CFT, SLE and phase boundaries

Maxim Kontsevich
IHES, 35 Route de Chartres,
91440 Bures-sur-Yvette, France

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1 Motivation: two pictures of the Ising model near critical temperature

I will explain the motivation for my talk in the context of the 2-dimensional Ising model, the main playground for statistical mechanics of random fields. Let $U \subset \mathbf{Z}^2$ be a finite set, a *configuration of spins* in U (or simply a configuration) is a map $\sigma : U \to \{+1, -1\}$. There are $2^{\#U}$ configurations. The *energy* of a configuration σ is defined as

$$E(\sigma) := \sum_{x,y \in U,\, |x-y|=1} (-\sigma(x)\sigma(y))\,.$$

The minimal energy is achieved on configurations in which all spins are up ($\sigma(x) = +1$ for $\forall x \in U$), or down ($\sigma(x) = -1$). Let us fix a real number $T > 0$ called the temperature of the system. The Gibbs distribution at temperature T is the probability distribution on the set of configurations defined by the following formula

$$Prob(\sigma) = Z^{-1}\, exp(-E(\sigma)/T)$$

where the numerical factor $Z > 0$ (the *partition function*) is defined by the condition that the total sum of probabilities of elementary events is equal to 1,

$$Z = \sum_{\sigma} exp(-E(\sigma)/T).$$

We can then study the behavior of the system in the thermodynamic limit, $\#U \to \infty$, $\#\partial U / \#U \to 0$ (here $\partial U \subset U$ is the "boundary" of U consisting of vertices $x \in U$ adjacent to at least one vertex outside of U). The behavior of the system changes drastically if the temperature crosses a special critical value $T_{crit} > 0$ (known explicitly for the Ising model). Namely, for $T > T_{crit}$ there is only one phase, the typical configuration in a large box has approximately half of the spins in positions up and down, respectively, also the correlators

$$\langle \sigma(x)\sigma(y) \rangle := \sum_{\sigma} \sigma(x)\sigma(y) Prob(\sigma)$$

for given $x, y \in \mathbf{Z}^2$ converge in the thermodynamic limit to a function of x, y which decays exponentially as $|x-y| \to \infty$. In the case $T < T_{crit}$ one has two phases, which means that with probability close to $1/2$ the configuration of spins in a large box will have a considerable majority of $+1$'s, or there will be the opposite case with the majority of -1's. The thermodynamic limit of the two point correlator $\langle \sigma(x)\sigma(y) \rangle$ will be a translation-invariant function of $x, y \in \mathbf{Z}^2$ which is equal to a strictly positive constant (depending on T) plus an exponentially small term as $|x-y| \to \infty$. Finally, at the critical temperature there will be one phase, but the two-point correlator will decay with the power law:

$$\langle \sigma(x)\sigma(y) \rangle = const \cdot (1 + o(1)) \cdot |x-y|^{-1/4}.$$

This is the most interesting situation.

1.1 Conformal Field Theory prediction

It was a great discovery in mid 80-ies by Belavin, Polyakov and Zamolodchikov (see [3]) that the critical behavior of two-dimensional systems is governed by so-called Conformal Field Theories (CFTs for short). I will later give a pre-definition of CFT, as a collection of real-analytic functions on moduli spaces of curves. In many cases CFT is completely known. For example, the critical Ising model corresponds to so called minimal model with central charge $c = 1/2$. The relation between the lattice model and CFT goes as follows: Let ψ_i, $i = 1, \dots, N$ be a collection of complex-valued functions on configurations in $\mathbf{Z}^2$, we assume that each function ψ_i is cylindrical, i.e. it depends only on the restriction of σ to a finite box. Then we consider (the thermodynamic limit of) the multiple correlator

$$\langle \prod_i (\text{shift by } \epsilon^{-1} v_i)^* (\psi_i)(\sigma) \rangle$$

where we apply translations by (lattice approximations to) large vectors $\epsilon^{-1} v_i$ to the observables ψ_i. Here $v_i \in \mathbf{R}^2$ are pairwise distinct vectors and $\epsilon > 0$ is a small parameter. Then one can show that there are certain *fields* with conformal dimensions $\Delta_i \geq 0$ in the corresponding CFT, denoted by ψ_i^{CFT}, such that in the limit $\epsilon \to 0$ one has

