The Milnor Conjecture.

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December 1996

Contents

[1]Partly supported by NSF

1 Introduction.

The goal of this paper is to prove the following conjecture:

Conjecture 1.1 (Milnor) *Let k be a field of characteristic not equal to 2. Then the norm residue homomorphisms $K_n^M(k)/2 \to H_{et}^n(k, \mathbf{Z}/2)$ are isomorphisms for all $n \geq 0$.*

As an application we deduce a number of results about motivic cohomology, etale cohomology and algebraic K-theory with $\mathbf{Z}/2^m$-coefficients. Conjecture 1.1 was formulated by as a question in [13] and was generalized to the case of $\mathbf{Z}/l$-coefficients for $l > 2$ in [2, p.118]. It was further generalized and reinterpreted in terms of a comparison conjecture for motivic cohomology in Zariski and etale topologies by A. Beilinson and S. Lichtenbaum ([7]). The following particular cases of this conjecture were known:

1. for $n = 0, 1$, l-arbitrary it is a corollary of the classical Theorem Hilbert 90
2. for $n = 2$, $l = 2$ it was proven by A. Merkurjev in [8]
3. for $n = 2$ and l arbitrary it was proven by A. Merkurjev and A. Suslin in [9]
4. for $n = 3$ and $l = 2$ it was proven by A. Merkurjev and A. Suslin in [10] and, in a different way, by M.Rost in [17]
5. for $n = 4$ and $l = 2$ it was announced by M. Rost (unpublished)

The paper is divided into four parts. In the first one we recall the definition of motivic complexes given in [22, p.2] and reduce Conjecture 1.1 for fields of characteristic zero to the existence of algebraic varieties $X_{\underline{a}}$ with certain properties (Theorem 2.25). Except for the extensive use of motivic cohomology and results of [22] the techniques of this part of the paper are just reformulations of the techniques developed by A. Merkurjev and A. Suslin in [10] for the proof of Conjecture 1.1 in dimension 3. Our main innovation here is the use of simplicial schemes $\check{C}(X)$ (see below) in the formulation of Theorem 2.25.

The goal of the second part of the paper is to prove Theorem 3.25. We recall first the construction of the homotopy category of algebraic varieties given in [14] and describe motivic cohomology as the theory represented by an explicitly given Eilenberg-Maclane spectrum. We also define in this section the *motivic Steenrod algebra* and the *algebraic cobordisms*. For a variety X over k let $\check{C}(X)$ be the simplicial scheme with terms $\check{C}(X)_n = X^{n+1}$ and faces and degeneracy morphisms given by partial projections and diagonals respectively. Assume that our base field k is embedded into the field $\mathbf{C}$ of complex numbers. If X is a smooth proper variety over k we denote by $s(X)$ the characteristic number of the complex variety $X(\mathbf{C})$ given by

$$s(X) = -dim(X)!ch(T_X)[X]$$

where $ch(T_X)$ is the Chern character of the tangent bundle of $X(\mathbf{C})$ and $ch(T_X)[X]$ is the value of the highest degree component of $ch(T_X)$ on the fundamental class of X. We say that X is a (ν_n, l)-variety if $dim(X) = l^n - 1$, $s(X) \neq 0(mod\ l^2)$ and for any integral chracteristic number c of X one has $c = 0(mod\ l)$. It turns out that if X is a (ν_n, l)-variety for some n and l then the motivic cohomology groups of $\check{C}(X)$ with $\mathbf{Z}/l$-coefficients considered as a module over the motivic Steenrod algebra has some very special properties.

Both the first and the second part of the paper deal with motivic cohomology of varieties over fields of characteristic zero with coefficients in $\mathbf{Z}/l$ where l is an arbitrary prime.

In the third part we specialize to the case $l = 2$. It is a well known fact that for any sequence $(a_1, \ldots, a_n)$ of invertible elements of a field k of characteristic not equal to 2 the quadric $Q_{\underline{a}}$ given by the equation

$$<< a_1, \ldots, a_{n-1} >>= a_n t^2$$

satisfies the first condition of Theorem 2.25. One verifies easily that $Q_{\underline{a}}$ is a $(\nu_{n-1}, 2)$-variety which allows us to use Theorem 3.25 to replace the second condition of Theorem 2.25 by the vanishing of the motivic cohomology group $H^{2^n-1}(\check{C}(Q_{\underline{a}}), \mathbf{Z}_{(2)}(2^{n-1}))$. We prove that this group is zero using two theorems of Markus Rost ([18],[19]) which finishes the proof of Conjecture 1.1 for fields of characteristic zero.

In the fourth part we deduce Conjecture 1.1 for all fields of characteristic not equal to 2 and prove a number of corollaries. Using similar techniques one can also prove the second part of Milnor Conjecture which asserts that the Milnor ring modulo 2 is isomorphic to the graded Witt ring of quadratic forms. The proof of this result together with more detailed computations of motivic cohomology groups of Pfister quadrics will appear in [16].

All through the paper we use the Nisnevich topology [15] instead of the Zariski one. We would like to mention that since all the complexes of sheaves considered in this paper have transfers and homotopy invariant cohomology sheaves [24, Theorem 5.6] implies that one can repalce Nisnevich hypercohomology by Zariski ones everywhere in the paper without changing the answers.

I am glad to be able to use this opportunity to thank all the people who answered a great number of my questions during my work on Milnor Conjecture. First of all I want to thank Andrei Suslin who taught me the techniques used in [9],[10]. Quite a few of the ideas of the first part of the paper are due to numerous conversations with him. Bob Thomason made a lot of comments on the preprint [23] and in particular explained to me why algebraic K-theory with $\mathbf{Z}/2$-coefficients has no multiplicative structure, which helped to eliminate the assumption $\sqrt{-1} \in k^*$ in Theorem 4.1. Jack Morava and Mike Hopkins answered a lot of my (mostly meaningless) topological questions and I am in debt to them for not being afraid of things like Steenrod algebra anymore. The same applies to Markus Rost and Alexander Vishik and me not being afraid anymore of the theory of quadratic forms. Dmitry Orlov guessed the form of the distinguished triangle in Theorem 4.4 which was a crucial step to the understanding of the structure of motives of Pfister quadrics. I would also like to thank Fabien Morel, Chuck Weibel, Bruno Kahn and Rick Jardine for a number of discussions which helped me to finish this work.

The mathematics of this paper was invented when I was a Junior Fellow of Harvard Society of Fellows and I wish to express my deep gratitude to the society for providing a unique opportunity to work for three years without having to think of things earthy. In its present form this paper was written during my stay in the Max-Planck Institute in Bonn.

