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Actions of "nilpotent tori" on G-varieties

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Introduction

Let G be a connected simply-connected semisimple algebraic group with Lie algebra $\mathfrak g$ and let $e \in \mathfrak g$ be a regular nilpotent element. Denote by G_e the centralizer of e in G. It is well known that $G_e \simeq G_e^o \times Z(G)$, where $Z(G)$ is the centre of G and G_e^o is a commutative unipotent group of dimension $\mathrm{rk}\, G$, the rank of G. R.K. Brylinski used the name "nilpotent torus" for the Lie algebra $\mathfrak g_e := \mathrm{Lie}\, G_e$ [5]. We shall apply the name "nilpotent torus" to the groups G_e^o and G_e, too.

The aim of this paper is to study the algebras of invariants $k[X]^{G_e^o}$ and $k[X]^{G_e}$, where X is an affine variety acted upon by G (= G-variety). The motivation (and an explanation of terminology) comes from the article of R.K. Brylinski [4]. She proved that $\mathfrak g_e$ is the limit of Cartan subalgebras in a very precise sense and that, for any irreducible G-module V, $V^{\mathfrak g_e}$ together with some additional data keeps information on q-analogs of the weight multiplicities in V. Therefore knowledge of the structure of the above algebras of invariants may provide some information on q-analogs for G-modules involved in $k[X]$. A useful class of G-varieties which deserves a special attention in this regard is that of HV-varieties, see [13, § 5] and sect. 3. In sect. 1, I tried to gather as many results of general nature on invariants of nilpotent tori as possible. Among them the assertion that $\mathrm{Spec}\, k[G]^{G_e^o}$ has rational singularities. It implies that $\mathrm{Spec}\, k[X]^{G_e^o}$ has rational singularities whenever X does so. In sect. 2, we describe the null-cone for G_e-invariants in terms of 1-parameter subgroups. This was inspired by the work of M. Brion on the null-cone for a maximal unipotent subgroup of G [2, IV]. In our case the answer turns out to be better and it has an interesting connection with the null-cone for a maximal torus. A surprising consequence is that the null-cone relative to G_e is never reduced. Actions of nilpotent tori on HV-varieties are considered in sect. 3. We describe all the cases when the algebra of G_e-invariants is polynomial and prove some curious results on G_e^o-invariants for $G = \mathbf{A}_{\mathrm p}$. In sect. 4, "$G_e$-theory" is used for getting some results on the diagonal G-action on $X \times \mathfrak g$. First, an estimate of embedding dimension of algebras of the form $k[X \times \mathfrak g]^G$ is given. As a by-product, we recover results of F. Levstein [11]. Second, it is proven that the null-cone in $X \times \mathfrak g$ relative to G is never reduced.

Notation. The ground field k is algebraically closed and of characteristic zero. Lie algebras of algebraic groups are denoted by the respective lower-case Gothik letters. Throughout the paper, X is an affine G-variety and $k[X]$ is the algebra of regular functions on X. For a subgroup $A \subset G$, $k[X]^A$ is the subalgebra of A-invariant functions. We write $X/\!\!/A$ for $\mathrm{Spec}\, k[X]^A$ whenever $k[X]^A$ is finitely generated.

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1 Invariants of nilpotent tori on G-varieties

Most properties of G_e-invariants come up through a relationship with G-invariants and the nilpotent cone $\mathfrak{N} \subset \mathfrak{g}$. Therefore we begin with recalling basic properties of the latter.

1.1 Theorem.
(i) [10] $\mathfrak{N}$ *is an irreducible normal complete intersection in* $\mathfrak{g}$ *and* $\mathit{codim}_{\mathfrak{N}}(\mathfrak{N}\setminus Ge) = 2$*;*
(ii) [9] $\mathfrak{N}$ *has rational singularities.* $\square$

It follows that $k[\mathfrak{N}] = k[Ge] = k[G]^{G_e}$, i.e., $k[G]^{G_e}$ is finitely generated and $\mathfrak{N} = \operatorname{Spec} k[G]^{G_e}$. Consider the larger algebra $k[G]^{G_e^o}$. Being a finite $k[G]^{G_e}$-module, $k[G]^{G_e^o}$ is finitely generated, too. Obviously, it is integrally closed. Hence $\tilde{\mathfrak{N}} := \operatorname{Spec} k[G]^{G_e^o}$ is a normal affine variety. The dense G-orbit in $\tilde{\mathfrak{N}}$ is isomorphic to G/G_e^o and its complement is of codimension 2. Let $\tilde{e}$ be a point in this G-orbit whose stabilizer is G_e^o.

1.2 Theorem. $\tilde{\mathfrak{N}}$ *has rational singularities.*

Proof. We follow the same way as in [12], where it was proved that the normalization of the closure of any nilpotent orbit has rational singularities. The reader may consult that paper for more details concerning the following exposition. Let $\mathcal{K}_Y$ and ω_Y denote, respectively, the Grauert-Riemenschneider canonical sheaf and the Grothendieck dualizing sheaf on a normal variety Y. By Flenner's theorem, $\tilde{\mathfrak{N}}$ has rational singularities whenever $\omega_{\tilde{\mathfrak{N}}}$ is invertible and $\mathcal{K}_{\tilde{\mathfrak{N}}} = \omega_{\tilde{\mathfrak{N}}}$. Here we have $\omega_{\tilde{\mathfrak{N}}} \simeq \mathcal{O}_{\tilde{\mathfrak{N}}}$ for the same reason as for the closure of any nilpotent orbit (cf. [12]): the dense orbit $G\cdot\tilde{e}$ has a symplectic structure and its complement is of codimension 2. Let $\tilde{\tau} : Z \to \tilde{\mathfrak{N}}$ be a G-equivariant resolution of singularities. In our setting, the equality $\mathcal{K}_{\tilde{\mathfrak{N}}} = \omega_{\tilde{\mathfrak{N}}}$ will follow from the fact that the pull-back of the symplectic 2-form on $G\cdot\tilde{e}$ under $\tilde{\tau}$ extends to a regular 2-form on Z. Let B be the unique Borel subgroup such that $e \in \mathfrak{b}$ and let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{b}$. Recall the famous Springer resolution $\tau : G *_B \mathfrak{u} = G *_B \overline{B\cdot e} \to \mathfrak{N}$. We may construct Z such that the following diagram exists and is commutative:

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{\tau}} & \tilde{\mathfrak{N}} \\ \downarrow & & \downarrow \\ G *_B \overline{B\cdot e} & \xrightarrow{\tau} & \mathfrak{N} \end{array} .$$

