any deflation is a deflation, then it follows from (the argument of) 1.11.2.1 and from 1.11.2.2 that the condition (e') holds if we take as \mathfrak{E}_X the class of all strict epimorphisms which coincide with their coimage and as \mathfrak{W}_X the class of all morphisms with trivial kernel.

2.0.2. Examples. (i) Suppose that the category C_X is additive. Then the class $\text{Iso}(C_X)^\wedge$ of all morphisms with a trivial kernel coincides with the class of all monomorphisms of the category C_X and a morphism has a kernel iff it has a 2-kernel. It follows also that the coimage of a morphism is the same as its image. Thus, if the class $\mathfrak{E}_X^\mathfrak{s}$ of all strict epimorphisms of the category C_X is closed under pull-backs, then $(C_X, \mathfrak{E}_X^\mathfrak{s}, \text{Iso}(C_X))$ satisfies all the conditions of 2.0.

(ii) Let C_X be the category Alg_k of associative unital k -algebras (see 1.4.1). Then, similarly to the additive case, the class of all strict epimorphism is stable under base change, the class of morphisms with trivial kernel coincides with the class of monomorphisms of the category C_X (see 1.9.3(c)) and $(C_X, \mathfrak{E}_X^\mathfrak{s}, \text{Iso}(C_X))$ satisfies all the conditions of 2.0.

2.1. Systems. We call a class of arrows \mathcal{S} of C_X a *system* in $(C_X, \bar{\mathfrak{E}}_X)$ if

- (a) \mathcal{S} is closed under pull-backs,
- (b) $\mathcal{W}_X \circ \mathcal{S} \circ \mathcal{W}_X = \mathcal{S} \supseteq \mathcal{W}_X$,

We denote the set of systems in $(C_X, \bar{\mathfrak{E}}_X)$ by $\mathfrak{S}(X, \bar{\mathfrak{E}}_X)$ and regard it as a preorder with respect to the inclusion. The smallest system, \mathcal{W}_X , will be referred as *trivial*.

We denote by $\mathfrak{S}^\wedge(X, \bar{\mathfrak{E}}_X)$ the subpreorder of $\mathfrak{S}(X, \bar{\mathfrak{E}}_X)$ formed by *stable* systems of deflations, i.e. systems \mathcal{S} such that $\mathcal{S} = \mathfrak{E}_X \cap \mathcal{S}^\wedge$.

2.1.1. Proposition. *The set $\mathfrak{S}(X, \bar{\mathfrak{E}}_X)$ of systems and the set $\mathfrak{S}^\wedge(X, \bar{\mathfrak{E}}_X)$ of stable systems in $(C_X, \bar{\mathfrak{E}}_X)$ are closed under compositions and arbitrary unions and intersections.*

Proof. The inclusion $\mathcal{T} \subseteq \mathcal{W}_X \circ \mathcal{T} \circ \mathcal{W}_X$ holds for any class of arrows \mathcal{T} of the category C_X . If $\{\mathcal{S}_i \mid i \in J\}$ is a set of systems, then

$$\mathcal{W}_X \subseteq \bigcap_{i \in J} \mathcal{S}_i \subseteq \mathcal{W}_X \circ \left(\bigcap_{i \in J} \mathcal{S}_i \right) \circ \mathcal{W}_X \subseteq \bigcap_{i \in J} \mathcal{S}_i$$

and, evidently,

$$\mathcal{W}_X \subseteq \bigcup_{i \in J} \mathcal{S}_i \subseteq \mathcal{W}_X \circ \left(\bigcup_{i \in J} \mathcal{S}_i \right) \circ \mathcal{W}_X = \bigcup_{i \in J} (\mathcal{W}_X \circ \mathcal{S}_i \circ \mathcal{W}_X) = \bigcup_{i \in J} \mathcal{S}_i.$$

The fact that each \mathcal{S}_i is invariant under base change implies that $\bigcup_{i \in J} \mathcal{S}_i$ and $\bigcap_{i \in J} \mathcal{S}_i$ have the same property. Therefore $\bigcup_{i \in J} \mathcal{S}_i$ and $\bigcap_{i \in J} \mathcal{S}_i$ are systems.

The similar assertion for stable systems follows from 1.10.1(b). ■

2.2. Right and left divisible systems. We call a system \mathcal{S} in $(C_X, \bar{\mathfrak{E}}_X)$ *right* (resp. *left*) *divisible* if $\mathfrak{s} \in \mathcal{S}$ (resp. $\mathfrak{t} \in \mathcal{S}$) whenever $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$.

We say that a system \mathcal{S} is *right* (resp. *left*) *divisible in* \mathfrak{E}_X if $\mathfrak{s} \in \mathcal{S}$ (resp. $\mathfrak{t} \in \mathcal{S}$) whenever $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$ and $\mathfrak{s} \in \mathfrak{E}_X$.

It follows that the class of right (resp. left) divisible systems is stable under arbitrary unions and intersections. Similarly for systems which are right (resp. left) divisible in \mathfrak{E}_X .

2.2.1. Proposition. *Suppose that the category C_X has pull-backs. Then, for any right (resp. left) divisible system \mathcal{S} , the system \mathcal{S}^\wedge is right (resp. left) divisible.*

If the system \mathcal{S} is left divisible in \mathfrak{E}_X , then \mathcal{S}^\wedge is left divisible in \mathfrak{E}_X .

Proof. (a) In fact, let $\mathcal{M} \xrightarrow{u} \mathcal{N}$ be an arrow from \mathcal{S}^\wedge , that is its pull-back along some arrow $\mathcal{L} \xrightarrow{f} \mathcal{N}$ belongs to \mathcal{S} . Let $u = \mathfrak{t} \circ \mathfrak{s}$. Then, since pull-backs exist in C_X , we have the decomposition

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{s}}} & \widetilde{\mathcal{K}} & \xrightarrow{\widetilde{\mathfrak{t}}} & \mathcal{L} \\ f' \downarrow & \text{cart} & \downarrow & \text{cart} & \downarrow f \\ \mathcal{M} & \xrightarrow{\mathfrak{s}} & \mathcal{K} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \end{array} \quad (1)$$

of the pull-back of u along f into cartesian squares. So, if \mathcal{S} is right (resp. left) divisible, then $\widetilde{\mathfrak{s}} \in \mathcal{S}$ (resp. $\widetilde{\mathfrak{t}} \in \mathcal{S}$), hence $\mathfrak{s} \in \mathcal{S}^\wedge$ (resp. $\mathfrak{t} \in \mathcal{S}^\wedge$).

(b) Suppose now that \mathcal{S} is a system which is left divisible in \mathfrak{E}_X , that is if $u = \mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$ and \mathfrak{s} is a deflation, then $\mathfrak{t} \in \mathcal{S}$. Consider a pull-back of such u which belongs to \mathcal{S} and consider its decomposition described by the diagram (1) above. Since, by hypothesis, \mathfrak{s} is a deflation, the arrow $\widetilde{\mathfrak{s}}$ in (1) is a deflation. Therefore, $\widetilde{\mathfrak{t}} \in \mathcal{S}$ which implies that $\mathfrak{t} \in \mathcal{S}^\wedge$. ■

2.3. Orthogonal complements. For a class of arrows Σ containing \mathcal{W}_X , we define the *orthogonal complement* Σ^\perp of Σ as the union of all right divisible systems \mathcal{S} such that $\mathcal{S} \cap \Sigma = \mathcal{W}_X$. In other words, Σ^\perp is the largest right divisible system having the trivial intersection with Σ .

2.3.1. Proposition. *Let $(C_X, \bar{\mathfrak{E}}_X)$ be a filtered right exact category with a stable class of weak equivalences; and let \mathcal{S} be a class of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ closed under pull-backs. If the category C_X has pull-backs, then $(\mathcal{S}^\perp)^\wedge = \mathcal{S}^\perp$.*

Proof. It follows from 1.6.1 and 1.7 that

$$\mathcal{S} \cap (\mathcal{S}^\perp)^\wedge \subseteq \mathfrak{E}_X \cap (\mathcal{S}^\wedge \cap (\mathcal{S}^\perp)^\wedge) = \mathfrak{E}_X \cap (\mathcal{S} \cap \mathcal{S}^\perp)^\wedge = \mathfrak{E}_X \cap \mathcal{W}_X^\wedge.$$

By hypothesis, the system of weak equivalences is stable, that is $\mathfrak{E}_X \cap \mathcal{W}_X^\wedge = \mathcal{W}_X$.

Since \mathcal{S}^\perp is right divisible and the category C_X has pull-backs, it follows from 2.2.1 that the system $(\mathcal{S}^\perp)^\wedge$ is right divisible. Therefore, $\mathcal{S}^\perp = (\mathcal{S}^\perp)^\wedge$. ■

2.3.2. Example. Let C_X be a category with an initial object \mathfrak{r} and kernels of morphisms. For a strict subcategory \mathcal{T} of the category C_X containing initial objects, we set

$$\Sigma_{\mathcal{T}} = \{\mathfrak{s} \in \text{Hom}C_X \mid \text{Ker}(\mathfrak{s}) \in \text{Ob}\mathcal{T}\}.$$

It follows from the general properties of kernels that $\Sigma_{\mathcal{T}}$ is stable under base change.

Suppose that the kernel of any morphism $M \xrightarrow{\mathfrak{e}} N$ of C_X with $M \in \text{Ob}\mathcal{T}$ belongs to the subcategory \mathcal{T} . Then $\Sigma_{\mathcal{T}}$ is a right divisible system.

This observation follows from the commutative diagram

$$\begin{array}{ccccccc} \text{Ker}(\mathfrak{s}) & \xrightarrow{\mathfrak{k}(\mathfrak{s})} & N & \xrightarrow{\mathfrak{s}} & M & \xrightarrow{\mathfrak{t}} & L \\ \wr \uparrow & & \mathfrak{k}(\mathfrak{s}) \uparrow & \text{cart} & \uparrow & & \uparrow \\ \text{Ker}(\tilde{\mathfrak{s}}) & \xrightarrow{\mathfrak{k}(\tilde{\mathfrak{s}})} & \text{Ker}(\mathfrak{t}\mathfrak{s}) & \xrightarrow{\tilde{\mathfrak{s}}} & \text{Ker}(\mathfrak{t}) & \longrightarrow & \mathfrak{r} \end{array} \quad (1)$$

with the cartesian central square.

Suppose that $\mathcal{W}_X = \mathfrak{E}_X^{\otimes}$ – the class of all deflations with trivial kernels (cf. 1.9.3(b)). Let $\mathcal{S} = \mathfrak{E}_{X,\mathcal{T}} \stackrel{\text{def}}{=} \Sigma_{\mathcal{T}} \cap \mathfrak{E}_X$. One can see that the *orthogonal complement* \mathcal{S}^{\perp} to \mathcal{S} coincides with $\Sigma_{\mathbb{T}}$, where \mathbb{T} is the full subcategory of C_X determined by

$$\text{Ob}\mathbb{T} = \{M \in \text{Ob}C_X \mid \text{Ker}(M \rightarrow \mathcal{N}) \notin \text{Ob}\mathcal{T} - \{\text{initial objects}\} \text{ for any arrow } M \rightarrow \mathcal{N}\}.$$

In the case of an abelian category C_X , this description means that $\text{Ob}\mathbb{T}$ consists of all \mathcal{T} -torsion free objects of the category C_X .

2.4. Topologizing systems of deflations.

2.4.1. Conventions. We assume that $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ is a svelte right exact category with a stable class of weak equivalences satisfying the condition 2.0(e); that is there exists a multiplicative class \mathfrak{D}_X of arrows of the category C_X which includes all deflations and \mathcal{W}_X^{\wedge} and a map which assigns to every $\mathfrak{s} \in \mathfrak{D}_X$ a decomposition $\mathfrak{s} = \gamma_{\mathfrak{s}} \circ \mathfrak{e}_{\mathfrak{s}}$, where $\mathfrak{e}_{\mathfrak{s}}$ is a strict epimorphism, such that $\gamma_{\mathfrak{s}} \in \mathcal{W}_X^{\wedge}$ and $\mathfrak{e}_{\mathfrak{s}} \in \mathfrak{E}_X$, whenever $\mathfrak{s} \in \mathfrak{E}_X \circ \mathcal{W}_X^{\wedge}$.

In some cases (like 2.4.4 below), we need a stronger assumption 2.0(e').

2.4.2. Left topologizing and right topologizing and topologizing systems of deflations. We call a system \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ *left topologizing* (resp. *right topologizing*, resp. *topologizing*) if it is left divisible (resp. right divisible, resp. divisible) in \mathfrak{E}_X and the following conditions hold:

(a) If all arrows of a cartesian or a cocartesian square belong to \mathcal{S} , then the composition of the consequent arrows of this square belongs to \mathcal{S} .

(b) The system \mathcal{S} is closed under push-forwards.

(c) For any $\mathfrak{s} \in \mathcal{S} \circ \mathcal{W}_X^{\wedge}$, the deflation $\mathfrak{e}_{\mathfrak{s}}$ in the decomposition $\mathfrak{s} = \gamma_{\mathfrak{s}} \circ \mathfrak{e}_{\mathfrak{s}}$ belongs to the system \mathcal{S} . In particular, $\mathcal{S} \circ \mathcal{W}_X^{\wedge} \subseteq \mathcal{W}_X^{\wedge} \circ \mathcal{S}$.

2.4.3. Proposition. *Suppose that the class of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is left divisible and the class of weak equivalences stable. Then every left topologizing (resp. right topologizing, resp. topologizing) system is stable.*

Proof. Since the class of weak equivalences is stable, it contains the class $\mathfrak{E}_X^\circledast$ of deflations with a trivial kernel. By definition, every (left or/and right) topologizing system is closed under push-forwards and composition with weak equivalences. Therefore, the assertion follows from 1.10.2. ■

We denote the preorder (with respect to the inclusion) of all left topologizing systems of $(C_X, \bar{\mathfrak{E}}_X)$ by $\mathfrak{T}_\ell(X, \bar{\mathfrak{E}}_X)$ and the preorder of topologizing systems by $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$.

It follows that the class \mathcal{W}_X of weak equivalences is the smallest topologizing system.

One can see that $\mathfrak{T}_\ell(X, \bar{\mathfrak{E}}_X)$ and $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ are closed under arbitrary intersections and filtered (with respect to the inclusion) unions.

2.4.4. Proposition. *Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ be a right exact category with weak equivalences and left divisible system of deflations (see 2.0(c)). Suppose that the condition 2.0(e') holds. Then*

(a) *The composition of left topologizing systems is a left topologizing system.*

(b) *Suppose that the class \mathcal{W}_X of weak equivalences is stable (the condition 2.0(b)) and the class \mathfrak{E}_X of deflations is weakly right divisible (the condition 2.0(d)). Then the class $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ of topologizing systems is closed under composition.*

Proof. (a) Let \mathcal{S}, \mathcal{T} be left topologizing systems of deflations. By 2.1.1, the composition of two systems is a system. In particular, $\mathcal{T} \circ \mathcal{S}$ is a system. Since a push-forward of a composition of two arrows is the composition of the corresponding push-forwards, the system $\mathcal{T} \circ \mathcal{S}$ is closed under push-forwards.

The system $\mathcal{T} \circ \mathcal{S} \circ \mathcal{W}_X^\wedge$ is preserved by the map $\mathbf{u} \mapsto \mathbf{e}_\mathbf{u}$.

In fact, let $\mathbf{u} = \mathbf{t} \circ \mathbf{s} \circ \mathbf{w}$, where $\mathbf{t} \in \mathcal{T}$, $\mathbf{s} \in \mathcal{S}$, and $\mathbf{w} \in \mathcal{W}_X^\wedge$. Then

$$\mathbf{u} = \mathbf{t} \circ (\gamma_{\mathbf{s}\mathbf{w}} \circ \mathbf{e}_{\mathbf{s}\mathbf{w}}) = \gamma_{\mathbf{t}\gamma_{\mathbf{s}\mathbf{w}}} \circ (\mathbf{e}_{\mathbf{t}\gamma_{\mathbf{s}\mathbf{w}}} \circ \mathbf{e}_{\mathbf{s}\mathbf{w}}),$$

where $\mathbf{e}_{\mathbf{t}\gamma_{\mathbf{s}\mathbf{w}}} \in \mathcal{T}$ and $\mathbf{e}_{\mathbf{s}\mathbf{w}} \in \mathcal{S}$, because the systems \mathcal{T} and \mathcal{S} are left topologizing, and $\gamma_{\mathbf{t}\gamma_{\mathbf{s}\mathbf{w}}} \in \mathfrak{W}_X \subseteq \mathcal{W}_X^\wedge$. By the condition 2.0(e'), the representation of a morphism as a product $\gamma \circ \mathbf{e}$ of $\gamma \in \mathfrak{W}_X$ and $\mathbf{e} \in \mathfrak{E}_X$ is unique up to isomorphism. In particular, the arrows $\mathbf{e}_{\mathbf{t}\gamma_{\mathbf{s}\mathbf{w}}} \circ \mathbf{e}_{\mathbf{s}\mathbf{w}} \in \mathcal{T} \circ \mathcal{S}$ and $\mathbf{e}_\mathbf{u}$ are isomorphic.

It remains to show that the system $\mathcal{T} \circ \mathcal{S}$ is left divisible in \mathfrak{E}_X . Let

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\mathbf{s}} & \mathcal{L} \\ \mathbf{u} \downarrow & & \downarrow \mathbf{t} \\ \mathcal{M} & \xrightarrow{\mathbf{v}} & \mathfrak{N} \end{array} \quad (4)$$

be a commutative square whose all arrows are deflations with $\mathbf{s} \in \mathcal{S}$ and $\mathbf{t} \in \mathcal{T}$. Since \mathcal{S} is closed under push-forwards, the diagram (4) is decomposed into the diagram

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{\mathbf{s}} & \mathcal{L} & & \\ \mathbf{t} \downarrow & \text{cocart} & \downarrow \mathbf{t}' & & \\ \mathcal{M} & \xrightarrow{\tilde{\mathbf{s}}} & \mathfrak{M} & \xrightarrow{\lambda} & \mathfrak{N} \end{array} \quad (5)$$

with a cocartesian square, where $\mathcal{M} \xrightarrow{\tilde{s}} \mathfrak{M}$ is an element of \mathcal{S} and the morphism $\mathfrak{M} \xrightarrow{\lambda} \mathfrak{N}$ is uniquely determined by the equalities $\lambda \circ \tilde{s} = \mathfrak{v}$ and $\lambda \circ \mathfrak{t}' = \mathfrak{t}$. Since \mathfrak{t}' and \mathfrak{t} are deflations, $\mathfrak{t} \in \mathcal{T}$, and the system \mathcal{T} is left divisible in \mathfrak{E}_X , it follows that $\lambda \in \mathcal{T}$. So, $\mathfrak{v} = \lambda \circ \tilde{s} \in \mathcal{T} \circ \mathcal{S}$, which shows that the system $\mathcal{T} \circ \mathcal{S}$ is left divisible in \mathfrak{E}_X .

(b) Suppose now that the systems \mathcal{T} and \mathcal{S} are topologizing (that is left topologizing and divisible) and the conditions 2.0(b) and 2.0(d) hold. The claim is that these conditions imply that the system $\mathcal{T} \circ \mathcal{S}$ is right divisible (hence divisible) in \mathfrak{E}_X .

In fact, consider again the commutative square (4). This time, we decompose it pulling back the arrow $\mathfrak{t} \in \mathcal{T}$ along $\mathcal{M} \xrightarrow{\mathfrak{v}} \mathfrak{N}$; that is we consider the diagram

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{\gamma} & \widetilde{\mathfrak{M}} & \xrightarrow{\xi_{\mathfrak{s}}} & \mathcal{L} \\ & & \widehat{\mathfrak{t}} \downarrow & \text{cart} & \downarrow \mathfrak{t} \\ & & \mathcal{M} & \xrightarrow{\mathfrak{v}} & \mathfrak{N} \end{array} \quad (6)$$

with cartesian square and morphism $\mathcal{N} \xrightarrow{\gamma} \widetilde{\mathfrak{M}}$ uniquely determined by the equalities $\xi_{\mathfrak{s}} \circ \gamma = \mathfrak{s}$ and $\widehat{\mathfrak{t}} \circ \gamma = \mathfrak{u}$. Since $\xi_{\mathfrak{s}}$ and \mathfrak{s} are deflations, it follows from the condition 2.0(b) that $\gamma = \mathfrak{w} \circ \mathfrak{e}_{\gamma}$, where $\mathfrak{w} \in \mathcal{W}_X^{\wedge}$ and \mathfrak{e}_{γ} is a deflation. Thus, $\mathfrak{u} = (\widehat{\mathfrak{t}} \circ \mathfrak{w}) \circ \mathfrak{e}_{\gamma}$. Since, by hypothesis, the system of deflations \mathfrak{E}_X is left divisible and $\mathfrak{u} \in \mathfrak{E}_X$, it follows from this equality that $\widehat{\mathfrak{t}} \circ \mathfrak{w}$ is a deflation. On the other hand, it belongs to $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\wedge})$ and, since \mathcal{T} is a topologizing system, $\mathcal{T} \circ \mathcal{W}_X^{\wedge} \subseteq \mathcal{W}_X^{\wedge} \circ \mathcal{T}$ (see 2.4.2(c)). Therefore,

$$\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\wedge}) \subseteq \mathfrak{E}_X \cap (\mathcal{W}_X^{\wedge} \circ \mathcal{T}) \subseteq (\mathfrak{E}_X \cap \mathcal{W}_X^{\wedge}) \circ \mathcal{T} = \mathcal{W}_X \circ \mathcal{T} = \mathcal{T},$$

because the class of weak equivalences \mathcal{W}_X is, by hypothesis, stable, i.e. $\mathcal{W}_X = \mathcal{W}_X^{\wedge} \cap \mathfrak{E}_X$.

Altogether shows that $\widehat{\mathfrak{t}} \circ \mathfrak{w} \in \mathcal{T}$. The class of deflations \mathfrak{E}_X being left divisible, the fact that $\mathfrak{s} = \xi_{\mathfrak{s}} \circ \gamma = (\xi_{\mathfrak{s}} \circ \mathfrak{w}) \circ \mathfrak{e}_{\gamma}$ implies that $\xi_{\mathfrak{s}} \circ \mathfrak{w}$ is a deflation. Since $\mathfrak{s} \in \mathcal{S}$ and the system \mathcal{S} is right divisible, it follows from the equality $\mathfrak{s} = (\xi_{\mathfrak{s}} \circ \mathfrak{w}) \circ \mathfrak{e}_{\gamma}$ that $\mathfrak{e}_{\gamma} \in \mathcal{S}$. Therefore, $\mathfrak{u} = (\widehat{\mathfrak{t}} \circ \mathfrak{w}) \circ \mathfrak{e}_{\gamma} \in \mathcal{T} \circ \mathcal{S}$.

This shows that the system $\mathcal{T} \circ \mathcal{S}$ is right divisible. Since, by (a) above, $\mathcal{T} \circ \mathcal{S}$ is left topologizing, it is topologizing. ■

2.4.5. Proposition. (a) Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite family of systems of deflations which are right divisible in \mathfrak{E}_X (that is if $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}_i$ and $\mathfrak{s} \in \mathfrak{E}_X$, then $\mathfrak{s} \in \mathcal{S}_i$). Suppose that the class \mathcal{W}_X of weak equivalences is stable and the condition 2.0(d1) holds. Then

$$\bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i) = \mathcal{T} \circ \left(\bigcap_{i \in J} \mathcal{S}_i \right)$$

for any topologizing system \mathcal{T} .

(b) Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite family of systems which are left divisible in \mathfrak{E}_X (that is if $\mathfrak{s} \circ \mathfrak{e} \in \mathcal{S}_i$ and $\mathfrak{e} \in \mathfrak{E}_X$, then $\mathfrak{s} \in \mathcal{S}_i$). Suppose that \mathcal{T} is a system of deflations such that for any pair $\mathcal{L} \xleftarrow{\mathfrak{s}} \mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ of arrows of \mathcal{T} , there exists a cocartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \\ \mathfrak{t} \downarrow & \text{cocart} & \downarrow \tilde{\mathfrak{t}} \\ \mathcal{N} & \xrightarrow{\mathfrak{s}'} & \mathfrak{N} \end{array}$$

with $\tilde{\mathfrak{t}} \circ \mathfrak{s} \in \mathcal{T}$. Then

$$\bigcap_{i \in J} (\mathcal{S}_i \circ \mathcal{T}) = \left(\bigcap_{i \in J} \mathcal{S}_i \right) \circ \mathcal{T}.$$

Proof. The inclusions

$$\bigcap_{i \in J} (\mathcal{S}_i \circ \mathcal{T}) \supseteq \left(\bigcap_{i \in J} \mathcal{S}_i \right) \circ \mathcal{T} \quad \text{and} \quad \bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i) \supseteq \mathcal{T} \circ \left(\bigcap_{i \in J} \mathcal{S}_i \right)$$

hold for class of arrows \mathcal{T} and any family of classes of arrows $\{\mathcal{S}_i \mid i \in J\}$. The claim is that the inverse inclusions hold under respective conditions (a) and (b).

(a) Let \mathfrak{v} be an element of $\bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i)$, that is $\mathfrak{u} = \mathfrak{t}_i \circ \mathfrak{s}_i$, where $\mathfrak{s}_i \in \mathcal{S}_i$, $\mathfrak{t}_i \in \mathcal{T}$ and i runs through J . So that for any $i, j \in J$, we have a commutative square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{s}_i} & \mathcal{M}_i \\ \mathfrak{s}_j \downarrow & & \downarrow \mathfrak{t}_i \\ \mathcal{M}_j & \xrightarrow{\mathfrak{t}_j} & \mathcal{N} \end{array}$$

which is decomposed into the diagram

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{\gamma} & \mathfrak{M} & \xrightarrow{\tilde{\mathfrak{t}}_i} & \mathcal{M}_i \\ & & \mathfrak{t}'_j \downarrow & \text{cart} & \downarrow \mathfrak{t}_i \\ & & \mathcal{M}_j & \xrightarrow{\mathfrak{t}_j} & \mathcal{N} \end{array}$$

with a cartesian square, where the morphism $\mathcal{M} \xrightarrow{\gamma} \mathfrak{M}$ is uniquely determined by the equalities $\mathfrak{t}'_j \circ \gamma = \mathfrak{s}_j$ and $\tilde{\mathfrak{t}}_i \circ \gamma = \mathfrak{s}_i$. Since \mathfrak{s}_i and \mathfrak{s}_j are deflations, it follows from the condition 2.0(d1) that $\gamma = \mathfrak{w} \circ \mathfrak{e}$, where $\mathfrak{e} \in \mathfrak{E}_X$ and $\mathfrak{w} \in \mathcal{W}_X^\wedge$. Set $\mathfrak{u}_i = \tilde{\mathfrak{t}}_i \circ \mathfrak{w}$ and $\mathfrak{u}_j = \mathfrak{t}'_j \circ \mathfrak{w}$. Since $\mathfrak{s}_i = \mathfrak{u}_i \circ \mathfrak{e}$, $\mathfrak{s}_j = \mathfrak{u}_j \circ \mathfrak{e}$ and the classes \mathcal{S}_i and \mathcal{S}_j are right divisible in \mathfrak{E}_X , the deflation \mathfrak{e} belongs to $\mathcal{S}_i \cap \mathcal{S}_j$. The composition $\mathfrak{t} = \mathfrak{t}_i \circ \tilde{\mathfrak{t}}_i$ belongs to \mathcal{T} and $\mathfrak{t}_i \circ \mathfrak{u}_i = \mathfrak{t} \circ \mathfrak{w}$ is a deflation which belongs to $\mathcal{T} \circ \mathcal{W}_X^\wedge$. But, by the argument 2.4.4(b), $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge) = \mathcal{T}$. Thus, the element $\mathfrak{u} = \mathfrak{t}_i \circ \mathfrak{s}_i$ equals to the composition $(\mathfrak{t} \circ \mathfrak{w}) \circ \mathfrak{e}$, where $\mathfrak{e} \in \mathcal{S}_i \cap \mathcal{S}_j$ and $\mathfrak{t} \circ \mathfrak{w} \in \mathcal{T}$. This proves the inclusion $\bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i) \subseteq \mathcal{T} \circ \left(\bigcap_{i \in J} \mathcal{S}_i \right)$ in the case when $|J| = 2$. By an induction argument, it follows for an arbitrary finite J .