$$\text{correlator as above} = \epsilon^{-\sum_i \Delta_i} \cdot \left(\langle \prod_i \psi_i^{CFT}(v_i) \rangle + o(1) \right)$$

where $\langle \prod_i \psi_i^{CFT}(v_i) \rangle \in \mathbf{C}$ is a CFT correlator, which is an explicitly known real-analytic function on $(\mathbf{R}^2)^N \setminus \{\text{all diagonals}\}$. Numbers Δ_i belong to a discrete subset of $\mathbf{R}_{\geq 0}$, the conformal spectrum of CFT, which for the critical Ising model looks as $\{0, 1/16, 1/8, 1/2, \dots\}$. Vector space of cylindrical functions in the lattice model carries a decreasing filtration by finite-codimensional subspaces indexed by CFT spectrum, the associated graded space is the space of CFT fields.

1.2 Phase boundaries at temperatures slightly below T_{crit}

Let us consider the Ising model at the temperature $T = T_{crit} - \delta T$ where $\delta T > 0$ is small. Then in the thermodynamic limit we will have with probability 1/2 a "sea" of spins +1 with "islands" of spins −1, or vice versa. Typical "large" islands will have size $\simeq (\delta T)^{-\mu}$ for some critical exponent $\mu > 0$ (in fact one can derive from CFT an explicit value of μ). Inside islands of, say spins −1 the system is confused about the global phase, and then one expects that there will again smaller second order islands of spins +1, etc. Passing to the limit $\delta T \rightarrow 0$ rescaling simultaneously the distance on $\mathbf{R}^2 \supset \mathbf{Z}^2$ by a factor $(\delta T)^{\mu}$, one obtains (conjecturally) a random collection of closed pairwise disjoint Jordan curves on $\mathbf{R}^2$, called phase boundaries. This collection is with probability 1 everywhere dense, there will be very few curves of a large size ≥ 1, and many curves of size $\simeq 1$, covering a positive part of the total area. Now let us consider the behavior of phase boundaries at small distances, i.e. rescale again the distance in $\mathbf{R}^2$. Here there are general heuristic arguments which show that the limiting distribution of curves is not degenerate, i.e. there are many curves of size (diameter) $\simeq 1$, and the distribution is now *scale* invariant.

Here a natural question arises: can one relate the limiting scale invariant random field of phase boundary curves with CFT? The goal of my talk is to propose such a construction of random curves starting from CFT and explain the related geometry of the moduli spaces of complex algebraic curves. In fact, there will be two completely different constructions, one for random intervals ending on the boundary of the surface, and another one for random loops.

2 Moduli spaces of curves and CFT

The content of this section is standard, I include it for self-containedness and also in order to introduce some notations.

2.1 Virasoro uniformization

Let $\mathcal{M}_{g,1}$ denote the moduli stack of smooth compact complex curves of genus g with one marked point. It is well-known that in a sense $\mathcal{M}_{g,1}$ is a double coset space (see [2],[9]). Namely, denote by $\mathcal{M}_{g,\hat{1}}$ the moduli space of triples (C, p, z) where C is a curve of genus g with marked point $p \in C$, and z is a formal coordinate on C near p, $z(p) = 0$. Obviously $\mathcal{M}_{g,\hat{1}}$ is fibered over $\mathcal{M}_{g,1}$, moreover the latter space is the quotient of $\mathcal{M}_{g,\hat{1}}$ (which is a projective limit of an infinite tower of finite-dimensional complex manifolds) by the action of pro-algebraic group of formal diffeomorphisms

$$z \rightarrow f(z) = \sum_{n \geq 1} a_n z^n \subset z \cdot (\mathbf{C}[[z]])^{\times} \ .$$

An easy calculation from deformation theory shows that the tangent space to $\mathcal{M}_{g,\hat{1}}$ at point (C, p, z) can be naturally identified with the quotient of two Lie algebras

$$\mathbf{C}((z))\frac{\partial}{\partial z}\Big/\left(\text{ image of } \Gamma(C\setminus\{p\},T_C) \text{ in } \mathbf{C}((z))\frac{\partial}{\partial z}\right)$$