2 Motivic cohomology in the etale topology and splitting varieties.

2.1 *Beilinson and Lichtenbaum motivic cohomology groups.*

Let us recall first the definition of motivic complexes $\mathbf{Z}(n)$ given in [22]. For a smooth variety X over k denote by $L(X)$ the sheaf on the category of smooth schemes over k which sends a connected smooth scheme U to the free abelian group generated by closed irreducible subsets of $U \times X$ which are finite and surjective over U. Consider the sheaf $L((\mathbf{A}^1 - \{0\})^n)$ and let F_n be its subsheaf which is the sum of images of homomorphisms $L((\mathbf{A}^1 - \{0\})^{n-1}) \to L((\mathbf{A}^1 - \{0\})^n)$ given by the embeddings $i_m : (\mathbf{A}^1 - \{0\})^{n-1} \to (\mathbf{A}^1 - \{0\})^n$ of the form

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{m-1}, 1, x_m, \ldots, x_{n-1}).$$

Let $\Delta^\bullet$ be the standard cosimplicial object in Sm/k. For any sheaf F of abelian groups we may consider the complex of sheaves $\underline{C}_*(F)$ given by

$$\underline{C}_n(F)(U) = F(U \times \Delta^n).$$

Denote by $\mathbf{Z}(n)$ the complexes of sheaves given by

$$\mathbf{Z}(n) = \begin{cases} \underline{C}_*(L((\mathbf{A}^1 - \{0\})^n)/F_n)[-n] & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

The complex $\mathbf{Z}(n)$ is called the *motivic complex of weight* n. One defines *Beilinson motivic cohomology groups* of a smooth variety X as the hypercohomology groups

$$H^{p,q}_B(X, \mathbf{Z}) = \mathbf{H}^p_{Nis}(X, \mathbf{Z}(q))$$

(here "Nis" refers to the Nisnevich topology, see [15]) and the *Lichtenbaum motivic cohomology groups* as the hypercohomology groups

$$H^{p,q}_L(X, \mathbf{Z}) = \mathbf{H}^p_{et}(X, \mathbf{Z}(q))$$

Similarly, using the complexes $\mathbf{Z}(n) \otimes \mathbf{Z}/m$ one defines Beilinson and Lichtenbaum motivic cohomology with finite coefficients. Since motivic cohomology are defined as hypercohomology with coefficients in certain complexes of sheaves on Sm/k we can extend this definition to smooth *simplicial* schemes in the usual way.

The main technical tools in the study of the basic properties of motivic cohomology are the theory of (pre-)sheaves with transfers developed in [24] and the theory of relative equidimensional cycles and their "cohomology" developed in [5] and [21]. In what follows we will use freely the terminology and notations of [24] and [25]. The statements 2.1-2.4 below summarize the elementary properties of the Beilinson motivic cohomology groups used in this paper and the statements 2.5-2.9 summarize the elementary properties of Lichtenbaum motivic cohomology groups. For any X one has a canonical homomorphism

$$H_B^{*,*}(X, \mathbf{Z}) \to H_L^{*,*}(X, \mathbf{Z})$$

As we show below for any field k and any prime l not equal to the characteristic of k there are canonical isomorphisms

$$H_L^{n,n}(K, \mathbf{Z}/l) = H_{et}^n(k, \mu_l^{\otimes n})$$

$$H_B^{n,n}(k, \mathbf{Z}/l) = K_n^M(k)/l$$

which shows that Conjecture 1.1 can be reformulated as a comparison statement between motivic cohomology in Nisnevich and etale topology. The general form of this comparison conjecture is discussed in 2.10-2.15.

Let $\mathbf{Z}_{Nis}(\mathbf{A}^1 - \{0\})$ be the free sheaf of abelian groups on $(Sm/k)_{Nis}$ generated by the sheaf represented by $\mathbf{A}^1 - \{0\}$ and let $\tilde{\mathbf{Z}}_{Nis}(\mathbf{A}^1 - \{0\})$ be the kernel of the canonical homomorphism $\mathbf{Z}_{Nis}(\mathbf{A}^1 - \{0\}) \to \mathbf{Z}$. For any complex of sheaves K denote by K_{-1} the internal Hom-object $\underline{Hom}(\tilde{\mathbf{Z}}_{Nis}(\mathbf{A}^1 - \{0\}), K)$ in the derived category of sheaves of abelian groups on $(Sm/k)_{Nis}$. Observe, that there is a canonical morphism $\mathbf{Z}(n-1)[-1] \to \mathbf{Z}(n)_{-1}$. The following result is a form of the Cancellation Theorem ([25, Theorem 4.3.1]).

Theorem 2.1 *For any field k of characteristic zero there the canonical morphisms $\mathbf{Z}(n-1)[-1] \to \mathbf{Z}(n)_{-1}$ are quasi-isomorphisms.*

Corollary 2.2 *Let $\mathcal{X}$ be a (simplicial) smooth scheme. The Beilinson motivic cohomology group $H_B^{p,q}(\mathcal{X}, \mathbf{Z})$ is zero in the following cases:*

1. *if $q < 0$*
2. *if $q = 0$ and $p < 0$*

3. if $q = 1$ and $p \leq 0$

Proof: One can easily compute the complexes $\mathbf{Z}(q)$ explicitly for $q \leq 1$ obtaining the following ([22, Proposition 2.2]):

$$\mathbf{Z}(q) = \begin{cases} 0 & \text{for } q < 0 \\ \mathbf{Z} & \text{for } q = 0 \\ \mathbf{G}_m[-1] & \text{for } q = 1 \end{cases}$$

The statement of the corollary is obvious modulo this computation.

Corollary 2.3 *Let X be a smooth scheme over k. Then $H_B^{p,q}(X, \mathbf{Z})$ is zero in the following cases:*

1. *if $p - q > dim(X)$*
2. *if $p > 2q$*

Proof: By definition of $\mathbf{Z}(q)$ the cohomology sheaves $\underline{H}^i(\mathbf{Z}(q))$ of this complex are zero for $i > q$. Thus the first statement is an immediate corollary of the cohomological dimension theorem in Nisnevich topology (see [14] or [15]). To prove the second statement it is sufficient to show that for a smooth X one has $H^n(X, \underline{H}^{p-n}(\mathbf{Z}(q))) = 0$ for $p > 2q$. Using the Gersten resolutions for homotopy invariant sheaves with transfers described in [24, Theorem 4.36] we have only to show that one has $\underline{H}^{p-n}(\mathbf{Z}(q))_{-n} = 0$ for $p > 2q$. By [24, Proposition 4.33] the functor $K \mapsto K_{-1}$ on the category of complexes of sheaves with transfers with homotopy invariant cohomology sheaves commutes with taking cohomology sheaves, thus by Theorem 2.1 we have

$$\underline{H}^{p-n}(\mathbf{Z}(q))_{-n} = \underline{H}^{p-2n}(\mathbf{Z}(q-n))$$

If $q < n$ this sheaf is zero since $\mathbf{Z}(q-n) = 0$ and if $q \geq n$ and $p > 2q$ we have $p - 2n > q - n$ and the sheaf $\underline{H}^{p-2n}(\mathbf{Z}(q-n))$ is zero since $\mathbf{Z}(q-n)$ does not have cohomology sheaves in dimension greater than $q - n$.

Denote by $\underline{K}_n^M$ the sheaf on Sm/k such that for a connected smooth scheme X over k the group $\underline{K}_n^M(X)$ is the subgroup in the n-th Milnor K-group of the function field of X which consists of elements with all residues in points of codimension 1 being zero.