(b) Suppose now that the conditions (b) hold. Let \mathfrak{u} be an element of $\bigcap_{i \in J} (\mathcal{S}_i \circ \mathcal{T})$, that is $\mathfrak{u} = \mathfrak{s}_i \circ \mathfrak{t}_i$, where $\mathfrak{s}_i \in \mathcal{S}$, $\mathfrak{t}_i \in \mathcal{T}$ and i runs through J . Thus, for any $i, j \in J$, we have a commutative square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{t}_i} & \mathcal{M}_i \\ \mathfrak{t}_j \downarrow & & \downarrow \mathfrak{s}_i \\ \mathcal{M}_j & \xrightarrow{\mathfrak{s}_j} & \mathcal{N} \end{array}$$

which is decomposed into the diagram

$$\begin{array}{ccccc}
\mathcal{M} & \xrightarrow{\mathfrak{t}_i} & \mathcal{M}_i & & \\
\mathfrak{t}_j \downarrow & \text{cocart} & \downarrow \mathfrak{t}'_i & & \\
\mathcal{M}_j & \xrightarrow{\tilde{\mathfrak{t}}_j} & \mathcal{L} & \xrightarrow{\lambda} & \mathcal{N}
\end{array}$$

with a cocartesian square, where the morphism $\mathcal{L} \xrightarrow{\lambda} \mathcal{N}$ is uniquely determined by the equalities $\lambda \circ \tilde{\mathfrak{t}}_j = \mathfrak{s}_j$ and $\lambda \circ \mathfrak{t}'_i = \mathfrak{s}_i$. Since $\tilde{\mathfrak{t}}_j$ and \mathfrak{t}'_i are deflations and, by hypothesis, the classes \mathcal{S}_i and \mathcal{S}_j are left divisible in \mathfrak{E}_X , the morphism λ belongs to $\mathcal{S}_i \cap \mathcal{S}_j$. On the other hand, the composition $\tilde{\mathfrak{t}}_j \circ \mathfrak{t}_j$ belongs to \mathcal{T} , because \mathcal{T} is a topologizing system and both \mathfrak{t}_i and \mathfrak{t}_j are its elements. Thus, $\mathfrak{s}_i \circ \mathfrak{t}_i = \lambda \circ (\tilde{\mathfrak{t}}_j \circ \mathfrak{t}_j) = \mathfrak{s}_j \circ \mathfrak{t}_j$. The rest of the proof is the standard induction argument. ■

2.4.6. Proposition. *Suppose that $(C_X, \mathfrak{E}_X, \mathcal{W}_X)$ is such that $\mathcal{W}_X^{\bar{\cdot}} \circ \mathfrak{E}_X = \text{Hom}C_X$, the class \mathcal{W}_X of weak equivalences is stable and the condition 2.0(d1) holds. Then*

$$\bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i) = \mathcal{T} \circ \left(\bigcap_{i \in J} \mathcal{S}_i \right)$$

for any topologizing system \mathcal{T} and any finite set $\{\mathcal{S}_i \mid i \in J\}$ of classes of morphisms which are right divisible in \mathfrak{E}_X .

Proof. The argument is similar to that of 2.4.5(a). Details are left to the reader. ■

2.5. Thick systems of deflations. We call a system of deflations \mathcal{S} of $(C_X, \bar{\mathfrak{E}}_X)$ *thick* if it is left and right divisible in \mathfrak{E}_X , closed under compositions and stable. We denote by $\mathfrak{M}(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to the inclusion) of thick systems of $(C_X, \bar{\mathfrak{E}}_X)$.

It follows that \mathcal{W}_X is the smallest thick system of $(C_X, \bar{\mathfrak{E}}_X)$.

2.5.1. Example. Suppose that C_X has an initial object, \mathfrak{r} . Let \mathcal{T} be a strictly full subcategory of C_X containing initial objects and

$$\mathcal{S} = \mathfrak{E}_{X, \mathcal{T}} \stackrel{\text{def}}{=} \Sigma_{\mathcal{T}} \cap \mathfrak{E}_X = \{\mathfrak{s} \in \mathfrak{E}_X \mid \text{Ker}(\mathfrak{s}) \in \text{Ob}\mathcal{T}\}$$

(see 2.3.2). Suppose that the kernel of any deflation $M \xrightarrow{\mathfrak{e}} N$ with $M \in \text{Ob}\mathcal{T}$ belongs to the subcategory \mathcal{T} . Then it follows from the diagram 2.3.2(1) that the system $\mathcal{S} = \mathfrak{E}_{X, \mathcal{T}}$ is right divisible in \mathfrak{E}_X . Notice that the cartesian square (2) gives rise to a cartesian square

$$\begin{array}{ccc}
\text{Ker}(\mathfrak{t} \circ \mathfrak{s}') & \longrightarrow & \text{Ker}(\mathfrak{t}) \\
\downarrow & \text{cart} & \downarrow \\
\text{Ker}(\mathfrak{s}) & \longrightarrow & \mathfrak{r}
\end{array} \tag{3}$$

with all arrows in $\mathfrak{E}_{X, \mathcal{T}}$. The condition of 2.4.2 holds iff for each cartesian square (2) with arrows from $\mathfrak{E}_{X, \mathcal{T}}$, the composition $\text{Ker}(\mathfrak{t} \circ \mathfrak{s}') \longrightarrow \mathfrak{r}$ of consecutive arrows of (3) belongs

to $\mathfrak{E}_{X,\mathcal{T}}$. Notice that the object $(\text{Ker}(\mathfrak{t} \circ \mathfrak{s}'), \text{Ker}(\mathfrak{t} \circ \mathfrak{s}') \rightarrow \mathfrak{x})$ of the category C_X/\mathfrak{x} is the product of $(\text{Ker}(\mathfrak{s}), \text{Ker}(\mathfrak{s}) \rightarrow \mathfrak{x})$ and $(\text{Ker}(\mathfrak{t}), \text{Ker}(\mathfrak{t}) \rightarrow \mathfrak{x})$.

2.5.2. Proposition. *Suppose that each deflation has a coimage which is also a deflation, every morphism to an initial object is a deflation, and the class of deflations \mathfrak{E}_X is left divisible.*

(a) *The system $\mathfrak{E}_{X,\mathcal{T}}$ is topologising iff the subcategory \mathcal{T}/\mathfrak{x} is closed under finite products (taken in C_X/\mathfrak{x}) and for any deflation $\mathcal{M} \xrightarrow{\epsilon} \mathcal{N}$ with $\mathcal{M} \in \text{Ob}\mathcal{T}/\mathfrak{x}$, both $\text{Ker}(\epsilon)$ and \mathcal{N} are objects of \mathcal{T}/\mathfrak{x} .*

(b) *The system $\mathfrak{E}_{X,\mathcal{T}}$ is thick iff for any deflation $M \xrightarrow{\epsilon} N$ such that N has arrows to initial objects, M is an object of the subcategory \mathcal{T} then and only then the objects N and $\text{Ker}(\epsilon)$ belong to \mathcal{T} .*

Proof. The argument for (a) follows from the discussion above. The proof of (b) uses the commutative diagram 2.3.2(1). Details are left to the reader. ■

2.5.3. The case of a pointed category. If \mathfrak{x} is also a final object of C_X , then the categories C_X/\mathfrak{x} and C_X are naturally isomorphic and, therefore, $K(\mathfrak{t} \circ \mathfrak{s}')$ is isomorphic to the product of $\text{Ker}(\mathfrak{t})$ and $\text{Ker}(\mathfrak{s})$.

2.5.4. Topologizing and thick subcategories of exact and abelian categories. It follows that if (C_X, \mathfrak{E}_X) is an exact category, then $\mathfrak{E}_{X,\mathcal{T}}$ is a topologizing system iff the subcategory \mathcal{T} is closed under finite products and admissible subquotients. In particular, if (C_X, \mathfrak{E}_X) is an abelian category, then $\mathfrak{E}_{X,\mathcal{T}}$ is a topologizing system iff \mathcal{T} is a topologizing subcategory of C_X in the sense of Gabriel.

It follows from 2.5.2 that any thick subcategory of an exact category (C_X, \mathfrak{E}_X) is topologizing. If (C_X, \mathfrak{E}_X) is abelian, then thick categories are thick in the usual sense.

2.6. Serre systems.

Fix a svelte right exact category with weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$.

2.6.1. The closure. For a class \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$, let $\mathfrak{R}_{\mathcal{S}}$ denote the set of all systems of deflations Σ divisible in \mathfrak{E}_X such that any *non-trivial* right divisible subsystem Σ' of Σ has a non-trivial intersection with \mathcal{S} (that is $\mathcal{S} \cap \Sigma' - \mathcal{W}_X$ is non-empty). We denote by \mathcal{S}^- the union of all $\Sigma \in \mathfrak{R}_{\mathcal{S}}$ and call it the *closure* of \mathcal{S} .

2.6.2. Proposition. (a) \mathcal{S}^- belongs to $\mathfrak{R}_{\mathcal{S}}$ (hence it is the largest element of $\mathfrak{R}_{\mathcal{S}}$).

(b) $(\mathcal{S}^-)^- = \mathcal{S}^-$.

(c) The system \mathcal{S}^- is closed under the composition.

(d) Suppose that the class \mathcal{W}_X of weak equivalences is stable. Then the system \mathcal{S}^- is stable, that is $\mathcal{S}^- = \mathfrak{E}_X \cap (\mathcal{S}^-)^\wedge$.

Proof. (a) Since divisible systems are closed under arbitrary unions, \mathcal{S}^- is a divisible system. Let Σ be a non-trivial right divisible subsystem of \mathcal{S}^- . Then there exists $\Sigma' \in \mathfrak{R}_{\mathcal{S}}$ such that $\Sigma' \cap \Sigma$ is a non-trivial right divisible system. Since it is a subsystem of Σ' and $\Sigma' \in \mathfrak{R}_{\mathcal{S}}$, the intersection $\Sigma' \cap \Sigma \cap \mathcal{S}$ is non-trivial. In particular, $\Sigma \cap \mathcal{S}$ is non-trivial.

(b) It follows from the argument (a) that $\mathfrak{R}_{\mathcal{S}} = \mathfrak{R}_{\mathcal{S}^-}$; hence $(\mathcal{S}^-)^- = \mathcal{S}^-$.

(c) Let Σ be a non-trivial right divisible in \mathfrak{E}_X system contained in $\mathcal{S}^- \circ \mathcal{S}^-$. Let $\mathfrak{t}, \mathfrak{s}$ be elements of \mathcal{S}^- such that $\mathfrak{t} \circ \mathfrak{s} \in \Sigma - \mathcal{W}_X$. Since Σ is right divisible, it contains \mathfrak{s} . Suppose that \mathfrak{s} is non-trivial, that is $\mathfrak{s} \notin \mathcal{W}_X$. Take any right divisible subsystem of Σ containing element \mathfrak{s} and denote by $\tilde{\Sigma}$ its intersection with \mathcal{S}^- . Thus, $\tilde{\Sigma}$ is a non-trivial right divisible subsystem of $\mathcal{S}^- \cap \Sigma$, hence it has a non-trivial intersection with \mathcal{S} . If $\mathfrak{s} \in \mathcal{W}_X$, then Σ contains $\mathfrak{t} \circ \mathfrak{s}$, and we apply the argument above to $\mathfrak{t} \circ \mathfrak{s}$ itself. This shows that $\mathcal{S}^- \circ \mathcal{S}^- \in \mathfrak{F}_S$, or, equivalently, $\mathcal{S}^- \circ \mathcal{S}^- = \mathcal{S}^-$.

(d) It follows from the definition of \mathcal{S}^- that it coincides with the union of all divisible systems of deflations \mathcal{T} such that $\mathcal{T} \cap \mathcal{S}^\perp = \mathcal{W}_X$. One can consider only *stable* systems \mathcal{T} , because, by 2.3.1,

$$(\mathfrak{E}_X \cap \mathcal{T}^\perp) \cap \mathcal{S}^\perp = \mathfrak{E}_X \cap (\mathcal{T}^\perp \cap (\mathcal{S}^\perp)^\perp) = \mathfrak{E}_X \cap (\mathcal{T} \cap \mathcal{S}^\perp)^\perp = \mathfrak{E}_X \cap \mathcal{W}_X^\perp = \mathcal{W}_X.$$

It follows from 1.6.1 that the union of stable systems is a stable system, hence \mathcal{S}^- is a stable system. ■

2.6.2.1. Note. One can see that $\mathcal{W}_X^\perp = \mathcal{W}_X$. In fact, \mathcal{W}_X^\perp coincides with $\text{Hom}C_X$, whence the equality $\mathcal{T} \cap \mathcal{W}_X^\perp = \mathcal{W}_X$ for a system \mathcal{T} means precisely that $\mathcal{T} = \mathcal{W}_X$.

2.6.3. Serre systems of deflations. We call a class \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ a *Serre system* of deflations if $\mathcal{S}^- = \mathcal{S}$. We denote by $\mathfrak{S}\mathfrak{e}(X, \mathfrak{E}_X)$ the preorder (with respect to the inclusion) of all Serre systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$.

It follows from this definition and 2.6.2 that Serre systems of deflations are thick.

2.6.4. Proposition. *Let (C_X, \mathfrak{E}_X) be a svelte right exact category.*

(a) *The intersection of any family of Serre systems of deflations of (C_X, \mathfrak{E}_X) is a Serre system.*

(b) *Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite set of right divisible systems of deflations of (C_X, \mathfrak{E}_X) . Then $\bigcap_{i \in J} \mathcal{S}_i^- = \left(\bigcap_{i \in J} \mathcal{S}_i\right)^-$.*

Proof. (a) Let $\{\Sigma_j \mid j \in \mathfrak{J}\}$ be a set of Serre systems of deflations of (C_X, \mathfrak{E}_X) . Let \mathcal{S} be a divisible system of deflations such that every non-trivial right divisible subsystem \mathfrak{G} of \mathcal{S} has a non-trivial intersection with $\bigcap_{j \in \mathfrak{J}} \Sigma_j$. In particular, $\mathfrak{G} \cap \Sigma_j$ is non-trivial for every $j \in \mathfrak{J}$. Since $\Sigma_j = \Sigma_j^-$ for all $j \in \mathfrak{J}$, it follows that $\mathcal{S} \subseteq \Sigma_j$ for all $j \in \mathfrak{J}$; that is $\mathcal{S} \subseteq \bigcap_{j \in \mathfrak{J}} \Sigma_j$. This shows that $\bigcap_{j \in \mathfrak{J}} \Sigma_j = \left(\bigcap_{j \in \mathfrak{J}} \Sigma_j\right)^-$.

(b) Let $\{\mathcal{S}_i \mid i \in J\}$ be a set of right divisible systems of deflations of (C_X, \mathfrak{E}_X) . Then, evidently, $\bigcap_{i \in J} \mathcal{S}_i^- \supseteq \left(\bigcap_{i \in J} \mathcal{S}_i\right)^-$. If J is finite, then the inverse inclusion holds.

Since, by (a) above, $\bigcap_{i \in J} \mathcal{S}_i^-$ is a Serre system of deflations, it suffices to show that any non-trivial right divisible subsystem of $\bigcap_{i \in J} \mathcal{S}_i^-$ has a non-trivial intersection with $\bigcap_{i \in J} \mathcal{S}_i$.

Let $J = \{1, 2, \dots, n\}$, and let \mathcal{T} be a non-trivial right divisible subsystem of $\bigcap_{i \in J} \mathcal{S}_i^-$.

In particular, \mathcal{T} is a non-trivial right divisible subsystem of \mathcal{S}_1^- . Therefore, $\mathcal{T} \cap \mathcal{S}_1$ is a non-trivial right divisible subsystem of $\bigcap_{2 \leq i \leq n} \mathcal{S}_i^-$. By a standard induction argument, this

implies that $(\mathcal{T} \cap \mathcal{S}_1) \cap \left(\bigcap_{2 \leq i \leq n} \mathcal{S}_i \right) = \mathcal{T} \cap \left(\bigcap_{i \in J} \mathcal{S}_i \right)$ is a non-trivial right divisible system.

Therefore, $\mathcal{T} \subseteq \left(\bigcap_{i \in J} \mathcal{S}_i \right)^-$. In particular, $\bigcap_{i \in J} \mathcal{S}_i^- \subseteq \left(\bigcap_{i \in J} \mathcal{S}_i \right)^-$. ■

2.6.5. The lattice of Serre systems. Fix a svelte right exact category (C_X, \mathfrak{E}_X) . For any pair Σ_1, Σ_2 of Serre systems of deflations, we denote by $\Sigma_1 \vee \Sigma_2$ the smallest Serre system containing Σ_1 and Σ_2 .

2.6.5.1. Proposition. *Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite set of Serre systems of deflations of (C_X, \mathfrak{E}_X) . Then $\Sigma \vee \left(\bigcap_{i \in J} \mathcal{S}_i \right) = \bigcap_{i \in J} (\Sigma \vee \mathcal{S}_i)$ for any Serre system of deflations Σ .*

Proof. There are the equalities

$$\bigcap_{i \in J} (\Sigma \vee \mathcal{S}_i) = \bigcap_{i \in J} (\Sigma \cup \mathcal{S}_i)^- = \left(\bigcap_{i \in J} (\Sigma \cup \mathcal{S}_i) \right)^- = \left(\left(\bigcap_{i \in J} \mathcal{S}_i \right) \cup \Sigma \right)^- = \left(\bigcap_{i \in J} \mathcal{S}_i \right) \vee \Sigma.$$

Here the second equality follows from 2.6.4. ■

2.7. Serre subcategories of a right exact category with initial objects.

Suppose that the category C_X has an initial object, \mathfrak{x} and all morphisms to \mathfrak{x} are deflations. Let \mathcal{S} be a class of deflations of (C_X, \mathfrak{E}_X) . We denote by $\tilde{\mathcal{T}}_{\mathcal{S}}$ the full subcategory of the category C_X generated by all $M \in \text{Ob}C_X$ having the following property: for any pair of deflations $M \xrightarrow{\epsilon} L \xrightarrow{\mathfrak{t}} \mathfrak{x}$ such that \mathfrak{t} is non-trivial (i.e. $\mathfrak{t} \notin \mathcal{W}_X$), there exists a decomposition $\mathfrak{t} = \mathfrak{u} \circ \mathfrak{s}$, where \mathfrak{u} and \mathfrak{s} are deflations and \mathfrak{s} is a non-trivial element of \mathcal{S} . We denote by $\mathcal{T}_{\mathcal{S}}$ the full subcategory of C_X generated by all $M \in \text{Ob}C_X$ such that for any pair of deflations $M \xrightarrow{\mathfrak{u}} L \xrightarrow{\mathfrak{t}} \mathfrak{x}$ the object $\text{Ker}(\mathfrak{u})$ belongs to the subcategory $\tilde{\mathcal{T}}_{\mathcal{S}}$.

It follows from the definition of $\mathcal{T}_{\mathcal{S}}$ that if M is an object of $\mathcal{T}_{\mathcal{S}}$ and $M \xrightarrow{\epsilon} L \xrightarrow{\mathfrak{t}} \mathfrak{x}$ are deflations, then $L \in \text{Ob}\mathcal{T}_{\mathcal{S}}$.

In fact, let $L \xrightarrow{\mathfrak{u}} N \xrightarrow{\mathfrak{t}} \mathfrak{x}$ be deflations. Then we have a commutative diagram

$$\begin{array}{ccccccc} \text{Ker}(\mathfrak{u} \circ \epsilon) & \xrightarrow{\epsilon'} & \text{Ker}(\mathfrak{u}) & \xrightarrow{\lambda_{\mathfrak{u}}} & \mathfrak{x} & & \\ \downarrow & \text{cart} & \downarrow & \text{cart} & \downarrow & & \\ M & \xrightarrow{\epsilon} & L & \xrightarrow{\mathfrak{u}} & N & \longrightarrow & \mathfrak{x} \end{array}$$

in which all horizontal arrows are deflations and $\text{Ker}(\mathfrak{u} \circ \epsilon)$ is an object of $\tilde{\mathcal{T}}_{\mathcal{S}}$. Therefore, $\text{Ker}(\mathfrak{u}) \in \text{Ob}\tilde{\mathcal{T}}_{\mathcal{S}}$, which implies that $L \in \text{Ob}\mathcal{T}_{\mathcal{S}}$.

By 2.3.2, the latter property implies that $\Sigma_{\mathcal{T}_{\mathcal{S}}}$ is a left divisible system.

2.7.1. Proposition. Let $(C_X, \bar{\mathfrak{E}}_X)$ be a right exact category with an initial object \mathfrak{x} and the class of weak equivalences coinciding with $\mathfrak{E}_X^{\otimes} = \{\mathfrak{e} \in \mathfrak{E}_X \mid Ker(\mathfrak{e}) \simeq \mathfrak{x}\}$.

If \mathcal{S} is a class of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ closed under pull-backs, then $\Sigma_{\mathcal{T}_{\mathcal{S}}} = \mathcal{S}^-$.

Proof. (a) The system $\Sigma_{\mathcal{T}_{\mathcal{S}}}$ belongs to $\mathfrak{R}_{\mathcal{S}}$; in particular, $\Sigma_{\mathcal{T}_{\mathcal{S}}} \subseteq \mathcal{S}^-$.

In fact, let $M \xrightarrow{\mathfrak{t}} N$ be an element of $\Sigma_{\mathcal{T}_{\mathcal{S}}} - \mathcal{W}_X$. By condition, this means that $Ker(\mathfrak{t})$ is non-trivial (i.e. it is not an initial) object of the subcategory $\mathcal{T}_{\mathcal{S}}$. Therefore, the canonical morphism $Ker(\mathfrak{t}) \xrightarrow{\mathfrak{e}_{\mathfrak{t}}} \mathfrak{x}$ is the composition of a non-trivial arrow $Ker(\mathfrak{t}) \xrightarrow{\mathfrak{s}} L$ of \mathcal{S} and a deflation $L \rightarrow \mathfrak{x}$. This shows that any right divisible system containing the arrow $M \xrightarrow{\mathfrak{t}} N$ has a non-trivial morphism from \mathcal{S} ; hence $\Sigma_{\mathcal{T}_{\mathcal{S}}} \subseteq \mathcal{S}^-$.

(b) It remains to show that $\mathcal{S}^- \subseteq \Sigma_{\mathcal{T}_{\mathcal{S}}}$.

Suppose this is not true, and let $M \xrightarrow{\mathfrak{s}} L$ be an arrow from \mathcal{S}^- which does not belong to $\Sigma_{\mathcal{T}_{\mathcal{S}}}$; that is $Ker(\mathfrak{s})$ is not an object of $\mathcal{T}_{\mathcal{S}}$, which means that the canonical deflation $Ker(\mathfrak{s}) \xrightarrow{\mathfrak{e}_{\mathfrak{s}}} \mathfrak{x}$ factors through a deflation $Ker(\mathfrak{s}) \xrightarrow{\mathfrak{v}} N$ such that $Ker(\mathfrak{v}) \xrightarrow{\lambda_{\mathfrak{v}}} \mathfrak{x}$ is a composition of two deflations, $Ker(\mathfrak{v}) \xrightarrow{\mathfrak{u}} L$ and $L \xrightarrow{\mathfrak{t}} \mathfrak{x}$, where \mathfrak{t} is non-trivial and \mathcal{S} -torsion free in the sense that if $\mathfrak{t} = \mathfrak{t}' \circ \gamma$ and $\gamma \in \mathcal{S}$, then $\gamma \in \mathcal{W}_X$. Since \mathcal{S}^- is a left divisible system of deflations, the deflation $L \xrightarrow{\mathfrak{t}} \mathfrak{x}$ belongs to \mathcal{S}^- .

Consider the smallest right divisible system generated by the morphism $L \xrightarrow{\mathfrak{t}} \mathfrak{x}$. It consists of all deflations $\mathcal{M} \xrightarrow{\mathfrak{e}} \tilde{\mathcal{N}}$ such that there is a deflation $\tilde{\mathcal{N}} \xrightarrow{\mathfrak{w}} \mathcal{N}$ and a cartesian square

$$\begin{array}{ccc} L & \xrightarrow{\mathfrak{t}} & \mathfrak{x} \\ \uparrow & \text{cart} & \uparrow \\ \mathcal{M} & \xrightarrow{\mathfrak{w} \circ \mathfrak{e}} & \mathcal{N} \end{array}$$

Since the composition of cartesian squares is a cartesian square, we have a decomposition

$$\begin{array}{ccccc} L & \longrightarrow & \mathcal{M} & \longrightarrow & L \\ \mathfrak{t} \downarrow & \text{cart} & \mathfrak{w} \downarrow & \text{cart} & \downarrow \mathfrak{t} \\ \mathfrak{x} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathfrak{x} \end{array} \quad \text{of} \quad \begin{array}{ccc} L & \xrightarrow{id} & L \\ \mathfrak{t} \downarrow & & \downarrow \mathfrak{t} \\ \mathfrak{x} & \xrightarrow{id} & \mathfrak{x} \end{array}$$

and a decomposition

$$\begin{array}{ccccc} L & \xrightarrow{\tilde{\mathfrak{e}}} & Ker(\mathfrak{w}) & \xrightarrow{\mathfrak{e}_{\mathfrak{w}}} & \mathfrak{x} \\ \downarrow & \text{cart} & \downarrow & \text{cart} & \downarrow \\ \mathcal{M} & \xrightarrow{\mathfrak{e}} & \tilde{\mathcal{N}} & \xrightarrow{\mathfrak{w}} & \mathcal{N} \end{array} \quad \text{of} \quad \begin{array}{ccc} L & \xrightarrow{\mathfrak{t}} & \mathfrak{x} \\ \downarrow & \text{cart} & \downarrow \\ \mathcal{M} & \xrightarrow{\mathfrak{w} \circ \mathfrak{e}} & \mathcal{N} \end{array}$$

If $\mathcal{M} \xrightarrow{\mathfrak{e}} \tilde{\mathcal{N}}$ is a non-trivial element of \mathcal{S} , then, since (by hypothesis, $Ker(\mathfrak{e})$ is non-trivial and) $Ker(\tilde{\mathfrak{e}}) \simeq Ker(\mathfrak{e})$, the arrow $L \xrightarrow{\tilde{\mathfrak{e}}} Ker(\mathfrak{w})$ is a non-trivial element of \mathcal{S} , which contradicts to the condition on $L \xrightarrow{\mathfrak{t}} \mathfrak{x}$. ■

2.8. Coreflective systems and Serre systems. Let $(C_X, \bar{\mathfrak{E}}_X)$ be a right exact category with weak equivalences and \mathcal{S} a class of its deflations containing \mathcal{W}_X . We call the

class \mathcal{S} *coreflective* if every deflation $M \xrightarrow{\epsilon} L$ is the composition of an arrow $M \xrightarrow{\mathfrak{s}_\epsilon} N$ of \mathcal{S} and a deflation $N \xrightarrow{\gamma_\epsilon} L$ such that any other decomposition $M \xrightarrow{\mathfrak{t}} \mathfrak{N} \xrightarrow{\mathfrak{u}} L$ of ϵ with $\mathfrak{t} \in \mathcal{S}$ factors through $M \xrightarrow{\mathfrak{s}_\epsilon} N \xrightarrow{\gamma_\epsilon} L$. The latter means that there exists a deflation $\mathfrak{N} \xrightarrow{\mathfrak{v}} N$ such that $\mathfrak{s}_\epsilon = \mathfrak{v} \circ \mathfrak{t}$ and $\mathfrak{u} = \gamma_\epsilon \circ \mathfrak{v}$. Since \mathfrak{t} is an epimorphism, the first equality implies that \mathfrak{v} is uniquely defined.