Indeed, take a B-equivariant resolution of singularities $R \to \overline{B\cdot\tilde{e}} \subset \tilde{\mathfrak{N}}$ and set $Z = G *_B R$. Next, the symplectic 2-form on $G\cdot\tilde{e}$ is the pull-back of the Kirillov form on $G\cdot e$ and the latter extends to a regular 2-form on $G *_B \overline{B\cdot e}$, see [12, lemma 2]. ¿From this diagram it then follows that the symplectic form on $G\cdot\tilde e$ extends to a regular 2-form on Z. $\Box$

Remarks. 1. Making use of Graham's results [7], one can describe an explicit model of $\tilde{\mathfrak N}$. Let $\chi_1,\dots,\chi_r$ be the nonzero minuscule dominant weights of G and V_{χ_i} $(i=1,\dots,r)$ the corresponding irreducible G-modules. Let $v_i\in V_{\chi_i}$ be a B-eigenvector. Consider the point $\tilde e=(e,v_1,\dots,v_r)\in\mathfrak g\oplus V_{\chi_1}\oplus\ldots\oplus V_{\chi_r}$. Clearly, $G_{\tilde e}=B_{\tilde e}=G_e^o$. I claim that $\overline{G\cdot\tilde e}$, the closure of G-orbit of $\tilde e$ in $\mathfrak g\oplus V_{\chi_1}\oplus\ldots\oplus V_{\chi_r}$ is isomorphic to $\tilde{\mathfrak N}$. Because $\tilde{\mathfrak N}$ is normal and the complement of the dense orbit is of codimension 2, the identity map of dense orbits gives rise to a dominant regular map $\tilde{\mathfrak N}\to\overline{G\cdot\tilde e}$. In other words, $k[\overline{G\cdot\tilde e}]$ is a subalgebra of $k[\tilde{\mathfrak N}]$. By [7, 1.2,3.4], each minuscule representation occurs in $k[\tilde{\mathfrak N}]$ exactly once and $k[\tilde{\mathfrak N}]$ is generated by these representations as $k[\mathfrak N]$-module. By construction, $k[\overline{G\cdot\tilde e}]$ contains all the stuff just described. Hence it must be equal to $k[\tilde{\mathfrak N}]$.
2. Using this model, it is easy to see that $\overline{B\cdot\tilde e}$ is isomorphic to the product of $\overline{T\cdot\tilde e}$ and an affine space (here T is a maximal torus of B). The closure $\overline{T\cdot\tilde e}$ is not normal whenever $\tilde{\mathfrak N}\ne\mathfrak N$, but its normalization has rational singularities. Therefore, it is enough to take the normalization of $\overline{B\cdot\tilde e}$ in place of R in the previous proof.

1.3 Proposition. *For any affine G-variety X, we have:*

(i) *$k[X]^{G_e}\simeq k[X\times\mathfrak N]^G$ and $k[X]^{G_e^o}\simeq k[X\times\tilde{\mathfrak N}]^G$, where the isomorphisms are given by restriction of G-invariant functions to $X\times\{e\}$ (resp. $X\times\{\tilde e\}$). In particular, both algebras are finitely generated;*

(ii) *If X has rational singularities, then so do $X/\!\!/G_e$ and $X/\!\!/G_e^o$.*

Proof. (i) This is a particular case of [8, 1.2]; see also [15, thm. 4]. What is important here is that $k[\mathfrak N]=k[G]^{G_e}$ and $k[\tilde{\mathfrak N}]=k[G]^{G_e^o}$.
(ii) By (1.1)(ii), (1.2), and Boutot's theorem [1]. $\Box$

Let $\{e_+,h,e_-\}$ be an $\mathfrak{sl}_2$-triple containing $e=e_+$ (i.e., $[h,e]=2e$, etc.). The subalgebra $\mathfrak s=\langle e_+,h,e_-\rangle$ is a principal TDS in Kostant's terminology. Denote by S the corresponding connected subgroup of G. Let V be an arbitrary (finite-dimensional) G-module. By SL_2-theory, the eigenvalues of h on V are integral and those on $V^{\mathfrak g_e}$ are nonnegative. Hence the action of h yields a $\mathbb Z_{\ge 0}$-grading on $k[X]^{G_e^o}$ (and, of course, on $k[X]^{G_e}$). The n^{th} graded piece is $k[X]^{G_e^o}_n=\{f\in k[X]^{G_e^o}\mid h{\cdot}f=nf\}$ $(n=0,1,\ldots)$. Here we regard h as an algebraic vector field on X.

1.4 Proposition.

(1) $k[X]^{G_e^o}_0=k[X]^{G_e}_0=k[X]^G$;

(2) *$k[X]^{G_e^o}_n$ is a finite $k[X]^{G_e^o}_0$-module for each $n\in\mathbb N$;*

(3) *$k[X]^{G_e}_n=0$ unless n is even.*

Proof. 1. By SL_2-theory, any function in $k[X]^{G_e^o}_0$ is S-invariant. It then suffices to prove that the Lie algebra generated by $\mathfrak g_e$ and $\mathfrak s$ is equal to $\mathfrak g$. This again follows from

SL_2-theory: successively applying $\mathrm{ad}\, e_-$ to a basis of $\mathfrak{g}_e$ consisting of h-eigenvectors, we may obtain a basis of $\mathfrak{g}$.
2. Associated to h, there is a 1-parameter subgroup (1-PS) $\nu : k^* \to S \subset G$ such that $d\nu(1) = h$. Because $\nu(k^*)$ normalizes G_e, it acts on $k[X]^{G_e^o}$ and $k[X]^{G_e^o}_n = \{f \in k[X]^{G_e^o} \mid \nu(t){\cdot}f = t^n f\}$. So, (2) becomes a standard fact on covariants of reductive groups.
3. For irreducible V, $V^{G_e} \neq 0$ if and only if $Z(G)$ acts trivially on V. Since the centre of S lies in $Z(G)$, $V|_S$ is the sum of odd-dimensional SL_2-modules in this case. Thus, all the eigenvalues of h are even. □

Remark. If $X = G$ and the G-action is given by right multiplication, then $k[G]^{G_e} = k[\mathfrak{N}]$ possesses two gradings. The first one is as above (h-grading) and the second one is inherited from $k[\mathfrak{g}]$. The relation between them has been observed in [6, 2.4]: $(2n)^{th}$ piece in h-grading is equal to n^{th} piece in $k[\mathfrak{g}]$-grading. In our setting, this easily follows from (1.3)(i).

Interest in $k[X]^{G_e}$ is also explained by the fact that $k[X]^T$, where T is a maximal torus, is a deformation of it. This is implicit in [4]. Our next goal is to present accurately this relationship. First, we need to match T and G_e. Having fixed $e \in \mathfrak{N}$ and hence G_e, we set $T = Z_G(h)$.

