Irreducibility criterion for the set of two matrices

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Abstract

We give the criterion for the irreducibility, the Schur irreducibility and the indecomposability of the set of two $n \times n$ matrices Λ_n and A_n in terms of the subalgebra associated with the "support" of the matrix A_n, where Λ_n is a diagonal matrix with different non zeros eigenvalues and A_n is an arbitrary one. The list of all maximal subalgebras of the algebra $\mathrm{Mat}(n, \mathbb{C})$ and the list of the corresponding invariant subspaces connected with these two matrices is also given. The properties of the corresponding subalgebras are expressed in terms of the graphs associated with the support of the second matrix.

For arbitrary n we describe all minimal subsets of the elementary matrices E_{km} that generate the algebra $\mathrm{Mat}(n, \mathbb{C})$.

Key words: matrix algebra, representation, irreducible, Schur irreducible, indecomposable representation, invariant subspace, graph theory, oriented graph, orientationally connected graph,
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Contents

1 Introduction

In the *representation theory* of different objects (groups, rings, algebras etc.) the problem of the *ireducibility* of the concrete *representations* (modules) sometimes reduce to the irreducibility of the algebra, generated by two operators or by two matrices if the representation is finite dimensional.

In the case of the discrete group generated by two elements this is exactly the problem one need to solve. The most popular examples are the following: the free group $\mathbb{F}_2$ generated by two elements, the Artin braid group B_3 on three strands, the group $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\pm 1$.

We give the criteria of the *irreducibility* and the *Schur irreducibility* (see below the definitions) of the set of two complex $n \times n$ matrices Λ_n and A_n in terms of the "support" of the matrix A_n, where Λ_n is a diagonal matrix with different non zeros eigenvalues and A_n is an arbitrary one (Theorem 5). The list of *all invariant subspaces* for this two matrices is also given (Theorem 6).

This criterion allows us to study completely in [2] the irreducibility of some family of representations depending on the parameters of the braid group B_3 in any dimensions.

There are three different notion connected with the *irreducibility* of the representations T of the group G in a complex vector space V

$$G \ni g \mapsto T_g \in \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ is the group of the linear invertible operators in the space V. They are as follows: 1) *irreducible*, 2) *Schur irreducible*, 3) *indecomposable.*

Definition 1 *We say that the representation is* irreducible *(resp.* Schur irreducible*) when there are no nontrivial invariant closed* subspaces *for all operators of the representation (resp.* there are no nontrivial *bounded* operators commuting *with all operators of the representation). The representation is* indecomposable *if* it can not be presented as the direct sum *of the subrepresentations.*

Remark 2 *It is well known that the relations between the mentioned notions when the space V is finite dimensional are as follows:* 1) $\Rightarrow$ 2) $\Rightarrow$ 3).

Remark 3 *The notions of irreducibility and Schur irrereducibility coincides for the unitary representation of an arbitrary group G (hence, for an arbitrary representation of a compact group, using the "Wayl trick" [7]).*

Counterexample 1. 2) $\not\Rightarrow$ 1). Let us consider the subalgebra of the algebra $\mathrm{Mat}(2,\mathbb{C})$ consisting of the matrices

$$\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right), a, b, c \in \mathbb{C}.$$

This subalgebra is subspace reducible (the subspace in $\mathbb{C}^2$ generated by the vector $(1,0)$ is invariant) but the algebra is operator irreducible.

Counterexample 2. 3) $\not\Rightarrow$ 2). The classical example of the operator reducible but the indecomposable representation of the additive group of $\mathbb{C}$ is as follows:

$$\mathbb{C} \ni z \mapsto \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C}). \tag{1}$$

1.1 Irreducibility criteria

Let $\mathrm{Mat}(n,\mathbb{C})$ be the algebra of all complex matrices over the field of complex numbers $\mathbb{C}$ and let Λ_n (resp. A_n) be a diagonal (resp. an arbitrary) matrix in $\mathrm{Mat}(n,\mathbb{C})$:

$$\Lambda_n = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n),\ \ A_n = (a_{km})_{1\leq k,m\leq n} \in \mathrm{Mat}(n,\mathbb{C}).$$

We shall call the *support of the matrix A* the following subset

$$\mathrm{Supp}(A_n) = \{(k,m) \in \{1,2,...,n\}^2 \mid a_{km} \neq 0\}. \tag{2}$$

Remark 4 *It is well known that the algebra* $\mathrm{Mat}(n,\mathbb{C})$ *acting in the space* $\mathbb{C}^n$ *is irreducible (and hence* Schur) irreducible*.*

Notation. Let us denote by E_{km} the matrix units i.e. the matrix $E_{km} = (a_{ij})$ such that $a_{ij} = \delta_{ki}\delta_{mj}$ where δ_{ij} are the Kronecker symbols. Obviously $E_{km}E_{pq} = \delta_{mp}E_{kq}$.