Corollary 2.4 *Let X be a smooth scheme over k. Then for any p, q there is a canonical homomorphism $H_B^{p,q}(X, \mathbf{Z}) \to H^{p-q}(X, \underline{K}_q^M)$ which is an isomorphism in the following cases*

1. *if $p \geq q + dim(X)$*
2. *if $p \geq 2q - 2$*

Proof: Consider the cohomology sheaf $\underline{H}^n(\mathbf{Z}(n))$. Since $\underline{H}^i(\mathbf{Z}(n)) = 0$ for $i > n$ there are canonical homomorphisms $H_B^{p,q}(X, \mathbf{Z}) \to H^{p-q}(X, \underline{H}^q(\mathbf{Z}(q)))$. Proceeding as in the proof of 2.3 one concludes that these homomorphisms are isomorphisms for values of p and q considred in 2.4. It remains to show that $\underline{H}^n(\mathbf{Z}(n)) = \underline{K}_n^M$. Since $\underline{H}^n(\mathbf{Z}(n))$ is a homotopy invariant sheaf with transfers for any smooth connected X the restriction homomorphism

$$H^0(X, \underline{H}^n(\mathbf{Z}(n))) \to H^0(Spec(k(X)), \underline{H}^n(\mathbf{Z}(n))$$

is a monomorphism ([24, Corollary 4.17]). It was shown in [22, Prop. 4.1] that for any field k one has canonical isomorphisms $H_B^{n,n}(k, \mathbf{Z}) = K_n^M(k)$. In particular for any X the group $H^0(X, \underline{H}^n(\mathbf{Z}(n)))$ is a subgroup in $K_n^M(k(X))$ and one verifies easily that it coincides with the subgroup $\underline{K}_n^M(X)$ of elements with all residues being zero.

Theorem 2.5 *The canonical homomorphisms*

$$H_B^{p,q}(-, \mathbf{Q}) \to H_L^{p,q}(-, \mathbf{Q})$$

are isomorphisms.

Proof: By the Leray spectral sequence associated to the obvious morphism of sites $(Sm/k)_{et} \to (Sm/k)_{Nis}$ it is sufficient to show that for any $i, j \in \mathbf{Z}$ and any smooth scheme X one has

$$H_{et}^i(X, \underline{H}^j(\mathbf{Q}(q))_{et}) = H_{Nis}^i(X, \underline{H}^j(\mathbf{Q}(q))).$$

Since $\underline{H}^j(\mathbf{Z}(q))$ are homotopy invariant sheaves with transfers it is a particular case of [24, Prop.5.22 and Prop.5.26].

For n prime to the characteristic of the base field Lichtenbaum motivic cohomology groups with $\mathbf{Z}/n$ coefficients are closely related to the "usual" etale cohomology.

Theorem 2.6 *Let k be a field and n be an integer prime to characteristic of k. Denote by μ_n the etale sheaf of n-th roots of unit on Sm/k and let $\mu_n^{\otimes q}$ be the n-th tensor power of μ_n in the category of $\mathbf{Z}/n$-modules. Then there is a canonical isomorphism $H_L^{p,q}(-,\mathbf{Z}/n) = H_{et}^p(-,\mu_n^{\otimes q})$.*

Proof: We have to show that the complex $\mathbf{Z}/n(q)$ is canonically quasi-isomorphic in the etale topology to the sheaf $\mu_n^{\otimes q}$. In the category of complexes of sheaves with transfers of $\mathbf{Z}/n$-modules in the etale topology with homotopy invariant cohomology sheaves $DM_-^{eff}(k,\mathbf{Z}/n,et)$ we have

$$\mathbf{Z}/n(q) = (\mathbf{Z}/n(1))^{\otimes q}$$

Since $\mathbf{Z}(1) = \mathbf{G}_m[-1]$ in the etale topology we have $\mathbf{Z}/n(1) = \mu_n$ which proves our claim.

For a smooth variety X over k denote by $\check{C}(X)$ the simplicial smooth scheme with terms $\check{C}_n(X) = X^{n+1}$ and faces and degeneracy morphisms given by partial projections and diagonals respectively.

Proposition 2.7 *Let X be a nonempty smooth scheme over k. Then the homomorphisms*

$$H_L^{p,q}(Spec(k),\mathbf{Z}) \to H_L^{p,q}(\check{C}(X),\mathbf{Z})$$

induced by the canonical morphism $\check{C}(X) \to Spec(k)$ are isomorphisms for all $p,q \in \mathbf{Z}$.

Proof: For a smooth scheme X let $\mathbf{Z}_{et}(X)$ be the sheaf of abelian groups on $(Sm/k)_{et}$ freely generated by the sheaf of sets represented by X. Similarly, for a simplicial scheme $\mathcal{X}$ let $\mathbf{Z}_{et}(\mathcal{X})$ be the complex of sheaves with terms $\mathbf{Z}_{et}(\mathcal{X}_n)$ and the differential being given by the alternated sums of morphisms given by the face maps. One can easily see that for any complex of etale sheaves K there are canonical isomorphisms

$$Hom_{\mathcal{D}}(\mathbf{Z}_{et}(\mathcal{X}),K[n]) = \mathbf{H}^n_{et}(\mathcal{X},K)$$

where the group of morphisms on the left hand side is considered in the derived category of etale sheaves on Sm/k. One can easily observe now that for any nonempty X the morphism of complexes

$$\mathbf{Z}_{et}(\check{C}(X)) \to \mathbf{Z}_{et}(Spec(k))$$

is a quasi-isomorphism which proves our claim.

Let $\pi : (Sm/k)_{et} \to (Sm/k)_{Nis}$ be the obvious morphism of sites. Consider the complex $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ of Nisnevich sheaves with transfers on Sm/k. We have

$$H_L^{p,q}(X, \mathbf{Z}) = \mathbf{H}^p_{Nis}(X, \mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))).$$

Lemma 2.8 *For a field k of characteristic zero the complex $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ has homotopy invariant cohomology sheaves.*

Proof: Since the sheaf associated with a homotopy invariant presheaf with transfers is homotopy invariant ([24, Propositions 4.25, 5.5]) it is sufficient to show that the functors $U \mapsto H_L^{p,q}(U, \mathbf{Z})$ are homotopy invariant i.e. that for any smooth U the homomorphism

$$H_L^{p,q}(U \times \mathbf{A}^1, \mathbf{Z}) \to H^{p,q}(U, \mathbf{Z})$$

given by the restriction to $U \times \{0\}$ is an isomorphism. Consider the long exact sequence relating Lichtenbaum motivic cohomology with $\mathbf{Z}$ coefficients to Lichtenbaum motivic cohomology with $\mathbf{Q}$-coefficients and $\mathbf{Q}/\mathbf{Z}$-coefficients. The cohomology with $\mathbf{Q}$-coefficients are homotopy invariant by Theorem 2.5. The cohomology with $\mathbf{Q}/\mathbf{Z}$-coefficients are isomorphic to the etale cohomology by Theorem 2.6 and therefore homotopy invariant as well. Our claim follows now from the five lemma.

Remark: For fields k of positive characteristic the statement Lemma 2.8 is false. Instead one can show that the complex $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q))) \otimes \mathbf{Z}[1/char(k)]$ has homotopy invariant cohomology sheaves while the complex
$\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q))) \otimes (\mathbf{Z}[1/char(k)]/\mathbf{Z})$ is contractible, i.e. represents zero in the motivic category.

Lemma 2.9 *Let k be a field of characteristic zero. Then for any $q \geq 0$ there is a canonical quasi-isomorphism*

$$\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))_{-1} = \mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q-1)))[-1].$$

Proof: We have to show that the canonical homomorphisms

$$H_L^{p-1,n-1}(X, \mathbf{Z}) \oplus H^{p,n}(X, \mathbf{Z}) \to H_L^{p,n}(X \times (\mathbf{A}^1 - \{0\}), \mathbf{Z})$$

are isomorphisms. Consider again the cases of rational and $\mathbf{Q}/\mathbf{Z}$-coefficients separately. For rational coefficients our result follows from Theorem 2.5 and Theorem 2.1. For $\mathbf{Q}/\mathbf{Z}$-coefficients it follows from Theorem 2.6 and the fact that the corresponding homomorphisms for the etale cohomology with $\mu_m^{\otimes n}$-coefficients are isomorphisms.