1.5 Theorem. *Let X be an arbitrary affine G-variety. Then there is an ascending filtration $J = \{J^n\}$ in $k[X]^T$ such that $\mathrm{gr}_J k[X]^T \simeq k[X]^{G_e}$.*

Proof. Set $J^{-1} = 0$ and $J^n = \{x \in k[X]^T \mid e^{n+1} \cdot x = 0\}$ for $n = 0, 1 \ldots$. This is well-defined, since each x belong to a finite-dimensional G-submodule and therefore e is a locally-nilpotent endomorphism of $k[X]$. Then $\mathrm{gr}_J k[X]^T := \oplus_{n\ge 0} J^n/J^{n-1}$. Define the linear map $\psi : \mathrm{gr}_J k[X]^T \to k[X]$ by: $\overline{x} \in J^n/J^{n-1} \mapsto \psi(\overline{x}) = \frac{1}{n!} e^n \cdot x$. From [4, 2.7] it follows that the image of ψ is $k[X]^{G_e}$ and ψ is injective. Because e acts as a derivation of $k[X]$, an application of the Leibniz rule shows ψ is an algebra homomorphism. □

The theorem implies that $k[X]^T$ is a deformation of $k[X]^{G_e}$ in a very precise sense. Practically, this means whenever $k[X]^{G_e}$ has a property 'of open type', $k[X]^T$ also has this property. See [15, §5,6] for a thorough treatment of deformations (contractions) and properties of open type.

1.6 Remark. G/T and G/G_e are two representatives of the finite (up to isomorphism) set of regular orbits in $\mathfrak{g}$. More generally, if G/A is an arbitrary regular orbit, then the analogue of (1.5) holds with A in place of T. Actually, all the varieties of the form $X/\!\!/A$ can be organized as the fibres of an equidimensional map onto an affine space. Endow $X \times \mathfrak{g}$ with the diagonal G-action. Associated to the projection $\vartheta : X \times \mathfrak{g} \to \mathfrak{g}$, there is the map of quotient varieties $\bar\vartheta : (X \times \mathfrak{g})/\!\!/G \to \mathfrak{g}/\!\!/G \simeq \mathbb{A}^{\mathrm{rk}\, G}$. Given $\xi \in \mathfrak{g}/\!\!/G$, denote by A_ξ the stabilizer of a point in the regular G-orbit over ξ. Then $\bar\vartheta^{-1}(\xi) = (X \times \pi_{\mathfrak{g}}^{-1}(\xi))/\!\!/G \simeq X/\!\!/A_\xi$. The latter isomorphism follows by [8, 1.2], since $k[\pi_{\mathfrak{g}}^{-1}(\xi)] = k[G/A_\xi]$ [10]. Therefore $\bar\vartheta$ is equidimensional, the fibres being of dimension $\dim X - \operatorname{rk} G$. Thus, $\bar\vartheta$ is flat whenever $(X \times \mathfrak{g})/\!\!/G$ is Cohen-Macaulay. A sufficient condition for it is that X has rational singularities.

2 The null-cone for G_e-invariants

Suppose X is an affine G-variety having a fixed point, say x_0. For a subgroup $H \subset G$, we set

$$\mathfrak{N}_H(X) = \{x \in X \mid f(x) = f(x_0) \text{ for any } f \in k[X]^H\} .$$

If $k[X]^H$ is finitely generated, then one may think of $\mathfrak{N}_H(X)$ as the (reduced) fibre containing x_0 of the quotient map $\pi_{X,H} : X \to X/\!\!/H$. By analogy with reductive group actions, we shall say that $\mathfrak{N}_H(X)$ is the *null-cone*. However, it fails to be true in general that x belongs to $\mathfrak{N}_H(X)$ if and only if the closure of Hx contains x_0; for instance, if H is unipotent then all H-orbits are closed. The Hilbert-Mumford (=HM) criterion provides a description of $\mathfrak{N}_H(X)$ in terms of 1-PS whenever H is reductive. Because this does not appeal to invariants, it is helpful to have a similar description in the nonreductive case. As a sample, we mention Brion's theorem on the null-cone for a maximal unipotent subgroup [2, IV]. Fortunately, the same approach applies to nilpotent tori as well and the answer in terms of 1-PS turns out to be even better, since no preliminary conjugation is required.

Recall that $T = Z_G(h)$. Here is the place to put a bit more notation forth. Let Δ be the set of roots relative to T. The unique Borel subalgebra $\mathfrak{b}$ containing e (and $\mathfrak{t}$) determines the subset of positive roots Δ_+. Let $\alpha_1, \dots, \alpha_p$ $(p = \operatorname{rk} G)$ be the corresponding simple roots. Then one may choose corresponding root vectors e_{α_i} $(i = 1, \dots, p)$ in such a way that $e = e_{\alpha_1} + \ldots + e_{\alpha_p}$. The fundamental weights are denoted by $\varphi_1, \dots, \varphi_p$. In the next theorem, $(\ |\ )$ stands for the natural pairing between the characters and the 1-parameter subgroups of T. If μ is a character and $\gamma : k^* \to T$ is a 1-PS, then $(\mu \circ \gamma)(t) =: t^{(\mu|\gamma)}$, $t \in k^*$.

Note that $\mathfrak{N}_{G_e^o}(X) = \mathfrak{N}_{G_e}(X)$, because $|G_e : G_e^o| < \infty$.

2.1 Theorem. *Notation as above, the following conditions are equivalent for $x \in X$:*

(i) $x \in \mathfrak{N}_{G_e}(X)$*;*

(ii) *there is a one-parameter subgroup $\tau : k^* \to T$ such that $(\alpha_i|\tau) > 0$ $(i = 1, \dots, p)$ and $\lim_{t\to 0} \tau(t) \cdot x = x_0$.*

Proof. The guiding idea is to consider the G-action on $X \times \mathfrak{N}$ in place of the G_e-action on X, $\mathfrak{N}_{G_e}(X)$ being replaced by $\mathfrak{N}_G(X \times \mathfrak{N})$. The latter is determined by the fixed point $(x_0, 0)$. The advantage is that one may apply the usual HM-criterion to the G-action. Later on, a 1-PS $\tau : k^* \to T$ is said to be *strictly dominant* whenever $(\alpha_i|\tau) > 0$ for $i = 1, \dots, p$.

(ii)$\Rightarrow$(i): By (1.3)(i), it is enough to prove that $(x,e)\in\mathfrak{N}_G(X\times\mathfrak{N})$. Since $e=\sum_i e_{\alpha_i}$, $\lim_{t\to 0}\tau(t)\cdot e=0$. Therefore the closure of $G\cdot(x,e)$ contains $(x_0,0)$ and the assertion follows.