Hence it looks like a tangent space to a homogeneous space. One can show that the identification above leads to a formally transitive action of $\mathbf{C}((z))\frac{\partial}{\partial z}$, the Lie algebra of vector fields on punctured formal disc, on $\mathcal{M}_{g,\hat{1}}$. The action of its subalgebra

$$z\mathbf{C}[[z]]\frac{\partial}{\partial z}$$

exponentiates to an action of the pro-algebraic group of formal diffeomorphisms on formal coordinates, the generator $L_{-1} := -\frac{\partial}{\partial z}$ corresponds in a sense to the moving of the marked point p, and the generators $L_{-n} := -z^{1-n}\frac{\partial}{\partial z}$ actually change the complex structure of the curve C itself. This action is a formal version of the action of the Lie algebra of complex-valued vector fields on the circle

$$S^1 = \{z \in \mathbf{C}\,|\,|z| = 1\}$$

on the moduli spaces of surfaces with complex structure and one boundary component parameterized by S^1. If we consider $\mathcal{M}_{g,\hat{1}}$ as a *real* infinite-dimensional manifold, then the holomorphic action of $\mathbf{C}((z))\frac{\partial}{\partial z}$ gives rise to a homomorphism of the direct sum of two complex Lie algebras

$$\mathbf{C}((z))\frac{\partial}{\partial z} \oplus \mathbf{C}((\overline{z}))\frac{\partial}{\partial \overline{z}}$$

to the complex Lie algebra of $\mathbf{C}$-valued smooth vector fields on $\mathcal{M}_{g,\hat{1}}$.

2.2 Determinant bundle

Let Σ be a closed surface with conformal structure. I define the *determinant line* $|det|_\Sigma$, an oriented one-dimensional vector space over $\mathbf{R}$, as follows. Any smooth metric g on Σ compatible with the conformal structure gives a positive point (base vector) in $|det|_\Sigma$, denoted by $[g]$. For two metrics g_1, g_2 the ratio of the corresponding vectors is given by

$$[g_1]/[g_2] := exp(S_{Liouv}(g_1, g_2))$$

where the *Liouville action* for two metrics with the same conformal structure is given by

$$S_{Liouv}(g_1, g_2) = \frac{1}{48\pi i}\int_\Sigma (\phi_1 - \phi_2)\partial\overline{\partial}(\phi_1 + \phi_2)\,.$$

Here $g_i = exp(\phi_i)|dz|^2$ where z is an arbitrary local complex coordinate on Σ compatible with the conformal structure. The main property of the Liouville action which ensures that det_Σ is well-defined is the following cocycle identity:

$$S_{Liouv}(g_1, g_3) = S_{Liouv}(g_1, g_2) + S_{Liouv}(g_2, g_3)\,.$$

Deforming the conformal structure on Σ we obtain an oriented real line bundle $|det|$ on the moduli stack of conformal structures on Σ. Also the determinant line can be defined for surfaces with boundary, one just uses the same formulas as above for metrics (compatible with a given conformal structure) which are flat near the boundary and such that the boundary is geodesic.

For any real c we can define the c-th tensor power $|det|^{\otimes c}$ of the line bundle $|det|$. One can easily show that the action of the Lie algebra $\mathbf{C}((z))\frac{\partial}{\partial z}$ on $\mathcal{M}_{g,\hat{1}}$ (or, more precisely, holomorphic and anti-holomorphic copies of it) extends canonically to an action of the Virasoro algebra (see the next subsection) with central charge c on sections of the pullback of $|det|^{\otimes c}$.

2.3 Pre-definition of CFTs

I will give here a description of a CFT which will contain all the data but not the whole list of axioms. A CFT is given by real number c (called the central charge), a countably-dimensional vector space $\mathcal{H}$ over $\mathbf{C}$ (whose elements are called *fields*), together with an action of two commuting Virasoro algebras (central extensions of $\mathbf{C}((z))\frac{\partial}{\partial z}$) with central charge c:

$$L_n, \overline{L}_n : \mathcal{H} \to H, \ \ n \in \mathbf{Z}\,,$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m,0} \cdot c \cdot Id_{\mathcal{H}}\,,$$

$$\text{the same for } \overline{L}_n\,,$$

$$[L_n, \overline{L}_m] = 0\,,$$

and correlators

$$\langle\ \rangle_{g,n} : \mathcal{H}^{\otimes n} \to \Gamma(\mathcal{M}_{g,\hat{n}}, |det|^{\otimes c}), \ \ g, n \geq 0\,.$$

Here the moduli space $\mathcal{M}_{g,\hat{n}}$ is analogous to $\mathcal{M}_{g,\hat{1}}$, except that now we have n distinct marked points on the curve and n formal local coordinates at these points. The main axiom says that $\langle\ \rangle_{g,n}$ is equivariant with respect to the action of n copies of Vir and $\overline{Vir}$, and also of the symmetric group S_n. In the case $n = 0$ we get a section of the bundle $|det|^{\otimes c}$ on $\mathcal{M}_g$, it is called the partition function in CFT, and its value at a point represented by some surface Σ is denoted by Z_Σ. The vector Z_Σ is related to the asymptotics of the lattice model partition function in the thermodynamic limit. Probabilities of events in CFT are obtained by division of certain vectors in $|det|^{\otimes c}_\Sigma$ by Z_Σ.