2.8.1. Proposition. *Every coreflective system of deflations which is stable under base change and closed under compositions is a Serre system.*

Proof. In fact, each deflation $M \xrightarrow{\epsilon} L$ has the biggest decomposition $M \xrightarrow{\mathfrak{s}_\epsilon} N \xrightarrow{\gamma_\epsilon} L$, where $\mathfrak{s}_\epsilon \in \mathcal{S}$. Since \mathcal{S} is closed under composition, γ_ϵ has only a trivial decomposition. Therefore, $\mathcal{S}^- = \mathcal{S}$. ■

3. The spectra related with topologizing, thick and Serre systems.

Fix a svelte right exact category with a stable class of weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$. Recall that $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ denotes the preorder of all topologizing systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$, $\mathfrak{S}\mathfrak{e}(X, \bar{\mathfrak{E}}_X)$ the preorder of all Serre systems and $\mathfrak{M}(X, \bar{\mathfrak{E}}_X)$ the preorder of all thick systems of $(C_X, \bar{\mathfrak{E}}_X)$. We denote by $\mathfrak{M}_{\mathfrak{T}}(X, \bar{\mathfrak{E}}_X)$ (resp. by $\mathfrak{S}\mathfrak{e}_{\mathfrak{T}}(X, \bar{\mathfrak{E}}_X)$) the subpreorder of all thick (resp. Serre) *topologizing* systems. That is

$$\mathfrak{M}_{\mathfrak{T}}(X, \bar{\mathfrak{E}}_X) = \mathfrak{M}(X, \bar{\mathfrak{E}}_X) \cap \mathfrak{T}(X, \bar{\mathfrak{E}}_X) \quad \text{and} \quad \mathfrak{S}\mathfrak{e}_{\mathfrak{T}}(X, \bar{\mathfrak{E}}_X) = \mathfrak{S}\mathfrak{e}(X, \bar{\mathfrak{E}}_X) \cap \mathfrak{T}(X, \bar{\mathfrak{E}}_X).$$

3.1. The support in topologizing systems. For any class \mathcal{S} of the deflations of $(C_X, \bar{\mathfrak{E}}_X)$ containing the class \mathcal{W}_X of weak equivalences, we denote by $\text{Supp}_{\mathfrak{T}}(\mathcal{S})$ the subpreorder of $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ formed by all topologizing systems which do not contain \mathcal{S} , and call it *the support of \mathcal{S} in topologizing systems*.

We denote by $\widehat{\mathcal{S}}$ the union of all systems of $\text{Supp}_{\mathfrak{T}}(\mathcal{S})$. It follows that the inclusion $\mathcal{S}_1 \subseteq \mathcal{S}_2$ implies that $\widehat{\mathcal{S}}_1 \subseteq \widehat{\mathcal{S}}_2$. If \mathcal{S}_2 is topologizing, then the inverse implication holds: if $\widehat{\mathcal{S}}_1 \subseteq \widehat{\mathcal{S}}_2$, then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ (because, if $\mathcal{S}_1 \not\subseteq \mathcal{S}_2$, then $\mathcal{S}_2 \subseteq \widehat{\mathcal{S}}_1$, but, $\mathcal{S}_2 \not\subseteq \widehat{\mathcal{S}}_2$). Let $[\mathcal{S}]$ denote the smallest topologizing system containing \mathcal{S} . It is clear that $\widehat{\mathcal{S}} = [\widehat{\mathcal{S}}]$.

Finally, notice that if $\mathcal{S} \supseteq \mathcal{S}_1 \not\subseteq \widehat{\mathcal{S}}$, then $\widehat{\mathcal{S}} \subseteq \widehat{\mathcal{S}}_1 \subseteq \widehat{\mathcal{S}}$, that is $\widehat{\mathcal{S}}_1 = \widehat{\mathcal{S}}$.

The system $\widehat{\mathcal{S}}$ is the largest element of $\text{Supp}_{\mathfrak{T}}(\mathcal{S})$ whenever $\widehat{\mathcal{S}}$ is topologizing. The following assertion provides sufficient conditions for this occurrence.

3.1.1. Lemma. *Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ be a svelte right exact category with a stable class of weak equivalences (condition 2.0(b)) and with a left divisible a weakly right divisible class \mathfrak{E}_X of deflations (conditions 2.0(c) and 2.0(d)). Suppose also that the condition 2.0(e') holds. If \mathcal{S} is a class of deflations such that the system $\widehat{\mathcal{S}}$ is multiplicative, then $\widehat{\mathcal{S}}$ is topologizing.*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2$ be topologizing systems from the support of \mathcal{S} . If $\widehat{\mathcal{S}}$ is multiplicative, then $\mathcal{T}_1 \circ \mathcal{T}_2 \subseteq \widehat{\mathcal{S}}$. By 2.4.4, the system $\mathcal{T}_1 \circ \mathcal{T}_2$ is topologizing and it contains \mathcal{T}_1 and \mathcal{T}_2 . This shows that the support of \mathcal{S} is filtered, hence $\widehat{\mathcal{S}}$ is a topologizing system. ■

3.2. The spectrum $\mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X)$. The elements of the spectrum $\mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X)$ are all topologizing systems \mathcal{S} such that $\widehat{\mathcal{S}}$ is a Serre system, i.e. $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}^-$. We endow $\mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X)$ with the preorder \supseteq called (with a good reason) the *specialization* preorder.

3.3. The spectra $\mathbf{Spec}_t^1(X, \bar{\mathfrak{E}}_X)$ and $\mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X)$. For any system \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$, let \mathcal{S}^* denote the intersection of all topologizing systems properly containing \mathcal{S} . We denote by $\mathbf{Spec}_t^1(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to \supseteq) of all thick topologizing systems of deflations Σ such that $\Sigma^* \neq \Sigma$ and set

$$\mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X) = \mathbf{Spec}_t^1(X, \bar{\mathfrak{E}}_X) \cap \mathfrak{S}\mathfrak{e}(X, \bar{\mathfrak{E}}_X).$$

Thus, the spectrum $\mathbf{Spec}_t^1(X, \bar{\mathfrak{E}}_X)$ is the disjoint union of

$$\begin{aligned} \mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X) &= \{\Sigma \in \mathfrak{T}(X, \bar{\mathfrak{E}}_X) \mid \Sigma = \Sigma^- \subsetneq \Sigma^*\} \quad \text{and} \\ \mathbf{Spec}_t^{1,0}(X, \bar{\mathfrak{E}}_X) &= \{\Sigma \in \mathfrak{M}_{\mathfrak{T}}(X, \bar{\mathfrak{E}}_X) \mid \Sigma \neq \Sigma^* \subseteq \Sigma^-\}. \end{aligned}$$

3.4. Proposition. *Suppose that $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ is a svelte right exact category with a stable class of weak equivalences (condition 2.0(b)) and with a left divisible a weakly right divisible class \mathfrak{E}_X of deflations (conditions 2.0(c) and 2.0(d)). Then there is a natural isomorphism*

$$\mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X) \xrightarrow{\sim} \mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X).$$

Proof. Consider the map which assigns to each $\Sigma \in \mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X)$ the union Σ_* of all right divisible in \mathfrak{E}_X subsystems of Σ^* which have trivial intersection with Σ . Notice that, since Σ is a Serre system, the right divisible system Σ_* is a non-trivial. The claim is that the topologizing system $[\Sigma_*]$ spanned by Σ_* (which is a topologizing subsystem of the topologizing system Σ^*) is an element of the spectrum $\mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X)$.

(i) Observe that $\Sigma \subseteq \widehat{\Sigma}_*$, because the system Σ is topologizing and the equality $\Sigma_* \cap \Sigma = \mathcal{W}_X$ combined with the non-triviality of the system Σ_* implies that $\Sigma_* \not\subseteq \Sigma$.

(ii) On the other hand, if \mathcal{S} is a topologizing system of deflations which is not contained in the system Σ , then $\Sigma_* \subseteq \mathcal{S}$.

In fact, suppose that $\mathcal{S} \not\subseteq \Sigma$. Then, by 2.4.4(b), the composition $\mathcal{S} \circ \Sigma$ is a topologizing system properly containing Σ . Therefore,

$$\Sigma_* \subseteq (\mathcal{S} \circ \Sigma) \cap \Sigma^\perp \subseteq \mathcal{S} \circ (\Sigma \cap \Sigma^\perp) = \mathcal{S} \circ \mathcal{W}_X = \mathcal{S}.$$

In other words, if \mathcal{S} is a topologizing system which does not contain Σ_* , then $\mathcal{S} \subseteq \Sigma$. This proves the inverse inclusion, $\widehat{\Sigma}_* \subseteq \Sigma$, hence the equality $\Sigma^- = \Sigma = \widehat{\Sigma}_*$. As it is observed in 3.2, $\widehat{\Sigma}_* = \widehat{[\Sigma_*]}$; so that $[\Sigma_*]$ is an element of the spectrum $\mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X)$. Thus, we obtained a map

$$\mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X) \longrightarrow \mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X), \quad \Sigma \longmapsto [\Sigma_*]. \quad (1)$$

Let now $\mathcal{S} \in \mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X)$, that is \mathcal{S} is a topologizing system such that $\widehat{\mathcal{S}}$ is a Serre system. Since, by 2.6.2(c), any Serre system is multiplicative, it follows from 3.1.1 that the system $\widehat{\mathcal{S}}$ is topologizing. If Σ is a topologizing system properly containing $\widehat{\mathcal{S}}$, then Σ contains \mathcal{S} . This shows that $\widehat{\mathcal{S}}^*$ coincides with the smallest topologizing system containing $\widehat{\mathcal{S}} \cup \mathcal{S}$; in particular, $\widehat{\mathcal{S}}^- = \widehat{\mathcal{S}} \neq \widehat{\mathcal{S}}^*$, i.e. $\widehat{\mathcal{S}}$ is an element of the spectrum $\mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X)$. One can see that the map

$$\mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X) \longrightarrow \mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X), \quad \mathcal{S} \longmapsto \widehat{\mathcal{S}},$$

is inverse to the map (1). ■

3.5. Remark. It follows from the argument of 3.4 that the map

$$\Sigma \longmapsto \Sigma_*, \quad \Sigma \in \mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X),$$

gives a canonical realization of $\mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X)$ as the preorder of systems of deflations \mathcal{S} which are characterized by the following properties:

- (a) $\widehat{\mathcal{S}}$ is a Serre system and $\mathcal{S} \cap \widehat{\mathcal{S}} = \mathcal{W}_X$;
- (b) if a system \mathcal{T} of deflations is such that $\widehat{\mathcal{T}} = \widehat{\mathcal{S}}$ and $\mathcal{T} \cap \widehat{\mathcal{S}} = \mathcal{W}_X$, then $\mathcal{T} \subseteq \mathcal{S}$.

Notice that for every such system \mathcal{S} , the corresponding Serre system $\widehat{\mathcal{S}}$ coincides with the union $\widehat{\mathcal{S}}$ of all topologizing systems of deflations Σ such that $\mathcal{S} \cap \Sigma = \mathcal{W}_X$.

3.6. Local right exact 'spaces' and categories with weak equivalences. Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ be a svelte right exact category with weak equivalences. We call $(C_X, \bar{\mathfrak{E}}_X)$ (and the *right exact* 'space' $(X, \bar{\mathfrak{E}}_X)$ it represents) *local* if there is the smallest non-trivial topologizing system, or, equivalently, the intersection \mathcal{W}_X^* of all non-trivial topologizing systems of (C_X, \mathfrak{E}_X) is non-trivial.

It follows that a right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ is *local* iff the spectrum $\mathbf{Spec}_t(X, \bar{\mathfrak{E}}_X)$ has a unique closed point, and this closed point belongs to the support of any non-trivial divisible system of (C_X, \mathfrak{E}_X) .

3.7. The spectrum $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$. The elements of this spectrum are all Serre systems Σ such that the intersection Σ^* of all thick systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ properly containing Σ is not equal to Σ . Equivalently, $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ consists of all Serre systems Σ such that $\Sigma_* \stackrel{\text{def}}{=} \Sigma^* \cap \Sigma^{bot}$ is non-trivial. As all other spectra, the spectrum $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ is endowed with the *specialization* preorder \supseteq .

One of the most essential properties of the spectrum $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ is the following.

3.7.1. Proposition. *Let $(X, \bar{\mathfrak{E}}_X)$ be a right exact 'space' and $\Sigma \in \mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$. For any finite family $\{\mathcal{S}_i \mid i \in \mathfrak{J}\}$ of right divisible in \mathfrak{E}_X systems of deflations, $\mathcal{S}_i \not\subseteq \Sigma$ for all $i \in \mathfrak{J}$ iff $\bigcap_{i \in \mathfrak{J}} \mathcal{S}_i \not\subseteq \Sigma$.*

Proof. By 2.6.4, $\bigcap_{i \in J} \mathcal{S}_i^- = (\bigcap_{i \in J} \mathcal{S}_i)^-$, and by 2.6.5.1, $\Sigma \vee (\bigcap_{i \in J} \mathcal{S}_i^-) = \bigcap_{i \in J} (\Sigma \vee \mathcal{S}_i^-)$.

Therefore,

$$\Sigma \vee (\bigcap_{i \in J} \mathcal{S}_i)^- = \bigcap_{i \in J} (\Sigma \vee \mathcal{S}_i^-). \quad (2)$$

If $\mathcal{S}_i \not\subseteq \Sigma$ for all $i \in J$, then each of the strongly closed systems $\mathcal{S}_i^- \vee \Sigma$ contains Σ properly. Since Σ is an element of the spectrum $\mathbf{Spec}_{sc}^{1,1}(X, \mathfrak{E}_X)$, the intersection of $\mathcal{S}_i^- \vee \Sigma$, $i \in J$, contains Σ properly. Then it follows from the equality (5) that the intersection $\bigcap_{i \in J} \mathcal{S}_i$ is not contained in Σ . ■

Since the spectrum $\mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X)$ is contained in $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$, the elements of $\mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X)$ have the property described in 3.7.1.

4. Semitopologizing systems and the related spectral theory.

The topological systems defined in 2.4 might be inconvenient in some situations, because they require invariance of deflations under push-forwards, which is not necessarily available in right exact, or even exact categories. There is a different setting based on the notion of a *semitopological system*, which does not require push-forwards and still recovers the abelian theory. It is sketched below.

4.0. Conventions. We fix a svelte right exact category with a stable class of weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ such that $\mathcal{W}_X \circ \mathcal{W}_X^\wedge = \mathcal{W}_X^\wedge$. We assume that the category C_X has fiber products.

4.1. Strongly stable, cartesian complete and semitopologizing systems.

(i) A class \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ will be called *strongly stable* if it is invariant under pull-backs, stable (that is $\mathcal{S} = \mathfrak{E}_X \cap \mathcal{S}^\wedge$) and, in addition,

$$\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge). \quad (1)$$

(ii) We call a class of deflations \mathcal{S} *cartesian complete* if, for any cartesian square with arrows in \mathcal{S} , the composition of two consecutive arrows belongs to \mathcal{S} .

(iii) We call a system of deflations *right semitopologizing* (resp. *left semitopologizing*) if it is cartesian complete, strongly stable, and right (resp. left) divisible in \mathfrak{E}_X .

We say that a system *semitopologizing* if it is both left and right semitopologizing.

4.1.1. Topological and semitopological systems. Suppose that the class \mathfrak{E}_X of deflations is *left divisible* in the following sense: if $\mathfrak{t} \circ \mathfrak{s} \in \mathfrak{E}_X \ni \mathfrak{s}$, then $\mathfrak{t} \in \mathfrak{E}_X$. Then every left (resp. right) topologizing system of deflations is left (resp. right) semitopologizing.

In fact, any left (resp. right) topologizing system is, by definition, cartesian complete and, by 2.4.3 (or 1.10.2), stable. If \mathcal{T} is a left (or/and right) topologizing system, then $\mathcal{T} \circ \mathcal{W}_X^\wedge \subseteq \mathcal{W}_X^\wedge \circ \mathcal{T}$ (see 2.4.2(c)), so that

$$\mathcal{T} \subseteq \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge) \subseteq \mathfrak{E}_X \cap (\mathcal{W}_X^\wedge \circ \mathcal{T}) = (\mathfrak{E}_X \cap \mathcal{W}_X^\wedge) \circ \mathcal{T} = \mathcal{W}_X \circ \mathcal{T} = \mathcal{T},$$

whence $\mathcal{T} = \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge)$. Here the first equality is due to the left divisibility of \mathfrak{E}_X and the second one to the stability of \mathcal{W}_X .

4.1.2. About cartesian completeness. Let C_X have an initial object, \mathfrak{r} , and let \mathbb{T} be a full subcategory of the category C_X/\mathfrak{r} . Consider the system $\mathcal{S}_{\mathbb{T}}$ of all deflations \mathfrak{s}

of $(C_X, \bar{\mathfrak{E}}_X)$ such that $(Ker(\mathfrak{s}), Ker(\mathfrak{s}) \longrightarrow \mathfrak{r})$ is an object of \mathbb{T} . The class of arrows $\mathcal{S}_{\mathbb{T}}$ is cartesian complete iff \mathbb{T} is a category with finite products.

This follows from the observation that to every cartesian square

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{s'} & \mathcal{M} \\ \tilde{\mathfrak{t}} \downarrow & \text{cart} & \downarrow \mathfrak{t} \\ \mathcal{L} & \xrightarrow{s} & \mathcal{N} \end{array} \quad (2)$$

there corresponds a cartesian square

$$\begin{array}{ccc} Ker(\mathfrak{s} \circ \tilde{\mathfrak{t}}) & \longrightarrow & Ker(\mathfrak{t}) \\ \downarrow & \text{cart} & \downarrow \\ Ker(\mathfrak{s}) & \longrightarrow & \mathfrak{r} \end{array}$$

obtained via pulling back the square (2) along the unique arrow $\mathfrak{r} \longrightarrow \mathcal{N}$.

4.1.3. Proposition. (a) Let \mathcal{S} be a system of deflations satisfying the equality $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})$. Then the stable envelope $\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$ of the system \mathcal{S} has this property; that is the class $\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$ is strongly stable.

(b) The family of strongly stable classes of deflations is closed under arbitrary intersections and unions.

(c) The family of cartesian complete classes of deflations is closed under arbitrary intersections and filtered (with respect to the inclusion) unions. Similarly for left or/and right semitopologizing systems.

Proof. (a) Since, by hypothesis, the category C_X has fiber products, for any pair of classes of arrows \mathcal{S}, \mathcal{T} , there is an obvious inclusion $\mathcal{S}^{\bar{\wedge}} \circ \mathcal{T}^{\bar{\wedge}} \subseteq (\mathcal{S} \circ \mathcal{T}^{\bar{\wedge}})^{\bar{\wedge}}$. In particular, $\mathcal{S}^{\bar{\wedge}} \circ \mathcal{W}_X^{\bar{\wedge}} \subseteq (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})^{\bar{\wedge}}$. Therefore, for any system \mathcal{S} such that $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})$, we obtain the following:

$$\begin{aligned} \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}} &\subseteq \mathfrak{E}_X \cap ((\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}) \circ \mathcal{W}_X^{\bar{\wedge}}) \subseteq \mathfrak{E}_X \cap (\mathcal{S}^{\bar{\wedge}} \circ \mathcal{W}_X^{\bar{\wedge}}) \subseteq \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})^{\bar{\wedge}} = \\ &= \mathfrak{E}_X \cap (\mathfrak{E}_X^{\bar{\wedge}} \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})^{\bar{\wedge}}) = \mathfrak{E}_X \cap (\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}}))^{\bar{\wedge}} = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}. \end{aligned}$$

(b) Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of classes of arrows such that $\mathcal{T}_i = \mathfrak{E}_X \cap (\mathcal{T}_i \circ \mathcal{W}_X^{\bar{\wedge}})$ for all $i \in J$. Then

$$\bigcap_{i \in J} \mathcal{T}_i \subseteq \mathfrak{E}_X \cap \left(\left(\bigcap_{i \in J} \mathcal{T}_i \right) \circ \mathcal{W}_X^{\bar{\wedge}} \right) \subseteq \mathfrak{E}_X \cap \left(\bigcap_{i \in J} \mathcal{T}_i \circ \mathcal{W}_X^{\bar{\wedge}} \right) = \bigcap_{i \in J} (\mathfrak{E}_X \cap (\mathcal{T}_i \circ \mathcal{W}_X^{\bar{\wedge}})) = \bigcap_{i \in J} \mathcal{T}_i.$$

Similarly,

$$\bigcup_{i \in J} \mathcal{T}_i \subseteq \mathfrak{E}_X \cap \left(\left(\bigcup_{i \in J} \mathcal{T}_i \right) \circ \mathcal{W}_X^{\bar{\wedge}} \right) = \mathfrak{E}_X \cap \left(\bigcup_{i \in J} \mathcal{T}_i \circ \mathcal{W}_X^{\bar{\wedge}} \right) = \bigcup_{i \in J} (\mathfrak{E}_X \cap (\mathcal{T}_i \circ \mathcal{W}_X^{\bar{\wedge}})) = \bigcup_{i \in J} \mathcal{T}_i.$$

By 2.1, the class of (right or/and left) stable systems is closed under arbitrary intersections and unions.

(c) The assertion follows from (b). ■

4.1.4. Proposition. (a) For any system of deflations \mathcal{T} , the intersection

$$\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge)^\wedge$$

is the smallest strongly stable system containing \mathcal{T} .

(b) If the system \mathcal{T} is cartesian complete, then the smallest strongly stable system containing \mathcal{T} is cartesian complete.

(c) Suppose that the condition 2.0(d) holds. Then, for any right divisible system of deflations \mathcal{T} , the smallest strongly stable system containing \mathcal{T} is right divisible.

In particular, if \mathcal{T} is a right divisible and cartesian complete system, then its strongly stable envelope $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge)^\wedge$ is a right semitopological system.

Proof. (a) By 1.6.1(ii), if a system \mathcal{S} satisfies the equality $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{S}^\wedge)$, then the class of arrows \mathcal{S}^\wedge is multiplicative. In particular, the class of arrows \mathcal{W}_X^\wedge is closed under composition. So that, for any system \mathcal{S} , we have

$$\begin{aligned} \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge) &\subseteq \mathfrak{E}_X \cap ((\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge)) \circ \mathcal{W}_X^\wedge) \subseteq \\ \mathfrak{E}_X \cap ((\mathcal{S} \circ \mathcal{W}_X^\wedge) \circ \mathcal{W}_X^\wedge) &= \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge), \end{aligned}$$

which shows that the system $\mathcal{T} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge)$ satisfies the equality $\mathcal{T} = \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge)$. Evidently, \mathcal{T} is the smallest system containing \mathcal{S} and satisfying this equality.

By 4.1.2(a), the system $\mathfrak{E}_X \cap \mathcal{T}^\wedge$ is the smallest strongly stable system containing \mathcal{T} . Notice that

$$\mathfrak{E}_X \cap \mathcal{T}^\wedge = \mathfrak{E}_X \cap (\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge)^\wedge) = \mathfrak{E}_X \cap (\mathfrak{E}_X^\wedge \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge)^\wedge) = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge)^\wedge$$

hence the assertion.

(b1) If a class of deflations \mathcal{T} is cartesian complete, then $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge)$ has this property.

In fact, for any pair of arrows $\mathfrak{t}_i \circ \mathfrak{w}_i \in \mathfrak{E}_X$, $i = 1, 2$, such that $\mathfrak{t}_i \in \mathcal{T}$ and $\mathfrak{w}_i \in \mathcal{W}_X^\wedge$, $i = 1, 2$, we have diagram

$$\begin{array}{ccccc} \mathfrak{L} & \xrightarrow{\mathfrak{w}_2''} & \mathfrak{M} & \xrightarrow{\mathfrak{t}_2''} & \mathcal{L}_1 \\ \tilde{\mathfrak{w}}_1 \downarrow & \text{cart} & \downarrow \mathfrak{w}'_1 & \text{cart} & \downarrow \mathfrak{w}_1 \\ \mathfrak{L}' & \xrightarrow{\mathfrak{w}_2'} & \mathfrak{M}' & \xrightarrow{\mathfrak{t}_2'} & \mathcal{M}_1 \\ \tilde{\mathfrak{t}}_1 \downarrow & \text{cart} & \downarrow \mathfrak{t}'_1 & \text{cart} & \downarrow \mathfrak{t}_1 \\ \mathcal{L}_2 & \xrightarrow{\mathfrak{w}_2} & \mathcal{M}_2 & \xrightarrow{\mathfrak{t}_2} & \mathcal{N} \end{array}$$

built out of cartesian squares. So that

$$(\mathfrak{t}_1 \circ \mathfrak{w}_1) \circ (\mathfrak{t}_2'' \circ \mathfrak{w}_2'') = (\mathfrak{t}_1 \circ \mathfrak{t}_2') \circ (\mathfrak{w}'_1 \circ \mathfrak{w}_2'') \in \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge),$$

because, by hypothesis, \mathcal{T} is cartesian complete and $\mathcal{W}_X^\wedge \circ \mathcal{W}_X^\wedge = \mathcal{W}_X^\wedge$.