(i)$\Rightarrow$(ii): By (1.3)(i), $(x,e)\in\mathfrak{N}_G(X\times\mathfrak{N})$. Therefore, by the HM-criterion, there is a 1-PS $\gamma: k^*\to G$ such that $\lim_{t\to 0}\gamma(t)\cdot(x,e)=(x_0,0)$.

Step 1: Without loss of generality, one may assume that $\gamma(k^*)\subset B$.
For, $P:=\{g\in G\mid \lim_{t\to 0}\gamma(t)g\gamma(t)^{-1} \text{ exists in } G\}$ is a parabolic subgroup. Therefore $B\cap P$ contains a maximal torus, say T', of G. Since $\gamma(k^*)$ is a 1-dimensional torus in P, there is $g\in P$ such that $g^{-1}\gamma(k^*)g\subset T'\subset B$. Then having written

$$g^{-1}\gamma(t)g\cdot(x,e)=g^{-1}[\gamma(t)g\gamma(t)^{-1}]\gamma(t)\cdot(x,e)\ ,$$

we see that this aggregate tends to $(x_0,0)$ as t tends to 0. Thus, one may replace γ by $g^{-1}\gamma g$.

Step 2:
Since $\gamma(k^*)\subset B$, there is $u\in U$ (the unipotent radical of B) such that $\tau(t):=u\gamma(t)u^{-1}\in T$ for all $t\in k^*$. Hence $\lim_{t\to 0}\tau(t)u\cdot(x,e)=(x_0,0)$. In particular, $\lim_{t\to 0}\tau(t)\cdot(u\cdot e)=0$. Because $u\cdot e$ is still a regular nilpotent element in $\mathfrak{u}\subset\mathfrak{b}$, τ must be strictly dominant. Next, we have $x_0=\lim_{t\to 0}\tau(t)u\cdot x=\lim_{t\to 0}[\tau(t)u\tau(t)^{-1}]\tau(t)\cdot x$. Since $u\in U$ and we already know that τ is strictly dominant, $\lim_{t\to 0}\tau(t)u\tau(t)^{-1}=1_G$ (the unit in G). Hence, $\lim_{t\to 0}\tau(t)x=x_0$, and we are done. □

Denote by W the Weyl group $N_G(T)/T$.

2.2 Corollary.
(i) *$\mathfrak{N}_{G_e}(X)\subset\mathfrak{N}_T(X)$ and $\mathfrak{N}_{G_e}(X)$ is B-stable;*
(ii) *$\mathfrak{N}_T(X)=W\cdot\mathfrak{N}_{G_e}(X)$. In particular,* $\dim\mathfrak{N}_T(X)=\dim\mathfrak{N}_{G_e}(X)$.

Proof. (i) Obvious.
(ii) By the first part, notation $W\cdot\mathfrak{N}_{G_e}(X)$ makes sense. Let us say a 1-PS γ is *regular* if $(\alpha|\gamma)\ne 0$ for each $\alpha\in\Delta$. Obviously, each regular 1-PS is W-conjugate to a strictly dominant one. It remains to observe that if $x\in\mathfrak{N}_T(X)$, i.e., $\lim_{t\to 0}\gamma(t)\cdot x=x_0$ for some 1-PS γ, then there is a regular 1-PS γ' with the same property. □

The corollary yields the following recipe: Consider the set of irreducible components of $\mathfrak{N}_T(X)$. The group W acts on it. Choose the 'dominant' representative in each W-orbit. The resulting set is precisely the set of irreducible components of $\mathfrak{N}_{G_e}(X)$.

Another curious consequence of the properties of the null-cone is that the fibre of π_{X,G_e} containing x_0 is never reduced (in the scheme-theoretic sense). For the quotient map $\pi_{X,G_e}: X\to X/\!\!/G_e$, denote by $\bar{\mathfrak{m}}_{0,X}$ the maximal ideal of the point $\pi_{X,G_e}(x_0)$. That is, $\bar{\mathfrak{m}}_{0,X}=\{f\in k[X]^{G_e}\mid f(x_0)=0\}$.

2.3 Theorem.
The ideal $k[X]\bar{\mathfrak{m}}_{0,X}\subset k[X]$ is not equal to the radical $\mathrm{Rad}\,(k[X]\bar{\mathfrak{m}}_{0,X})$.

Proof. The idea is quite simple. By (2.2)(i), any $f \in k[X]^T$ such that $f(x_0) = 0$ lies in $\mathrm{Rad}\,(k[X]\bar{\mathfrak{m}}_{0,X})$. On the other side, theorem (1.5) suggests it is unlikely that all such functions lie in $k[X]\bar{\mathfrak{m}}_{0,X}$. Implementation of this program is easy, if $k[X]$ is graded. Fortunately, the problem can always be reduced to this case. Let $\mathcal{O}$ be a minimal G-orbit containing x_0 in its closure, $\mathcal{O} \neq \{x_0\}$. Then the stabilizer of a (any) point in $\mathcal{O}$ contains a maximal unipotent subgroup of G. (The proof of [3, Lemme on p.401] applies. The only difference in the assumptions is that we do not assume that X can be embedded in an irreducible G-module.) Such varieties have been studied in [16]. If $C := \overline{\mathcal{O}}$ is normal, then it is so-called HV-variety of G. In any case, $k[C]$ has a $\mathbb{Z}_{\geq 0}$-grading such that the sum of all pieces of positive grade is the ideal of $x_0 \in C$. Let $f \in k[C]^T$ be an invariant of the least *positive* degree, say n. Since $k[C]^T \cap k[C]^{G_e} = k[C]^G = k$, we have $f \notin k[C]^{G_e}$. It follows from (1.5) that the least positive degree of G_e-invariants is also n. Hence $f \notin k[C]\bar{\mathfrak{m}}_{0,C}$. Now, we can extend f to a T-invariant function $\hat{f}$ on X. (This is possible, for T is reductive.) Obviously, then $\hat{f} \notin k[X]\bar{\mathfrak{m}}_{0,X}$. □

3 Actions of nilpotent tori on HV-varieties

This section can be viewed as a supplement to my paper [14], where T-actions on HV-varieties were studied. Recall that an HV-variety $C(\lambda)$ is the closure of the orbit of highest weight vectors in the irreducible G-module V_λ with highest weight λ. Interest in G_e-invariants on HV-varieties is explained by their relations to Lusztig's q-analogs of weight multiplicities (see [13, §5]).