The following fundamental conjecture is due to S. Lichtenbaum ([7, p.130]):

Conjecture 2.10 *Let k be a field and X be a smooth variety over k. Then the canonical homomorphism $H_B^{p,q}(X,\mathbf{Z}) \to H_L^{p,q}(X,\mathbf{Z})$ is an isomorphism for $p \leq q+1$.*

Since $\mathbf{Z}(0) = \mathbf{Z}$ and $\mathbf{Z}(1) = \mathbf{G}_m[-1]$ one can see that this conjecture holds for $q = 0$ for obvious reasons and for $q = 1$ because it becomes equivalent to the classical Theorem Hilbert 90.

Let us say that $H90(n,l)$ holds in characteristic zero if for any field k of characteristic zero and any $q \leq n$ one has $H_L^{q+1,q}(Spec(k), \mathbf{Z}_{(l)}) = 0$. Clearly, H90(n,l) is a particular case of Conjecture 2.10 for $q = n$. In particular it holds in characteristic zero for $n = 0, 1$ and all l. Our proof of $H90(n,2)$ in Section 4 goes by induction on n and in order to be able to make the inductive step we will need to know some corollaries of $H90(n-1,l)$.

Let $L(q)$ be the canonical truncation of complex $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ on level $q+1$ i.e. $L(q)$ is the subcomplex of sheaves in $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ whose cohomology sheaves $\underline{H}^i(L(q))$ are the same as for $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ for $i \leq q+1$ and zero for $i > q+1$. Since $\underline{H}^i(\mathbf{Z}(q)) = 0$ for $i > q$ the canonical morphism $\mathbf{Z}(q) \to \mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ factors through $L(q)$. Let $K(q)$ be the complex of sheaves with transfers on $(Sm/k)_{Nis}$ defined by the distinguished triangle $\mathbf{Z}(q) \to L(q) \to K(q) \to \mathbf{Z}(q)[1]$.

One can easily see that Conjecture 2.10 is equivalent to vanishing of the complex $K(q)$. By Lemma 2.8 the cohomology sheaves of $K(q)$ are homotopy invariant sheaves with transfers which are in particular separated ([24, Corollary 4.17]). Thus H90(q,l) is equivalent to the vanishing of the sheaf $\underline{H}^{q+1}(K(q)) \otimes \mathbf{Z}_{(l)}$. The following result is essentially the main theorem of [22].

Theorem 2.11 *Assume that $H90(n,l)$ holds in characteristic zero. Then the complex $K(n) \otimes \mathbf{Z}_{(l)}$ is acyclic.*

Proof: By the assumption that H90(n,l) holds we know that $\underline{H}^{n+1}(K(n)\otimes \mathbf{Z}_{(l)})=0$. In order to show that $K(n)\otimes\mathbf{Z}_{(l)}$ is quasi-isomorphic to zero it remains to verify that for any smooth scheme X over k and any $p\leq n$ the homomorphism $H^{p,n}_B(X,\mathbf{Z}_{(l)})\rightarrow H^{p,n}_L(X,\mathbf{Z}_{(l)})$ is an isomorphism. Consider the long exact sequences of motivic cohomology groups associated with the distinguished triangles $\mathbf{Z}_{(l)}(n)\rightarrow\mathbf{Q}(n)\rightarrow\mathbf{Q}/\mathbf{Z}_{(l)}(n)\rightarrow\mathbf{Z}_l(n)[1]$ considered in both Nisnevich and etale topologies. It follows from Theorem 2.5 that in order to prove the theorem it is sufficient to verify that the homomorphisms

$$H^{p,n}_B(X,\mathbf{Q}/\mathbf{Z}_{(l)})\rightarrow H^{p,n}_L(X,\mathbf{Q}/\mathbf{Z}_{(l)})$$

are isomorphisms for $p\leq n$. Since we assume H90(n,l), Theorem 2.6 implies that for any field k of characteristic zero the Bockstein homomorphisms

$$H^n_{et}(k,\mu_{\mathbf{l^m}}^{\otimes\mathbf{n}})\rightarrow H^{n+1}_{et}(k,\mu_{\mathbf{l}}^{\otimes\mathbf{n}})$$

are zero and therefore [22, Proposition 7.1] and [22, Theorem 5.9] imply that homomorphisms in question are isomorphisms.

Corollary 2.12 *Assume that H90(n,l) holds in characteristic zero. Then for any field k of characteristic zero and and complex K of Nisnevich sheaves with transfers such that*

1. *cohomology of K are concentrated in homologically nonnegative degrees*
2. *K is exact in the etale topology*

one has $Hom_{DM}(K,\mathbf{Z}_{(l)}(q)[p])=0$ for $q\leq n$, $p\leq q+2$.

Proof: Any complex K whose cohomology sheaves are concentrated in non-negative homological degrees is quasi-isomorphic to a complex K' such that $K'_n=0$ for $n<0$ and K_i is a sum of sheaves of the form $L(X)$ for $i\geq 0$. Theorem 2.11 implies that for any K like that and any p,q satisfying the conditions of the corollary the canonical map

$$Hom_{DM}(K,\mathbf{Z}_{(l)}(q)[p])\rightarrow Hom_{DM}(K,\mathbf{R}\pi_*(\pi^*(\mathbf{Z}_{(l)}(q)[p])))$$

is a monomorphism. On the other hand the right hand side group is isomorphic to the group of morphisms $Hom(K_{et},\pi^*(\mathbf{Z}_{(l)}(q)[p]))$ in the derived category of etale sheaves with transfers which is zero since K_{et} is quasi-isomorphic to zero.

Corollary 2.13 *Assume that $H90(n,l)$ holds in characteristic zero. Then for any simplicial smooth scheme $\mathcal{X}$ one has*

1. *the homomorphisms*

$$H_B^{p,q}(\mathcal{X}, \mathbf{Z}_{(l)}) \to H_L^{p,q}(\mathcal{X}, \mathbf{Z}_{(l)})$$

are isomorphisms for $p-1 \leq q \leq n$ and monomorphisms for $p = q+2$ and $q \leq n$

2. *the homomorphisms*

$$H_B^{p,q}(\mathcal{X}, \mathbf{Z}/l) \to H_L^{p,q}(\mathcal{X}, \mathbf{Z}/l)$$

are isomorphisms for $p \leq q \leq n$ and monomorphisms for $p = q+1$ and $q \leq n$

Corollary 2.14 *Assume that $H90(n,l)$ holds in characteristic zero. Then for any field k of characteristic zero any $q \leq n$ and any cyclic extension E/F of degree l the sequence*

$$K_q^M(E) \stackrel{1-\sigma}{\to} K_q^M(E) \stackrel{N_{E/k}}{\to} K_q^M(F)$$

(where σ is a generator of $Gal(E/k)$) is exact.