(c) Consider a commutative square

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\mathfrak{w}} & \mathcal{M} \\ \mathfrak{s}_1 \downarrow & & \downarrow \mathfrak{t} \\ \mathcal{L} & \xrightarrow{\mathfrak{s}_2} & \mathfrak{M} \end{array}$$

where $\mathfrak{t} \in \mathcal{T}$, $\mathfrak{w} \in \mathcal{W}_X$, and $\mathfrak{t} \circ \mathfrak{w}$, \mathfrak{s}_1 and \mathfrak{s}_2 are deflations. The square is decomposed into the diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\mathfrak{f}} & \tilde{\mathcal{L}} & \xrightarrow{\mathfrak{w}_2} & \mathcal{M} \\ & & \mathfrak{t}_1 \downarrow & \text{cart} & \downarrow \mathfrak{t} \\ & & \mathcal{L} & \xrightarrow{\mathfrak{s}_2} & \mathfrak{M} \end{array}$$

with cartesian square such that $\tilde{\mathfrak{s}}_2 \circ \mathfrak{f} = \mathfrak{w}$ and $\mathfrak{t}_1 \circ \mathfrak{f} = \mathfrak{s}_1$. By the condition 2.0(d), the latter equality implies that $\mathfrak{f} = \mathfrak{w}_1 \circ \mathfrak{e}$, where \mathfrak{e} is a deflation and $\mathfrak{w}_1 \in \mathcal{W}_X^\wedge$. It follows from the fact that $\mathfrak{w} = (\mathfrak{w}_2 \circ \mathfrak{w}_1) \circ \mathfrak{e} \in \mathcal{W}_X$ and \mathfrak{e} is a deflation that $\mathfrak{w}_2 \circ \mathfrak{w}_1 \in \mathcal{W}_X$ and $\mathfrak{e} \in \mathcal{W}_X$. Therefore, $\mathfrak{s}_1 = \mathfrak{t}_1 \circ (\mathfrak{w}_1 \circ \mathfrak{e})$, where $\mathfrak{t}_1 \in \mathcal{T}$ and $\mathfrak{w}_1 \circ \mathfrak{e} \in \mathcal{W}_X^\wedge$. ■

4.2. Strongly thick systems. We call a system of deflations \mathcal{S} *strongly thick* if it is divisible in \mathfrak{E}_X , stable, and $\mathcal{S} \circ \mathcal{S}^\wedge = \mathcal{S}^\wedge$.

We denote by $\mathfrak{M}_s(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to the inclusion) of all strongly thick systems of $(C_X, \bar{\mathfrak{E}}_X)$. It follows from our assumptions (see 4.0) that the class \mathcal{W}_X of weak equivalences is the smallest element of $\mathfrak{M}_s(X, \bar{\mathfrak{E}}_X)$.

4.2.1. Observations. (a) Thanks to the existence of fiber products in C_X , for any class of arrows \mathcal{S} which is invariant under pull-backs, the inclusion $\mathcal{S} \circ \mathcal{S}^\wedge \subseteq \mathcal{S}^\wedge$ is equivalent to the multiplicativity of \mathcal{S}^\wedge , that is the inclusion $\mathcal{S}^\wedge \circ \mathcal{S}^\wedge \subseteq \mathcal{S}^\wedge$.

In particular, a system \mathcal{S} is strongly thick iff it is stable and $\mathcal{S}^\wedge \circ \mathcal{S}^\wedge = \mathcal{S}^\wedge$.

(b) Every strongly thick system of deflation \mathcal{S} is multiplicative, because

$$\mathcal{S} \circ \mathcal{S} \subseteq \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{S}^\wedge) = \mathfrak{E}_X \cap \mathcal{S}^\wedge = \mathcal{S}.$$

(c) Every strongly thick system \mathcal{S} is semitopologizing, because it is multiplicative and

$$\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^\wedge) \subseteq \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{S}^\wedge) = \mathfrak{E}_X \cap \mathcal{S}^\wedge = \mathcal{S}.$$

(d) A stable, divisible in \mathfrak{E}_X system of deflations \mathcal{S} is strongly thick, when it satisfies the condition

(#) If in the commutative diagram

$$\begin{array}{ccccc} \tilde{\mathcal{L}} & \xrightarrow{\tilde{\mathfrak{s}}} & \mathcal{K} & & \\ \mathfrak{j}' \downarrow & \text{cart} & \downarrow \mathfrak{j} & & \\ \mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{K} \end{array} \quad (3)$$

with cartesian square $t \circ j = id_{\mathcal{N}}$ and morphisms t and \tilde{s} belong to \mathcal{S} , then $t \circ s \in \mathcal{S}$.

Indeed, if the condition $(\#)$ holds, then, by 1.6.1(ii), the class $\mathcal{S}^{\bar{\cdot}}$ is multiplicative.

(e) Evidently, the class of deflations \mathfrak{E}_X is strongly thick iff $\mathfrak{E}_X \circ \mathfrak{E}_X^{\bar{\cdot}} = \mathfrak{E}_X^{\bar{\cdot}}$.

(f) It follows from 4.1.2 that the preorder $\mathfrak{M}_s(X, \bar{\mathfrak{E}}_X)$ of strongly thick systems is closed under arbitrary intersections and filtered unions.

4.2.2. Note. One can see that the condition $(\#)$ from 4.2.1(d) holds for \mathfrak{E}_X , if the class \mathfrak{E}_X is left divisible, because $(t \circ s) \circ j' = \tilde{s} \in \mathfrak{E}_X$ (see the diagram (3) above).

In general, the condition $(\#)$ provides an effective tool for finding if a system is stable or not. The following assertion shows that in most of cases of interest the condition $(\#)$ is a criterium.

4.2.3. Proposition. *Suppose that the class \mathfrak{E}_X of deflations satisfies the condition $(\#)$ (say, it is left divisible). Let \mathcal{S} be a stable class of deflations invariant under pull-backs. Then the following conditions are equivalent:*

(a) $\mathcal{S}^{\bar{\cdot}} \circ \mathcal{S}^{\bar{\cdot}} \subseteq \mathcal{S}^{\bar{\cdot}}$,

(b) \mathcal{S} satisfies the condition $(\#)$.

Proof. (a) \Rightarrow (b). The morphism $t \circ s$ in the condition $(\#)$ belongs to the intersection of $\mathcal{S} \circ \mathcal{S}^{\bar{\cdot}}$ and \mathfrak{E}_X due to the fact that the arrows \tilde{s} and t in the condition $(\#)$ are deflations and \mathfrak{E}_X satisfies $(\#)$. Therefore, if the condition (b) holds and \mathcal{S} is stable, $t \circ s$ belongs to $\mathcal{S}^{\bar{\cdot}} \cap \mathfrak{E}_X = \mathcal{S}$.

The implication (b) \Rightarrow (a) follows from 1.6.1(ii). ■

4.2.4. Proposition. *Suppose that $\mathfrak{E}_X^{\bar{\cdot}} \subseteq \mathcal{W}_X^{\bar{\cdot}} \circ \mathfrak{E}_X$. Then a strongly stable divisible in \mathfrak{E}_X system of deflations \mathcal{S} is strongly thick iff it is multiplicative.*

In other words, a thick system \mathcal{S} is strongly thick iff $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\cdot}})$.

Proof. By 4.2.1(b), any strongly thick system is multiplicative.

The claim is that any multiplicative strongly stable right divisible in \mathfrak{E}_X class of deflations satisfies the condition $(\#)$.

In fact, the inclusion $\mathfrak{E}_X^{\bar{\cdot}} \subseteq \mathcal{W}_X^{\bar{\cdot}} \circ \mathfrak{E}_X$ allows to replace the diagram in $(\#)$ by the diagram

$$\begin{array}{ccccccc}
 \tilde{\mathcal{L}} & \xrightarrow{\tilde{s}_1} & \mathcal{K}_1 & \xrightarrow{\tilde{w}} & \mathcal{K} & & \\
 j' \downarrow & \text{cart} & \downarrow j_1 & \text{cart} & \downarrow j & & \\
 \mathcal{L} & \xrightarrow{s_1} & \mathcal{M}_1 & \xrightarrow{w} & \mathcal{M} & \xrightarrow{t} & \mathcal{K}
 \end{array} \tag{4}$$

with cartesian squares, where $\tilde{w} \circ \tilde{s}_1 = \tilde{s}$, $w \circ s_1 = s$, $s_1 \in \mathfrak{E}_X$, and $w \in \mathcal{W}_X^{\bar{\cdot}}$.

Since the system of deflations \mathfrak{E}_X is left divisible in \mathfrak{E}_X , the morphism \tilde{w} is a deflation (hence it belongs to \mathcal{W}_X thanks to the stability of \mathcal{W}_X). The composition $t \circ w$ is a deflation, because, by hypothesis, the system of deflations satisfies $(\#)$ and both \tilde{w} and t are deflations (look at the diagram (3) ignoring its left square). The equality $\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\cdot}}) = \mathcal{S}$ implies that $t \circ w \in \mathcal{S}$. Since the system \mathcal{S} is right divisible in \mathfrak{E}_X , the arrow \tilde{s} belongs to \mathcal{S} , whence $s_1 \in \mathfrak{E}_X \cap \mathcal{S}^{\bar{\cdot}} = \mathcal{S}$ (thanks to the stability of \mathcal{S}). Finally, the multiplicativity of \mathcal{S} implies that $t \circ s = (t \circ w) \circ s_1 \in \mathcal{S}$. ■

4.2.5. Note. For any class of arrows \mathcal{S} invariant under pull-backs and such that $\mathcal{W}_X \circ \mathcal{S} = \mathcal{S}$, there is the inclusion $\mathcal{W}_X^\wedge \circ \mathcal{S} \subseteq \mathcal{S}^\wedge$. In particular, $\mathcal{W}_X^\wedge \circ \mathfrak{E}_X \subseteq \mathfrak{E}_X^\wedge$. Therefore, the inclusion $\mathfrak{E}_X^\wedge \subseteq \mathcal{W}_X^\wedge \circ \mathfrak{E}_X$ used in 4.2.4 is equivalent to the equality $\mathfrak{E}_X^\wedge = \mathcal{W}_X^\wedge \circ \mathfrak{E}_X$.

The equality $\mathfrak{E}_X^\wedge = \mathcal{W}_X^\wedge \circ \mathfrak{E}_X$ holds when \mathfrak{E}_X coincides with the class of all strict epimorphisms of the category C_X , in C_X , there exist 2-coimages of arbitrary arrows (see 2.0.1(d)&(e)), and \mathcal{W}_X^\wedge contains all monomorphisms of C_X (say, C_X has initial objects).

Indeed, in this case $\mathcal{W}_X^\wedge \circ \mathfrak{E}_X$ (hence \mathfrak{E}_X^\wedge) coincides with $\text{Hom}C_X$.

4.3. Strongly closed systems.

4.3.1. The strong closure. For a class of deflations \mathcal{S} of $(C_X, \bar{\mathfrak{E}}_X)$, let $\mathfrak{R}_\mathcal{S}^s$ denote the set of all systems \mathcal{T} divisible in \mathfrak{E}_X such that $\mathcal{T} = \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge)$ and $\mathcal{T} \cap \mathcal{S}^\perp = \mathcal{W}_X$. We denote by \mathcal{S}^\dagger the union of all $\mathcal{T} \in \mathfrak{R}_\mathcal{S}^s$.

The construction $\mathcal{S} \mapsto \mathcal{S}^\dagger$ has the properties similar to those of the closure $\mathcal{S} \mapsto \mathcal{S}^-$.

4.3.2. Proposition. (a) For any class of deflations \mathcal{S} , the system \mathcal{S}^\dagger belongs to $\mathfrak{R}_\mathcal{S}^s$ (hence it is the largest element of $\mathfrak{R}_\mathcal{S}^s$).

(b) $(\mathcal{S}^\dagger)^\dagger = \mathcal{S}^\dagger$.

(c) The system \mathcal{S}^\dagger is closed under the composition.

(d) The system \mathcal{S}^\dagger is stable (hence strongly stable), that is $\mathcal{S}^\dagger = \mathfrak{E}_X \cap (\mathcal{S}^\dagger)^\wedge$.

Proof. (a) The assertion follows from 4.1.2(b).

(b) It follows that $\mathfrak{R}_{\mathcal{S}^\dagger}^s \subseteq \mathfrak{R}_\mathcal{S}^s$. On the other hand, $\mathcal{S}^\dagger \in \mathfrak{R}_{\mathcal{S}^\dagger}^s$. Hence the equality.

(c) The argument is similar to that of 2.6.2(c).

(d) If $\mathcal{T} \in \mathfrak{R}_\mathcal{S}^s$, then the associated stable system, $\mathfrak{E}_X \cap \mathcal{T}^\wedge$, belongs to $\mathfrak{R}_\mathcal{S}^s$.

In fact, by 4.1.2(a), the system $\mathfrak{E}_X \cap \mathcal{T}^\wedge$ is strongly stable. On the other hand, the equality $\mathcal{S}^\perp = (\mathcal{S}^\perp)^\wedge$ implies that

$$(\mathfrak{E}_X \cap \mathcal{T}^\wedge) \cap \mathcal{S}^\perp = \mathfrak{E}_X \cap \mathcal{T}^\wedge \cap (\mathcal{S}^\perp)^\wedge = \mathfrak{E}_X \cap (\mathcal{T} \cap \mathcal{S}^\perp)^\wedge = \mathfrak{E}_X \cap \mathcal{W}_X^\wedge = \mathcal{W}_X.$$

This shows that $\mathfrak{E}_X \cap \mathcal{T}^\wedge$ belongs to $\mathfrak{R}_\mathcal{S}^s$. In particular, $\mathfrak{E}_X \cap (\mathcal{S}^\dagger)^\wedge$ belongs to $\mathfrak{R}_\mathcal{S}^s$, which implies the stability of \mathcal{S}^\dagger . ■

4.3.3. Observations. (a) One can see that \mathcal{S}^\dagger is the largest divisible in \mathfrak{E}_X strongly stable subsystem of \mathcal{S}^- . So that $\mathcal{S}^- = \mathcal{S}^\dagger$ iff $\mathfrak{E}_X \cap (\mathcal{S}^- \circ \mathcal{W}_X^\wedge) = \mathcal{S}^-$.

It follows from 4.1.1 that if the class of deflatons \mathfrak{E}_X is left divisible (in the sense of 4.1.1) and \mathcal{S}^- is a topological system, then $\mathcal{S}^- = \mathcal{S}^\dagger$.

In particular, $\mathcal{S}^- = \mathcal{S}^\dagger$ in the case of an abelian category $(C_X, \bar{\mathfrak{E}}_X)$.

(b) Since, by hypothesis, the class \mathcal{W}_X of weak equivalences is strongly stable and, by 2.6.2.1, $\mathcal{W}_X^- = \mathcal{W}_X$, it follows that $\mathcal{W}_X = \mathcal{W}_X^\dagger$.

4.3.4. Proposition. Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite set of right divisible in \mathfrak{E}_X systems of deflations of (C_X, \mathfrak{E}_X) . Then $\bigcap_{i \in J} \mathcal{S}_i^\dagger = (\bigcap_{i \in J} \mathcal{S}_i)^\dagger$.

Proof. The argument is similar to the proof of 2.6.4(b). ■

4.3.5. Strongly closed systems of deflations. We call a class \mathcal{S} of deflations of (C_X, \mathfrak{E}_X) a *strongly closed system* if $\mathcal{S} = \mathcal{S}^\dagger$. Strongly closed systems of (C_X, \mathfrak{E}_X) form a preorder with respect to the inclusion, which we denote by $\mathfrak{S}\mathfrak{e}_5(X, \mathfrak{E}_X)$.

By 4.3.2, strongly closed systems are strongly stable and multiplicative; and, by definition, they are divisible in \mathfrak{E}_X . Therefore, every strongly closed system is semitopological.

4.3.6. Proposition. (a) *The intersection of any set of strongly closed systems of deflations of (C_X, \mathfrak{E}_X) is a strongly closed system.*

(b) *Suppose that $\mathfrak{E}_X^\wedge = \mathcal{W}_X^\wedge \circ \mathfrak{E}_X$. Then every strongly closed system of deflations is strongly thick.*

Proof. (a) The argument is similar to that of 2.6.4(a).

(b) The assertion follows from the multiplicativity of strongly closed systems (see 4.3.2(c)) and 4.2.4. ■

4.3.7. The lattice of strongly closed systems. Fix a svelte right exact category (C_X, \mathfrak{E}_X) . For any pair Σ_1, Σ_2 of strongly closed systems of deflations, we denote by $\Sigma_1 \sqcup \Sigma_2$ the smallest strongly closed system containing Σ_1 and Σ_2 .

4.3.7.1. Proposition. *Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite set of strongly closed systems of deflations of (C_X, \mathfrak{E}_X) . Then $\Sigma \sqcup \left(\bigcap_{i \in J} \mathcal{S}_i \right) = \bigcap_{i \in J} (\Sigma \sqcup \mathcal{S}_i)$ for any strongly closed system of deflations Σ .*

Proof. There are the equalities

$$\bigcap_{i \in J} (\Sigma \sqcup \mathcal{S}_i) = \bigcap_{i \in J} (\Sigma \cup \mathcal{S}_i)^\dagger = \left(\bigcap_{i \in J} (\Sigma \cup \mathcal{S}_i) \right)^\dagger = \left(\left(\bigcap_{i \in J} \mathcal{S}_i \right) \cup \Sigma \right)^\dagger = \left(\bigcap_{i \in J} \mathcal{S}_i \right) \sqcup \Sigma.$$

Here the second equality follows from 2.6.4. ■

4.4. Spectra. For every class of deflations \mathcal{S} , we denote by \mathcal{S}^{st} the intersection of all semitopologizing systems properly containing \mathcal{S} and by \mathcal{S}^{sc} the intersection of all *strongly stable* thick systems properly containing the class \mathcal{S} .

We denote by $\mathbf{Spec}_{\text{sc}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to \supseteq) formed by those strongly closed systems of deflations Σ for which $\Sigma^{\text{sc}} \neq \Sigma$, or, equivalently, the intersection $\Sigma_{\text{sc}} \stackrel{\text{def}}{=} \Sigma^{\text{sc}} \cap \Sigma^\perp$ is a non-trivial system of deflations.

Similarly, we define the spectrum $\mathbf{Spec}_{\text{st}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ as the preorder formed by all strongly closed systems Σ for which $\Sigma^{\text{st}} \neq \Sigma$, or, what is the same, the system of deflations

$$\Sigma_{\text{st}} \stackrel{\text{def}}{=} \Sigma^{\text{st}} \cap \Sigma^\perp$$

is non-trivial. It follows from the definitions that the spectrum $\mathbf{Spec}_{\text{st}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ is a subpreorder of the spectrum $\mathbf{Spec}_{\text{sc}}^{1,1}(X, \bar{\mathfrak{E}}_X)$.

The following useful fact is a direct analog of 3.7.1.

4.4.1. Proposition. *Let $\Sigma \in \mathbf{Spec}_{\text{sc}}^{1,1}(X, \bar{\mathfrak{E}}_X)$. For any finite family $\{\mathcal{S}_i \mid i \in \mathfrak{J}\}$ of right divisible in \mathfrak{E}_X systems of deflations, $\mathcal{S}_i \not\subseteq \Sigma$ for all $i \in \mathfrak{J}$ iff $\bigcap_{i \in \mathfrak{J}} \mathcal{S}_i \not\subseteq \Sigma$.*

Proof. The argument below is similar to the proof of 3.7.1.

By 4.3.4, $\bigcap_{i \in J} \mathcal{S}_i^\dagger = (\bigcap_{i \in J} \mathcal{S}_i)^\dagger$, and by 4.3.7.1, $\Sigma \sqcup (\bigcap_{i \in J} \mathcal{S}_i^\dagger) = \bigcap_{i \in J} (\Sigma \sqcup \mathcal{S}_i^\dagger)$. Therefore,

$$\Sigma \sqcup (\bigcap_{i \in J} \mathcal{S}_i)^\dagger = \bigcap_{i \in J} (\Sigma \sqcup \mathcal{S}_i^\dagger). \quad (5)$$

If $\mathcal{S}_i \not\subseteq \Sigma$ for all $i \in J$, then each of the strongly closed systems $\mathcal{S}_i^\dagger \sqcup \Sigma$ contains Σ properly. Since Σ is an element of the spectrum $\mathbf{Spec}_{sc}^{1,1}(X, \bar{\mathfrak{E}}_X)$, the intersection of $\mathcal{S}_i^\dagger \sqcup \Sigma$, $i \in J$, contains in Σ properly. Then it follows from the equality (5) that the intersection $\bigcap_{i \in J} \mathcal{S}_i$ is not contained in Σ . ■

5. Strongly 'exact' functors and localizations.

5.0. Strongly 'exact' functors. Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ and $(C_Y, \mathfrak{E}_Y, \mathcal{W}_Y)$ be right exact categories with weak equivalences. Recall that an 'exact' functor from (C_Y, \mathfrak{E}_Y) to $(C_X, \bar{\mathfrak{E}}_X)$ is given by a functor $C_Y \rightarrow C_X$ which maps deflations to deflations, weak equivalences to weak equivalences and preserves pull-backs of deflations.

We say that an 'exact' functor $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_X, \bar{\mathfrak{E}}_X)$ is *strongly 'exact'* if any cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{f}} & \widetilde{\mathcal{N}} \\ \mathfrak{s}' \downarrow & \text{cart} & \downarrow \mathfrak{s} \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N} \end{array}$$

whose left vertical arrow is a deflation can be completed by a pull-back of this deflation

$$\begin{array}{ccccc} \widetilde{\mathcal{L}} & \xrightarrow{\widetilde{\xi}} & \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{f}} & \widetilde{\mathcal{N}} \\ \mathfrak{s}'' \downarrow & \text{cart} & \mathfrak{s}' \downarrow & \text{cart} & \downarrow \mathfrak{s} \\ \mathcal{L} & \xrightarrow{\xi} & \mathcal{M} & \xrightarrow{f} & \mathcal{N} \end{array}$$

such that F maps the outer cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{L}} & \xrightarrow{\widetilde{f \circ \xi}} & \widetilde{\mathcal{N}} \\ \mathfrak{s}'' \downarrow & \text{cart} & \downarrow \mathfrak{s} \\ \mathcal{L} & \xrightarrow{\xi \circ f} & \mathcal{N} \end{array}$$

to a cartesian square.

In particular, any functor $C_Y \xrightarrow{F} C_X$ which maps deflations to deflations and preserves cartesian squares having at least one deflation among its arrows is strongly 'exact'.

5.0.1. Strong 'exactness' and preserving kernels. This seemingly technical notion has a transparent meaning in the case the category C_X has initial objects and morphisms to initial objects are deflations. In this case,

an 'exact' functor $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_X, \bar{\mathfrak{E}}_X)$ is strongly exact iff the functor $C_Y \xrightarrow{F} C_X$ preserves kernels of arrows.

5.1. Proposition. Let $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{F} (C_Y, \bar{\mathfrak{E}}_Y)$ be a strongly 'exact' functor.

(a) $F(\mathcal{S}^\wedge) \subseteq F(\mathcal{S})^\wedge$ for any class of deflations \mathcal{S} of the category C_X .

In particular, $F(\mathcal{W}_X^\wedge) \subseteq F(\mathcal{W}_X)^\wedge \subseteq \mathcal{W}_Y^\wedge$ and $F(\mathfrak{E}_X^\wedge) \subseteq F(\mathfrak{E}_X)^\wedge \subseteq \mathfrak{E}_Y^\wedge$.

(b) The map $\mathcal{T} \mapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ transfers stable, strongly stable, thick and semi-topologizing systems of deflations to systems of deflations of the same kind.

(c) Suppose that one of the following conditions holds:

(i) If in the commutative diagram

$$\begin{array}{ccccc} \tilde{\mathcal{L}} & \xrightarrow{\tilde{\mathfrak{s}}} & \mathcal{K} & & \\ j' \downarrow & \text{cart} & \downarrow j & & \\ \mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{K} \end{array}$$

in C_X or C_Y the square is cartesian, $\mathfrak{t} \circ j = id_{\mathcal{K}}$ and morphisms \mathfrak{t} and $\tilde{\mathfrak{s}}$ are deflations, then the composition $\mathfrak{t} \circ \mathfrak{s}$ is a deflation.

(ii) $\mathfrak{E}_X^\wedge = \mathcal{W}_X^\wedge \circ \mathfrak{E}_X$.

Then the map $\mathcal{T} \mapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ preserves strongly thick systems.

Proof. (a) The inclusions follow from definitions.

(b1) Suppose that \mathcal{T} is a class of arrows of C_Y satisfying $\mathcal{T} = \mathfrak{E}_Y \cap (\mathcal{T} \circ \mathcal{W}_Y^\wedge)$. Then

$$\begin{aligned} F(\mathfrak{E}_X \cap ((\mathfrak{E}_X \cap F^{-1}(\mathcal{T})) \circ \mathcal{W}_X^\wedge)) &\subseteq \mathfrak{E}_Y \cap F(F^{-1}(\mathcal{T})) \circ \mathcal{W}_X^\wedge \subseteq \\ \mathfrak{E}_Y \cap (\mathcal{T} \circ F(\mathcal{W}_X^\wedge)) &\subseteq \mathfrak{E}_Y \cap (\mathcal{T} \circ \mathcal{W}_Y^\wedge) = \mathcal{T}, \end{aligned}$$

which implies the inclusion

$$\mathfrak{E}_X \cap ((\mathfrak{E}_X \cap F^{-1}(\mathcal{T})) \circ \mathcal{W}_X^\wedge) \subseteq \mathfrak{E}_X \cap F^{-1}(\mathcal{T}).$$

Since the inverse inclusion holds (for any class of arrows \mathcal{T}), we obtain the equality

$$\mathfrak{E}_X \cap ((\mathfrak{E}_X \cap F^{-1}(\mathcal{T})) \circ \mathcal{W}_X^\wedge) = \mathfrak{E}_X \cap F^{-1}(\mathcal{T}).$$

(b2) Similarly with the stability: if $\mathcal{T} = \mathfrak{E}_Y \cap \mathcal{T}^\wedge$, then

$$\begin{aligned} F(\mathfrak{E}_X \cap (\mathfrak{E}_X \cap F^{-1}(\mathcal{T}))^\wedge) &= F(\mathfrak{E}_X \cap (\mathfrak{E}_X^\wedge \cap (F^{-1}(\mathcal{T}))^\wedge)) = \\ F(\mathfrak{E}_X \cap (F^{-1}(\mathcal{T}))^\wedge) &\subseteq \mathfrak{E}_Y \cap F((F^{-1}(\mathcal{T}))^\wedge) \subseteq \mathfrak{E}_Y \cap \mathcal{T}^\wedge = \mathcal{T}, \end{aligned}$$

which implies the inclusions

$$\mathfrak{E}_X \cap F^{-1}(\mathcal{T}) \subseteq \mathfrak{E}_X \cap (\mathfrak{E}_X \cap F^{-1}(\mathcal{T}))^\wedge \subseteq \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$$

equivalent to the stability of the class $\mathfrak{E}_X \cap F^{-1}(\mathcal{T})$.

(b3) The fact that the map $\mathcal{T} \mapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ respects strong stability implies, obviously, that it maps semitopologizing systems to semitopologizing systems.