First, we say some words on description of the null-cone of G_e on HV-varieties. Because $C(\lambda)$ consists of finitely many B-orbits and $\mathfrak{N}_{G_e}(C(\lambda))$ is B-stable, we need only to indicate a finite collection of points generating these B-orbits. Let $v_\lambda \in V_\lambda$ be a highest weight vector. ¿From the Bruhat decomposition it then follows that
$C(\lambda) = (\bigsqcup_{w \in W^\lambda} B \cdot v_{w\lambda}) \cup \{0\}$, where W^λ is the set of representatives of minimal lehgth for the coset space W/W_λ. In this case $\dim B \cdot v_{w\lambda} = l(w)$, the length of $w \in W^\lambda$. The following is an immediate consequence of (2.1).

3.1 Lemma. *$v_{w\lambda} \in \mathfrak{N}_{G_e}(C(\lambda))$ if and only if $(w\lambda, \varphi_i) > 0$ for some $i \in [1, p]$ if and only if $w\lambda \notin \sum_{i=1}^p \mathbb{Q}_{\geq 0}(-\alpha_i)$.* □

3.2 Example. Suppose $\lambda = \Lambda$, the highest root. To be in consent with preceding notation, root vectors in $\mathfrak{g}$ are denoted by e_α, $\alpha \in \Delta$. Here the condition of the lemma reduces to the following: $e_{w\Lambda} \in \mathfrak{N}_{G_e}(C(\Lambda))$ if and only if $w\Lambda \in \Delta_+$. It is easy to see that the maximal B-orbits in the null-cone correspond to the *simple* roots of the form $w\Lambda$. That is, *the number of irreducible components of $\mathfrak{N}_{G_e}(C(\Lambda))$ is equal to the number of long simple roots.* Furthermore, an easy computation shows (if a simple root α_i is long!)

$$\dim B \cdot e_{\alpha_i} = \frac{1}{2}(\#\{\alpha \in \Delta \mid (\alpha, \alpha_i) > 0\} + 1) = \frac{1}{2} \dim C(\Lambda) \ .$$

Thus, $\dim \mathfrak{N}_{G_e}(C(\Lambda)) = \frac{1}{2}\dim C(\Lambda)$. (It also follows from the above discussion that $\mathfrak{N}_{G_e}(C(\Lambda)) = C(\Lambda)\cap\mathfrak{u}$.) Making use of it, one can prove that $\dim C(\Lambda) \ge 2\mathrm{rk}\, G$ and the equality holds if and only if the quotient map $\pi_{C(\Lambda),G_e}$ is equidimensional:

$$\dim C(\Lambda) \le \dim C(\Lambda)/\!\!/G_e + \dim \mathfrak{N}_{G_e}(C(\Lambda)) = \dim C(\Lambda) - \mathrm{rk}\, G + \tfrac{1}{2}\dim C(\Lambda).$$

Since $k[C(\lambda)]^T$ is a deformation of $k[C(\lambda)]^{G_e}$, it follows that properties of the latter are in general worse. However, as we shall see, no change for the worse actually happens for the property of algebras of being polynomial (if G is simple).

3.3 Theorem. *Suppose G is a simple algebraic group. Then $k[C(\lambda)]^{G_e}$ is polynomial if and only if $k[C(\lambda)]^T$ is.*

Proof. The "only if" part follows by standard deformation-type argument. The real job is required for the "if"-part. *Unfortunately*, the proof is case-by-case. The table of all λ's such that $C(\lambda)/\!\!/T$ is an affine space is found in [14, 5.3]. It then suffices to prove that $C(\lambda)/\!\!/G_e$ is an affine space in all these cases. *Fortunately*, this will be derived from some general results on algebras of G_e-invariants when $G = \mathbf{A}_{\rm p}$. The dominant weights in [loc. cit.] break into the two parts. The first part includes the weights (and some of their multiples) corresponding to the simplest representations of the classical groups. The fact that $k[C(\lambda)]^{G_e}$ is polynomial for them is proved in [13, 5.9]. The second part consists—up to the symmetry of Dynkin diagram—of two infinite series of weights for $G = \mathbf{A}_{\rm p}$: $\varphi_1+\varphi_p+n(p+1)\varphi_1$, $(p-1)\varphi_1+\varphi_2+n(p+1)\varphi_1$, $n = 0,1,\ldots$. Here is the job we are to do. Actually, we shall show there is a vast area in the Weyl chamber, including these weights, such that $C(\lambda)/\!\!/T$ and $C(\lambda)/\!\!/G_e$ are (noncanonically) isomorphic, see (3.7). □

Before switching to the $\mathbf{A}_{\rm p}$-case, we state an easy general result on representations:

3.4 Proposition. *Let P be a parabolic subgroup in G with the unipotent radical N. Let P_- be an opposite parabolic subgroup. Let V be an irreducible G-module. Then there is a unique P_--stable complement to V^N. Furthermore, this complement is the unique maximal proper P_--submodule of V. In particular, the P_--equivariant projection $pr_N : V \to V^N$ is well-defined.*
(Hint: consider V as a $P\cap P_-$-module and take the sum of all submodules distinct from V^N, cf. [14, 2.3].) □

Until the end of section, $G = \mathbf{A}_{\rm p} = SL_{p+1}$ with the obvious choices of B—as the uppertriangular matrices—and T—as the diagonal matrices. Recall that $e = e_{\alpha_1}+\ldots+e_{\alpha_p}$. Let L be the simple subgroup of G whose simple roots are $\alpha_2,\ldots,\alpha_p$. Denote by P the parabolic subgroup containing B whose semisimple part is L. The unipotent radical N is p-dimensional, it corresponds to the roots $\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\ldots+\alpha_p$.[1] Define P_- to be the standard opposite to P parabolic subgroup.

[1]This is the only case, among all simple groups, where $\dim N = \mathrm{rk}\, G$. To a great extent, the following results are explained by this equality.

3.5 Proposition. *1.* $\mathfrak{p}_- \oplus \mathfrak{g}_e = \mathfrak{g}$ *and hence* $P_-G_e^o$ *is dense in* G*;*
2. Let V *be an irreducible* G*-module. Then the restriction onto* $V^{G_e^o}$ *of the* P_-*-equivariant projection* $pr_N : V \to V^N$ *is injective. In particular,* $\dim V^{G_e^o} \le \dim V^N$*.*

Proof. 1. Straightforward.
2. Denote the vector $pr_N(v)$ by v_N. Let $v \in V^{G_e^o}$. Then $V = \langle G\cdot v\rangle = \langle P_-G_e^o \cdot v\rangle = \langle P_- \cdot v\rangle$. That is, v does not belong to a P_--stable subspace of V. By (3.4), it then forces $v_N \ne 0$. □

Recall that $k[C(\lambda)]$ is graded, the piece of grade n being $V_{n\lambda^*}$. Here λ^* stands for the highest weight of the dual G-module. While dealing with $C(\lambda)$ and $k[C(\lambda)]$, it is unavoidable to get somewhere an abundance of '*' . To minimize the total amount, the variety will frequently be denoted as $C(\lambda^*)$.