Theorem 5 *Let the eigevnalues λ_k of Λ_n are different and non zeros, then*
1) the family of two operators (Λ_n, A_n) is irreducible *if and only if the set $(E_{km} \mid (k,m) \in \mathrm{Supp}(A_n))$ generates the algebra* $\mathrm{Mat}(n, \mathbb{C})$*;*
2) the family (Λ_n, A_n) is Schur irreducible *if and only if the set $(E_{km} \mid (k,m) \in \mathrm{Supp}(A_n) \bigcup \mathrm{Supp}(A_n^t))$ generates the algebra* $\mathrm{Mat}(n, \mathbb{C})$ *;*
3) the family (Λ_n, A_n) is indecomposable *if and only if the set $(E_{km} \mid (k,m) \in$* $\mathrm{Supp}(A_n))$ generate the indecomposable subalgebra *in* $\mathrm{Mat}(n, \mathbb{C})$.

If A and B be complex $n \times n$ matrices. When they have 1) a *common eigenvectors*; 2) a *common invariant subspace of dimension* k, $(2 \leq k < n)$?
In 1984 Dan Shemesh [5] shows that the criteria for 1) is: $\bigcap_{k,l=1}^{n-1} \ker[A^k, B^l] \neq 0$. In [4], under the additional assumption that at least one of the matrix A and B has distinct eigenvalues, were given some sufficient conditions for 2) in terms of kth *compound* matrix $C_k(A)$ and $C_k(B)$ of the matrix A and B (for definition see e.f. [3], chap I,§ 4). Namely, 2) holds if the matrix $C_k(A)$ and $C_k(B)$ have a common invariant vector.

The *advantage of our approach* is that in the case where one of the matrices is diagonal, we give the *criterion for* 2) in terms of the support of the second matrix. The list of *all invariant subspaces* for this two matrices is also given (Theorem 6). In Section 3 we reformulate theorems 5 and 6 in terms of the *graph* associated with the support of the second matrix. It allows us to make use of *graph theory* (which is well developed).

1.2 Irreducibility

PROOF. 1) The sufficiency part $\Leftarrow$ is obvious using the Remark 4. Indeed let us denote by $\mathfrak{A}_n$ the algebra generated by operators Λ_n and A_n. Since λ_k are different and non zeros we conclude that $E_{kk} \in \mathfrak{A}_n$, $0 \leq k \leq n$. Further, since

$$E_{kk} A_n E_{mm} = a_{km} E_{km},$$

we conclude that $E_{km} \in \mathfrak{A}_n$ if $a_{km} \neq 0$ i.e. if $(k,m) \in \mathrm{Supp}(A_n)$.

To prove the necessary part $\Rightarrow$ for any fixed $n = 1, 2, ...$, let us suppose that the set $(E_{km} \mid (k,m) \in \mathrm{Supp}(A_n))$ does not generate the whole algebra $\mathrm{Mat}(n, \mathbb{C})$, but only some *proper subalgebra* $s(n)$ of the following form

$$s(n) = (x \in \mathrm{Mat}(n, \mathbb{C}) \mid x = \sum_{(k,m) \in S(n)} x_{km} E_{km}), \tag{3}$$

corresponding to some subset of indices $S(n) \subset \{1, ..., n\}^2$. We can suppose that this subalgebra is *maximal* proper subalgebra of the form (3). Indeed, if we can find the invariant subspace V for the maximal subalgebra hence this subspace would be also invariant one for any of its subalgebra. By Theorem 6 the list of the maximal proper subalgebras $s(n)$ in $\mathrm{Mat}(n,\mathbb{C})$ of the form (3) is as follows:

$$s_{\mathbf{i}}(n) = \Big(x = (x_{km})_{1\leq k,m\leq n} \in \mathrm{Mat}(n,\mathbb{C}) \mid x_{km} = 0,\ k \in \hat{\mathbf{i}},\ \ m \in \mathbf{i}\Big), \qquad (4)$$

where $\mathbf{i} = \{i_1, i_2, ..., i_k\} \subseteq \{1, 2, ..., n\},\ \ k \leq n$ and $\hat{\mathbf{i}} = \{1, 2, ..., n\} \setminus \mathbf{i}$.