(c1) Suppose that the class of deflations \mathfrak{E}_X satisfies the condition (i). Let \mathcal{S} be a stable class of deflations of $(C_Y, \bar{\mathfrak{E}}_Y)$ invariant under pull-backs and satisfying the condition

(#) If in the commutative diagram

$$\begin{array}{ccccc} \tilde{\mathcal{L}} & \xrightarrow{\tilde{s}} & \mathcal{K} & & \\ j' \downarrow & \text{cart} & \downarrow j & & \\ \mathcal{L} & \xrightarrow{s} & \mathcal{M} & \xrightarrow{t} & \mathcal{K} \end{array} \quad (1)$$

with cartesian square $t \circ j = id_{\mathcal{K}}$ and morphisms t and \tilde{s} belong to \mathcal{S} , then $t \circ s \in \mathcal{S}$.

Then the class $\mathfrak{E}_X \cap F^{-1}(\mathcal{S})$ satisfies this condition.

In fact, let (1) be a diagram whose square is cartesian and \tilde{s} and t are arrows from $\mathfrak{E}_X \cap F^{-1}(\mathcal{S})$. Since the functor F is strongly 'exact' and the arrow \tilde{s} in the diagram (1) is a deflation, there exists a pull-back of \tilde{s} along some arrow $\mathfrak{K} \xrightarrow{\gamma} \mathcal{K}$ such that F maps the pull-back of $\mathcal{L} \xrightarrow{s} \mathcal{M}$ along $\mathfrak{K} \xrightarrow{j \circ \gamma} \mathcal{M}$ to a pull-back of $F(s)$ along $F(j \circ \gamma)$.

By hypothesis, the class of deflations \mathfrak{E}_X satisfies the condition (#). So that the composition $t \circ s$ is a deflation. Taking a pull-back of the deflation $t \circ s$ along the morphism $\mathfrak{K} \xrightarrow{\gamma} \mathcal{K}$, we obtain the diagram

$$\begin{array}{ccccccc} \tilde{\mathcal{L}} & \xrightarrow{\tilde{s}} & \mathfrak{K} & & & & \\ \lambda' \downarrow & \text{cart} & \downarrow \lambda & & & & \\ \mathcal{L} & \xrightarrow{s} & \mathfrak{M} & \xrightarrow{\bar{t}} & \mathfrak{K} & & \\ \gamma'' \downarrow & \text{cart} & \downarrow \gamma' & \text{cart} & \downarrow \gamma & & \\ \mathcal{L} & \xrightarrow{s} & \mathcal{M} & \xrightarrow{t} & \mathcal{K} & & \end{array} \quad (2)$$

built of cartesian squares, where the morphism $\mathfrak{K} \xrightarrow{\lambda} \mathfrak{M}$ is uniquely determined by the equalities $\gamma' \circ \lambda = j \circ \gamma$ and $\bar{t} \circ \lambda = id_{\mathfrak{K}}$. Since the functor F is 'exact' and the arrows t and $t \circ s$ are deflations, F preserves pull-backs of this arrows which implies that it maps the lower two cartesian squares of the diagram (2) to cartesian squares. Since, by construction, F preserves the pull-back of the arrow $\mathcal{L} \xrightarrow{s} \mathcal{M}$ along the morphism $\gamma' \circ \lambda = j \circ \gamma$, it follows that F maps the upper square of (2) to a cartesian square as well. By the condition (#), $F(\bar{t} \circ \bar{s}) = F(\bar{t}) \circ F(\bar{s}) \in \mathcal{S}$, that is $\bar{t} \circ \bar{s} \in F^{-1}(\mathcal{S}) \cap \mathfrak{E}_X$. Since, by hypothesis, the class \mathcal{S} is stable and $\bar{t} \circ \bar{s}$ is a pull-back of the deflation $t \circ s$, it follows from the assertion (b) that $t \circ s \in F^{-1}(\mathcal{S}) \cap \mathfrak{E}_X$.

(c2) Suppose that the condition (i) holds, that is the both classes of deflations, \mathfrak{E}_X and \mathfrak{E}_Y , satisfy the condition (#). The fact that \mathfrak{E}_Y satisfies (#) implies, by 4.2.3, that any class \mathcal{S} of deflations of $(C_Y, \bar{\mathfrak{E}}_Y)$ invariant under pull-backs and such that $\mathcal{S}^{\bar{\cdot}}$ is multiplicative satisfies the condition (#). If, in addition, the class \mathcal{S} is stable, then, by (c1) above, the class $\mathfrak{E}_X \cap F^{-1}(\mathcal{S})$ satisfies the condition (#), which implies the multiplicativity

of the class $(\mathfrak{E}_X \cap F^{-1}(\mathcal{S}))^\wedge$. Since, by (b), the map $\mathcal{T} \mapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ preserves strongly stable systems, we obtain that it preserves strongly thick systems.

(c3) If $\mathfrak{E}_X^\wedge = \mathcal{W}_X^\wedge \circ \mathfrak{E}_X$, then, by 4.2.4, a strongly stable divisible in \mathfrak{E}_X system is *strongly thick* iff it is multiplicative. Evidently, the $\mathcal{T} \mapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ maps multiplicative systems to multiplicative systems. Therefore, it maps strongly stable multiplicative systems (in particular, strongly thick systems) to strongly thick systems. ■

5.2. Proposition. *Let $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_Z, \bar{\mathfrak{E}}_Z)$ be a strongly 'exact' functor. Set $\mathfrak{E}_{Y,F} = \Sigma_F \cap \mathfrak{E}_Y = \{\mathfrak{s} \in \mathfrak{E}_Y \mid F(\mathfrak{s}) \in \text{Iso}(C_Z)\}$.*

(a) *Suppose that the class of deflations \mathfrak{E}_Y satisfies the condition (i) of 5.1. Then the class $(\mathfrak{E}_{Y,F})^\wedge$ is multiplicative.*

(b) *If all deflations of $(C_Z, \bar{\mathfrak{E}}_Z)$ having a trivial kernel are isomorphisms, then $\mathfrak{E}_{Y,F}$ is a stable class, that is $\mathfrak{E}_{Y,F} = (\mathfrak{E}_{Y,F})^\wedge \cap \mathfrak{E}_Y$.*

Proof. (a) It suffices to show that if $\mathcal{L} \xrightarrow{\mathfrak{s}} \mathcal{M}$ and $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ are morphisms of C_X such that $\mathfrak{s} \in \mathfrak{E}_{Y,F}^\wedge$ and $\mathfrak{t} \in \mathfrak{E}_{Y,F}$, then $\mathfrak{t} \circ \mathfrak{s} \in \mathfrak{E}_{Y,F}^\wedge$. Since $\mathfrak{s} \in \mathfrak{E}_{Y,F}^\wedge$ and the functor F is strongly 'exact', there is a cartesian square

$$\begin{array}{ccc} \tilde{\mathcal{L}} & \xrightarrow{\tilde{\mathfrak{s}}} & \mathcal{K} \\ j' \downarrow & \text{cart} & \downarrow j \\ \mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} \end{array} \quad (3)$$

such that $\tilde{\mathfrak{s}} \in \mathfrak{E}_{Y,F}$ and the functor F maps it to a cartesian square. Taking pull-back of $\mathcal{L} \xrightarrow{\mathfrak{t} \circ \mathfrak{s}} \mathcal{N}$ along the morphism $\mathcal{K} \xrightarrow{\mathfrak{t} \circ j} \mathcal{N}$, we obtain a diagram

$$\begin{array}{ccccc} \tilde{\mathcal{L}} & \xrightarrow{\tilde{\mathfrak{s}}} & \mathcal{K} & & \\ \beta' \downarrow & \text{cart} & \downarrow \beta & & \\ \mathfrak{L} & \xrightarrow{\bar{\mathfrak{s}}} & \mathfrak{M} & \xrightarrow{\bar{\mathfrak{t}}} & \mathcal{K} \\ \gamma'' \downarrow & \text{cart} & \downarrow \gamma' & \text{cart} & \downarrow \gamma = \mathfrak{t} \circ j \\ \mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{K} \end{array} \quad (4)$$

built of cartesian squares with the arrow $\mathcal{K} \xrightarrow{\beta} \mathfrak{M}$ uniquely determined by the equalities $\bar{\mathfrak{t}} \circ \beta = id_{\mathcal{K}}$, $\gamma' \circ \beta = j$ (and with $\gamma'' \circ \beta'$ equal to the left vertical arrow j' in the cartesian square (3)). Since $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{K}$ is a deflation, the functor F , being 'exact', maps the right cartesian square of the diagram (4) to a cartesian square. In particular, $\bar{\mathfrak{t}} \in \mathfrak{E}_{Y,F}$. By construction, F transfers the square (3) to a cartesian square, which implies that it maps the upper square of the diagram (4) to a cartesian square. The equality $\bar{\mathfrak{t}} \circ \beta = id_{\mathcal{K}}$ together with the fact that $F(\bar{\mathfrak{t}})$ is an isomorphism, implies that $F(\beta)$ is an isomorphism. Therefore, $F(\beta')$ is an isomorphism. Since $\tilde{\mathfrak{s}} \in \mathfrak{E}_{Y,F}$ by construction, we obtain that $F(\tilde{\mathfrak{s}})$ is an isomorphism. So that $F(\bar{\mathfrak{t}} \circ \tilde{\mathfrak{s}})$ is an isomorphism. By hypothesis, the class of deflations \mathfrak{E}_Y satisfies the condition (i) of 5.1, which implies that $\bar{\mathfrak{t}} \circ \tilde{\mathfrak{s}}$ is a deflation. Since $\bar{\mathfrak{t}} \circ \tilde{\mathfrak{s}}$ is a pull-back of $\mathfrak{t} \circ \mathfrak{s}$, the latter belongs to $\mathfrak{E}_{Y,F}^\wedge$.

(b) For any strongly 'exact' functor $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_Z, \bar{\mathfrak{E}}_Z)$ and any class of deflations \mathcal{S} of $(C_Y, \bar{\mathfrak{E}}_Y)$, we have, by 5.1(a), the inclusions

$$F(\mathcal{S}^\wedge) \subseteq (F(\mathcal{S}))^\wedge \quad \text{and} \quad F(\mathcal{S}^\wedge \cap \bar{\mathfrak{E}}_Y) \subseteq (F(\mathcal{S}))^\wedge \cap \bar{\mathfrak{E}}_Z.$$

So that if $\mathcal{S} \subseteq \bar{\mathfrak{E}}_{Y,F} = \{\mathfrak{s} \in \bar{\mathfrak{E}}_Y \mid F(\mathfrak{s}) \in \text{Iso}(C_Z)\}$, then $F(\mathcal{S}^\wedge \cap \bar{\mathfrak{E}}_Y)$ is contained in the class $\bar{\mathfrak{E}}_Z^\oplus \stackrel{\text{def}}{=} \text{Iso}(C_Z)^\wedge \cap \bar{\mathfrak{E}}_Z$ of deflations with a trivial kernel. Therefore, if $\bar{\mathfrak{E}}_Z^\oplus = \text{Iso}(C_Z)$, then $\mathcal{S} \subseteq \mathcal{S}^\wedge \cap \bar{\mathfrak{E}}_Y \subseteq \Sigma_F \cap \bar{\mathfrak{E}}_Y = \bar{\mathfrak{E}}_{Y,F}$, hence $\bar{\mathfrak{E}}_{Y,F}$ is a stable class of deflations. ■

5.3. 'Exact' and strongly 'exact' localizations. An 'exact' (resp. strongly 'exact') functor $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{q^*} (C_X, \bar{\mathfrak{E}}_X)$ will be called an '*exact*' (resp. *strongly 'exact'*) *localization*, if $C_Y \xrightarrow{q^*} C_X$ is a localization and the essential image of $\bar{\mathfrak{E}}_Y$ (resp. the essential image of \mathcal{W}_Y) coincides with $\bar{\mathfrak{E}}_X$ (resp. with \mathcal{W}_X).

5.3.1. Note. Since $C_Y \xrightarrow{q^*} C_X$ is a localization functor, it is determined by the class of arrows

$$\Sigma_{q^*} \stackrel{\text{def}}{=} \{\mathfrak{s} \in \text{Hom}C_Y \mid q^*(\mathfrak{s}) \in \text{Iso}(C_X)\}.$$

The fact that $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{q^*} (C_X, \bar{\mathfrak{E}}_X)$ is a 'exact' localization means that the class of deflations $\bar{\mathfrak{E}}_{Y,q^*} \stackrel{\text{def}}{=} \Sigma_{q^*} \cap \bar{\mathfrak{E}}_Y$ is invariant under pull-backs.

5.4. Strongly 'exact' saturation. Every strongly 'exact' functor

$$(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_Z, \bar{\mathfrak{E}}_Z)$$

factors through a strongly 'exact' localization $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{q^*} (C_X, \bar{\mathfrak{E}}_X)$ uniquely determined by the equality $\Sigma_{q^*} = \Sigma_F$. This implies that for any class of arrows \mathcal{S} of a right exact category $(C_Y, \bar{\mathfrak{E}}_Y)$, there exists the smallest strongly 'exact' localization $q^*_\mathcal{S}$ which maps all arrows of \mathcal{S} to isomorphisms.

In fact, we consider the family $\Xi_\mathcal{S}$ of all strongly 'exact' functors from $(C_Y, \bar{\mathfrak{E}}_Y)$ which map all arrows of \mathcal{S} to isomorphisms. Since the category C_Y is svelte, the family $\{\Sigma_F \mid F \in \Xi_\mathcal{S}\}$ is a set. Therefore, there is a subset $\tilde{\Xi}_\mathcal{S}$ of $\Xi_\mathcal{S}$ such that $\{\Sigma_F \mid F \in \tilde{\Xi}_\mathcal{S}\} = \{\Sigma_F \mid F \in \Xi_\mathcal{S}\}$. The set of 'exact' functors $\tilde{\Xi}_\mathcal{S}$ defines an 'exact' functor $\Phi_\mathcal{S}$ to the product of the corresponding right exact categories. Evidently, $\Sigma_{\Phi_\mathcal{S}} = \bigcap_{F \in \tilde{\Xi}_\mathcal{S}} \Sigma_F$.

We denote $\Sigma_{q^*_\mathcal{S}} = \Sigma_{\Phi_\mathcal{S}}$ by $\bar{\mathcal{S}}$ and call it the *strongly 'exact' saturation* of \mathcal{S} .

5.5. Saturated multiplicative classes of deflations. We call a class of deflations \mathcal{S} of a right exact category with weak equivalences $(C_X, \bar{\mathfrak{E}}_X)$ *saturated* if $\bar{\mathcal{S}} \cap \bar{\mathfrak{E}}_X = \mathcal{S}$.

It follows that, for any class of deflations \mathcal{S} , the intersection $\bar{\mathcal{S}} \cap \bar{\mathfrak{E}}_X$ is the smallest saturated class of deflations containing \mathcal{S} .

Since the localization at $\bar{\mathcal{S}}$ is an 'exact' functor, in particular it maps deflations to deflations, the class $\bar{\mathcal{S}}$ is left and right divisible in $\bar{\mathfrak{E}}_X$ in the sense that if $\mathfrak{s} \circ \mathfrak{e} \in \bar{\mathcal{S}}$ and \mathfrak{e}

is a deflation, then both \mathfrak{s} and \mathfrak{e} are elements of \mathcal{S} . In particular, the system of deflations $\bar{\mathcal{S}} \cap \mathfrak{E}_X$ is divisible in \mathfrak{E}_X .

5.5.1. Proposition. (a) Suppose that the class of deflations \mathfrak{E}_X satisfies the condition (i) of 5.1. Then, for any saturated system \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$, the class \mathcal{S}^\wedge is multiplicative.

(b) If the system \mathcal{S} is stable (that is $\mathcal{S} = \mathcal{S}^\wedge \cap \mathfrak{E}_X$), then deflations with trivial kernel of the quotient right exact category $(C_{\bar{\mathcal{S}}^{-1}X}, \bar{\mathfrak{E}}_{\bar{\mathcal{S}}^{-1}X})$ are isomorphisms.

Proof. (a) Let $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{\mathfrak{q}^*} (C_Z, \bar{\mathfrak{E}}_Z)$ be the localization at the saturation $\bar{\mathcal{S}}$ of \mathcal{S} . Since, by definition of $\bar{\mathcal{S}}$, the functor \mathfrak{q}^* is strongly 'exact', it follows from 5.2 that the system \mathcal{S}^\wedge is multiplicative.

(b) Let $M \xrightarrow{s'} N$ be a deflation of $(C_Z, \bar{\mathfrak{E}}_Z)$ with a trivial kernel. The latter means that there exists a cartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{t'} & \mathcal{N} \\ \mathfrak{f}' \downarrow & \text{cart} & \downarrow \mathfrak{f} \\ M & \xrightarrow{s'} & N \end{array} \quad (1)$$

whose upper horizontal arrow is an isomorphism. Since \mathfrak{q}^* is an 'exact' *localization* functor, every arrow of \mathfrak{E}_Z is isomorphic to the image of an arrow of \mathfrak{E}_Y , there is a deflation $\mathfrak{M} \xrightarrow{s} \mathfrak{N}$ and an arrow $\tilde{\mathfrak{M}} \xrightarrow{\phi} \mathfrak{N}$ such that the pair of arrows $\mathfrak{q}^*(\mathfrak{M} \xrightarrow{s} \mathfrak{N} \xleftarrow{\phi} \tilde{\mathfrak{M}})$ is isomorphic to the pair of arrows $M \xrightarrow{s'} N \xleftarrow{\mathfrak{f}} N$. Therefore, the functor \mathfrak{q}^* maps the cartesian square

$$\begin{array}{ccc} \tilde{\mathfrak{M}} & \xrightarrow{\tilde{s}} & \tilde{\mathfrak{N}} \\ \phi' \downarrow & \text{cart} & \downarrow \phi \\ \mathfrak{M} & \xrightarrow{s} & \mathfrak{N} \end{array} \quad (2)$$

to a square isomorphic to the cartesian square (1). In particular, \tilde{s} is a deflation which \mathfrak{q}^* maps to an isomorphism; that is $\tilde{s} \in \mathcal{S}$. Since, by hypothesis, \mathcal{S} is stable, the lower horizontal arrow of (2), $\mathfrak{M} \xrightarrow{s} \mathfrak{N}$, belongs to \mathcal{S} too. Therefore, the arrow $M \xrightarrow{s'} N$ in the diagram (1) is an isomorphism. ■

5.6. Stable saturated classes. For a svelte right exact category with weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$, we denote by $\mathfrak{M}_s(X, \mathfrak{E}_X)$ the preorder (with respect to the inclusion) formed by stable saturated classes of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ and by $\tilde{\mathfrak{M}}_s(X, \mathfrak{E}_X)$ the (isomorphic to $\mathfrak{M}_s(X, \mathfrak{E}_X)$) preorder formed by the strongly 'exact' saturations $\{\bar{\mathcal{S}} \mid \mathcal{S} \in \mathfrak{M}_s(X, \mathfrak{E}_X)\}$ of these classes.

It follows that every stable saturated class of deflations containing the class \mathcal{W}_X of weak equivalences is thick. Notice that, since any saturated class of deflations is stable and contains all isomorphisms, each element of $\mathfrak{M}_s(X, \bar{\mathfrak{E}}_X)$ automatically contains \mathcal{W}_X , if the latter consists of deflations with trivial kernels.

6. Functorial properties of spectra.

6.1. Proposition. Let $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{u^*} (C_U, \bar{\mathfrak{E}}_U)$ be an 'exact' localization having the following properties:

(1) $\Sigma_{u^*} \subseteq (\Sigma_{u^*} \cap \bar{\mathfrak{E}}_X)^\wedge$,

(2) Σ_{u^*} is closed under push-forwards of deflations along arrows of Σ_{u^*} ,

(3) If $\mathcal{M} \xrightarrow{\epsilon} \mathcal{L} \xleftarrow{t} \mathcal{N}$ are deflations such that the arrows $u^*(\epsilon) = u^*(t) \circ \phi$ for some isomorphism ϕ , then there exists a pull-back $\tilde{\mathcal{M}} \xrightarrow{\tilde{\epsilon}} \tilde{\mathcal{L}} \xleftarrow{\tilde{t}} \tilde{\mathcal{N}}$ of these arrows along some morphism $\tilde{\mathcal{L}} \rightarrow \mathcal{L}$ and a commutative diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\mathfrak{s}} & \tilde{\mathcal{M}} \\ \mathfrak{s}' \downarrow & & \downarrow \tilde{\epsilon} \\ \tilde{\mathcal{N}} & \xrightarrow{\tilde{t}} & \tilde{\mathcal{L}} \end{array}$$

where $\mathfrak{s}, \mathfrak{s}'$ are arrows of Σ_{u^*} and \mathfrak{s} is a deflation.

Let Σ be a system of deflations containing $\bar{\mathfrak{E}}_X \cap \Sigma_{u^*}$. Then for any strongly stable system \mathcal{T} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$, there is the equality

$$\mathcal{T} \cap \Sigma^\perp = \bar{\mathfrak{E}}_X \cap u^{*-1}([u^*(\mathcal{T} \cap \Sigma^\perp)]) \cap \Sigma^\perp. \quad (2)$$

Proof. The inclusion $\mathcal{T}' \subseteq u^{*-1}([u^*(\mathcal{T}')])$ for any class of arrows \mathcal{T}' imply, in particular, that

$$\mathcal{T} \cap \Sigma^\perp \subseteq \bar{\mathfrak{E}}_X \cap (u^{*-1}([u^*(\mathcal{T} \cap \Sigma^\perp)]) \cap \Sigma^\perp). \quad (3)$$

The claim is that the inverse inclusion holds.

In fact, let $\mathcal{L} \xrightarrow{\xi} \mathcal{M}$ be an element of $\bar{\mathfrak{E}}_X \cap u^{*-1}([u^*(\mathcal{T} \cap \Sigma^\perp)]) \cap \Sigma^\perp$. This means that $\xi \in \bar{\mathfrak{E}}_X \cap \Sigma^\perp$ and there exists an isomorphism $u^*(\xi) \simeq u^*(t)$ for some $t \in \mathcal{T} \cap \Sigma^\perp$. This isomorphism is represented by a diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\xi} & \mathcal{M} \\ \sigma \uparrow & & \uparrow \gamma \\ \mathcal{L}' & & \mathcal{M}' \\ \sigma' \downarrow & & \downarrow \gamma' \\ \mathcal{L} & \xrightarrow{t} & \mathcal{M} \end{array}$$

whose vertical arrows belong to Σ_{u^*} and, in addition, the upper vertical arrows, σ and γ are deflations. Using the fact that ξ and t are deflations, we can form two cartesian squares

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\xi} & \mathcal{M} \\ \tilde{\gamma} \uparrow & \text{cart} & \uparrow \gamma \\ \mathcal{L} & \xrightarrow{\xi'} & \mathcal{M}' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L}' & \xrightarrow{t'} & \mathcal{M}' \\ \tilde{\gamma}' \downarrow & \text{cart} & \downarrow \gamma' \\ \mathcal{L} & \xrightarrow{t} & \mathcal{M} \end{array}$$

whose vertical arrows belong to Σ_{u^*} and, besides, all arrows of the left square are deflations.

The arrow $\mathcal{L}' \xrightarrow{t'} \mathcal{M}'$ belongs to $\mathfrak{T} \cap \Sigma^\perp$, because $t \in \mathfrak{T} \cap \Sigma^\perp$ and both \mathfrak{T} and Σ^\perp are base change invariant. One can see that the arrows $u^*(\xi')$ and $u^*(t')$ are isomorphic.

By hypothesis, there exists a pull-back $\tilde{\mathcal{L}}' \xrightarrow{\tilde{t}'} \tilde{\mathcal{M}}' \xleftarrow{\tilde{\xi}} \tilde{\mathcal{L}}$ of these two arrows along some morphism $\tilde{\mathcal{M}}' \rightarrow \mathcal{M}'$ and an isomorphism between them which can be represented by a commutative diagram

$$\begin{array}{ccc} \mathcal{L}'' & \xrightarrow{s} & \tilde{\mathcal{L}} \\ s' \downarrow & & \downarrow \tilde{\xi} \\ \tilde{\mathcal{L}}' & \xrightarrow{\tilde{t}'} & \tilde{\mathcal{M}}' \end{array} \quad (4)$$

whose upper horizontal and left vertical arrows belong to Σ_{u^*} and both horizontal and the right vertical arrows are deflations. In particular, there exists a kernel pair

$$Ker_2(s) = \tilde{\mathcal{L}} \prod_{s, s} \tilde{\mathcal{L}} \xrightarrow[p_2]{p_1} \tilde{\mathcal{L}}.$$

of the morphism $\hat{\mathcal{L}} \xrightarrow{s} \mathcal{L}$. Since s is a deflation, there exists a cocartesian square

$$\begin{array}{ccc} \tilde{\mathcal{L}} & \xrightarrow{s} & \mathcal{L} \\ s' \downarrow & \text{cocart} & \downarrow s'' \\ \mathcal{L}' & \xrightarrow{e} & \mathfrak{M}' \end{array}$$

whose arrows belong to Σ_{u^*} . It is easy to see that the arrow $\mathcal{L}' \xrightarrow{e} \mathfrak{M}'$ is the cokernel of the pair

$$Ker_2(s) \xrightarrow[s'p_2]{s'p_1} \mathcal{L}'.$$

It follows from the commutativity of the diagram (4) that $\tilde{t}' = \bar{t} \circ e$ and $\tilde{\xi} = \bar{t} \circ s''$ for a uniquely determined morphism $\mathfrak{M}' \xrightarrow{\bar{t}} \mathcal{M}'$. Since $t' \in \mathfrak{T}$ and the system \mathfrak{T} is left divisible in \mathfrak{E}_X , the morphism \bar{t} belongs to \mathfrak{T} . This shows that an appropriate pull-back of the morphism ξ belongs to $\mathfrak{T} \circ \Sigma_{u^*}$, that is $\xi \in (\mathfrak{T} \circ \Sigma_{u^*})^\wedge$. So that we obtained the inclusion $u^{*-1}([u^*(\mathfrak{T})]) \subseteq (\mathfrak{T} \circ \Sigma_{u^*})^\wedge$ which implies the inclusion

$$\mathfrak{E}_X \cap (u^{*-1}([u^*(\mathfrak{T})])) \cap \Sigma^\perp \subseteq \mathfrak{E}_X \cap ((\mathfrak{T} \circ \Sigma_{u^*})^\wedge) \cap \Sigma^\perp. \quad (5)$$

By 2.3.1, $\Sigma^\perp = (\Sigma^\perp)^\wedge$. Therefore, we have

$$\begin{aligned} \mathfrak{E}_X \cap ((\mathfrak{T} \circ \Sigma_{u^*})^\wedge) \cap \Sigma^\perp &= \mathfrak{E}_X \cap ((\mathfrak{T} \circ \Sigma_{u^*})^\wedge) \cap (\Sigma^\perp)^\wedge = \\ \mathfrak{E}_X \cap ((\mathfrak{T} \circ \Sigma_{u^*}) \cap \Sigma^\perp)^\wedge &= \mathfrak{E}_X \cap ((\mathfrak{T} \circ \Sigma_{u^*}) \cap \Sigma^\perp)^\wedge. \end{aligned}$$

Since Σ^\perp is right divisible,

$$(\mathfrak{T} \circ \Sigma_{u^*}) \cap \Sigma^\perp = \mathfrak{T} \circ (\Sigma_{u^*} \cap \Sigma^\perp) \cap \Sigma^\perp. \quad (6)$$

By hypothesis, $\Sigma_{u^*} \subseteq (\Sigma_{u^*} \cap \mathfrak{E}_X)^\wedge$; and, by the condition (i) which we assume here, the inclusion $(\Sigma_{u^*} \cap \mathfrak{E}_X) \subseteq \Sigma$ holds for all $i \in \mathcal{I}$. Therefore,

$$\Sigma_{u^*} \cap \Sigma^\perp \subseteq (\Sigma_{u^*} \cap \mathfrak{E}_X)^\wedge \cap \Sigma^\perp \subseteq \Sigma^\wedge \cap \Sigma^\perp \subseteq (\Sigma \cap \Sigma^\perp)^\wedge = \mathcal{W}_X^\wedge. \quad (7)$$

The last inclusion, $\Sigma^\wedge \cap \Sigma^\perp \subseteq (\Sigma \cap \Sigma^\perp)^\wedge$, is due to the fact that if \mathfrak{s} is an element of the intersection $\Sigma^\wedge \cap \Sigma^\perp$, then some pull-back of \mathfrak{s} is an element of $\Sigma \cap \Sigma^\perp = \mathcal{W}_X$.