Proof: For any finite separable field extension $p : Spec(k') \to Spec(k)$ the presheaf with transfers $L(Spec(k'))$ on Sm/k is canonically isomorphic to the direct image $p_*(\mathbf{Z})$. In particular there is a complex of presheaves with transfers of the form

$$0 \to L(k) \to L(E) \stackrel{1-\sigma}{\to} L(E) \to L(k) \to 0$$

Denote this complex with the last $L(k)$ placed in degree zero by $\underline{K}$. It is clearly exact in the etale topology and therefore Lemma 2.12 implies that

$$Hom(\underline{K}, \mathbf{Z}_{(l)}(n)[n+2]) = 0$$

in the derived category of Nisnevich sheaves with transfers. The standard spectral sequence which converges to this group from morphisms

$$Hom(L(-), \mathbf{Z}_{(l)}(n)[j]) = H_B^{j,n}(-, \mathbf{Z}_{(l)})$$

together with the comparison result 2.4 relating motivic cohomology to Milnor K-theory implies the statement of the corollary.

Proposition 2.15 *Assume that H90(n,l) holds in characteristic zero. Let X be a smooth scheme over k and U be a dense open subscheme in X. Then the homomorphisms $\mathbf{H}^*_{Nis}(X, K(n+1)\otimes \mathbf{Z}_{(l)}) \to \mathbf{H}^*_{Nis}(U, K(n+1)\otimes \mathbf{Z}_{(l)})$ are isomorphisms.*

Proof: By obvious induction it is sufficient to show that the statement of the lemma holds for $U = X - Z$ where Z is a smooth closed subscheme in X. Consider $K(n+1)\otimes \mathbf{Z}_{(l)}$ as a complex of Nisnevich sheaves with transfers with homotopy invariant cohomology sheaves, i.e. as an object of $DM^{eff}_-(k)$. We have a distinguished triangle ([25, Proposition 3.5.4])

$$M(Z)(c)[2c-1] \to M(U) \to M(X) \to M(Z)(c)[2c]$$

where c is the codimension of Z and therefore it is sufficient to show that $Hom_{DM^{eff}}(M(Z)(c)[2c], K(n+1)[*]) = 0$ for any $c > 0$. For any $c > 0$ we have

$$Hom(M(Z)(c)[2c], K(n+1)[*]) = Hom(M(Z)(c-1)[2c-1], K(n+1)_{-1}[*])$$

By Lemma 2.9 we conclude that $K(n+1)_{-1} = K(n)[-1]$ and therefore the statement of the proposition follows from Theorem 2.11.

2.2 *Some computations with Galois cohomology*

In this section we are dealing only with classical objects namely Milnor K-theory and etale cohomology. More general motivic cohomology do not appear here. The only result of this section which we will directly use below is Theorem 2.23. One may observe that in the case of $\mathbf{Z}/2$-coefficients it can be proven in a much easier way, but we decided to include the case of general l for possible future use. Everywhere below k is a field and l is a fixed prime not equal to the characteristic of k.

Definition 2.16 *Let k be a field, l be a prime not equal to the characteristic of k and $n \geq 0$ an integer. We say that $BK(n,l)$ holds over k if for any field K over k and any $m \leq n$ one has:*

1. *the norm residue homomorphism $K^M_m(K)/l \to H^m_{et}(K, \mu_l^{\otimes m})$ is an isomorphism*

2. *for any cyclic extension E/K of degree l the sequence*

$$K_m^M(E) \stackrel{1-\sigma}{\to} K_m^M(E) \stackrel{N_{E/K}}{\to} K_m^M(K)$$

where σ is a generator of $Gal(E/K)$ is exact.

Remark: Note that by 2.13 and 2.14 the validity of $H90(n,l)$ in characteristic zero implies the validity of $BK(n,l)$ in characteristic zero.

Proposition 2.17 *Let k be a field which has no extensions of degree prime to l. Assume that $BK(n,l)$ holds over k. Then for any cyclic extension E/k of k of degree l there is an exact sequence of the form*

$$H_{et}^n(E,\mathbf{Z}/l) \stackrel{N_{E/k}}{\to} H_{et}^n(k,\mathbf{Z}/l) \stackrel{-\wedge[a]}{\to} H_{et}^{n+1}(k,\mathbf{Z}/l) \to H_{et}^{n+1}(E,\mathbf{Z}/l)$$

where $[a] \in H_{et}^1(k,\mathbf{Z}/l)$ is the class which corresponds to E/k.

Proof: In order to prove the proposition we will need to do some preliminary computations. Fix an algebraic closure $\bar{k}$ of k. Since k has no extensions of degree prime to l there exists a primitive root of unit $\xi \in k$ of degree l. Let $E \subset \bar{k}$ be a cyclic extension of k of degree l. We have $E = k(b)$ where $b^l = a$ for an element a in k^*. Denote by σ_ξ the generator of the Galois group $G_b = Gal(E/k)$ which acts on b by multiplication by ξ and by $[a]_\xi$ the class in $H_{et}^1(k,\mathbf{Z}/l)$ which corresponds to the homomorphism $Gal(\bar{k}/k) \to G \to \mathbf{Z}/l$ which takes σ_ξ to the canonical generator of $\mathbf{Z}/l$ (one can easily see that this class is determined by a and ξ and does notdepend on b).

Let $p : Spec(E) \to Spec(k)$ be the projection. Consider the etale sheaf $F = p_*(\mathbf{Z}/l)$. The group G acts on F in the natural way. Denote by F_i the kernel of the homomorphism $(1-\sigma)^i : F \to F$. One can verify easily that $F_i = Im(1-\sigma)^{l-i}$ and that as a $\mathbf{Z}/l[Gal(\bar{k}/k)]$-module F_i has dimension i. In particular we have $F = F_l$. Note that the extension $0 \to \mathbf{Z}/l \to F_2 \to \mathbf{Z}/l \to 0$ represents the element $[a]_\xi$ in $H_{et}^1(k,\mathbf{Z}/l) = Ext^1_{\mathbf{Z}/l[Gal(\bar{k}/k)]}(\mathbf{Z}/l,\mathbf{Z}/l)$

Let α_i be the element in $H^2(k,\mathbf{Z}/l) = Ext^2_{\mathbf{Z}/l[Gal(\bar{k}/k)]}(\mathbf{Z}/l,\mathbf{Z}/l)$ defined by the exact sequence

$$0 \to \mathbf{Z}/l \to F_i \stackrel{u_i}{\to} F_i \to \mathbf{Z}/l \to 0$$

where $u_i = 1-\sigma$ and $Im(u_i) = F_{i-1}$.

Lemma 2.18 *One has $\alpha_l = c([a]_\xi \wedge [\xi]_\xi)$ where c is an invertible element of $\mathbf{Z}/l$ and $\alpha_i = 0$ for $i < l$.*

Proof: The fact that $\alpha_i = 0$ for $i < l$ follows from the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 \to & \mathbf{Z}/l & \to & F_{i+1} & \to & F_{i+1} & \to & \mathbf{Z}/l & \to 0 \\ & 0\downarrow & & \downarrow & & \downarrow & & \downarrow Id & \\ 0 \to & \mathbf{Z}/l & \to & F_i & \to & F_i & \to & \mathbf{Z}/l & \to 0 \end{array}$$

To compute α_l note first that since the action of $Gal(\bar{k}/k)$ on $F = F_l$ factors through $G = Gal(E/k)$ it comes from a well defined element in $H^2(G, \mathbf{Z}/l)$. This element is not zero for trivial reasons. On the other hand the group

$$H^2(G, \mathbf{Z}/l) = H^2(\mathbf{Z}/l, \mathbf{Z}/l) = \mathbf{Z}/l$$

is generated by the element $\beta(\gamma)$ where γ is the canonical generator of $H^1(G, \mathbf{Z}/l)$ and β is the Bockstein homomorphism. Thus we conclude that up to multiplication by an invertible element of $\mathbf{Z}/l$ our class α_l equals $\beta([a]_\xi)$. It remains to show that $\beta([a]_\xi) = c[a]_\xi \wedge [\xi]_\xi$ which follows by simple explicite computations from the fact that $[a]_\xi$ has a lifting to an element of $H^1_{et}(Spec(k), \mu_{l^2})$.