Notation. For the general n let $V_{i_1 i_2 ... i_k}(n) = \langle e_{i_1}, e_{i_2}, ...e_{i_k}\rangle$ be the linear subspace in $\mathbb{C}^n$ generated by the vectors $e_{i_1}, e_{i_2}, ...e_{i_k},\ 1 \leq i_1 < i_2 < ... < i_k \leq n$, where $e_k = (\delta_{rk})_{r=1}^n \in \mathbb{C}^n,\ 1 \leq k \leq n$.

The subspace $V_{\mathbf{i}}(n) := V_{i_1 i_2 ... i_k}(n)$ is invariant subspace for the algebra $s_{\mathbf{i}}(n)$.

2) To prove the Schur irreducibility we note that the commutant $(\Lambda_n)'$ of the operator Λ_n has the following form

$$(\Lambda_n)' = (B \in \mathrm{Mat}(n,\mathbb{C}) \mid B = \mathrm{diag}(b_k)_{k=1}^n)$$

hence, the relation $[A_n, B] = 0$ is equivalent with

$$a_{km} b_m = b_k a_{km},\ 1 \leq k, m \leq n. \qquad (5)$$

We say that we can *connect* k and m where $1 \leq k, m \leq n$ if $a_{km} \neq 0$ or $a_{mk} \neq 0$ i.e. $(k, m) \in \mathrm{Supp}(A_n)$ or $(k, m) \in \mathrm{Supp}(A_n^t)$. In this case $b_k = b_m$. To show that all b_k coincide (i.e. that $B = bI$) we should be able to connect step by step all k and m i.e. for any $(k, m) \in \{1, 2, ..., n\}^2$ we should be able to find the sequence $(k_r, m_r)_{r=1}^l \in \mathrm{Supp}(A_n) \bigcup \mathrm{Supp}(A_n^t)$, such that

$$E_{km} = E_{k_1,m_1} E_{k_r,m_r} ... E_{k_l,m_l}. \qquad (6)$$

This proves the sufficiency part of the second part of the theorem. We say in this case that the set $\{1, 2, ..., n\}$ is *connected*.

To prove the sufficiency part let us suppose that the set $J = \{1, 2, ..., n\}$ is not *connected* i.e. it consists of l connected components J_r i.e. $J = \bigcup_{r=1}^l J_r$. In this case $b_k = b_m$ for $k, m \in J_r$ and the operator $B = \oplus_{r=1}^l b_r I_r$ where $I_r = \sum_{k\in J_r} E_{kk}$, commute with A_n, i.e. $[A_n, B] = 0$ hence, the representation is Schur reducible. Part 3) is evident. $\Box$

2 Maximal proper subalgebras of $\mathrm{Mat}(n,\mathbb{C})$

We give the *complete list* of subsets of indices $S(n) \subset \{1, 2, ..., n\}^2$ such that the *subspace* $s(n) \in \mathrm{Mat}(n,\mathbb{C})$ defined by

$$s(n) = (x \in \mathrm{Mat}(n,\mathbb{C}) \mid x = \sum_{(k,m)\in S(n)} x_{km} E_{km}) \qquad (7)$$

is a *maximal proper subalgebra* in $\mathrm{Mat}(n,\mathbb{C})$.

Theorem 6 *The list of maximal proper subalgebras $s(n)$ in $\mathrm{Mat}(n,\mathbb{C})$ is as follows*

$$s_{\mathbf{i}}(n)=\Big(x=(x_{km})_{1\leq k,m\leq n}\in\mathrm{Mat}(n,\mathbb{C})\mid x_{km}=0,\ k\in\hat{\mathbf{i}},\ m\in\mathbf{i}\Big), \qquad (8)$$

where $\mathbf{i}=\{i_1,i_2,...,i_k\}\subseteq\{1,2,...,n\}$, $k<n$ and $\hat{\mathbf{i}}=\{1,2,...,n\}\setminus\mathbf{i}$. The corresponding invariant subspace is $V_{\mathbf{i}}(n):=V_{i_1i_2\dots i_k}(n)$.