3.6 Theorem. *For any dominant weight* λ*,* $k[C(\lambda^*)]^{G_e^o}$ *is isomorphic to a graded subalgebra of* $k[C(\lambda^*)]^N$*.*

Proof. Subalgebras of $k[C(\lambda^*)]$ are assumed to have the inherited grading. For each n, we have the injective linear map $v \in (V_{n\lambda})^{G_e^o} \mapsto v_N \in (V_{n\lambda})^N$. Putting them together, we obtain an injective linear map of algebras of regular functions. Let us check that this map respects multiplication. Take $v_i \in (V_{n_i\lambda})^{G_e^o}$, $i = 1, 2$. Write $v_i = (v_i)_N + \tilde v_i$, where $\tilde v_i$ lies in the maximal proper P_--submodule of $V_{n_i\lambda}$. Then $v_1v_2 = (v_1)_N(v_2)_N + \tilde v$ and it is easy to see that $\tilde v$ belongs to a proper P_--submodule of $V_{(n_1+n_2)\lambda}$. ¿From (3.4) it then follows that $(v_1v_2)_N = (v_1)_N(v_2)_N$. □

Thus, whenever one knows that—for some reason—dimensions of spaces $(V_{n\lambda})^{G_e^o}$ and $(V_{n\lambda})^N$ are equal for all $n \in \mathbb{N}$, it will imply the corresponding algebras of invariants on $C(\lambda^*)$ are isomorphic. Surprisingly enough, this possibility is not an imaginary one. Recall from [14, sect. 4] the following condition on a dominant weight $\lambda = \sum_{i=1}^p l_i\varphi_i$:

$$(\diamond) \qquad l_1 \ge (p-1)l_2 + (p-2)l_3 + \ldots + l_p \quad .$$

3.7 Theorem. *Let* $\lambda = \sum_{i=1}^p l_i\varphi_i$ *be a dominant weight of* SL_{p+1} *satisfying* $(\diamond)$*. Then*

(i) $k[C(\lambda^*)]^{G_e^o} \simeq k[C(\lambda^*)]^N$*;*
(ii) $k[C(\lambda^*)]^T \simeq k[C(\lambda^*)]^{G_e}$*.*

Proof. (i) We are to show $\dim(V_{n\lambda})^{G_e^o} = \dim(V_{n\lambda})^N$ for all n. Since this relies on inequality $(\diamond)$, which is linear, it suffices to handle the case $n = 1$. By [14, 2.2], $(V_\lambda)^N$ is an irreducible L-module whose highest weight λ' is obtained from λ by a truncation. That is, $\dim(V_\lambda)^N = \dim V_{\lambda'}$. This is true regardless of $(\diamond)$. By a result of W. Graham [7, 1.6], $\dim(V_\lambda)^{G_e^o}$ is dimension of the largest weight space in V_λ. This space corresponds to the unique minuscule dominant weight that is a weight of V_λ. For $\mathbf{A}_{\rm p}$, the muniscule weights are $\varphi_0 = 0$ and the fundamental weights φ_i, $i = 1, \ldots, p$. A straightforward calculation shows that condition "φ_i is a weight of V_λ" is equivalent to "$l_1-(p-1)l_2-(p-2)l_3-\ldots-l_p\equiv i \bmod (p+1)$". That is, $l_1=(p-1)l_2+\ldots+l_p+i+m(p+1)$. Then $(\diamond)$ means $m\ge 0$. Lemma 4.1 in [14] says that, under a certain constraint on weight μ, $\dim V_\lambda^\mu=\dim V_{\lambda'}$. We shall prove this applies to $\mu=\varphi_i$. That constraint has to do with presentation of weights in terms of partitions. Namely, if $\underline{\lambda}=(\lambda_1,\ldots,\lambda_p,0)$ is the partition associated to λ, then the condition is: each part of the partition associated to φ_i must be $\ge\lambda_2$. By definition, $\lambda_2=l_2+\ldots+l_p$ and the above formula for l_1 shows that the partition corresponding to φ_i has i parts of size $l_2+\ldots+l_p+m+1$ and $p+1-i$ parts of size $l_2+\ldots+l_p+m$. Thus, $\dim(V_\lambda)^{G_e^o}=\dim V_\lambda^{\varphi_i}=\dim V_{\lambda'}$ whenever $m\ge 0$.

(ii) Both $(V_{n\lambda})^T$ and $(V_{n\lambda})^{G_e}$ are nonzero if and only if $n\lambda$ lies in the root lattice Q. Hence it is enough to treat the case $\lambda\in Q$. Then there is no difference between G_e- and G_e^o-invariants, and an isomorphism of $k[C(\lambda^*)]^T$ and $k[C(\lambda^*)]^N$ follows from [14, 2.3,4.4]. $\square$

Remarks. 1. We have $C(\lambda)/\!\!/H\simeq C(\lambda^*)/\!\!/H$, if $H=T$ or G_e or G_e^o. But this is not true for N, since P is not conjugate to P_-.
2. Coming back to the proof of (3.3), we see that $(\diamond)$ holds for the two series of weights mentioned therein. This completes that proof.

Examples. 1. For SL_3, condition $(\diamond)$ reads $l_1\ge l_2$. Whence by symmetry of the situation, we conclude $C(\lambda)/\!\!/T\simeq C(\lambda)/\!\!/G_e$ and $C(\lambda)/\!\!/G_e^o\simeq C(\lambda)/\!\!/N$ for any λ.

2. In some cases, information on the graded structure of $k[C(\lambda^*)]^{G_e}$ allows us to write explicitly down the q-analogs of zero-weight multiplicity in the modules $V_{n\lambda}$, $n=1,2\ldots$. Now, we can add (at least) one more example to the collection of formulas in [13, 5.9,5.11]. First, recall some relevant notions. The algebra $k[C(\lambda^*)]^{G_e}$ is always $(\mathbb{Z}_{\ge 0})^2$-graded: together with the grading coming from $k[C(\lambda^*)]$ it possesses the $\frac{1}{2}h$-grading. The precise definition is that $k[C(\lambda^*)]^{G_e}_{l,n}:=\{f\in(V_{n\lambda})^{G_e}\mid h\cdot f=2lf\}$ (our present h is not the same as in [13], the transition factor is $1/2$). The Poincaré series is defined by