The complete axiomatics include a so-called OPE (operator product expansion), plus more technical analytic constraints which are kind of known for so-called unitary CFT (which exists only for $c \geq 0$), and less clear in general (work in progress).

2.4 Boundary CFTs

In the Ising model (at critical temperature) one can consider random configurations of spins with various boundary conditions e.g. one can force all spins to be equal to $+1$ or -1 on some large parts of the boundary of a large box U. The corresponding formalism in CFT was proposed by J. Cardy (see [4]). Namely, in the boundary CFT one has not only a central charge c and a space $\mathcal{H}$ of fields as before, but also a certain set BC of possible (local and diffeomorphism invariant) boundary conditions, and for any ordered pair of its elements $\alpha, \beta \in BC$ a representation $\mathcal{H}_{\alpha,\beta}$ of just *one* copy of Virasoro algebra with central charge c (the space of boundary fields). Correlators are defined for collections

$$(\Sigma, p_1, \dots, p_n, q_1, \dots, q_m; z_1, \dots, z_n, w_1, \dots, w_m; bc)\,,$$

where Σ is an oriented surface with conformal structure and possibly with boundary, $(p_i)_{i=1,\dots,n}$ is a collection of pairwise distinct points in the interior $int(\Sigma)$, $(q_j)_{j=1,\dots,m}$ is a collection of pairwise distinct points on the boundary $\partial\Sigma$, (z_i) is a collection of formal local holomorphic complex coordinates on Σ at points p_i, (w_j) are formal real positively oriented local coordinates on $\partial\Sigma$ at points q_j, and

$$bc : \partial\Sigma \setminus \{q_1, \dots, q_m\} \to BC$$

is a locally constant map. The correlators for a given collection as above is a linear functional on the tensor product

$$\mathcal{H}^{\otimes n} \otimes \bigotimes_{j=1,\dots,m} \mathcal{H}_{\alpha_j^{left}, \alpha_j^{right}}$$

where α_j^{left} (resp. α_j^{right}) are boundary conditions on the left (resp. on the right) of point q_j. There is an obvious extension of the Virasoro uniformization to the case of surfaces with boundaries, one has just to consider complex algebraic curves with anti-holomorphic involutions. The axiom again says that the correlator gives an equivariant map to the space of sections of the bundle $|det|^{\otimes c}$.

In the case of the critical Ising model, the set BC consists of 3 elements, namely $+$, $-$, and the free boundary condition (see [5]). Also the spaces $\mathcal{H}_{\alpha,\beta}$ and correlators are explicitly known.

3 Prediction for phase separating intervals

Here I will describe results of a joint work with R. Friedrich and J. Kalkkinen. We propose a way to construct a probability measure on paths on a given surface starting at a given point on the boundary, using correlators for some boundary field. Conjecturally our probability distribution describes intervals separating phases, growing from a point on the boundary of the surface.

3.1 Degenerate highest vectors and canonical differential operators

Let V be an arbitrary representation of the Virasoro algebra with central charge c. A vector $v \in V$ is called a highest vector of weight $h \in \mathbf{R}$ iff $L_{\geq 1}(v) = 0$ and $L_0(v) = h \cdot v$. A direct calculation shows that if v is such a vector, and $\lambda \in \mathbf{R}$ is a parameter, then the new vector

$$(\lambda L_{-1}^2 - L_{-2})(v)$$

is also a highest vector if and only if

$$c = (3 - 2\lambda)(3 - 2/\lambda), \quad h = (3/\lambda - 2)/4\,.$$

Let us fix parameters λ, c, h related as above. Then we construct a canonical second-order differential operator