PROOF. For $n=2$ we have only one subset $G(2)=\{(1,2),(2,1)\}$ and two subsets $S(2)$, namely $\{(1,1),(1,2),(2,2)\}$ and $\{(1,1),(2,1),(2,2)\}$. We shall use the following *notations* for the set $G(2)$ and the algebra $s(2)$:

$$G(2):=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\quad s(2):\quad \left(\begin{smallmatrix}*&*\\0&*\end{smallmatrix}\right),\ \left(\begin{smallmatrix}*&0\\*&*\end{smallmatrix}\right).$$

Notation. In general we shall write the subset $G(n)$ as the matrix $G(n)=(g_{km})$ with matrix elements $g_{km}=1$ (resp. $g_{km}=0$) if the corresponding $(k,m)\in G(n)$ (resp. $(k,m)\notin G(n)$).

The mentioned subalgebras $s(2)$ have respectively the invariant subspaces: $V_1(2)=\langle e_1=(1,0)\rangle$ and $V_2(2)=\langle e_2=(0,1)\rangle$. For $n=3$ the list of subsets $G(3)$ and the subalgebra $s(3)$ are as follows:

$$\begin{aligned}G(3):\quad G_1(3)=\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}\right),\ G_2(3)=\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right),\ G_3(3)=\left(\begin{smallmatrix}0&1&0\\1&0&1\\0&1&0\end{smallmatrix}\right),\\ G_4(3)=\left(\begin{smallmatrix}0&1&1\\1&0&0\\1&0&0\end{smallmatrix}\right),\ G_5(3)=\left(\begin{smallmatrix}0&0&1\\0&0&1\\1&1&0\end{smallmatrix}\right),\end{aligned} \qquad (9)$$

$$s(3):\quad \left(\begin{smallmatrix}*&*&*\\0&*&*\\0&*&*\end{smallmatrix}\right),\ \left(\begin{smallmatrix}*&0&*\\*&*&*\\*&0&*\end{smallmatrix}\right),\ \left(\begin{smallmatrix}*&*&0\\*&*&0\\*&*&*\end{smallmatrix}\right);\ \left(\begin{smallmatrix}*&0&0\\*&*&*\\*&*&*\end{smallmatrix}\right),\ \left(\begin{smallmatrix}*&*&*\\0&*&0\\*&*&*\end{smallmatrix}\right),\ \left(\begin{smallmatrix}*&*&*\\*&*&*\\0&0&*\end{smallmatrix}\right). \qquad (10)$$

The mentioned subalgebras $s(3)$ have respectively the following invariant subspaces: $V_1(3),\ V_2(3),\ V_3(3);\ V_{23}(3),\ V_{13}(3)$ and $V_{12}(3)$.

To obtain the list of subalgebra $s(n+1)$ from the list of $s(n)$ we consider two projectors $P^{(0)}_{n,n+1}$ and $P^{(1)}_{n,n+1}$ defined as follows

$$P^{(r)}_{n,n+1}:\mathrm{Mat}(n+1,\mathbb{C})\mapsto\mathrm{Mat}(n,\mathbb{C}),$$

$$\sum_{1\leq k,m\leq n+1}x_{km}E_{km}=x\mapsto P^{(r)}_{n,n+1}(x)=\sum_{r+1\leq k,m\leq n+r}x_{km}E_{km},$$

$$\left(\begin{smallmatrix}*&*&\dots&*&*\\&&\dots&&\\*&*&\dots&*&*\\*&*&\dots&*&*\end{smallmatrix}\right)\overset{P^{(0)}_{n,n+1}}{\longrightarrow}\left(\begin{smallmatrix}*&*&\dots&*&0\\&&\dots&&\\*&*&\dots&*&0\\0&0&\dots&0&0\end{smallmatrix}\right),\quad \left(\begin{smallmatrix}*&*&\dots&*&*\\*&*&\dots&*&*\\&&\dots&&\\*&*&\dots&*&*\\*&*&\dots&*&*\end{smallmatrix}\right)\overset{P^{(1)}_{n,n+1}}{\longrightarrow}\left(\begin{smallmatrix}0&0&\dots&0&0\\0&*&\dots&*&*\\&&\dots&&\\0&*&\dots&*&*\end{smallmatrix}\right).$$