$$F_\lambda(q,t)=\sum_{l,n\ge 0}\dim k[C(\lambda^*)]^{G_e}_{l,n}\,q^l t^n\ .$$

If we write it as $F_\lambda(q,t)=\sum_n m^0_{n\lambda}(q)t^n$, then $m^0_{n\lambda}(q)$ is nothing but Lusztig's q-analog of zero-weight multiplicity. For SL_{p+1}, these were known before as Kostka-Foulkes polynomials $K_{n\lambda,0}$. Take $\lambda=(p-1)\varphi_1+\varphi_2$ for SL_{p+1}. We have proved that $k[C(\lambda^*)]^{G_e}$ is polynomial. It is generated by functions lying in V_λ, i.e., those having t-degree 1 (since this is true for $k[C(\lambda^*)]^T$, see [14]). It is not hard to compute the eigenvalues of $\frac{1}{2}h$ on $(V_\lambda)^{G_e}$. These are $\binom{p}{2}+i$, $i=0,1,\ldots,p-1$. Therefore $F_\lambda(q,t)=\dfrac{1}{\prod_{i=0}^{p-1}(1-q^{\binom{p}{2}+i}t)}$. Whence $m^0_{n\lambda}(q)=q^{\binom{p}{2}}\begin{bmatrix}p+n-1\\ p-1\end{bmatrix}$, where the brackets stand for a Gaussian polynomial in q.

4 Some applications

We come back to the general case, where X is an affine G-variety. Our applications concern some properties of the quotient map $\pi_G : X\times\mathfrak g\to (X\times\mathfrak g)/\!\!/G$, where $X\times\mathfrak g$ is equipped with the diagonal G-action.

Let $\mathcal A$ be a finitely generated k-algebra. The embedding dimension of $\mathcal A$ (or $\operatorname{Spec}\mathcal A$) is the minimal number of generators of $\mathcal A$. This will be denoted by $\operatorname{ed}\mathcal A$ (or $\operatorname{ed}\operatorname{Spec}\mathcal A$).

4.1 Theorem. *Let X be an affine G-variety. Then* $\operatorname{ed}(X\times\mathfrak g)/\!\!/G\le\operatorname{rk}G+\operatorname{ed}X/\!\!/G_e$.

Proof. The polynomial algebra $k[\mathfrak g]$ has the standard $\mathbb Z_{\ge0}$-grading. Give $k[X\times\mathfrak g]^G$ a $\mathbb Z_{\ge0}$-grading such that its n^{th} piece is $(k[X]\otimes k[\mathfrak g]_n)^G$. Let $f_1,\dots,f_p$ $(p=\operatorname{rk}G)$ be free homogeneous generators of $k[\mathfrak g]^G$ and $\{\bar g_1,\dots,\bar g_s\}$ a minimal homogeneous generating system of $k[X\times\mathfrak N]^G$. Recall that the last algebra is isomorphic to $k[X]^{G_e}$. Lift the $\bar g_i$'s to G-invariant functions on $X\times\mathfrak g$. That is, take for each i a $g_i\in k[X\times\mathfrak g]^G$ such that $g_i|_{X\times\mathfrak N}=\bar g_i$ and $\deg g_i=\deg\bar g_i$. Note that if $\deg\bar g_i=0$, then $\bar g_i\in k[X]^G$. In this case g_i is uniquely determined as the image of $\bar g_i$ under the injection $k[X]^G\to k[X\times\mathfrak g]^G$. I claim that $k[X\times\mathfrak g]^G$ is generated by $f_1,\dots,f_p,g_1,\dots,g_s$. The proof is by induction on degree. Let f be a homogeneous G-invariant. If $\deg f=0$, then f is a polynomial in g_i's such that $\deg g_i=0$. Suppose $\deg f>0$. Then $\bar f:=f|_{X\times\mathfrak N}=F(\bar g_1,\dots,\bar g_s)$, a polynomial in $\bar g_i$'s. Since $(f_1,\dots,f_p)=\operatorname{Rad}(f_1,\dots,f_p)$ [10], we have $f-F(g_1,\dots,g_s)=\sum_{i=1}^p r_if_i$. Applying the Reynolds operator, one can get $r_i\in k[X\times\mathfrak g]^G$. Because $\deg f_i>0$ and then $\deg r_i<\deg f$, the assertion follows by the induction hypothesis. □

4.2 Corollary. *1. If $X/\!\!/G_e\simeq\mathbb A^s$, then $(X\times\mathfrak g)/\!\!/G\simeq\mathbb A^{s+\operatorname{rk}G}$;*
2. If $X/\!\!/G_e$ is a hypersurface then $(X\times\mathfrak g)/\!\!/G$ is either an affine space or a hypersurface.
□

Examples. 1. Take $X=C(\lambda)$, an HV-variety. Denote by G_λ the stabilizer of $v_\lambda\in V_\lambda$. Because $k[C(\lambda)]=k[G/G_\lambda]$, we have $k[C(\lambda)\times\mathfrak g]^G\simeq k[\mathfrak g]^{G_\lambda}$. Thus, the following holds: *If $k[C(\lambda)]^{G_e}$ is a polynomial algebra, then so does $k[\mathfrak g]^{G_\lambda}$.* Then consulting with (3.3) and the table in [14, 5.3], one finds some admissible G_λ. The list obtained includes all (but is not exhausted by) the examples found by Levstein [11].
2. Let $G=SL_{p+1}$ and $X=V_{\varphi_1}\oplus V_{\varphi_p}$. Here $X/\!\!/G_e$ is a hypersurface. Therefore $(\mathfrak g\oplus V_{\varphi_1}\oplus V_{\varphi_p})/\!\!/SL_{p+1}$ is a hypersurface, for it is not an affine space.

¿From now on, we assume that X has a fixed point x_0. Then $X\times\mathfrak g$ has a fixed point $(x_0,0)$ and we shall compare various null-cones in X and $X\times\mathfrak g$.

4.3 Theorem.

(i) $\mathfrak N_G(X\times\mathfrak g)=G\cdot(\mathfrak N_{G_e}(X)\times\mathfrak u)$;

(ii) *The map $Y_i\mapsto G\cdot(Y_i\times\mathfrak u)$ sets up a one-to-one correspondence between the irreducible components of $\mathfrak N_{G_e}(X)$ and $\mathfrak N_G(X\times\mathfrak g)$;* $\dim G\cdot(Y_i\times\mathfrak u)=\dim Y_i+\dim\mathfrak N$.