Lemma 2.19 *Assume that $BK(n, l)$ holds over k. Then for all $m \leq n$ and all $i = 1, \ldots, l-1$ one has:*

1. *The sequence $H^m_{et}(k, \mathbf{Z}/l) \oplus H^m_{et}(k, F_{i+1}) \to H^m_{et}(k, F_{i+1}) \to H^m_{et}(k, \mathbf{Z}/l)$ where the first homomorphism is given on the second summand by $1-\sigma$ is exact.*

2. *The homomorphisms $\nu_{m,i} : H^m_{et}(k, \mathbf{Z}/l) \oplus H^m_{et}(k, F_{i+1}) \to H^m_{et}(k, F_i)$ given by the canonical morphisms $\mathbf{Z}/l \to F_i$, $F_{i+1} \to F_i$ are surjective.*

Proof: We proceed by induction on i. Consider first the case $i = l-1$. The first statement follows immediately the assumption that $BK(n, l)$ holds.

Let us prove the second one. The image of $H^m_{et}(k, F_{l-1})$ in $H^m_{et}(k, F_l) = H^m_{et}(E, \mathbf{Z}/l)$ coincides with the kernel of the norm homomorphism

$$H^m_{et}(E, \mathbf{Z}/l) \to H^m_{et}(k, \mathbf{Z}/l).$$

The first statement implies then that $H^m_{et}(k, \mathbf{Z}/l) \oplus H^m_{et}(k, F_l)$ maps surjectively to this image. It is sufficient therefore to show that an element $\gamma \in H^m_{et}(k, F_{l-1})$ which goes to zero in $H^m_{et}(k, F_l)$ belongs to the image of $\nu_{m,l-1}$. Any such element is a composition of a cohomology class in $H^{m-1}_{et}(k, \mathbf{Z}/l)$ with the canonical extension

$$0 \to F_{l-1} \to F_l \to \mathbf{Z}/l \to 0$$

Thus we may assume that $m = 1$ and γ is the element which corresponds to this extension. Let δ be the image of $c[\xi]_\xi$ (where c is as in Lemma 2.18) under the homomorphism $H^1_{et}(k, \mathbf{Z}/l) \to H^1_{et}(k, F_{l-1})$. The composition

$$H^1_{et}(k, \mathbf{Z}/l) \to H^1_{et}(k, F_{l-1}) \to H^2_{et}(k, \mathbf{Z}/l)$$

where the later homomorphism corresponds to the extension

$$0 \to \mathbf{Z}/l \to F_l \to F_{l-1} \to 0$$

equals to multiplication by $[a]_\xi$. We conclude now by Lemma 2.18 that the image of $\gamma - \delta$ in $H^2(k, \mathbf{Z}/l)$ is zero. Then it lifts to $H^1_{et}(k, F_l)$ which proves our Lemma in the case $i = l - 1$.

Suppose that the lemma is proven for all $i > j$. Let us show that it holds for $i = j$. The first statement follows immediately from the inductive assumption and the commutativity of the diagram

$$\begin{array}{ccc} F_{j+2} & \stackrel{1-\sigma}{\to} & F_{j+2} \\ \downarrow & & \downarrow \\ F_{j+1} & \stackrel{1-\sigma}{\to} & F_{j+1} \end{array}$$

The proof of the second one is now similar to the case $i = l - 1$ with a simplification due to the fact that $\alpha_i = 0$ for $i < l$ (Lemma 2.18).

The statement of the proposition follows immediately from Lemma 2.19.

Remark: For $l = 2$ Proposition 2.17 is a trivial corollary of the exactness of the sequence $0 \to \mathbf{Z}/2 \to F_2 \to \mathbf{Z}/2 \to 0$. In particular it holds without the $BK(n, l)$ assumption and not only in the context of Galois cohomology but for cohomology of any (pro-)finitegroup. For $l > 2$ this is not true anymore which one can see considering cohomology of $\mathbf{Z}/l$.

Lemma 2.20 *Let k be a field which has no extensions of degree prime to l and such that $BK(n,l)$ holds over k. Let further E/k be a cyclic extension of degree l such that the norm homomorphism $K_n^M(E) \to K_n^M(k)$ is surjective. Then the sequence*

$$K_{n+1}^M(E) \stackrel{1-\sigma}{\to} K_{n+1}^M(E) \stackrel{N_{E/k}}{\to} K_{n+1}^M(k)$$

where σ is a generator for $Gal(E/k)$ is exact.

Proof: It is essentially a version of the proof given in [20] for $n=2$ and in [10] for $p=3$. Let us define a homomorphism

$$\phi : K_{n+1}^M(k) \to K_{n+1}^M(E)/(Im(1-\sigma))$$

as follows. Let a be an element in $K_{n+1}^M(k)$ of the form $(a_0,\ldots,a_n)$ and let b be an element in $K_n^M(E)$ such that

$$N_{E/k}(b) = (a_0,\ldots,a_{n-1}).$$

We set $\phi(a) = b \wedge a_n$. Since $BK(n.l)$ holds over k the element $\phi(a)$ in $K_{n+1}^M(E)/(Im(1-\sigma))$ does not depend on the choice of b and one can easily see that ϕ is a homomorphism from $(k^*)^{\otimes n+1}$ to $K_{n+1}^M(E)/(Im(1-\sigma))$. To show that it is a homomorphism from $K_n^M(k)$ it is sufficient to verify that ϕ takes an element of the form $(a_0,\ldots,a_n)$ such that say $a_0+a_n=1$ to zero. Let b be a preimage of $(a_0,\ldots,a_{n-1})$ in $K_n^M(k)$. We have to show that $(b,a_n) \in (1-\sigma)K_{n+1}^M(E)$. Assume first that a_0 is not in $(k^*)^l$ and let c be an element in $\bar{k}^*$ such that $c^l = a_0$. Let further $F = k(c)$. Then by $BK(n.l)$ one has

$$b \wedge a_n = b \wedge (1-a_0) = N_{EF/E}(b_{EL} \wedge (1-c)) =$$

$$= N_{EF/E}((b-(c,a_1,\ldots,a_n-1)) \wedge (1-c)) \in (1-\sigma)K_{n+1}^M(E)$$

since $N_{EF/F}(b-(c,a_1,\ldots,a_n-1)) = 0$. The proof for the case when $a_0 \in (k^*)^l$ is similar.

Clearly ϕ is a section for the obvious morphism $K_{n+1}^M(E)/(Im(1-\sigma)) \to K_{n+1}^M(k)$. It remains to show that it is surjective. It follows immediately from the fact that under our assumption on k the group $K_{n+1}^M(E)$ is generated by symbols of the form $(b,a_1,\ldots,a_n)$ where $b \in E^*$ and $a_1,\ldots,a_n \in k^*$ (see [1]).