Notation. For arbitrary subset of indices $\mathbf{i}=\{i_1,i_2,...,i_k\}\subseteq\{1,2,...,n\}$ let us denote by $s^{(r)}_{\mathbf{i}}(n)=(P^{(r)}_{n,n+1})^{-1}(s_{\mathbf{i}}(n))$ the corresponding subspace in the algebra $\mathrm{Mat}(n+1,\mathbb{C})$, where we denote by $A^{-1}(H_0)=\{x\in H_1\mid Ax\in H_0\}$ the preimage of the subset $H_0\subset H_2$ for an operator $A:H_1\to H_2$.

We show how to obtain the list $s(3)$ from the list $s(2)$. Since the algebra $s(3)$ should be contained in the space $s^{(r)}(2) = (P_{2,3}^{(r)})^{-1}(s(2))$ for $r = 0, 1$ we get

$$s_1(2) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \leftarrow \begin{pmatrix} * & * & \times \\ 0 & * & \times \\ \times & \times & \times \end{pmatrix}, \begin{pmatrix} \times & \times & \times \\ \times & * & * \\ \times & 0 & * \end{pmatrix}; \ s_2(2) = \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \leftarrow \begin{pmatrix} * & 0 & \times \\ * & * & \times \\ \times & \times & \times \end{pmatrix}, \begin{pmatrix} \times & \times & \times \\ \times & * & 0 \\ \times & * & * \end{pmatrix}.$$

Since $E_{21} = E_{23}E_{31}$ and $E_{32} = E_{31}E_{12}$ we have only two subalgebras in $s_1^{(0)}(2)$ and two subalgebras in $s_1^{(1)}(2)$:

$$\begin{pmatrix} * & * & \times \\ 0 & * & \times \\ \times & \times & \times \end{pmatrix} \to \left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{smallmatrix}\right); \quad \begin{pmatrix} \times & \times & \times \\ \times & * & * \\ \times & 0 & * \end{pmatrix} \to \left(\begin{smallmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{smallmatrix}\right)$$

and since $E_{12} = E_{13}E_{32}$ and $E_{23} = E_{21}E_{13}$ we have only two subalgebras in $s_2^{(0)}(2)$ and two subalgebras in $s_2^{(1)}(2)$:

$$\begin{pmatrix} * & 0 & \times \\ * & * & \times \\ \times & \times & \times \end{pmatrix} \to \left(\begin{smallmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{smallmatrix}\right); \begin{pmatrix} \times & \times & \times \\ \times & * & 0 \\ \times & * & * \end{pmatrix} \to \left(\begin{smallmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{smallmatrix}\right).$$

Finally we obtain the list (10) of subalgebras $s(3)$. We see that

$$\begin{aligned} &s_1^{(0)}(2) \to s_1(3),\ s_{13}(3), \quad s_1^{(1)}(2) \to s_2(3),\ s_{12}(3), \\ &s_2^{(0)}(2) \to s_2(3),\ s_{23}(3), \quad s_2^{(1)}(2) \to s_3(3),\ s_{13}(3). \end{aligned} \tag{11}$$

The list of subalgebra $s(4)$ is as follows

$$s_i(4): \quad \left(\begin{smallmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & * & * \\ * & * & 0 & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{smallmatrix}\right),$$

$$s_{i_1i_2i_3}(4): \quad \left(\begin{smallmatrix} * & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & 0 \\ * & * & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{smallmatrix}\right),$$

$$s_{i_1i_2}(4): \quad \left(\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{smallmatrix}\right) \left(\begin{smallmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & * & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & 0 & 0 & * \\ * & * & * & * \\ * & * & * & * \\ * & 0 & 0 & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & 0 & * & 0 \\ * & * & * & * \\ * & 0 & * & 0 \\ * & * & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{smallmatrix}\right).$$

The corresponding invariant subspaces are $V_i(4)$, $1 \le i \le 4$; $V_{i_1i_2i_3}(4)$, $1 \le i_1 < i_2 < i_3 \le 4$; and $V_{i_1i_2}(4)$, $1 \le i_1 < i_2 \le 4$.