$$\Delta_\lambda : \Gamma(\mathcal{M}_{g,1_b}, |T_q^*\partial\Sigma|^{\otimes h} \otimes |det|^{\otimes c}) \rightarrow \Gamma(\mathcal{M}_{g,1_b}, |T_q^*\partial\Sigma|^{\otimes(h+2)} \otimes |det|^{\otimes c})$$

acting on sections of two different oriented real line bundles on the moduli space $\mathcal{M}_{g,1_b}$ of conformal structures on surfaces Σ of genus g with one boundary component and one marked point q on the boundary. The construction is the following one. We consider a modified moduli space $\mathcal{M}_{g,\hat{1}_b}$ by adding to the data also a formal real coordinate on $\partial\Sigma$ near q. On the space of sections of the pullback of the bundle $|det|^{\otimes c}$ the Virasoro algebra acts. The space of highest vectors with the weight h (resp. $h+2$) is canonically identified with the space of sections of a line bundle on the non-modified finite-dimensional moduli space. The operator Δ_λ is just a second order element $(\lambda L_{-1}^2 - L_{-2})$ from the universal envelopping algebra of Vir, acting on this representation.

3.2 Field ψ and random walk on moduli spaces

Our main prediction is that the shape of phase boundaries is governed, in a sense described later, by a certain distinguished field $\psi \in \mathcal{H}_{+,-}$ (and also a vector in $\mathcal{H}_{-,+}$ also denoted by ψ) which is a degenerate highest weight vector, with parameters h and c expressed via λ as in the previous subsection. The parameter λ seems to take values in semi-interval $(0, 1]$, which corresponds to the constraint $-\infty < c \leq 1$. The meaning of $\mathcal{H}_{+,-}$ is clear in the case of critical Ising model (with $c = 1/2$), presumably the whole prediction extends to other models with different values of c. Incidentally, the vector ψ seems always to be a vector in $\mathcal{H}_{+,-}$ with the lowest eigenvalue of L_0, i.e. with the lowest conformal dimension.

Now let us consider a moduli space $\mathcal{M}$ of conformal surfaces with boundary and $m = 2k$ marked points $(q_j)_{j=1,\dots,m}$ on the boundary, and a a boundary condition bc changing from $+$ to $-$ or vice versa at all marked points. The correlator $\langle \prod_j \psi(q_j) \rangle$ is a section of the (complexified) oriented line bundle L on $\mathcal{M}$ whose fiber is $\left(\bigotimes_j |T_{q_j}^* \partial\Sigma|^{\otimes h} \right) \otimes |det|_\Sigma$. By our assumption on ψ this correlator is annihilated by m second order operators $\Delta_\lambda^{(j)}$ acting from sections of L to sections of $L \otimes |T_{q_j}\partial\Sigma|^2$, constructed similarly to the operator Δ_λ from the previous subsection.

We expect that the correlator $\langle \prod_j \psi(q_j) \rangle$ is in fact real and everywhere positive. Let us consider the trivialization of L given by this correlator, and an *arbitrary* trivialization of $L \otimes |T^*_{q_j}\partial\Sigma|^2$. We obtain therefore a second-order differential operator acting on functions, vanishing on constants, and with (constant rank) non-negative symbol. Such an operator is always an operator of Kolmogorov type, i.e. locally it can be presented as

$$\sum_{i=1}^{N} v_i^2 + v_0$$

where $v_0, v_1, \ldots, v_N$ are some smooth real vector fields. In our case, the number N can be chosen to be equal 1.

Any operator of Kolmogorov type defines a diffusion process, i.e. a time independent continuous random walk on manifold $\mathcal{M}$. A theorem by Hörmander (see [8]) says that this operator is hypoelliptic (invertible on the space of distributions modulo smooth functions) if the Lie algebra generated by $(v_i)_{i=0,\ldots,N}$ acts infinitesimally transitively on the manifold. One can deduce easily from Virasoro uniformization and from the fact that the elements $L_{\geq -2}$ generate the whole Virasoro algebra, that our operators $\Delta_\lambda^{(j)}$ are hypoelliptic.

Fix the index j of a marked point q_j on the boundary. Let us denote by $P = P_j$ the subbundle of the tangent bundle to $\mathcal{M}$ generated by vector fields v_0, v_1 in the representation as above. This is a universal two-dimensional subbundle, independent of the choices made above, including the parameter λ. It contains a canonically defined subbundle of open half-planes $P_+ := \mathbf{R} \cdot v_1 + \mathbf{R}_{>0} \cdot v_0$. The random walk goes in a sense always in the direction lying in P_+. The distribution of planes P is not holonomic.