Applying the inclusion $\Sigma_{u^*} \cap \Sigma^\perp \subseteq \mathcal{W}_X^\wedge$ from (7) to (6), we obtain the inclusion

$$(\mathfrak{T} \circ \Sigma_{u^*}) \cap \Sigma^\perp \subseteq (\mathfrak{T} \circ \mathcal{W}_X^\wedge) \cap \Sigma^\perp. \quad (8)$$

It follows from (8) that

$$(\mathfrak{T} \circ \Sigma_{u^*}) \cap \Sigma^\perp \subseteq (\mathfrak{T} \circ \mathcal{W}_X^\wedge) \cap \Sigma^\perp. \quad (9)$$

Combining all above (starting with (5)) and using the stability of \mathfrak{T} and the equality $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge) = \mathcal{T}$, we obtain

$$\begin{aligned} \mathfrak{E}_X \cap (u^{*-1}([u^*(\mathfrak{T})])) \cap \Sigma^\perp &\subseteq \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge)^\wedge \cap \Sigma^\perp = \\ \mathfrak{E}_X \cap \mathfrak{E}_X^\wedge \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge)^\wedge \cap \Sigma^\perp &= \mathfrak{E}_X \cap (\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^\wedge))^\wedge \cap \Sigma^\perp = \\ (\mathfrak{E}_X \cap \mathcal{T}^\wedge) \cap \Sigma^\perp &= \mathcal{T} \cap \Sigma^\perp \end{aligned}$$

whence the equality (2). ■

6.2. Proposition. *Let $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{u^*} (C_U, \bar{\mathfrak{E}}_U)$ be a strongly 'exact' localization satisfying the conditions (1)–(3) of 6.1 and such that $\mathfrak{E}_X \cap \Sigma_{u^*}$ is a stable system.*

Then, for any $\mathcal{Q} \in \mathbf{Spec}_{\mathfrak{sc}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ such that $\Sigma_{u^} \cap \mathfrak{E}_X \subseteq \mathcal{Q}$, the system $[u^*(\mathcal{Q})]$ belongs to the spectrum $\mathbf{Spec}_{\mathfrak{sc}}^{1,1}(U, \bar{\mathfrak{E}}_U)$. If $\mathcal{Q} \in \mathbf{Spec}_{\mathfrak{st}}^{1,1}(X, \bar{\mathfrak{E}}_X)$, then $[u^*(\mathcal{Q})]$ belongs to the spectrum $\mathbf{Spec}_{\mathfrak{st}}^{1,1}(U, \bar{\mathfrak{E}}_U)$.*

Proof. $\mathcal{Q} \in \mathbf{Spec}_{\mathfrak{sc}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ and $\mathfrak{E}_X \cap \Sigma_{u^*} \subseteq \mathcal{Q}$. Let \mathbb{T} be a strongly stable thick system of deflations of $(C_U, \bar{\mathfrak{E}}_U)$ properly containing $[u^*(\mathcal{Q})]$. Since $\mathfrak{E}_U = [u^*(\mathfrak{E}_X)]$, this means precisely that $u^{*-1}(\mathbb{T})$ contains \mathcal{Q} properly, hence it contains $\mathcal{Q}^{\mathfrak{sc}}$. So that $[u^*(\mathcal{Q}^{\mathfrak{sc}})] \subseteq \mathbb{T}$. On the other hand, $[u^*(\mathcal{Q})] \subsetneq [u^*(\mathcal{Q}^{\mathfrak{sc}})]$, because, by 6.1,

$$\mathfrak{E}_X \cap u^{*-1}([u^*(\mathcal{Q} \cap \mathcal{Q}^\perp)]) \cap \mathcal{Q}^\perp = \mathcal{Q} \cap \mathcal{Q}^\perp = \mathcal{W}_X,$$

while

$$\mathfrak{E}_X \cap u^{*-1}([u^*(\mathcal{Q}^{\mathfrak{sc}} \cap \mathcal{Q}^\perp)]) \cap \mathcal{Q}^\perp = \mathcal{Q}^{\mathfrak{sc}} \cap \mathcal{Q}^\perp \stackrel{\text{def}}{=} \mathcal{Q}_{\mathfrak{sc}}$$

and, since $\mathcal{Q} \in \mathbf{Spec}_{\mathfrak{sc}}^{1,1}(X, \bar{\mathfrak{E}}_X)$, the system $\mathcal{Q}_{\mathfrak{sc}}$ is non-trivial.

Same argument (with \mathbb{T} a semitopological system) shows that $[u^*(\mathcal{Q})] \in \mathbf{Spec}_{\mathfrak{st}}^{1,1}(U, \bar{\mathfrak{E}}_U)$ for any $\mathcal{Q} \in \mathbf{Spec}_{\mathfrak{st}}^{1,1}(X, \bar{\mathfrak{E}}_X)$. ■

6.3. Covers. We call a set $\{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (U, \bar{\mathfrak{E}}_U) \mid i \in J\}$ of 'exact' localizations a *cover* of $(X, \bar{\mathfrak{E}}_X)$ if $\mathfrak{E}_X \cap \left(\bigcap_{i \in J} \Sigma_{u_i^*}\right) = \mathcal{W}_X$. Below we consider only covers which have finite subcovers whose elements satisfy the conditions of 6.1.

6.4. Proposition. Let $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (X, \bar{\mathfrak{E}}_X) \mid i \in J\}$ be a cover of the right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ by strongly 'exact' localizations which has a finite subcover $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathcal{I}\}$ with the following properties:

(1) $\Sigma_{u_i^*} \subseteq (\Sigma_{u_i^*} \cap \mathfrak{E}_X)^\wedge$,

(2) $\Sigma_{u_i^*}$ is closed under push-forwards of deflations along arrows of $\Sigma_{u_i^*}$,

(3) If $\mathcal{M} \xrightarrow{\epsilon} \mathcal{L} \xleftarrow{t} \mathcal{N}$ are deflations such that the arrows $u_i^*(\epsilon) = u_i^*(t) \circ \phi$ for some isomorphism ϕ , then there exists a pull-back $\tilde{\mathcal{M}} \xrightarrow{\tilde{\epsilon}} \tilde{\mathcal{L}} \xleftarrow{\tilde{t}} \tilde{\mathcal{N}}$ of these arrows along some morphism $\tilde{\mathcal{L}} \rightarrow \mathcal{L}$ and a commutative diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\mathfrak{s}} & \tilde{\mathfrak{M}} \\ \mathfrak{s}' \downarrow & & \downarrow \tilde{\epsilon} \\ \tilde{\mathfrak{N}} & \xrightarrow{\tilde{t}} & \tilde{\mathcal{L}} \end{array}$$

where $\mathfrak{s}, \mathfrak{s}'$ are arrows of $\Sigma_{u_i^*}$ and \mathfrak{s} is a deflation.

Then the following conditions on a Serre system Σ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ are equivalent:

(a) $\Sigma \in \mathbf{Spec}_{\mathfrak{s}t}^{1,1}(X, \bar{\mathfrak{E}}_X)$,

(b) $\Sigma \in \mathbf{Spec}_{\mathfrak{t}c}^{1,1}(X, \bar{\mathfrak{E}}_X)$ and $[u_i^*(\Sigma)] \in \mathbf{Spec}_{\mathfrak{s}t}^{1,1}(U_i, \bar{\mathfrak{E}}_{U_i})$ whenever $\mathfrak{E}_X \cap \Sigma_{u_i^*} \subseteq \Sigma$.

Proof. The implication (a) \Rightarrow (b) follows from C2.4.6.4.

(b) \Rightarrow (a). Fix a finite subcover $\mathfrak{U}_{\mathcal{I}} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathcal{I}\}$ of the cover \mathfrak{U} . Set $\mathcal{I}_{\Sigma} = \{j \in \mathcal{I} \mid \mathfrak{E}_{X, u_j^*} \subseteq \Sigma\}$. Let Σ be an element of $\mathbf{Spec}_{\mathfrak{m}t}^{1,1}(X, \bar{\mathfrak{E}}_X)$ such that $[u_i^*(\Sigma)] \in \mathbf{Spec}_{\mathfrak{t}c}^{1,1}(U_i, \bar{\mathfrak{E}}_{U_i})$ for every $i \in \mathcal{I}_{\Sigma}$. The claim is that $\Sigma \in \mathbf{Spec}_{\mathfrak{t}c}^{1,1}(X, \bar{\mathfrak{E}}_X)$.

For every $i \in \mathcal{I}_{\Sigma}$, we denote by $\tilde{\mathcal{S}}_i$ the intersection $\mathfrak{E}_X \cap u_i^{*-1}([u_i^*(\Sigma)]^*) \cap \Sigma^\perp$.

Recall that Σ^\perp is the largest right divisible system having the trivial intersection with Σ (cf. 2.2). Since Σ is a Serre system of deflations, the right divisible system of deflations $\tilde{\mathcal{S}}_i$ is non-trivial, and $\tilde{\mathcal{S}}_i \not\subseteq \Sigma$. By 4.4.1, this implies that $\tilde{\mathcal{S}} = \bigcap_{i \in \mathcal{I}_{\Sigma}} \tilde{\mathcal{S}}_i$ is not contained in

Σ . Since $\tilde{\mathcal{S}} \subseteq \Sigma^\perp$, this means precisely that $\tilde{\mathcal{S}}$ is a non-trivial system.

We consider each of the two cases: $\mathcal{I}_{\Sigma} = \mathcal{I}$ and $\mathcal{I}_{\Sigma} \neq \mathcal{I}$.

(i) Suppose that $\mathcal{I}_{\Sigma} = \mathcal{I}$. Set $\tilde{\mathcal{S}} = \bigcap_{i \in \mathcal{I}_{\Sigma}} \tilde{\mathcal{S}}_i$. The claim is that $\langle \tilde{\mathcal{S}} \rangle = \Sigma$ which im-

plies that $\Sigma \in \mathbf{Spec}_{\mathfrak{s}t}^{1,1}(X, \bar{\mathfrak{E}}_X)$. The equality $\langle \tilde{\mathcal{S}} \rangle = \Sigma$ means precisely that if \mathfrak{T} is a semitopologizing system of deflations of $(X, \bar{\mathfrak{E}}_X)$ such that $\tilde{\mathcal{S}} \not\subseteq \mathfrak{T}$, then $\mathfrak{T} \subseteq \Sigma$.

Since $\tilde{\mathcal{S}} \subseteq \Sigma^\perp$, the fact $\tilde{\mathcal{S}} \not\subseteq \mathfrak{T}$ is equivalent to $\tilde{\mathcal{S}} \not\subseteq \mathfrak{T} \cap \Sigma^\perp$.

It follows from 6.1 that

$$\begin{aligned} \mathfrak{T} \cap \Sigma^\perp &= \mathfrak{E}_X \cap u_i^{*-1}([u_i^*(\mathfrak{T} \cap \Sigma^\perp)]) \cap \Sigma^\perp \quad \text{and} \\ \tilde{\mathcal{S}} &= \tilde{\mathcal{S}} \cap \Sigma^\perp = \mathfrak{E}_X \cap u_i^{*-1}([u_i^*(\tilde{\mathcal{S}})]) \cap \Sigma^\perp \end{aligned} \tag{2}$$

for every $i \in \mathcal{I}$. The equality (2) implies that if $\tilde{\mathfrak{S}} \not\subseteq \mathfrak{T} \cap \Sigma^\perp$, then $[\mathbf{u}_i^*(\tilde{\mathfrak{S}})] \not\subseteq [\mathbf{u}_i^*(\mathfrak{T})]$. But, then $[\mathbf{u}_i^*(\mathfrak{T})] \subseteq [\mathbf{u}_i^*(\Sigma)]$, whence $\mathfrak{T} \subseteq \mathbf{u}_i^{*-1}([\mathbf{u}_i^*(\Sigma)]) \cap \mathfrak{E}_X = \Sigma$.

(ii) Suppose now that $\mathcal{I}_\Sigma \neq \mathcal{I}$. Set $\mathcal{I}^\Sigma = \mathcal{I} - \mathcal{I}_\Sigma$ and $\mathfrak{E}_X^\Sigma = \bigcap_{i \in \mathcal{I}^\Sigma} \mathfrak{E}_{X, \mathbf{u}_i^*}$. Since, by the definition of \mathcal{I}^Σ , $\mathfrak{E}_{X, \mathbf{u}_i^*} \not\subseteq \Sigma$ for all $i \in \mathcal{I}^\Sigma$, it follows from 4.4.1 that $\mathfrak{E}_X^\Sigma \not\subseteq \Sigma$.

Set $\mathfrak{S} = \tilde{\mathfrak{S}} \cap \mathfrak{E}_X^\Sigma$. The claim is that $\langle \mathfrak{S} \rangle = \Sigma$.

Indeed, if \mathfrak{T} is a semitopologizing system of deflations of $(X, \bar{\mathfrak{E}}_X)$ such that $\mathfrak{S} \not\subseteq \mathfrak{T}$, then it follows from the argument (i) above that $[\mathbf{u}_i^*(\mathfrak{S})] \not\subseteq [\mathbf{u}_i^*(\mathfrak{T})]$ for some $i \in \mathcal{I}$. Notice that this i belongs to \mathcal{I}_Σ , because $\mathfrak{S} \subseteq \mathfrak{E}_X^\Sigma$, hence $\mathbf{u}_i^*(\mathfrak{S}) \subseteq \text{Iso}(C_{U_i}) \subseteq [\mathbf{u}_i^*(\mathfrak{T})]$ for every $i \in \mathcal{I}^\Sigma$. Therefore, the end of the argument of (i) applies. ■

Similar fact (but, with additional assumptions) holds for the spectrum $\mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X)$.

6.5. Proposition. *Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ be a svelte right exact category with a stable class of weak equivalences (that is $\mathcal{W}_X = \mathcal{W}_X^\wedge \cap \mathfrak{E}_X$) and a left divisible and weakly right divisible class of deflations (conditions 2.0(c) and 2.0(d)). Let $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (X, \bar{\mathfrak{E}}_X) \mid i \in J\}$ be a cover of the right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ by strongly 'exact' localizations which has a finite subcover $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathcal{I}\}$ with the following properties:*

(1) $\Sigma_{\mathbf{u}_i^*} \subseteq (\Sigma_{\mathbf{u}_i^*} \cap \mathfrak{E}_X)^\wedge$,

(2) $\Sigma_{\mathbf{u}_i^*}$ is closed under push-forwards of deflations along arrows of $\Sigma_{\mathbf{u}_i^*}$,

(3) If $\mathcal{M} \xrightarrow{e} \mathcal{L} \xleftarrow{t} \mathcal{N}$ are deflations such that the arrows $\mathbf{u}_i^*(e) = \mathbf{u}_i^*(t) \circ \phi$ for some isomorphism ϕ , then there exists a pull-back $\tilde{\mathcal{M}} \xrightarrow{\tilde{e}} \tilde{\mathcal{L}} \xleftarrow{\tilde{t}} \tilde{\mathcal{N}}$ of these arrows along some morphism $\tilde{\mathcal{L}} \rightarrow \mathcal{L}$ and a commutative diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{s} & \tilde{\mathfrak{M}} \\ \mathfrak{s}' \downarrow & & \downarrow \tilde{e} \\ \tilde{\mathfrak{N}} & \xrightarrow{\tilde{t}} & \tilde{\mathfrak{L}} \end{array}$$

where \mathfrak{s} , \mathfrak{s}' are arrows of $\Sigma_{\mathbf{u}_i^*}$ and \mathfrak{s} is a deflation.

(4) The functors \mathbf{u}_i^* preserve push-forwards of deflations.

Then the following conditions on a Serre system Σ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ are equivalent:

(a) $\Sigma \in \mathbf{Spec}_t^{1,1}(X, \bar{\mathfrak{E}}_X)$,

(b) $\Sigma \in \mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ and $[\mathbf{u}_i^*(\Sigma)] \in \mathbf{Spec}_t^{1,1}(U_i, \bar{\mathfrak{E}}_{U_i})$ whenever $\mathfrak{E}_X \cap \Sigma_{\mathbf{u}_i^*} \subseteq \Sigma$.

Proof. The argument is similar to that of 6.4. Details are left to the reader. ■

6.6. Comments about the conditions on localizations. In the assertions of this section, we consider strongly 'exact' localizations $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{u^*} (C_U, \bar{\mathfrak{E}}_U)$ such that $\Sigma_{\mathbf{u}^*} \cap \mathfrak{E}_X$ is stable and the following properties hold:

(1) $\Sigma_{\mathbf{u}^*} \subseteq (\Sigma_{\mathbf{u}^*} \cap \mathfrak{E}_X)^\wedge$,

(2) $\Sigma_{\mathbf{u}^*}$ is closed under push-forwards of deflations along arrows of $\Sigma_{\mathbf{u}^*}$,

(3) If $\mathcal{M} \xrightarrow{\epsilon} \mathcal{L} \xleftarrow{t} \mathcal{N}$ are deflations such that $\mathbf{u}^*(\epsilon) = \mathbf{u}^*(t) \circ \phi$ for some isomorphism ϕ , then there exists a pull-back $\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\epsilon}} \widetilde{\mathcal{L}} \xleftarrow{\widetilde{t}} \widetilde{\mathcal{N}}$ of these arrows along some morphism $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ and a commutative diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\mathfrak{s}} & \widetilde{\mathcal{M}} \\ \mathfrak{s}' \downarrow & & \downarrow \widetilde{\epsilon} \\ \widetilde{\mathcal{N}} & \xrightarrow{\widetilde{t}} & \widetilde{\mathcal{L}} \end{array}$$

where \mathfrak{s} , \mathfrak{s}' are arrows of $\Sigma_{\mathbf{u}^*}$ and \mathfrak{s} is a deflation.

(1) The condition (1) in combination with the stability of $\mathfrak{E}_{X, \mathbf{u}^*} = \Sigma_{\mathbf{u}^*} \cap \mathfrak{E}_X$ implies that the system of deflations $\mathfrak{E}_{X, \mathbf{u}^*}$ is *saturated* (cf. 5.5).

(2) The condition (2) holds if $\Sigma_{\mathbf{u}^*}$ is closed under taking cokernels of pairs of arrows $\mathcal{M} \xrightarrow{t_1} \mathcal{N} \xrightarrow{t_2}$ such that $\mathbf{u}^*(t_1) = \mathbf{u}^*(t_2)$ (see the argument of 6.1).

The condition (2) holds, if the functor \mathbf{u}^* preserves push-forwards of deflations.

(3) It follows from the condition (1) that there exists a pull-back $\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\epsilon}} \widetilde{\mathcal{L}} \xleftarrow{\widetilde{t}} \widetilde{\mathcal{N}}$ of the deflations $\mathcal{M} \xrightarrow{\epsilon} \mathcal{L} \xleftarrow{t} \mathcal{N}$ along some morphism $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ such that the isomorphism ϕ is described by a diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\mathfrak{s}} & \widetilde{\mathcal{M}} \\ \mathfrak{s}' \downarrow & & \downarrow \widetilde{\epsilon} \\ \widetilde{\mathcal{N}} & \xrightarrow{\widetilde{t}} & \widetilde{\mathcal{L}} \end{array}$$

whose upper horizontal and left vertical arrows belong to $\Sigma_{\mathbf{u}^*}$, which the functor \mathbf{u}^* transforms to a commutative diagram. The condition (3) holds for sure if the category C_X is pointed (or, more generally, C_X has initial objects and, for any object of C_X , there is at most one morphism to an initial object): it suffices to take a pull-back of the square above along the unique arrow $\mathfrak{x} \longrightarrow \widetilde{\mathfrak{L}}$ from an initial object.

Notice that the conditions (1), (2), (3) stand finite intersections. So that one talk about covers and the corresponding pretopology. We shall not go into details of this here.

7. Spectra of right exact 'spaces' over a point.

We start with 'spaces' represented by right exact categories with stable class of weak equivalences and initial objects and gather together different facts and observations scattered in the previous sections.

7.0. Right exact 'spaces' over a point. A "point", \mathbf{x} , is represented by the *trivial* right exact category, that is the category $C_{\mathbf{x}}$ with only one (hence identical) arrow.

A right exact 'space' over a point \mathbf{x} is a pair $((X, \bar{\mathfrak{E}}_X), \gamma)$, where γ is a *continuous* morphism $(X, \bar{\mathfrak{E}}_X) \rightarrow \mathbf{x}$. Right exact 'spaces' over the point \mathbf{x} form a category in a standard way: morphisms from $((X, \bar{\mathfrak{E}}_X), \gamma)$ to $((Y, \bar{\mathfrak{E}}_Y), \tilde{\gamma})$ are given by morphisms of 'spaces' $X \xrightarrow{f} Y$ whose inverse image functor f^* is an 'exact' functor from $(C_Y, \bar{\mathfrak{E}}_Y)$ to $(C_X, \bar{\mathfrak{E}}_X)$ and such that $\tilde{\gamma} \circ f = \gamma$, which means that $f^* \circ \tilde{\gamma}^* \simeq \gamma^*$.

Recall that *continuous* means that an inverse image functor $C_{\mathbf{x}} \xrightarrow{\gamma^*} C_X$ of the morphism γ has a right adjoint. One can see that this condition means precisely that γ^* maps the unique object of the category $C_{\mathbf{x}}$ to an initial object of the category C_X . It follows that morphisms of right exact 'spaces' over a point are precisely those morphisms of right exact 'spaces' whose inverse image functor preserves initial objects.

7.0.1. Conventions. We fix a right exact 'space' $((X, \bar{\mathfrak{E}}_X), \gamma)$ over a point \mathbf{x} together with an inverse image functor the morphism γ . The latter means that we fix an initial object \mathfrak{r} of the category C_X . We assume that $(C_X, \bar{\mathfrak{E}}_X)$ has a stable class of weak equivalences and that all split epimorphisms of the category C_X are deflations. In particular, every morphism to an initial object is a deflation.

Since in a general right exact category deflations are not invariant under push-forwards, we look at the version of the spectral theory based on the notion of a *semitopological* system (see Sections 4 and 5). Fix an initial object \mathfrak{r} of the category C_X .

7.1. Stable systems of deflations and subcategories of C_X/\mathfrak{r} . Following general pattern, we consider the correspondence which assigns to any class of deflations \mathcal{S} of $(C_X, \bar{\mathfrak{E}}_X)$ the full subcategory $\mathbb{T}_{\mathcal{S}}$ of the category C_X/\mathfrak{r} whose objects are pairs $(M, M \xrightarrow{\mathfrak{s}} \mathfrak{r})$ with $\mathfrak{s} \in \mathcal{S}$. In other words, $\mathbb{T}_{\mathcal{S}}$ is generated by the kernels of arrows of \mathcal{S} . Here by a kernel of a morphism $\mathcal{M} \xrightarrow{f} \mathcal{N}$ we understand the pair $(Ker(f), Ker(f) \rightarrow \mathfrak{r})$ (– an object of C_X/\mathfrak{r}), where $Ker(\mathfrak{e}) \rightarrow \mathfrak{r}$ is the canonical morphism.

The stability of a class \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ means that

$$\mathcal{S} = \{\mathfrak{s} \in \bar{\mathfrak{E}}_X \mid Ker(\mathfrak{s}) \in \mathbb{T}_{\mathcal{S}}\}.$$

The correspondence $\mathcal{S} \mapsto \mathbb{T}_{\mathcal{S}}$ establishes an isomorphism between the preorder of stable systems invariant under pull-backs and the preorder formed by strictly full subcategories of the category C_X/\mathfrak{r} containing kernels of weak equivalences; in particular, they contain initial objects. The inverse maps assigns to a strictly full subcategory \mathcal{T} of the category C_X/\mathfrak{r} the class $\bar{\mathfrak{E}}_X^{\mathcal{T}}$ of all deflations \mathfrak{s} such that $Ker(\mathfrak{s}) \in \mathcal{T}$.

Given a strictly full subcategory \mathcal{T} of the category C_X/\mathfrak{r} , let $\mathfrak{K}_X^{\mathcal{T}}$ denote the class of all arrows of C_X which have a kernel from \mathcal{T} . By definition, $\bar{\mathfrak{E}}_X^{\mathcal{T}} = \bar{\mathfrak{E}}_X \cap \mathfrak{K}_X^{\mathcal{T}}$. It follows that $(\bar{\mathfrak{E}}_X^{\mathcal{T}})^{\wedge} = \mathfrak{K}_X^{\mathcal{T}}$. So that if \mathcal{S} is a stable system of deflations, then $\mathcal{S}^{\wedge} = \mathfrak{K}_X^{\mathbb{T}_{\mathcal{S}}}$, i.e. \mathcal{S}^{\wedge} consists of all arrows of C_X whose kernel exists and belongs to $\mathbb{T}_{\mathcal{S}}$.

7.2. Cartesian closedness and divisibility. A stable system \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is cartesian closed iff the corresponding full subcategory $\mathbb{T}_{\mathcal{S}}$ of the category C_X/\mathfrak{r} is closed under finite products (taken in C_X/\mathfrak{r}).

A system \mathcal{S} is *left divisible* iff for any $M \in \text{Ob}\mathbb{T}_{\mathcal{S}}$ and any deflation $M \rightarrow N$ (in C_X/\mathfrak{r}), the object N belongs to $\mathbb{T}_{\mathcal{S}}$. A system \mathcal{S} is *right divisible* if for any object M of $\mathbb{T}_{\mathcal{S}}$, the kernel of any deflation $M \xrightarrow{\epsilon} N$ belongs to $\mathbb{T}_{\mathcal{S}}$.

7.3. Strong stability. The class $\mathcal{W}_X^{\bar{\cdot}}$ contains all morphisms with trivial kernel, in particular, all monomorphisms. Therefore, the condition $\mathcal{S} = \bar{\mathfrak{E}}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\cdot}})$ (which makes a difference between the strong stability and stability) implies that the subcategory $\mathbb{T}_{\mathcal{S}}$ is closed under taking arbitrary (not only “admissible”) subobjects. If morphisms with trivial kernel are isomorphisms and all weak equivalences are isomorphisms, then the system \mathcal{S} is strongly stable iff the corresponding subcategory is closed under taking arbitrary subobjects.