To get $s(4)$ from $s(3)$ we show how this works only for two subalgebras $s_1(3)$ and $s_{13}(3)$

$$\left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right) \leftarrow \begin{pmatrix} * & * & * & \times \\ 0 & * & * & \times \\ 0 & * & * & \times \\ \times & \times & \times & \times \end{pmatrix}, \begin{pmatrix} \times & \times & \times & \times \\ \times & * & * & * \\ \times & 0 & * & * \\ \times & 0 & * & * \end{pmatrix}, \quad \left(\begin{smallmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{smallmatrix}\right) \leftarrow \begin{pmatrix} * & * & * & \times \\ 0 & * & 0 & \times \\ * & * & * & \times \\ \times & \times & \times & \times \end{pmatrix}, \begin{pmatrix} \times & \times & \times & \times \\ \times & * & * & * \\ \times & 0 & * & 0 \\ \times & * & * & * \end{pmatrix}.$$

Since we have only one possibility to obtain E_{21} and E_{31}, namely $E_{21} = E_{24}E_{41}$ and $E_{31} = E_{34}E_{41}$, we have only two subalgebras in $s_1^{(0)}(3)$ (case a)). Another cases are treated similarly. In the case *b*) $s_1^{(1)}(3)$ we have $E_{32} = E_{31}E_{12}$ and $E_{42} = E_{41}E_{12}$; in the case *c*) $s_{13}^{(0)}(3)$ we have $E_{21} = E_{24}E_{41}$ and $E_{23} = E_{24}E_{43}$; in the case *d*) $s_{13}^{(1)}(3)$ we have $E_{32} = E_{31}E_{12}$ and $E_{34} = E_{31}E_{14}$. Finally we get

$$a) \begin{pmatrix} * & * & * & \times \\ 0 & * & * & \times \\ 0 & * & * & \times \\ \times & \times & \times & \times \end{pmatrix} \to \left(\begin{smallmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & * & * & * \end{smallmatrix}\right); \quad b) \begin{pmatrix} \times & \times & \times & \times \\ \times & * & * & * \\ \times & 0 & * & * \\ \times & 0 & * & * \end{pmatrix} \to \left(\begin{smallmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{smallmatrix}\right)$$

$c)\ \begin{pmatrix} * & * & * & \times \\ 0 & * & 0 & \times \\ * & * & * & \times \\ \times & \times & \times & \times \end{pmatrix} \rightarrow \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}; \quad d)\ \begin{pmatrix} \times & \times & \times & \times \\ \times & * & * & * \\ \times & 0 & * & 0 \\ \times & * & * & * \end{pmatrix} \rightarrow \begin{pmatrix} * & 0 & * & 0 \\ * & * & * & * \\ * & 0 & * & 0 \\ * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & 0 \\ * & * & * & * \end{pmatrix}.$

So we have the following relations using (11) and the latter considerations

$$\begin{array}{ll} s_1^{(0)}(2) \rightarrow s_1(3),\ s_{13}(3), & s_1^{(0)}(3) \rightarrow s_1(4),\ s_{14}(4), \\ s_1^{(1)}(2) \rightarrow s_2(3),\ s_{12}(3), & s_1^{(1)}(3) \rightarrow s_2(4),\ s_{12}(4), \\ s_2^{(0)}(2) \rightarrow s_2(3),\ s_{23}(3), & s_{13}^{(0)}(3) \rightarrow s_{13}(4),\ s_{134}(4), \\ s_2^{(1)}(2) \rightarrow s_3(3),\ s_{13}(3), & s_{13}^{(1)}(2) \rightarrow s_{24}(4),\ s_{124}(4). \end{array}$$

To guess the general formula for arbitrary n we note also that

$$s_{14}^{(0)}(4) \rightarrow s_{14}(5),\ s_{145}(5).$$

The similar considerations explains us how to describe all the subalgebra $s(n+1)$ starting from the subalgebra $s(n)$. Namely we have

$$s_{\mathbf{i}}(n) \leftarrow s_{\mathbf{i}}^{(0)}(n),\ s_{\mathbf{i}}^{(1)}(n),$$

$$s_{\mathbf{i}}^{(0)}(n) \rightarrow s_{\mathbf{i}}(n+1),\ s_{\mathbf{i}_0}(n+1), \quad s_{\mathbf{i}}^{(1)}(n) \rightarrow s_{\mathbf{i+1}}(n+1),\ s_{\mathbf{i}_1}(n+1),$$