Lemma 2.21 *Let k be a field which has no extensions of degree prime to l and such that $BK(n,l)$ holds over k. Then the following two conditions are equivalent*

1. $K^M_{n+1}(k) = lK^M_{n+1}(k)$
2. *for any cyclic extension E/k the norm homomorphism $K^M_n(E) \to K^M_n(k)$ is surjective.*

Proof: The $2 \Rightarrow 1$ part is trivial. Since $BK(n,l)$ holds we conclude that there is a commutative square with surjective horizontal arrows of the form

$$\begin{array}{ccc} K^M_n(E) & \to & H^n(E, \mathbf{Z}/l) \\ \downarrow & & \downarrow \\ K^M_n(k) & \to & H^n(k, \mathbf{Z}/l) \end{array}$$

and thus the cokernel of the left vertical arrow is the same as the cokernel of the right one. By Lemma 2.17 it gives us an exact sequence

$$K^M_n(E) \to K^M_n(k) \to H^{n+1}(k, \mathbf{Z}/l)$$

and since the last arrow clearly factors through $K^M_{n+1}(k)/l$ it is zero. Lemma is proven.

Lemma 2.22 *Let k be a field which has no extensions of degree prime to l and such that $BK(n,l)$ holds over k. Assume further that $K^M_{n+1}(k) = lK^M_{n+1}(k)$. Then for any finite extension E/k one has $K^M_{n+1}(E) = lK^M_{n+1}(E)$.*

Proof: It is a variant of the proof given in [20] for $n = 1$. It is clearly sufficient to prove the lemma in the case of a cyclic extension E/k of degree l. By Lemma 2.17 and Lemma 2.20 we have an exact sequence

$$K^M_{n+1}(E) \stackrel{1-\sigma}{\to} K^M_{n+1}(E) \stackrel{N_{E/k}}{\to} K^M_{n+1}(k).$$

Let α be an element in $K^M_{n+1}(E)$ and let $\beta \in K^M_{n+1}(k)$ be an element such that $N_{E/k}(\alpha) = l\beta$. Then $N_{E/k}(\alpha - \beta_E) = 0$ and we conclude that the endomorphism

$$1 - \sigma : K^M_{n+1}(E)/l \to K^M_{n+1}(E)/l$$

is surjective. Since $(1-\sigma)^l = 0$ it implies that $K^M_{n+1}(E)/l = 0$.

Theorem 2.23 *Let k be a field which has no extensions of degree prime to l such that $BK(n,l)$ holds over k and $K_{n+1}^M(k) = lK_{n+1}^M(k)$. Then*

$$H_{et}^{n+1}(k, \mathbf{Z}/l) = 0.$$

Proof: Let α be an element of $H_{et}^{n+1}(k, \mathbf{Z}/l)$. We have to show that $\alpha = 0$. By Lemma 2.22 and obvious induction we may assume that α vanishes on a cyclic extension of k. Then by Lemma 2.17 it is of the form $\alpha_0 \wedge a$ where $a \in H_{et}^1(k, \mathbf{Z}/l)$ is the element which represents our cyclic extension. Thus since $BK(n,l)$ holds it belongs to the image of the homomorphism $K_{n+1}^M(k)/l \to H_{et}^{n+1}(k, \mathbf{Z}/l)$ and therefore is zero.

2.3 *Splitting varieties.*

Definition 2.24 *Let $\underline{a} = (a_1, \dots, a_n)$ be a sequence of invertible elements of k and l be a prime. A smooth variety X over k is called a (l-)splitting variety for $\underline{a}$ if the symbol $\underline{a}$ is divisible by l in $K_n^M(k(X))$ where $k(X)$ is the function field of X.*

Theorem 2.25 *Assume that $H90(n-1,l)$ holds in characteristic zero and suppose that for any field k of characteristic zero and any sequence $\underline{a} = (a_1, \dots, a_n)$ there exists a splitting variety $X_{\underline{a}}$ with the following properties:*

1. *$X_{\underline{a}} \times_{Spec(k)} Spec(k(X_{\underline{a}}))$ is rational over $k(X_{\underline{a}})$*
2. *$H_B^{n+1,n}(\check{C}(X_{\underline{a}}), \mathbf{Z}_{(l)}) = 0$*

Then $H90(n,l)$ holds in characteristic zero (see the definition of $\check{C}(X)$ right before Proposition 2.7).

Proof: Let k be a field of characteristic zero which has no extensions of degree prime to l and such that $K_n^M(k)$ is l-divisible. Then by Theorem 2.23 and Theorem 2.6 we have $H_L^{n+1,n}(Spec(k), \mathbf{Z}/l) = 0$ and therefore the group $H_L^{n+1,n}(Spec(k), \mathbf{Z}_{(l)})$ must be torsion free. On the other hand by Theorem 2.5 and Corollary 2.3(1) we have $H_L^{n+1,n}(Spec(k), \mathbf{Q}) = 0$ which implies that for any k satisfying these conditions we have $H_L^{n+1,n}(Spec(k), \mathbf{Z}_{(l)}) = 0$. For a finite extension E/k of degree prime to l the homomorphism $H_L^{n+1,n}(Spec(k), \mathbf{Z}_{(l)}) \to H_L^{n+1,n}(Spec(E), \mathbf{Z}_{(l)})$ is a monomorphism by the transfer argument. Thus in order to prove $H90(n,l)$ it is sufficient to show that for any element $\underline{a} \in K_n^M(k)$ there exists an extension K/k such that $\underline{a}$ is divisible by l in $K_n^M(K)$ and the homomorphism $H_L^{n+1,n}(Spec(k), \mathbf{Z}_{(l)}) \rightarrow H_L^{n+1,n}(Spec(K), \mathbf{Z}_{(l)})$ is a monomorphism. Since any element in $K_n^M(k)$ is a sum of symbols of the form $\underline{a} = (a_1, \dots, a_n)$ it is sufficient to show that for varieties $X_{\underline{a}}$ satisfying the conditions of the theorem the homomorphisms

$$H_L^{n+1,n}(Spec(k), \mathbf{Z}_{(l)}) \rightarrow H_L^{n+1,n}(Spec(k(X_{\underline{a}})), \mathbf{Z}_{(l)})$$

are monomorphisms. Let u be an element in the kernel of this homomorphism. By Proposition 2.7 and our assumption on $X_{\underline{a}}$ it is sufficient to show that the image u' of u in $H_L^{n+1,n}(\check{C}(X_{\underline{a}}), \mathbf{Z}_{(l)})$ belongs to the image of the group $H_B^{n+1,n}(\check{C}(X_{\underline{a}}), \mathbf{Z}_{(l)})$.

Let $K(n)$ be the complex of sheaves on $(Sm/k)_{Nis}$ defined at the end of Section 2.1. We have to verify that under our assumptions the image of u' in the hypercohomology group $\mathbf{H}_{Nis}^{n+1}(\check{C}(X_{\underline{a}}), K(n))$ is zero. Since u becomes zero in the generic point of $X_{\underline{a}}$ there exists a nonempty open subscheme U of $X_{\underline{a}}$ such that the restriction of u' to U is zero. Then by Proposition 2.15 the restriction of u' to $X_{\underline{a}}$ is zero. Using the standard spectral sequence which starts with the hypercohomology of $X_{\underline{a}}^i$ with coefficients in $K(n)$ and converges to hypercohomology of $\check{C}(X_{\underline{a}})$ with coefficients in $K(n)$ we see that it is sufficient to show that for $p < n+1$ the homomorphisms $\mathbf{H}^p(X_{\underline{a}}^i, K(n)) \rightarrow \mathbf{H}^p(X_{\underline{a}}, K(n))$ given by the diagonal embeddings $X_{\underline{a}} \rightarrow X_{\underline{a}}^i$ are isomorphisms. Since $X_{\underline{a}}$ is rational over $k(X_{\underline{a}})$ there exist an open neighborhood U of the generic point of $X_{\underline{a}}$ in $X_{\underline{a}}^i$ and an open neighborhood W of the generic point of $X_{\underline{a}} \times \{0\}$ in $X_{\underline{a}} \times \mathbf{A}^N$ such that the pair $(U, U \cap X_{\underline{a}})$ is isomorphic to the pair $(W, W \cap (X_{\underline{a}} \times \{0\}))$. Therefore the required homomorphisms are isomorphisms by Proposition 2.15 and homotopy invariance of hypercohomology with coefficients in $K(n)$ which follow from Lemma 2.8.