7.4. Semitopologizing systems and strongly topologizing subcategories. Summarizing all above, one can see that the map $\mathcal{S} \mapsto \mathbb{T}_{\mathcal{S}}$ induces an isomorphism between the preorder of semitopologizing systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ and the preorder of full subcategories \mathcal{T} of C_X which are closed under finite products and subobjects (taken in C_X) and such that for any deflation $\mathcal{M} \xrightarrow{\epsilon} \mathcal{N}$ with $\mathcal{M} \in \text{Ob}\mathcal{T}$, the object \mathcal{N} belongs to \mathcal{T} . We call such subcategories *strongly topologizing*.

7.4.1. Note. We use here *strongly topologizing*, because the name “topologizing subcategories” was given (years ago) to the most straightforward generalization of this notion for exact categories [R, Ch.5]. We recall it for completeness: a subcategory \mathcal{T} of an exact category is called *topologizing* if it is closed under finite products and for any deflation $\mathcal{M} \xrightarrow{\epsilon} \mathcal{N}$ with $\mathcal{M} \in \text{Ob}\mathcal{T}$, both \mathcal{N} and $\text{Ker}(\epsilon)$ are objects of the subcategory \mathcal{T} .

7.5. Thick systems and thick subcategories. A system \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is thick iff the corresponding subcategory $\mathbb{T}_{\mathcal{S}}$ is *thick* in the most expected, ordinary sense: if $M \xrightarrow{\epsilon} N$ is a deflation in C_X/\mathfrak{r} , then M is an object of $\mathbb{T}_{\mathcal{S}}$ iff both N and $\text{Ker}(\epsilon)$ are objects of $\mathbb{T}_{\mathcal{S}}$. In other words, the subcategory $\mathbb{T}_{\mathcal{S}}$ is topologizing and closed under extensions.

7.6. Strongly thick systems and strongly thick subcategories. A system of deflations \mathcal{S} is strongly thick iff the corresponding subcategory $\mathbb{T}_{\mathcal{S}}$ is strongly topologizing and closed under extensions; or, what is the same, strongly topologizing and thick.

7.7. Orthogonal complements. We call objects of the subcategory $\mathbb{T}_{\mathcal{W}_X}$ *trivial*. If \mathcal{W}_X consists of arrows with trivial kernel, then objects of $\mathbb{T}_{\mathcal{W}_X}$ are pairs $(V, V \rightarrow \mathfrak{r})$, where V runs through initial objects of C_X (i.e. $V \rightarrow \mathfrak{r}$ is an isomorphism).

Let \mathcal{T} be a strictly full subcategory of the category C_X/\mathfrak{r} containing $\mathbb{T}_{\mathcal{W}_X}$. We denote by \mathcal{T}^{\perp} the full subcategory of the category C_X generated by all objects \mathcal{M} of C_X such that the kernel of a deflation $\mathcal{M} \rightarrow \mathcal{N}$ belongs to \mathcal{T} iff it is trivial. It follows that, for any stable system of deflations \mathcal{S} , its orthogonal complement \mathcal{S}^{\perp} contains $\mathfrak{K}_X^{\mathbb{T}_{\mathcal{S}}^{\perp}}$ and is contained in $\mathfrak{K}_X^{\mathbb{T}_{\mathcal{S}}^{\perp}} \cup \{\text{morphisms of } C_X \text{ without kernel}\}$.

In particular, if the category C_X has kernels of all morphisms, then $\mathcal{S}^\perp = \mathfrak{K}_X^{\mathbb{T}_\mathcal{S}^\perp}$.

7.8. Serre systems of deflations and Serre subcategories of C_X/\mathfrak{r} . For any subcategory \mathcal{T} of the category C_X/\mathfrak{r} , let \mathcal{T}^- denote the full subcategory of C_X/\mathfrak{r} generated by all objects \mathcal{M} having the following property: if $\mathcal{M} \xrightarrow{\epsilon} \mathcal{N}$ is a non-trivial deflation (that is $\text{Ker}(\epsilon)$ is non-trivial), then there exists a non-trivial deflation $\text{Ker}(\epsilon) \xrightarrow{\xi} \mathcal{L}$ (in C_X/\mathfrak{r}) with $\text{Ker}(\xi) \in \text{Ob}\mathcal{T}$. We call a subcategory \mathcal{T} of C_X/\mathfrak{r} a *Serre subcategory* if $\mathcal{T} = \mathcal{T}^-$.

There is the equality $(\mathfrak{E}_{X,\mathcal{T}})^- = \mathfrak{E}_{X,\mathcal{T}^-}$.

In particular, a system of deflations \mathcal{S} of Σ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is a Serre system iff it is stable and $\mathbb{T}_\mathcal{S} = \mathbb{T}_\mathcal{S}^-$. This establishes an isomorphism between the preorders of Serre systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ and Serre subcategories of C_X/\mathfrak{r} .

7.9. Strongly closed systems of deflations and strongly closed subcategories of C_X/\mathfrak{r} . For any subcategory \mathcal{T} of the category C_X/\mathfrak{r} , let \mathcal{T}^\dagger denote the full subcategory of C_X/\mathfrak{r} generated by objects $\mathcal{M} \in \mathcal{T}^-$ such that for any morphism $L \rightarrow M$ from \mathcal{W}_X^\wedge , the object L belongs to \mathcal{T}^- .

We call a subcategory \mathcal{T} of C_X/\mathfrak{r} *strongly closed* if $\mathcal{T} = \mathcal{T}^\dagger$.

It follows from 7.8 and the observation 4.3.3(a) that $(\mathfrak{E}_{X,\mathcal{T}})^\dagger = \mathfrak{E}_{X,\mathcal{T}^\dagger}$. In particular, a system of deflations \mathcal{S} is strongly closed iff it is stable and $\mathbb{T}_\mathcal{S} = \mathbb{T}_\mathcal{S}^\dagger$.

7.10. Strongly 'exact' functors. An 'exact' functor $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{F} (C_Y, \bar{\mathfrak{E}}_Y)$ is *strongly 'exact'* iff it maps cartesian squares of the form

$$\begin{array}{ccc} \text{Ker}(f) & \longrightarrow & \mathfrak{r} \\ \downarrow & \text{cart} & \downarrow \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N} \end{array} \quad (1)$$

to cartesian squares. If F maps initial objects to initial objects, this condition means that F preserves kernels of arrows. Since localizations map initial objects to initial objects, an 'exact' localization is strongly 'exact' iff it preserves kernels.

7.10.1. Remark. Since morphisms to initial objects in C_X are deflations, it follows from the diagram (1) that, for a strongly 'exact' functor $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{F} (C_Y, \bar{\mathfrak{E}}_Y)$, the class of arrows $\Sigma_F = \{\mathfrak{s} \in \text{Hom}C_X \mid F(\mathfrak{s}) \text{ is invertible}\}$ is contained in $(\Sigma_F \cap \bar{\mathfrak{E}}_X)^\wedge = \bar{\mathfrak{E}}_{X,F}^\wedge$ iff all arrows of Σ_F have kernels. So that, in the case when all arrows of the category C_X have kernels, $\Sigma_F \subseteq \bar{\mathfrak{E}}_{X,F}^\wedge$ for any strongly 'exact' functor F .

7.10.2. Kernels of strongly 'exact' functors. Suppose that a strongly 'exact' functor $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{F} (C_Y, \bar{\mathfrak{E}}_Y)$ maps initial objects to initial objects. Then F induces a functor $(C_{X_\mathfrak{r}}, \bar{\mathfrak{E}}_{X_\mathfrak{r}}) \xrightarrow{F_\mathfrak{r}} (C_{Y_\mathfrak{r}}, \bar{\mathfrak{E}}_{Y_\mathfrak{r}})$, where $C_{X_\mathfrak{r}} = C_X/\mathfrak{r}$, $C_{Y_\mathfrak{r}} = C_Y/\mathfrak{r}$, $\mathfrak{r} = F(\mathfrak{r})$.

One can see that the subcategory $\mathbb{T}_{\mathfrak{E}_{X,F}}$ coincides with the kernel of the functor $C_X/\mathfrak{r} \xrightarrow{F_\mathfrak{r}} C_Y/\mathfrak{r}$, and the latter is naturally equivalent to the full subcategory of kernel of the functor F generated by all objects N of $\text{Ker}(F)$ having a morphism to \mathfrak{r} .

We denote the kernel of the functor $F_\mathfrak{r}$ by $\mathfrak{Ker}(F)$.

7.10.3. Covers by strongly 'exact' localizations. Elements of the covers we consider here are morphisms $(U, \mathfrak{E}_U) \xrightarrow{u} (X, \mathfrak{E}_X)$ whose inverse image functors are strongly 'exact' localizations such that all arrows of Σ_{u^*} have kernels and the intersection $\mathfrak{E}_{X, u^*} = \mathfrak{E}_X \cap \Sigma_{u^*}$ is a stable system of deflations. We call a set

$$\{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathcal{J}\}$$

of such morphisms a *cover* of the right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ if $\mathfrak{E}_X \cap \left(\bigcap_{i \in J} \Sigma_{u_i^*}\right) = \mathcal{W}_X$, or,

equivalently, $\bigcap_{i \in J} \mathbb{T}_{\mathfrak{E}_{X, u_i^*}} = \mathbb{T}_{\mathcal{W}_X}$. Taking into consideration the discussion and notation of

7.10.2, we can rewrite the latter equality as $\bigcap_{i \in J} \mathfrak{Ker}(u_i^*) = \mathbb{T}_{\mathcal{W}_X}$. If the class \mathcal{W}_X consists

of deflations with a trivial kernel (which is a standard choice), then the trivial subcategory $\mathbb{T}_{\mathcal{W}_X}$ is trivial in the usual sense: all its objects are initial.

7.11. The spectra. For every subcategory \mathcal{T} of the category C_X/\mathfrak{r} , we denote by \mathcal{T}^* the intersection of all strongly thick subcategories of C_X/\mathfrak{r} which contain properly the subcategory \mathcal{T} . We denote by \mathcal{T}_* the intersection $\mathcal{T}^* \cap \mathcal{T}^\perp$.

We denote by $\mathbf{Spec}_{\mathfrak{E}\mathfrak{c}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to the inverse inclusion) formed by all strongly closed subcategories \mathcal{T} of the category C_X/\mathfrak{r} for which $\mathcal{T}^* \neq \mathcal{T}$, or, equivalently, the subcategory \mathcal{T}_* is non-trivial.

Similarly, we denote by \mathcal{T}^* the intersection of all strongly topologizing subcategories of C_X/\mathfrak{r} properly containing \mathcal{T} and set $\mathcal{T}_* = \mathcal{T}^* \cap \mathcal{T}^\perp$. We denote by $\mathbf{Spec}_{\mathfrak{E}\mathfrak{s}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ the subpreorder of $\mathbf{Spec}_{\mathfrak{E}\mathfrak{c}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ formed by those strongly closed subcategories \mathcal{T} of the category C_X/\mathfrak{r} for which $\mathcal{T}^* \neq \mathcal{T}$, or, equivalently, \mathcal{T}_* is a non-trivial subcategory of C_X/\mathfrak{r} .

7.11.1. Proposition. *The map $\mathcal{S} \mapsto \mathbb{T}_{\mathcal{S}}$ induces isomorphisms*

$$\begin{aligned} \mathbf{Spec}_{\mathfrak{sc}}^{1,1}(X, \bar{\mathfrak{E}}_X) &\xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{E}\mathfrak{c}}^{1,1}(X, \bar{\mathfrak{E}}_X) \\ \mathbf{Spec}_{\mathfrak{st}}^{1,1}(X, \bar{\mathfrak{E}}_X) &\xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{E}\mathfrak{s}}^{1,1}(X, \bar{\mathfrak{E}}_X) \end{aligned} \quad (2)$$

between the spectra defined in terms of systems of deflations (4.4) and the spectra defined in terms of strongly closed subcategories.

Proof. The assertion follows from the sketched above dictionary between the stable systems of deflations of different kind and the subcategories of the category C_X/\mathfrak{r} . ■

7.11.2. Proposition. *Let $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (X, \bar{\mathfrak{E}}_X) \mid i \in J\}$ be a cover of the right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ by strongly 'exact' localizations which has a finite subcover $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{u_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathcal{J}\}$ such that, for every $i \in \mathcal{J}$, the subcategory $\mathfrak{Ker}(u_i^*)$ of C_X/\mathfrak{r} is invariant under push-forwards of deflations (which holds if $\mathfrak{Ker}(u_i^*)$ is invariant under cokernels of pairs of arrows).*

Then the following conditions on a strongly closed subcategory \mathcal{P} of the category C_X/\mathfrak{r} are equivalent:

$$(a) \mathcal{P} \in \mathbf{Spec}_{\mathfrak{E}\mathfrak{s}}^{1,1}(X, \bar{\mathfrak{E}}_X),$$

(b) $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{C}}^{1,1}(X, \bar{\mathfrak{C}}_X)$ and $[\mathbf{u}_i^*(\mathcal{P})] \in \mathbf{Spec}_{\mathfrak{C}}^{1,1}(U_i, \bar{\mathfrak{C}}_{U_i})$ whenever $\mathfrak{Ker}(\mathbf{u}_i^*) \subseteq \mathcal{P}$.

Proof. The claim is that the assertion follows from 6.4.

If fact, since, by the definition of covers, the arrows of $\Sigma_{\mathbf{u}_i^*}$ have kernels for all $i \in J$, the condition (1) of 6.4 holds: $\Sigma_{\mathbf{u}_i^*} \subseteq (\Sigma_{\mathbf{u}_i^*} \cap \mathfrak{C}_X)^\wedge$ for all $i \in J$ (see 7.10.1).

It follows from isomorphisms of 7.11.1 that one can, replacing the category C_X by C_X/\mathfrak{I} , assume that the category is pointed. Therefore, the condition (3) holds (see 6.6(3)).

Finally, the invariance of the subcategories $\mathfrak{Ker}(\mathbf{u}_i^*)$ under push-forwards of deflations is what remains of the condition (2) of 6.4. ■

8. Special cases, some examples.

8.1. The abelian case. Let $(C_X, \bar{\mathfrak{C}}_X)$ be an abelian category; that is C_X is an abelian category, deflations are arbitrary epimorphisms and weak equivalences are isomorphisms. Then (as it was already mentioned in the text) semitopological classes of deflations become topological and, therefore, the spectral theories outlined in Sections 3 and 4 coincide. To every class \mathcal{S} of epimorphisms of the category C_X , we assign a full subcategory $\mathbb{T}_{\mathcal{S}}$ of C_X whose objects are kernels of morphisms from \mathcal{S} . The correspondence $\mathcal{S} \mapsto \mathbb{T}_{\mathcal{S}}$ induces isomorphisms between the preorder of topological systems of deflations and the preorder (with respect to the inclusion) of topological subcategories of the category C_X in the sense of Gabriel (– full subcategories of C_X closed under taking subquotients and finite products). Similarly, $\mathcal{S} \mapsto \mathbb{T}_{\mathcal{S}}$ induces an isomorphism between the preorder of thick (resp. Serre) systems and the preorder of thick (resp. Serre) subcategories of C_X .

Strongly 'exact' functors between abelian categories are the same as 'exact' functors and the latter are just *exact* functors in the usual sense. And for any exact functor $C_X \xrightarrow{F} C_Y$, the class of arrows $\Sigma_F = \{\mathfrak{s} \in \mathit{Hom}C_X \mid F(\mathfrak{s}) \text{ is an isomorphism}\}$ satisfies all the conditions which appear in the main assertions of Section 6 (and are discussed in 6.6).

It follows from this isomorphisms and coincidences that the results of this work (translated into the language of topological, thick and Serre subcategories) recover all essential facts of [R5] (see Appendix 2).

8.2. Spectra of 'spaces' represented by exact categories. Suppose that $(C_X, \bar{\mathfrak{C}}_X)$ is an exact category. In this case, weak equivalences are isomorphisms and deflations are called sometimes *admissible epimorphisms*. Since, for a general exact category, deflations are not invariant under push-forwards, we look at the version of the spectral theory based on the notion of a *semitopological* system (see Sections 4 and 5).

Following general pattern, we consider the correspondence which assigns to any class of deflations \mathcal{S} the full subcategory $\mathbb{T}_{\mathcal{S}}$ of the category C_X generated by all objects M such that morphism from M to the zero object belongs to \mathcal{S} .

Notice that, since the category C_X is additive and weak equivalences are isomorphisms, the class \mathcal{W}_X^\wedge consists of all monomorphisms of the category C_X . Therefore, a system \mathcal{S} of deflations of $(C_X, \bar{\mathfrak{C}}_X)$ is strongly stable iff the subcategory $\mathbb{T}_{\mathcal{S}}$ is closed under taking arbitrary subobjects (cf. 7.3).

The map $\mathcal{S} \mapsto \mathbb{T}_{\mathcal{S}}$ induces an isomorphism between the preorder of semitopologizing systems of deflations of $(C_X, \bar{\mathfrak{C}}_X)$ and the preorder of *strongly topologizing* subcategories \mathcal{T} of C_X which are full subcategories of C_X closed under finite products and subobjects

(taken in C_X) and such that for any deflation $\mathcal{M} \xrightarrow{\epsilon} \mathcal{N}$ with $\mathcal{M} \in \text{Ob}\mathcal{T}$, the object \mathcal{N} belongs to \mathcal{T} (see 7.4).

8.3. A note about the spectral theory of the category of algebras. Let C_X be the category Alg_k of associative unital algebras over a commutative unital ring k , deflations are strict (that is surjective) epimorphisms of algebras and weak equivalences are isomorphisms. The k -algebra k is the canonical initial object of the category Alg_k and the category Alg_k/k is isomorphic to the category of augmented algebras. The category Alg_k/k of augmented k -algebras is naturally equivalent to the category Alg_k^1 of non-unital k -algebras. The equivalence is given by the functor $\text{Alg}_k^1 \xrightarrow{\iota_k^*} \text{Alg}_k/k$ which assigns to a non-unital k -algebra \mathcal{R} the augmented algebra $(k \oplus \mathcal{R}, k \oplus \mathcal{R} \rightarrow k)$. Its quasi-inverse functor maps an augmented algebra (A, ξ_A) to its augmentation ideal $\text{Ker}(\xi_A)$.

We have a commutative diagram of functors

$$\begin{array}{ccccc} \text{Alg}_k^1 & \xrightarrow{\iota_k^*} & \text{Alg}_k/k & \xrightarrow{j_k^*} & \text{Alg}_k \\ \tilde{f}^* \uparrow \downarrow \tilde{f}_* & & \bar{f}^* \uparrow \downarrow \bar{f}_* & & f^* \uparrow \downarrow f_* \\ k - \text{mod} & \xrightarrow{Id} & k - \text{mod} & \xrightarrow{Id} & k - \text{mod} \end{array} \quad (1)$$

where \tilde{f}_* is the forgetful functor, \tilde{f}^* its left adjoint which assigns to every k -module V the irrelevant ideal $T_k^{\geq 1}(V) = \bigoplus_{n \geq 1} V^{\otimes_k n}$ of the tensor algebra $T_k(V)$ of the module V ; j_k^* is

the canonical forgetful functor and the functor \bar{f}^* assigns to each k -module V the tensor algebra $T_k(V)$ with the canonical augmentation and f_* is the forgetful functor.

By 7.11.1, the spectral theory outlined here requires only the category Alg_k/k of augmented k -algebras. There is another pair of adjoint functors

$$k - \text{mod} \begin{array}{c} \xrightarrow{\varphi_*} \\ \xleftarrow{\varphi^*} \end{array} \text{Alg}_k/k, \quad (2)$$

where the functor φ_* assigns to each k -module V the k -algebra $k \oplus V$ with $V \cdot V = 0$. Its left adjoint functor, φ^* , assigns to every augmented k -algebra $(A, A \xrightarrow{\xi_A} k)$ the k -module $\text{Ker}(\xi_A)/\text{Ker}(\xi_A)^2$. The composition $\varphi^* \circ \varphi_*$ is the identical functor, which means that φ_* is a fully faithful functor and, therefore, φ^* is a localization functor. The functor φ_* is 'exact' and induces a natural equivalence between the category $k - \text{mod}$ and the topologizing subcategory \mathbb{T}_0 of the category Alg_k/k whose objects are those augmented algebras $(A, A \xrightarrow{\xi_A} k)$ for which $\text{Ker}(\xi_A)^2 = 0$. Thus, the spectrum of the 'space' $\mathbf{Sp}(k)$ (which is isomorphic to $\text{Spec}(k)$ – the prime spectrum of k , is embedded into $\mathbf{Spec}_{\mathbb{G}\mathbb{T}}^{1,1}(X_{\mathbb{T}})$.

The picture looks slightly simpler in terms of the category $\mathcal{C}_{\mathfrak{A}_X^1} = \text{Alg}_k^1$ of non-unital k -algebras. Namely, the pair of adjoint functors (2) corresponds to the functors

$$k - \text{mod} \begin{array}{c} \xrightarrow{\gamma_*} \\ \xleftarrow{\gamma^*} \end{array} \text{Alg}_k^1, \quad (2')$$

where γ_* assigns to each k -module V the same k -module with the zero multiplication and the functor γ^* maps each non-unital k -algebra A to the k -module A/A^2 . The kernel of

the localization functor γ^* is the full subcategory $\mathcal{C}_{\mathfrak{A}_X^f} = \text{Alg}_k^f$ of the category $\text{Alg}_k^1 = \mathcal{C}_{\mathfrak{A}_X^1}$ whose objects are so-called *firm* algebras A defined by the equality $A^2 = A$ (evidently, every unital algebra is firm). One can show that the spectrum $\mathbf{Spec}_{\mathfrak{S}\mathfrak{X}}^{1,1}(\mathfrak{A}_X^1, \bar{\mathfrak{E}}_{\mathfrak{A}_X^1})$ is the disjoint union of the image of the prime spectrum of k and the spectrum $\mathbf{Spec}_{\mathfrak{S}\mathfrak{X}}^{1,1}(\mathfrak{A}_X^f, \bar{\mathfrak{E}}_{\mathfrak{A}_X^f})$ of the right exact 'space' represented by the subcategory of firm k -algebras.

8.3.1. Generalizations. Let $\tilde{\mathcal{C}}_X = (\mathcal{C}_X, \odot, \mathbb{I})$ be a monoidal category with 'tensor' product \odot and the unit object \mathbb{I} . We assume that \mathcal{C}_X is a pointed category with countable colimits preserved by 'tensor' product; and tensoring any object by a zero object produces a zero object. Besides, \mathcal{C}_X is endowed with a right exact structure \mathfrak{E}_X with weak equivalences \mathcal{W}_X such that all split epimorphisms of \mathcal{C}_X are deflations, both respected by the 'tensor' product \odot . The latter means that $\alpha \odot \beta$ is a deflation (resp. a weak equivalence), if α and β are deflations (resp. weak equivalences).

Let $\mathcal{C}_{\mathfrak{A}_X}$ denote the category $\text{Alg}_{\tilde{\mathcal{C}}_X}$ of algebras in the $\tilde{\mathcal{C}}_X$ (in classical sources, like [ML], the objects of $\text{Alg}_{\tilde{\mathcal{C}}_X}$ are called monoids). Thanks to the existence of countable coproducts and the compatibility of 'tensor' product with them, the forgetful functor $\mathcal{C}_{\mathfrak{A}_X} \xrightarrow{f_*} \mathcal{C}_X$ has a left adjoint, f^* which assigns to every object V of the monoidal category $\tilde{\mathcal{C}}_X$ its *tensor algebra* $(T(V), \mu_V)$, where $T(V) = \bigoplus_{n \geq 0} V^{\odot n}$ and the multiplication μ_V is

given by the canonical isomorphisms $V^{\odot n} \odot V^{\odot m} \xrightarrow{\sim} V^{\odot n+m}$. Here $V^{\odot 0} \stackrel{\text{def}}{=} \mathbb{I}$.

The category of algebras has a natural initial object – the 'unit' algebra \mathbb{I} . We denote the category $\text{Alg}_{\tilde{\mathcal{C}}_X}/\mathbb{I}$ of *augmented* algebras by $\mathcal{C}_{\mathfrak{A}_X^1}$.

If the category \mathcal{C}_X is additive, then the category of augmented algebras is naturally equivalent to the category $\text{Alg}_{\tilde{\mathcal{C}}_X}^1$ of non-unital algebras.

Like in the case of k -algebras, we have a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{C}_{\mathfrak{A}_X^1} & \xrightarrow{j_X^*} & \mathcal{C}_{\mathfrak{A}_X} \\ \bar{f}^* \uparrow \downarrow \bar{f}_* & & f^* \uparrow \downarrow f_* \\ \mathcal{C}_X & \xrightarrow{Id} & \mathcal{C}_X \end{array} \quad (3)$$

where j_X^* is the functor which forgets augmentation.

We define deflations on the category $\mathcal{C}_{\mathfrak{A}_X}$ of algebras by setting $\mathfrak{E}_{\mathfrak{A}_X} = f_*^{-1}(\mathfrak{E}_X)$ and weak equivalences by $\mathcal{W}_{\mathfrak{A}_X} = f_*^{-1}(\mathcal{W}_X)$. The right exact structure on the category of augmented algebras induced via the forgetful functor j_X^* , so that $\mathfrak{E}_{\mathfrak{A}_X^1} = \bar{f}_*^{-1}(\mathfrak{E}_X)$ and $\mathcal{W}_{\mathfrak{A}_X^1} = \bar{f}_*^{-1}(\mathcal{W}_X)$ (see the diagram (3)).

There is also the embedding $\mathcal{C}_X \xrightarrow{\phi_*} \mathcal{C}_{\mathfrak{A}_X^1}$, which assigns to every object V of \mathcal{C}_X the algebra $(\mathbb{I} \oplus V, \mu_V^0)$, where μ_V^0 is the multiplication trivial on V . This embedding has a left adjoint, $\mathcal{C}_{\mathfrak{A}_X} \xrightarrow{\phi^*} \mathcal{C}_{\mathfrak{A}_X^1}$, which maps every augmented algebra $(A, A \xrightarrow{\xi_A} \mathbb{I})$ to the object $\text{Ker}(\xi_A)/(\text{Ker}(\xi_A))^2$. Notice that the object $\text{Ker}(\xi_A)$ exists because ξ_A is a split epimorphism, hence a deflation.

The embedding $\mathcal{C}_X \xrightarrow{\phi_*} \mathcal{C}_{\mathfrak{A}_X^1}$ induces an embedding of the spectrum $\mathbf{Spec}_{\mathfrak{G}\mathfrak{T}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ of the 'space' $(X, \bar{\mathfrak{E}}_X)$ into the spectrum $\mathbf{Spec}_{\mathfrak{G}\mathfrak{T}}^{1,1}(\mathfrak{A}_X^1, \bar{\mathfrak{E}}_{\mathfrak{A}_X^1})$ of the 'space' $(\mathfrak{A}_X^1, \bar{\mathfrak{E}}_{\mathfrak{A}_X^1})$ represented by the right exact category of augmented algebras.