If one changes the trivialization of the second line bundle $L \otimes |T^*_{q_j}\partial\Sigma|^2$, it results only in a certain time change on random paths in $\mathcal{M}$, i.e. the induced probability measure on the space of paths up to reparameterization remains the same.

3.3 Lifting of the random walk to self-avoiding walks on surface, relation to SLE

Our next goal is to relate random walks on $\mathcal{M}$ with self-avoiding random walks on the surface itself. The prototype of this correspondence is the following one. Let $x(t)$ be a smooth path on $\mathcal{M}$ starting at a given point $(\Sigma, \ldots)$, with velocity lying in P_+ for any moment of time. I claim that $x(t)$ is associated with a unique self-avoiding path $y(t)$ on Σ starting at q_j, smooth for $t > 0$ and such that $y(t)^2$ is also smooth and has non-vanishing first derivative at $t = 0$ (here for small t I use a local holomorphic coordinate on Σ near q_j identifying it with a neighborhood of 0 in the standard upper half-plane). Namely, for any $t > 0$

we can make a cut on Σ along the curve $y([0,t])$ and declare the end of this cut to be the new marked point q_j. In this way we obtain a surface with conformal structure and marked points depending on t, i.e. a path $x(t)$ on $\mathcal{M}$. One can easily check that the path $x(t)$ is a P_+-path, and an arbitrary P_+-path can be obtained uniquely in this way.

There was recently a very exciting development in probability theory, people constructed and studied so called SLE_κ processes (Stochastic Loewner evolution or Schramm-Loewner Evolution), a probability measure on fractal curves on the disk connecting two fixed points on the boundary, depending on parameter $\kappa > 0$ which can be identified as 4λ in our notations. For a review I recommend [12]. It can be argued from SLE_κ theory that the correspondence between P_+-paths on $\mathcal{M}$ and self-avoiding paths on Σ extends from smooth paths to fractal paths, in the case curves without self-touching , that is for $\kappa \leq 4$ (in which case the Hausdorff dimension of the path on Σ is $1 + \lambda/2 = 1 + \kappa/8$).

Our prediction is that self-avoiding random paths on Σ associated with random walks on $\mathcal{M}$ coming from correlators of the basic field ψ are exactly random phase boundaries. In the case of the disk with two marked points on the boundary the moduli space consist of only one point, therefore the correlator $\langle\psi(q_1)\psi(q_2)\rangle$ is known a priori, it depends only on the value of λ. Therefore the random path in this case depends only on λ, and one can argue that it is the $SLE_{4\lambda}$-path. The relation between operator $(\lambda L_{-1}^2 - L_{-2})$ and SLE paths on disc was independently proposed in [1].

3.4 Example of predictions for probabilities of crossing events

Let $z_1, \ldots, z_n$ be pairwise distinct points on the upper half plane. We are interested in calculating the probablity $P_{n,m}(z_1, \ldots, z_n)$ where $1 \leq m \leq n$ that the $SLE_{4\lambda}$ path connecting 0 and $i\infty$ separates the upper half plane in two domains with points $z_1, \ldots, z_m$ on the left, and points $z_{m+1}, \ldots, z_n$ on the right. This probability satisfies the hypoelliptic equation (independent on m)

$$\left(\lambda\left(\sum_{i=1}^{n}\left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \overline{z}_i}\right)\right)^2 - \sum_{i=1}^{n}\left(\frac{1}{z_i\partial z_i} + \frac{1}{\overline{z}_i\partial\overline{z}_i}\right)\right) P = 0\,.$$

The boundary conditions for the solution can be determined inductively by n, if some points (z_i) collide, or go to the real axis, the limiting probability is either expressed through a function P with a strictly smaller number of arguments, or equal to zero. Together with the obvious equality $P_{0,0}() = 1$ this gives a complete characterization of probabilities $P_{n,m}$. I do not know explicit formulas for $P_{n,m}$, but in a simpler case (see [7]) one can calculate the probabilities completely.

4 Prediction for phase separating loops

In the case of closed loops separating phases the procedure described above does not work, there is no canonical seed point from boundary to grow. Here I will describe briefly another approach (joint work with Y. Suhov).