or

$$\begin{array}{cccc} & & s_{\mathbf{i}}(n) & & \\ & s_{\mathbf{i}}^{(0)}(n) & & s_{\mathbf{i}}^{(1)}(n) & \\ s_{\mathbf{i}}(n+1) & s_{\mathbf{i}_0}(n+1) & & s_{\mathbf{i+1}}(n+1) & s_{\mathbf{i}_1}(n+1) \end{array} \tag{12}$$

where for $\mathbf{i} = \{i_1, i_2, ..., i_k\}$ we denote $\mathbf{i+1} = \{i_1+1, i_2+1, ..., i_k+1\}$, $\mathbf{i}_0 = \mathbf{i} \cup \{n+1\}$ and $\mathbf{i}_1 = \mathbf{i+1} \cup \{1\}$. $\square$

3 Generating sets, maximum subalgebra and the graph theory

Definition 7 *We say that the subset $G \subset \{1, 2, ..., n\}^2$ is* generating subset *if the set of matrices*

$$(E_{km} \mid (k, m) \in G)$$

generate the algebra $\mathrm{Mat}(n, \mathbb{C})$.

We would like to describe the *minimal* generating subsets G in terms of the graphs. It would be nice also to find the *complete list* $G(n)$ of the *minimal* generating subsets in $\{1, 2, ..., n\}^2$.

Definition 8 *We associate with the any subset $G \subset \{1, 2, ..., n\}^2$ an* directed (oriented) graph Γ *on n vertices in the following way: if $(k, m) \in G$ we draw the edge (arrow, arc) from the vertex k to the vertex m on the graph.*

Definition 9 *A directed graph* Γ *is called* orientationally connected *if starting from any vertex* k *on the graph we can arrive by arrows in any other vertex* m *on the graph.*

Definition 10 *An orientationally connected graph is called* minimal *if one can not extract any vertex without losing the property of being orientationally connected.*

Definition 11 *A directed graph* Γ *is called* symmetric if, *for every arc that belongs to* Γ, *the corresponding inverted arc also belongs to* Γ. *For a given directed graph* Γ *we call its* symmetric closure *the minimal symmetric graph* Γ^s *containing the initial graph.*

Lemma 12 *The subset* G *is* minimal generating *if and only if the corresponding graph* Γ *is* minimal orientationally connected.

PROOF. Use (6). □

Now we can reformulate the Theorem 5 in terms of the graph Γ_A associated with the support of $G_A = \mathrm{Supp}(A_n)$ of the matrix A_n.

Theorem 13 *1) The family* (Λ_n, A_n) *is* irreducible *if and only if the graph* Γ_A *is* orientationally connected*;*
2) the family (Λ_n, A_n) *is* Schur irreducible *if and only if the symmetric closure* Γ_A^s *of the graph* Γ_A *is* orientationally connected*;*
3) the family (Λ_n, A_n) *is* indecomposable *if and only if the graph* Γ_A is connected.

Definition 14 Adjacency matrix. *This is the* n *by* n *matrix* A*, where* n *is the number of vertices in the graph. If there is an edge from some vertex* x *to some vertex* y*, then the element* $a_{x,y}$ *is* 1 *(or in general the number of* xy *edges), otherwise it is* 0.

Let us use denote by $A_\Gamma = A_G$ the *adjacency matrix* of the graph Γ associated with the set G. We have the following correspondence:

$$\text{set } G \leftrightarrow \text{graph } \Gamma \leftrightarrow \text{adjacency matrix } A_G = A_\Gamma. \tag{13}$$

Definition 15 *For two subset* $G_1, G_2 \subseteq \{1, 2, ..., n\}^2$ *define the* product $G_3 = G_1 \circ G_2$ *as the subset*

$$G_1 \circ G_2 = \{(k, m) \mid (k, p) \in G_1, (p, m) \in G_2 \quad \textit{for some } p\}. \tag{14}$$

Let us denote for any subset $G \subset \{1, ..., n\}^2$ by g the corresponding subspace

$$g = (x \in \mathrm{Mat}(n, \mathbb{C}) \mid x = \sum_{(k,m)\in G} x_{km} E_{km}). \tag{15}$$

Let g_1 and g_2 be the subspaces corresponding (via (15)) to two subsets G_1 and G_2. We define the product $g_1 g_2$ as follows $g_1 g_2 = (z = xy \mid x \in g_1, y \in g_2)$. Obviously, we have

$$g_1 g_2 = (x \in \mathrm{Mat}(n, \mathbb{C}) \mid x = \sum_{(k,m)\in G_3} x_{km} E_{km}),$$

where $G_3 = G_1 \circ G_2$.