8.3.2. Example. Let R be an associative unital k -algebra and \mathcal{C}_X the monoidal category of R -bimodules endowed with the standard exact structure. Algebras in this monoidal category are associative unital k -algebras A endowed with an algebra morphism $R \xrightarrow{\psi_A} A$. In other words, the category of algebras is isomorphic to the category $R \backslash \text{Alg}_k$ of k -algebras over R . Augmented algebras are triples (ψ_A, A, ξ_A) , where $A \xrightarrow{\xi_A} R$ is the left inverse to ψ_A , that is $\xi_A \circ \psi_A = id_R$. The right exact structure on the category of (augmented) algebras over R is standard: deflations are surjective morphisms and weak equivalences are isomorphisms. We have the natural embedding of the spectrum $\mathbf{Spec}_{\mathfrak{G}\mathfrak{T}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ of the 'space' $(X, \bar{\mathfrak{E}}_X)$ represented by the category of R -bimodules into the spectrum $\mathbf{Spec}_{\mathfrak{G}\mathfrak{T}}^{1,1}(\mathfrak{A}_X^1, \bar{\mathfrak{E}}_{\mathfrak{A}_X^1})$ of the 'space' $(\mathfrak{A}_X^1, \bar{\mathfrak{E}}_{\mathfrak{A}_X^1})$ represented by the right exact category of augmented algebras. The complement to the image of $\mathbf{Spec}_{\mathfrak{G}\mathfrak{T}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ is the spectrum of the 'subspace' $(\mathfrak{A}_X^f, \bar{\mathfrak{E}}_{\mathfrak{A}_X^f})$ of the right exact 'space' $(\mathfrak{A}_X^1, \bar{\mathfrak{E}}_{\mathfrak{A}_X^1})$ represented by the subcategory of all augmented rings (ψ_A, A, ξ_A) such that $\text{Ker}(\xi_A)^2 = \text{Ker}(\xi_A)$. These augmented algebras correspond to the firm non-unital k -algebras over the algebra \mathcal{R} .

Appendix 1: some properties of kernels.

A1.1. Proposition. *Let $M \xrightarrow{f} N$ be a morphism of C_X which has a kernel pair, $M \times_N M \xrightarrow[p_2]{p_1} M$. Then the morphism f has a kernel iff p_1 has a kernel, and these two kernels are naturally isomorphic to each other.*

Proof. Suppose that f has a kernel, i.e. there is a cartesian square

$$\begin{array}{ccc} \text{Ker}(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ f' \downarrow & \text{cart} & \downarrow f \\ x & \xrightarrow{i_N} & N \end{array} \quad (1)$$

Then we have the commutative diagram

$$\begin{array}{ccccc} \text{Ker}(f) & \xrightarrow{\gamma} & M \times_N M & \xrightarrow{p_2} & M \\ f' \downarrow & & p_1 \downarrow & \text{cart} & \downarrow f \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array} \quad (2)$$

which is due to the commutativity of (1) and the fact that the unique morphism $x \xrightarrow{i_N} N$ factors through the morphism $M \xrightarrow{f} N$. The morphism γ is uniquely determined by the equality $p_2 \circ \gamma = \mathfrak{k}(f)$. The fact that the square (1) is cartesian and the equalities $p_2 \circ \gamma = \mathfrak{k}(f)$ and $i_N = f \circ i_M$ imply that the left square of the diagram (2) is cartesian, i.e. $\text{Ker}(f) \xrightarrow{\gamma} M \times_N M$ is the kernel of the morphism p_1 .

Conversely, if p_1 has a kernel, then we have a diagram

$$\begin{array}{ccccc} \text{Ker}(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_2} & M \\ p'_1 \downarrow & \text{cart} & p_1 \downarrow & \text{cart} & \downarrow f \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array}$$

which consists of two cartesian squares. Therefore the square

$$\begin{array}{ccc} \text{Ker}(p_1) & \xrightarrow{\mathfrak{k}(f)} & M \\ p'_1 \downarrow & \text{cart} & \downarrow f \\ x & \xrightarrow{i_N} & N \end{array}$$

with $\mathfrak{k}(f) = p_2 \circ \mathfrak{k}(p_1)$ is cartesian. ■

A1.2. Remarks. (a) Needless to say that the picture obtained in (the argument of) A1.1 is symmetric, i.e. there is an isomorphism $\text{Ker}(p_1) \xrightarrow{\tau'_f} \text{Ker}(p_2)$ which is an arrow

in the commutative diagram

$$\begin{array}{ccccc}
\text{Ker}(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_1} & M \\
\tau'_f \downarrow \wr & & \tau_f \downarrow \wr & & \downarrow id_M \\
\text{Ker}(p_2) & \xrightarrow{\mathfrak{k}(p_2)} & M \times_N M & \xrightarrow{p_2} & M
\end{array}$$

(b) Let a morphism $M \xrightarrow{f} N$ have a kernel pair, $M \times_N M \xrightarrow[p_2]{p_1} M$, and a kernel. Then, by A1.1, there exists a kernel of p_1 , so that we have a morphism $\text{Ker}(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M$ and the diagonal morphism $M \xrightarrow{\Delta_M} M \times_N M$. Since the left square of the commutative diagram

$$\begin{array}{ccccc}
x & \longrightarrow & \text{Ker}(p_1) & \xrightarrow{p'_1} & x \\
\downarrow & \text{cart} & \mathfrak{k}(p_1) \downarrow & & \downarrow \\
M & \xrightarrow{\Delta_M} & M \times_N M & \xrightarrow{p_1} & M
\end{array}$$

is cartesian and compositions of the horizontal arrows are identical morphisms, it follows that its left square is cartesian too. Loosely, one can say that the intersection of $\text{Ker}(p_1)$ with the diagonal of $M \times_N M$ is zero. If there exists a coproduct $\text{Ker}(p_1) \coprod M$, then the pair of morphisms $\text{Ker}(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M \xleftarrow{\Delta_M} M$ determine a morphism

$$\text{Ker}(p_1) \coprod M \longrightarrow M \times_N M.$$

If the category C_X is additive, then this morphism is an isomorphism, or, what is the same, $\text{Ker}(f) \coprod M \simeq M \times_N M$. In general, it is rarely the case, as the reader can find out looking at the examples of 1.4.

A1.3. Proposition. *Let*

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\
\widetilde{g} \downarrow & \text{cart} & \downarrow g \\
M & \xrightarrow{f} & N
\end{array} \tag{3}$$

be a cartesian square. Then $\text{Ker}(f)$ exists iff $\text{Ker}(\widetilde{f})$ exists, and they are naturally isomorphic to each other.

A1.4. The kernel of a composition and related facts. Fix a category C_X with an initial object x .

A1.4.1. The kernel of a composition. Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be morphisms such that there exist kernels of g and $g \circ f$. Then the argument similar to that of A1.3

shows that we have a commutative diagram

$$\begin{array}{ccccccc}
 Ker(gf) & \xrightarrow{\tilde{f}} & Ker(g) & \xrightarrow{g'} & x & & \\
 \mathfrak{k}(gf) \downarrow & & \text{cart} & \downarrow \mathfrak{k}(g) & \text{cart} & \downarrow i_N & \\
 L & \xrightarrow{f} & M & \xrightarrow{g} & N & &
 \end{array} \tag{1}$$

whose both squares are cartesian and all morphisms are uniquely determined by f , g and the (unique up to isomorphism) choice of the objects $Ker(g)$ and $Ker(gf)$.

Conversely, if there is a commutative diagram

$$\begin{array}{ccccccc}
 K & \xrightarrow{u} & Ker(g) & \xrightarrow{g'} & x & & \\
 t \downarrow & & \text{cart} & \downarrow \mathfrak{k}(g) & & \downarrow i_N & \\
 L & \xrightarrow{f} & M & \xrightarrow{g} & N & &
 \end{array}$$

whose left square is cartesian, then its left vertical arrow, $K \xrightarrow{t} L$, is the kernel of the composition $L \xrightarrow{g \circ f} N$.

A1.4.2. Remarks. (a) It follows from A1.3 that the kernel of $L \xrightarrow{f} M$ exists iff the kernel of $Ker(gf) \xrightarrow{\tilde{f}} Ker(g)$ exists and they are isomorphic to each other. More precisely, we have a commutative diagram

$$\begin{array}{ccccccc}
 Ker(\tilde{f}) & \xrightarrow{\mathfrak{k}(\tilde{f})} & Ker(gf) & \xrightarrow{\tilde{f}} & Ker(g) & \xrightarrow{g'} & x \\
 \wr \downarrow & & \mathfrak{k}(gf) \downarrow & & \text{cart} & \downarrow \mathfrak{k}(g) & \text{cart} \downarrow i_N \\
 Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & \xrightarrow{g} & N
 \end{array}$$

whose left vertical arrow is an isomorphism.

(b) Suppose that (C_X, \mathfrak{E}_X) is a right exact category (with an initial object x). If the morphism f above is a deflation, then it follows from this observation that the canonical morphism $Ker(gf) \xrightarrow{\tilde{f}} Ker(g)$ is a deflation too. In this case, $Ker(f)$ exists, and we have a commutative diagram

$$\begin{array}{ccccccc}
 Ker(\tilde{f}) & \xrightarrow{\mathfrak{k}(\tilde{f})} & Ker(gf) & \xrightarrow{\tilde{f}} & Ker(g) & & \\
 \wr \downarrow & & \mathfrak{k}(gf) \downarrow & & \text{cart} & \downarrow \mathfrak{k}(g) & \\
 Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & &
 \end{array}$$

whose rows are conflations.

The following observations is useful (and are used) for analysing diagrams.

A1.4.3. Proposition. (a) Let $M \xrightarrow{g} N$ be a morphism with a trivial kernel. Then a morphism $L \xrightarrow{f} M$ has a kernel iff the composition $g \circ f$ has a kernel, and these two kernels are naturally isomorphic one to another.

(b) Let

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \gamma \downarrow & & \downarrow g \\ \widetilde{M} & \xrightarrow{\phi} & N \end{array}$$

be a commutative square such that the kernels of the arrows f and ϕ exist and the kernel of g is trivial. Then the kernel of the composition $\phi \circ \gamma$ is isomorphic to the kernel of the morphism f , and the left square of the commutative diagram

$$\begin{array}{ccccccc} \text{Ker}(f) & \xrightarrow{\sim} & \text{Ker}(\phi\gamma) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M \\ & & \widetilde{\gamma} \downarrow & \text{cart} & \gamma \downarrow & & \downarrow g \\ & & \text{Ker}(\phi) & \xrightarrow{\mathfrak{k}(\phi)} & \widetilde{M} & \xrightarrow{\phi} & N \end{array}$$

is cartesian.

Proof. (a) Since the kernel of g is trivial, the diagram A1.4.1(1) specializes to the diagram

$$\begin{array}{ccccccc} \text{Ker}(gf) & \xrightarrow{\widetilde{f}} & x & \xrightarrow{id_x} & x & & \\ \mathfrak{k}(gf) \downarrow & \text{cart} & \downarrow \mathfrak{k}(g) & & \downarrow i_N & & \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N & & \end{array}$$

with cartesian squares. The left cartesian square of this diagram is the definition of $\text{Ker}(f)$. The assertion follows from A1.4.1.

(b) Since the kernel of g is trivial, it follows from (a) that $\text{Ker}(f)$ is naturally isomorphic to the kernel of $g \circ f = \phi \circ \gamma$. The assertion follows now from A1.4.1. ■

A1.4.4. Corollary. Let C_X be a category with an initial object x . Let $L \xrightarrow{f} M$ be a strict epimorphism and $M \xrightarrow{g} N$ a morphism such that $\text{Ker}(g) \xrightarrow{\mathfrak{k}(g)} M$ exists and is a monomorphism. Then the composition $g \circ f$ is a trivial morphism iff g is trivial.

A1.4.4.1. Note. The following example shows that the requirement " $\text{Ker}(g) \rightarrow M$ is a monomorphism" in A1.4.4 cannot be omitted.

Let C_X be the category Alg_k of associative unital k -algebras, and let \mathfrak{m} be an ideal of the ring k such that the epimorphism $k \rightarrow k/\mathfrak{m}$ does not split. Then the identical morphism $k/\mathfrak{m} \rightarrow k/\mathfrak{m}$ is non-trivial, while its composition with the projection $k \rightarrow k/\mathfrak{m}$ is a trivial morphism.

Appendix 2. Geometry of 'spaces' represented by abelian categories.

Below we sketch parts of the spectral theory of 'spaces' represented by svelte abelian categories which were serving as a guide for this work.

A2.1. Topologizing subcategories and the spectrum $\mathbf{Spec}(-)$. A full subcategory of an abelian category C_X is called *topologizing* if it is closed under finite coproducts and subquotients. For an object M , we denote by $[M]$ the smallest topologizing subcategory of C_X containing M . One can show that objects of $[M]$ are subquotients of finite coproducts of copies of M . The *spectrum* $\mathbf{Spec}(X)$ of the 'space' X consists of all nonzero $[M]$ such that $[L] = [M]$ for any nonzero subobject L of M . We endow $\mathbf{Spec}(X)$ with the preorder \supseteq which is called (with a good reason) the *specialization* preorder.

If M is a simple object, then the objects of $[M]$ are isomorphic to finite direct sums of copies of M and $[M]$ is a minimal element of $(\mathbf{Spec}(X), \supseteq)$. If C_X is the category of modules over a commutative unital ring R , then the map $p \mapsto [R/p]$ is an isomorphism between the prime spectrum of R with specialization preorder and $(\mathbf{Spec}(X), \supseteq)$.

A2.2. Local 'spaces'. An abelian category C_Y (and the 'space' Y) is called *local* if it has the smallest nonzero topologizing subcategory. It follows that this subcategory coincides with $[M]$ for any of its nonzero objects M ; so that it is the smallest element of $\mathbf{Spec}(Y)$. If a local category has a simple object, M , then this smallest category coincides with $[M]$. In particular, all simple objects of C_Y (if any) are isomorphic one to another. The category of modules over a commutative ring is local iff the ring is local.

A2.3. Serre subcategories and $\mathbf{Spec}^-(X)$. A topologizing subcategory of an abelian category C_X is called *thick* if it is closed under extensions. For any subcategory \mathcal{T} of C_X , let \mathcal{T}^- denote the full subcategory of C_X whose objects are characterized by the property: their subquotients has nonzero subobject from \mathcal{T} . One can show that $(\mathcal{T}^-)^- = \mathcal{T}^-$ and the subcategory \mathcal{T}^- is thick. We call a subcategory \mathcal{T} of C_X a *Serre subcategory* if $\mathcal{T} = \mathcal{T}^-$.

Let $\mathbf{Spec}^-(X)$ denote the set of all Serre subcategories \mathcal{P} such that the quotient category C_X/\mathcal{P} is local. One can show that if C_X is a locally noetherian Grothendieck category (more generally, a Grothendieck category with a Gabriel-Krull dimension), then $\mathbf{Spec}^-(X)$ is isomorphic to the Gabriel spectrum of C_X .

Let $\mathbf{Spec}_!^1(X)$ denote the set of all Serre subcategories of C_X such that the intersection \mathcal{P}^* of all topologizing subcategories properly containing \mathcal{P} is not equal to \mathcal{P} .

A2.4. Theorem. (a) $\mathbf{Spec}_!^1(X) \subseteq \mathbf{Spec}^-(X)$.

(b) The map which assigns to a topologizing subcategory \mathcal{Q} the union $\widehat{\mathcal{Q}}$ of all topologizing subcategories which do not contain \mathcal{Q} is a bijection $\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}_!^1(X)$.

(c) Let \mathcal{T} be a Serre subcategory of C_X and $C_X \xrightarrow{q^*} C_X/\mathcal{T}$ the localization functor.

(c1) If $\mathcal{T} \not\supseteq [M] \in \mathbf{Spec}(X)$, then $[q^*(M)] \in \mathbf{Spec}(X/\mathcal{T})$.

(c2) The map $\mathcal{P} \mapsto q^{*-1}(\mathcal{P})$ is a bijection from $\mathbf{Spec}^-(X/\mathcal{T})$ onto the subset $\{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{T} \subseteq \mathcal{P}\}$.

(d) Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories such that $\bigcap_{i \in J} \mathcal{T}_i = 0$. Then

$$(d1) \mathbf{Spec}^-(X) = \bigcup_{i \in J} \mathbf{Spec}^-(X/\mathcal{T}_i).$$

(d2) An element \mathcal{P} of $\mathbf{Spec}^-(X)$ belongs to $\mathbf{Spec}_t^1(X)$ iff $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}_t^1(X/\mathcal{T}_i)$ whenever $\mathcal{T}_i \subseteq \mathcal{P}$.

The assertion (c) can be extracted from [R, Ch.III]. The (most important) last assertion says that an element of $\mathbf{Spec}^-(X)$ belongs to $\mathbf{Spec}_t^1(X)$ (that is corresponds to an element of $\mathbf{Spec}(X)$) iff it belongs to $\mathbf{Spec}_t^1(X)$ *locally*. Two different proofs of this fact are in [R4] and [R5]. The argument in [R5] is based on some simple properties of Serre subcategories and the Gabriel multiplication of topologizing categories.

A2.5. Geometric center of a 'space' and reconstruction of commutative schemes. Recall that the *center* of the category C_Y is the (commutative) ring of endomorphisms of its identical functor. If C_Y is a category of left modules over an associative unital ring R , then the center of C_Y is naturally isomorphic to the center of R .

We endow the spectrum $\mathbf{Spec}(X)$ with *Zariski* topology (which we do not describe here). The map $\tilde{\mathcal{O}}_X$ which assigns to every open subset W of $\mathbf{Spec}(X)$ the center of the quotient category C_X/\mathcal{S}_W , where $\mathcal{S}_W = \bigcap_{\mathcal{Q} \in W} \hat{\mathcal{Q}}$, is a presheaf on $\mathbf{Spec}(X)$. We denote by \mathcal{O}_X its associated sheaf. One can show that the stalk of the sheaf \mathcal{O}_X at a point \mathcal{Q} of the spectrum is isomorphic to the center of the local category $C_X/\hat{\mathcal{Q}}$, and the center of a local category is a local commutative ring. The locally ringed space $(\mathbf{Spec}(X), \mathcal{O}_X)$ is called the *geometric center* (or *Zariski geometric center*) of the 'space' X .

One of the consequences of the theorem above is the following reconstruction theorem:

A2.6. Theorem. *If C_X is the category of quasi-coherent sheaves on a commutative quasi-compact quasi-separated scheme, then the geometric center $(\mathbf{Spec}(X), \mathcal{O}_X)$ of the 'space' X is naturally isomorphic to the scheme. So that any quasi-separated quasi-compact commutative scheme is canonically reconstructed, uniquely up to isomorphism, from its category of quasi-coherent sheaves.*

If the scheme is noetherian, this theorem recovers the Gabriel's reconstruction theorem [Gab], because it is easy to show that if C_X is the category of modules over a commutative noetherian ring, then the injective spectrum of C_X is naturally isomorphic to $\mathbf{Spec}(X)$.

A2.7. Geometric realization of a noncommutative scheme. Let C_X be an abelian category with enough objects of finite type. We have a contravariant pseudo-functor from the category of the Zariski open sets of the spectrum $\mathbf{Spec}(X)$ to Cat which assigns to each open set \mathcal{U} of $\mathbf{Spec}(X)$ the quotient category $C_X/\mathcal{S}_{\mathcal{U}}$, where $\mathcal{S}_{\mathcal{U}} = \bigcap_{\mathcal{Q} \in \mathcal{U}} \hat{\mathcal{Q}}$, and to each embedding $\mathcal{U} \hookrightarrow \mathcal{V}$ the corresponding localization functor. To this pseudo-functor, there corresponds (by a standard formalism) a fibred category over the Zariski topology of $\mathbf{Spec}(X)$. The associated stack, \mathfrak{F}_X^3 , is a stack of local categories: its stalk at each point \mathcal{Q} of $\mathbf{Spec}(X)$ is equivalent to the local category $C_X/\hat{\mathcal{Q}}$.

We regard the stack \mathfrak{F}_X^3 as a *geometric realization* of the abelian category C_X .

If X is a (noncommutative) scheme, then the stack \mathfrak{F}_X^3 is *locally affine*.

A2.8. Note: geometric realization of noncommutative schemes and ringed spaces. Taking the center of each fiber of the stack \mathfrak{F}_X^3 , we recover the presheaf of commutative rings $\tilde{\mathcal{O}}_X$, hence the geometric center of the 'space' X .

Notice that the stalks at points of a noncommutative scheme are local abelian categories, which only in exceptional cases are equivalent to categories of modules over rings. This explains why imposing that noncommutative schemes should be ringed topological spaces over injective (or some other) spectrum did not work.

A2.9. The spectrum $\mathbf{Spec}_c^0(X)$. If C_X is the category of quasi-coherent sheaves on a non-quasi-compact scheme, like, for instance, the flag variety of a Kac-Moody Lie algebra, or a noncommutative scheme which does not have a finite affine cover (say, the quantum flag variety of a Kac-Moody Lie algebra, or the corresponding quantum D-scheme), then the spectrum $\mathbf{Spec}(X)$ is not sufficient. It should be replaced by the spectrum $\mathbf{Spec}_c^0(X)$ whose elements are coreflective topologizing subcategories of C_X of the form $[M]_c$ (i.e. generated by the object M) such that if L is a nonzero subobject of M , then $[L]_c = [M]_c$.

There is a natural map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_c^0(X)$ which assigns to every $\mathcal{Q} \in \mathbf{Spec}(X)$ the smallest coreflective subcategory $[\mathcal{Q}]_c$ containing \mathcal{Q} . If the category C_X has enough objects of finite type, this canonical map is a bijection.

A2.10. Theorem. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of coreflective thick subcategories of an abelian category C_X such that $\bigcap_{i \in J} \mathcal{T}_i = 0$; and let $C_X \xrightarrow{u_i^*} C_X/\mathcal{T}_i$ be the localization functor. The following conditions on a nonzero coreflective topologizing subcategory \mathcal{Q} of C_X are equivalent:*

- (a) $\mathcal{Q} \in \mathbf{Spec}_c^0(X)$,
- (b) $[u_i^*(\mathcal{Q})]_c \in \mathbf{Spec}_c^0(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{Q} \not\subseteq \mathcal{T}_i$.

One of the consequences of this theorem is the following reconstruction theorem.

A2.11. Theorem. *Let C_X be the category of quasi-coherent sheaves on a commutative scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O})$. Suppose that there is an affine cover $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ of the scheme \mathbf{X} such that all immersions $\mathcal{U}_i \hookrightarrow \mathcal{X}$, $i \in J$, have a direct image functor (say, the scheme \mathbf{X} is quasi-separated). Then the geometric center $(\mathbf{Spec}_c^0(X), \mathcal{O}_X)$ is isomorphic to the scheme \mathbf{X} .*

If $\mathbf{X} = (\mathcal{X}, \mathcal{O})$ is a quasi-compact quasi-separated scheme, then the category C_X of quasi-coherent sheaves on \mathbf{X} has enough objects of finite type, hence the spectrum $\mathbf{Spec}_c^0(X)$ coincides with $\mathbf{Spec}(X)$. Thus, the reconstruction theorem for quasi-compact quasi-separated schemes is a special case of the theorem above.

References

- [Ba] H. Bass, Algebraic K-theory, Benjamin 1968
- [Gr] A. Grothendieck, Sur quelques points d'algèbre homologique. Tohoku Math. J., Ser. II, 9, 120–221 (1957)
- [GZ] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Springer Verlag, Berlin-Heidelberg-New York, 1967
- [Ke1] B. Keller, Derived categories and their uses, preprint, 1994.
- [KeV] B. Keller, D. Vossieck, Sous les catégories dérivées, C. R. Acad. Sci. Paris 305 (1987), 225–228.
- [KR1] M. Kontsevich, A.L. Rosenberg, Noncommutative smooth spaces, in “The Gelfand Mathematical Seminar 1996–1999” (2000), 87–109.
- [KR2] M. Kontsevich, A.L. Rosenberg, Noncommutative spaces and flat descent, preprint MPI, 2004(36), 108 pp.
- [KR3] M. Kontsevich, A.L. Rosenberg, Noncommutative spaces, preprint MPI, 2004(35), 79 pp.
- [KR4] M. Kontsevich, A. Rosenberg, Noncommutative stacks, preprint MPI, 2004(37), 55 pp.
- [KR5] M. Kontsevich, A. Rosenberg, Noncommutative Grassmannians and related constructions, to appear.
- [KR6] M. Kontsevich, A. Rosenberg, Orthogonals, in preparation.
- [KR7] M. Kontsevich, A. Rosenberg, Noncommutative 'spaces' and stacks, a monograph in preparation.
- [LR] V. Lunts, A.L. Rosenberg, Localization for quantum groups, Selecta Mathematica, New Series, 5 (1999), 123–159.
- [ML] S. Mac-Lane, Categories for the working mathematicians, Springer - Verlag; New York - Heidelberg - Berlin (1971)
- [MLM] S. Mac-Lane, L. Moerdijk, Sheaves in Geometry and Logic, Springer - Verlag; New York - Heidelberg - Berlin (1992)
- [Q] D. Quillen, Higher algebraic K-theory, LNM 341, 85–147 (1973)
- [R] A.L. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras, Kluwer Academic Publishers, Mathematics and Its Applications, v.330 (1995), 328 pages.
- [R1] A.L. Rosenberg, Noncommutative local algebra, Geometric and Functional Analysis (GAFA), v.4, no.5 (1994), 545-585
- [R2] A.L. Rosenberg, Noncommutative schemes, Compositio Mathematica 112 (1998), 93-125
- [R3] A.L. Rosenberg, Noncommutative spaces and schemes, preprint MPIM, 1999
- [R4] A.L. Rosenberg, Underlying spaces of noncommutative schemes, preprint MPIM, 2003 (111), 43 pp
- [R5] A.L. Rosenberg, Spectra of 'spaces' represented by abelian categories, preprint MPIM, 2004(115), 73 pp.
- [R6] A.L. Rosenberg, Spectra, associated points, and representations, preprint MPIM, 2007(10), 71 pp.
- [R7] A.L. Rosenberg, Spectra of noncommutative spaces, preprint MPIM, 2003(110)

- [R8] A.L. Rosenberg, Spectra related with localizations, preprint MPIM, 2003(112), 77 pp
- [R9] A.L. Rosenberg, Homological algebra of noncommutative 'spaces' I, to appear.
- [R10] A.L. Rosenberg, Trieste Lectures, to appear in Lecture Notes of the School on Algebraic K-Theory and Applications, ICTP, Italy, 92 pp.
- [R11] A.L. Rosenberg, Topics in noncommutative algebraic geometry, homological algebra and K-theory, preprint MPIM, 2008(57), 105 pp
- [R12] A.L. Rosenberg, Geometry of noncommutative 'spaces' and schemes, to appear.
- [S] J.-P. Serre, Faisceaux algébriques cohérents, *Annals of Math.*62, 1955
- [T] Tanisaki, T., The Beilinson-Bernstein correspondence for quantized enveloping algebras, *arXiv. Math. QA/0309349*, v1(2003).
- [Ve1] J.-L. Verdier, Catégories dérivées, Séminaire de Géométrie Algébrique 4 1/2, Cohomologie Étale, LNM 569, pp. 262–311, Springer, 1977
- [Ve2] J.-L. Verdier, Des catégories dérivées des catégories abéliennes, *Astérisque*, v.239, 1996