Proof. Note that $\mathfrak{N}$ is the null-cone in $\mathfrak{g}$. Therefore $\mathfrak{N}_G(X\times\mathfrak{g})=\mathfrak{N}_G(X\times\mathfrak{N})$.
(i) By (1.3)(i), $\mathfrak{N}_G(X\times\mathfrak{N})\cap(X\times\{e\})=\mathfrak{N}_{G_e}(X)\times\{e\}$. Since $\mathfrak{N}_{G_e}(X)$ is B-stable, we have $\overline{B\cdot(\mathfrak{N}_{G_e}(X)\times\{e\})}=\mathfrak{N}_{G_e}(X)\times\mathfrak{u}$. This proves inclusion "$\supset$". The proof of the converse uses the HM-criterion. If $(x,n)\in\mathfrak{N}_G(X\times\mathfrak{N})$, then there are a 1-PS $\gamma: k^*\to T$ and $g\in G$ such that $\lim_{t\to 0}\gamma(t)g\cdot(x,n)=(x_0,0)$. Without loss of generality, one may assume that γ is regular. Conjugating the point, one may assume that $g=1_G$ and then that γ is strictly dominant. In the last case, the condition $\lim_{t\to 0}\gamma(t)\cdot(x,n)=(x_0,0)$ implies $x\in\mathfrak{N}_{G_e}(X)$ and $n\in\mathfrak{u}$.
(ii) Let $\mathfrak{N}_{G_e}(X)=\cup_i Y_i$ be the irreducible decomposition. Then each $Z_i:=G\cdot(Y_i\times\mathfrak{u})$ is a closed subvariety of $\mathfrak{N}_G(X\times\mathfrak{N})$. Because $Z_i\cap(X\times\{e\})=Y_i\times\{e\}$, we see that Z_i's are the irreducible components of $\mathfrak{N}_G(X\times\mathfrak{N})$. The dimension formula is obvious. □

Next, I am going to prove that $\mathfrak{N}_G(X\times\mathfrak{g})$ is never reduced in the scheme-theoretic sense. It will be based on (2.3) and the following technical assertion:

4.4 Theorem. *Suppose X is normal and $f\in k[X]$ vanishes on $\mathfrak{N}_{G_e}(X)$. Then there exists $F\in k[X\times\mathfrak{N}]$ such that $F|_{X\times\{e\}}=f$ and $F|_{\mathfrak{N}_G(X\times\mathfrak{N})}=0$.*

Proof. First, extend f to a function $\hat{f}$ on $X\times\mathfrak{u}$ by $\hat{f}(x,n):=f(x)$. Then $\hat{f}|_{\mathfrak{N}_{G_e}(X)\times\mathfrak{u}}=0$. Let U_- be the opposite to U maximal unipotent subgroup, i.e., U_-B is the big cell in G. Consider the map $\delta: U_-\times X\times\mathfrak{u}\to X\times\mathfrak{N}$, $(u,x,n)\mapsto(u\cdot x,(\mathrm{Ad}\,u)n)$. Since $\mathfrak{N}=G\mathfrak{u}=\overline{U_-\mathfrak{u}}$, we have δ is dominant. Moreover, δ is birational. It follows from the easy fact that if $e'\in\mathfrak{u}$ is regular and $(\mathrm{Ad}\,u)e'\in\mathfrak{u}$ for $u\in U_-$, then $u=1_G$. Hence each polynomial function on $U_-\times X\times\mathfrak{u}$ gives rise to a rational function on $X\times\mathfrak{N}$. Define $\breve{f}\in k[U_-\times X\times\mathfrak{u}]$ by $\breve{f}(u,x,n):=\hat{f}(x,n)$. If $\breve{f}$ turns out to be polynomial on $X\times\mathfrak{N}$, then $F:=\breve{f}$ and we are done. (The equality $\mathfrak{N}_G(X\times\mathfrak{N})=\overline{U_-(\mathfrak{N}_{G_e}(X)\times\mathfrak{u})}$ guarantee us that F vanishes on $\mathfrak{N}_G(X\times\mathfrak{N})$.) If not, then $\breve{f}$ has the divisor of poles, which we denote by $(\breve{f})_\infty$. The map δ is an isomorphism over the open subset $X\times U_-B{\cdot}e\subset X\times\mathfrak{N}$. Hence $\mathrm{Supp}\,(\breve{f})_\infty\subset X\times D$, where $D=\mathfrak{N}\setminus U_-B\cdot e$. Let $g\in k[\mathfrak{N}]$ be a function such that $g(e)=1$ and $\mathrm{Supp}\,(g)_0\supset D$. (Actually, there is g such that $\mathrm{Supp}\,(g)_0=D$, for the divisor class group $\mathrm{Cl}\,(\mathfrak{N})$ is finite.) Then $\breve{f}g^l\in k[X\times\mathfrak{N}]$ for some $l>0$ and we may take $F=\breve{f}g^l$. □

The theorem can be restated as follows: *Let I_1 (resp. I_2) be the ideal of all functions in $k[X\times\mathfrak{N}]$ vanishing on $X\times\{e\}$ (resp. $\mathfrak{N}_G(X\times\mathfrak{N})$). Then $I_1+I_2=\mathrm{Rad}\,(I_1+I_2)$.*

4.5 Theorem. *Let X be a normal G-variety having a fixed point x_0. Then the ideal in $k[X\times\mathfrak{g}]$ (or in $k[X\times\mathfrak{N}]$) generated by all G-invariant functions vanishing at $(x_0,0)$ is not equal to its radical.*

Proof. By (2.3), there is $f\in k[X]$ such that $f\notin k[X]\bar{\mathfrak{m}}_{0,X}$ and f vanishes on $\mathfrak{N}_{G_e}(X)$. Let F be an extension of f to $X\times\mathfrak{N}$ given by (4.4). It then follows from (1.3)(i) that F does not belong to the ideal in $k[X\times\mathfrak{N}]$ generated by G-invariants vanishing at $(x_0,0)$.

Extending somehow F to a function $\hat{F}$ on $X \times \mathfrak{g}$, we obtain the required function in $k[X \times \mathfrak{g}]$. $\square$

Recall that X is said to be *conical*, if $k[X]$ is $\mathbb{Z}_{\geq 0}$-graded and the zero-grade piece is just k. The point corresponding to the maximal homogeneous ideal is called the vertex. It is clear what is called a *conical G-variety.* If X is a conical G-variety, then $X \times \mathfrak{g}$ has a conical structure, too.

4.6 Proposition. *X being a conical G-variety, the following conditions are equivalent:* (i) $\pi_{X,T} : X \to X/\!\!/T$ *is equidimensional;* (ii) $\pi_{X,G_e} : X \to X/\!\!/G_e$ *is equidimensional;* (iii) $\pi_G : X \times \mathfrak{g} \to (X \times \mathfrak{g})/\!\!/G$ *is equidimensional.*

Proof. In the conical case, the fibre of the quotient map containing the vertex, which is a fixed point, is of maximal dimension. Then equivalence of (i) and (ii) follows from (2.2)(ii). Equivalence of (ii) and (iii) follows from (4.3)(ii) by straightforward calculations of dimensions involved. $\square$

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