To define correctly the product of two adjacency matrix matrices A_{G_1} and A_{G_2}, corresponding to the subsets G_1 and G_2, we assume that the matrix elements of the matrix A_{G_i}, which are equal to 0 and 1, are in the semiring R.

Definition 16 *Define the* semiring R *consisting of two elements* 0 *and* 1 *with operations (see [1])*

$$0+0=0, \quad 0+1=1, \quad 1+1=1, \quad 0\times 0=0, \quad 0\times 1=0, \quad 1\times 1=1. \tag{16}$$

$$\text{We} \quad \text{have} \quad A_{G_1} A_{G_2} = A_{G_1 \circ G_2}. \tag{17}$$

Lemma 17 *The set* G *generate the algebra* $\mathrm{Mat}(n, \mathbb{C})$ *if and only if the powers* $G^k = G \circ ... \circ G,\ k = 1, 2, ..., n$ *covers the set* $\{1, 2, ..., n\}^2$.

Using Lemmas 6 and 12 we get

Lemma 18 *1) The number* $\sharp(s(n))$ *of the maximal proper subalgebra is equal to*

$$\sharp(s(n)) = \sum_{r=1}^{n-1} C_n^r = 2^n - 2,$$

the number of ordered subsets of the set $\{1, 2, ..., n\}$ *of the length between* 1 *and* $n-1$*;*
2) the numbers $\sharp(G(n))$ *of the generating subset* $G(n)$ *is equal to the numbers of the minimal orientationally connected graphs with* n *vertices.*

Problem 1. To find the number $\sharp(G(n))$.

We know that the first values of $\sharp(G(n))$ for $n = 2, 3, 4$ are $1, 5, 54$.

4 Appendix, some examples

Notations. For the sake of shortness we shall use the same notations for the set $G(n)$ and for the corresponding adjacency matrix $A_{G(n)}$. We shall denote both by $G(n)$.

Example 1. We show, using Lemma 17 , that the set $G(2)$ and sets $G(3)$ from the list (9) are generating. For $n = 2$ the set $G(2)$ is obviously unique. We get

$$G(2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G^2(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow G(2) \bigcup G(2)^2 = \{1,2\}^2.$$

For $n = 3$ we have $G_1(3) \bigcup G_1^2(3) \bigcup G_1^2(3) = \{1,2,3\}^2$. Indeed

$$G_1(3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; G_1^2(3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; G_1^3(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G_2(3) = G_1^2(3), \; G_2^2(3) = G_1^4(3) = G_1(3), \; G_2^3(3) = G_1^6(3) = G_1^3(3),$$
$$G_3(3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; G_3^2(3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow G_3(3) \bigcup G_3^2(3) = \{1,2,3\}^2,$$
$$G_4(3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; G_4^2(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow G_4(3) \bigcup G_4^2(3) = \{1,2,3\}^2,$$
$$G_5(3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \; G_5^2(3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow G_5(3) \bigcup G_5^2(3) = \{1,2,3\}^2.$$

Example 2. The list of $G(n)$ and $s(n)$ for $n \leq 4$ (for $n = 4$ only some $G(n)$).

$$G(2): \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; s(2): \quad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \; \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \tag{18}$$

$$G(3): \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{19}$$

$$s(3): \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}; \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \tag{20}$$

$$G(4): \; \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{21}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \tag{22}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{23}$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \tag{24}$$

$$s_i(4): \quad \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{pmatrix}, \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & * & * \\ * & * & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix},$$
$$s_{i_1 i_2 i_3}(4): \quad \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & 0 \\ * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

$$s_{i_1 i_2}(4): \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{pmatrix} \begin{pmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & * & * & * \end{pmatrix}, \begin{pmatrix} * & 0 & 0 & * \\ * & * & * & * \\ * & * & * & * \\ * & 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & 0 & * & 0 \\ * & * & * & * \\ * & 0 & * & 0 \\ * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

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