4.1 Malliavin measure on loops

For an oriented surface Σ with conformal structure define the space $Loops(\Sigma)$ to be the space of simple closed Jordan curves on Σ, endowed with the Hausdorff topology. It is a locally compact countably generated topological space. For any loop $\mathcal{L}$ there exists an open $U \supset \mathcal{L}$ such that U is homeomorphic to an annulus and $U \setminus \mathcal{L}$ is homeomorphic to two annuli.

Define a continuous oriented real line bundle $|det|$ on $Loops(\Sigma)$ as follows. Its fiber over $\mathcal{L}$ is defined as $|det|_U \otimes (|det|_{U\setminus\mathcal{L}})^{-1}$ where $U \supset \mathcal{L}$ is an arbitrary neighborhood as above. Here we use determinant lines for surfaces with boundary obtained by canonical conformal compactifications of open surfaces with conformal structure and with finite first Betti number.

One can easily deduce from the definition of $|det|$ in terms of metrics, and from the locality of the Liouville action, that different choices of neighborhoods U give canonically isomorphic lines.

Conjecture: *For any $c \leq 1$ there exists a unique (up to a factor) positive density μ_c on $Loops(\Sigma)$ with values in $|det|^{\otimes c}$, covariant with respect to open embeddings of surfaces with conformal structures.*

The search for the density μ_c was initiated by P. Malliavin in [11] (in a bit different terms), and physics of SLE_κ curves suggests strongly the existence and uniqueness of μ_c.

4.2 Coupling with CFT

Let Σ be a surface without marked points but possibly with boundary, and $bc : \partial\Sigma \to \{+,-\}$ be a locally constant map. Let us fix a loop $\mathcal{L}$ on Σ, and also attach the sign $+$ to one side of $\mathcal{L}$ in Σ, and the sign $-$ to the opposite side. We are interested in the probability that in the critical Ising model with boundary condition bc there will be a phase separating a loop close to $\mathcal{L}$ (with phases near it specified by our choices of signs). Denote by Σ' the complement $\Sigma \setminus \mathcal{L}$ with canonically attached boundary such that the canonical conformal structure in its interior extends smoothly to the boundary. We have a canonical choice of boundary conditions bc' for Σ'.

There is a canonical isomorphism between oriented real lines

$$|det|_{\mathcal{L}} \simeq |det|_\Sigma / |det|_{\Sigma'}$$

(it follows from the locality of S_{Liouv}). Hence the expression

$$\mu_c \times Z_{\Sigma',bc'}/Z_{\Sigma,bc}$$

can be interpreted as a density on $Loops(\Sigma)$ near point $\mathcal{L}$.

Our prediction is that this density coinsides with the infinitesimal density of phase separating loops.

This description generalizes easily to the case of several loops, and eventually produces a random countable everywhere dense collection of disjoint simple cooriented Jordan curves on Σ, satisfying a kind of Markov property.

4.3 Algebraic structure arising from μ_c, and new identities for CFT

The (conjectural) densities μ_c can be used to construct a nice algebraic structure called the modular operad. The prototype is the collection of the homology groups $H_*(\overline{\mathcal{M}}_{g,n})$ of the moduli stacks of stable curves with marked points. There are many polylinear operations on these spaces given by pushforward maps from the boundary strata of moduli stacks.

Let us (for given $c \in (-\infty, +1])$ define an infinite-dimensional real vector space $V_{g,n}$ as the space of (measurable?) sections of the line bundle $|det|^{\otimes c}$ on the moduli space of conformal structures on surfaces of genus g with n enumerated boundary components. Gluing of boundary components and integration with repsect to μ_c gives "operadic compositions" on the spaces $V_{g,n}$, e.g. a map

$$V_{g_1,n_1+1} \otimes V_{g_2,n_2+1} \longrightarrow V_{g_1+g_2,n_1+n_2}\,.$$

In particular, on the space $A := V_{0,2}$ (isomorphic to the space of functions of one real positive variable) we obtain a commutative associative product $\star$ (depending on the central charge c). An easy combinatorial argument shows heuristically that our perdiction for the phase boundaries implies the following identity. Let Z_{++} (and Z_{+-}) be elements of A corresponding to partition functions of the critical Ising model on the cylinder with boundary conditions $+,+$ on both boundary components ($+$ on one component and $-$ on the other component, respectively). Then one has

$$Z_{++} \star Z_{++} = Z_{+-} \star Z_{+-} + Z_{+-}$$

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