3 Motivic cohomology and algebraic cobordisms.

3.1 *A topological lemma.*

In this section we prove a purely topological result (Lemma 3.6) which will be used below to deduce information on the action of the motivic Steenrod algebra in the motivic cohomology of simplicial schemes of the form $\check{C}(X)$. It should be mentioned that this result is certainly both well known and very basic from the point of view of the modern homotopy theory.

Let $\mathcal{A}^*(l)$ be the Steenrod algebra of operations in ordinary cohomology with $\mathbf{Z}/l$-coefficients and $\mathcal{A}_*(l)$ its dual. For a CW-complex X and a pair (α_i, α^j) where $\alpha_i \in H_i(X, \mathbf{Z}/l)$ and $\alpha^j \in H^j(X, \mathbf{Z}/l)$ one assigns an element $(\alpha_i, \alpha^j) \in \mathcal{A}_{i-j}(l)$ which takes an operation $\phi \in \mathcal{A}^{i-j}(l)$ to the number given by the value of the pairing $H_i(X, \mathbf{Z}/l) \otimes H^i(X, \mathbf{Z}/l) \to \mathbf{Z}/l$ on $(\alpha_i, \phi(\alpha^j))$. Let β_i be the canonical generator of $H_i(B\mathbf{Z}/l, \mathbf{Z}/l)$, α be the canonical generator of $H^1(B\mathbf{Z}/l, \mathbf{Z}/l)$ and β be the canonical generator of $H^2(B\mathbf{Z}/l, \mathbf{Z}/l)$. Following [12, p. 159] one defines elements $\tau_i \in \mathcal{A}_{2l^i-1}(l)$ and $\xi_i \in \mathcal{A}_{2(l^i-1)}(l)$ as the ones represented by the pairs (β_{l^i}, α) and (β_{l^i}, β) respectively. Considered as a (graded) commutative algebra $\mathcal{A}_*(l)$ is isomorphic for odd l to $P_{\mathbf{Z}/l}[\xi_1, \ldots] \otimes_{\mathbf{Z}/l} E_{\mathbf{Z}/l}[\tau_0, \ldots]$ where $P_{\mathbf{Z}/l}$ and $E_{\mathbf{Z}/l}$ denote the polynomial and the exterior algebras over $\mathbf{Z}/l$ with generators ξ_i and τ_i of degrees $2(p^i - 1)$ and $2p^i - 1$ respectively and $\mathcal{A}_*(2)$ is isomorphic to $P_{\mathbf{Z}/2}[\tau_0, \ldots]$ where $deg(\tau_i) = 2^{i+1} - 1$ ([12, Theorem 2]). Observe that for $l = 2$ one has $\xi_i = (\tau_{i-1}^2)$. One defines the *Milnor elements* Q_i in $\mathcal{A}^{2l^i-1}(l)$ as the elements dual to τ_i with respect to the monomial basis in $\mathcal{A}_*(l)$. Let us also define operations $q_i \in \mathcal{A}^{2(l^i-1)}$ as the ones dual with respect to the monomial basis in $\mathcal{A}_*(l)$ to ξ_i.

These operations have the following well known properties which we list here mostly to provide a topological "background" for the similar result in motivic cohomology given Section 3.3.

Proposition 3.1 *For any ptime l one has:*

1. *Operations Q_i generate an exterior subalgebra in $\mathcal{A}^*(l)$, i.e. one has*
   (a) $Q_iQ_j + Q_jQ_i = 0$
   (b) $Q_i^2 = 0$
2. *For any $u \in H^*(X, \mathbf{Z}/l)$ and $v \in H^*(Y, \mathbf{Z}/l)$ one has*
$$Q_i(u \times v) = Q_i(u) \times v \pm u \times Q_i(v)$$
3. *$Q_0 = \beta$ is the Bockstein homomorphism and for any $i > 0$ one has $Q_i = [q_i, \beta]$*

4. for any $i > 0$ one has $Q_i = \pm[Q_{i-1}, P^{l^{i-1}}]$

Note that the fourth property of Q_i's may be used to define them inductively in terms of the Bockstein and the reduced power operations.

The following expression for the coproduct of q_i which will be used in the proof of Lemma 3.6.

Corollary 3.2 *For any $i \geq 1$ one has*

$$\Delta(q_i) = \begin{cases} q_i \otimes 1 \pm 1 \otimes q_i & \text{for } l \neq 2 \\ q_i \otimes 1 + 1 \otimes q_i + Q_{i-1} \otimes Q_{i-1} & \text{for } l = 2 \end{cases}$$

i.e. for any $x \in H^*(X, \mathbf{Z}/l)$, $y \in H^*(Y, \mathbf{Z}/l)$ one has

$$q_i(x \times y) = \begin{cases} q_i(x) \times y \pm x \times q_i(y) & \text{for } l \neq 2 \\ q_i(x) \times y + x \times q_i(y) + Q_{i-1}(x) \times Q_{i-1}(y) & \text{for } l = 2 \end{cases}$$

Finally let us prove the following result which describes the action of operations q_i in the cohomology ring of $\mathbf{CP}^\infty$.

Lemma 3.3 *Let $x \in H^2(\mathbf{CP}^\infty, \mathbf{Z}/2)$ be the canonical generator. Then $q_i(x) = x^{l^i-1}$.*

Proof: The canonical map $\mathbf{CP}^\infty \to B\mathbf{Z}/l$ gives a monomorphism on even cohomology groups and therefore it is sufficient to verify that $q_i(\beta) = \beta^{l^i-1}$. That follows immediately from our definition of q_i as a dual to ξ_i and ξ_i as the element represented by the pair (β_{l^i}, β).

Let $\mathbf{MU}$ be the spectrum representing the complex cobordisms theory and $\tau : \mathbf{MU} \to \mathbf{H}_{\mathbf{Z}/l}$ be the Thom class with values in ordinary cohomology with $\mathbf{Z}/l$-coefficients. For any cohomological operation $P \in \mathcal{A}^{2n}(l)$ we may consider the cohomological class $P(\tau) \in H^*(\mathbf{MU}, \mathbf{Z}/l) = H^*(BGL, \mathbf{Z}/l)$. Let $s_i \in H^{2i}(BGL, \mathbf{Z}/l)$ be the element which corresponds to the characteristic class defined by the properties:

1. $s_i(E \oplus F) = s_i(E) + s_i(F)$
2. $s_i(L) = c_1(L)^i$ for a line bundle L.

Lemma 3.4 *One has $q_i(\tau) = s_{l^i-1}$ in $H^*(\mathbf{MU}, \mathbf{Z}/l)$.*

Proof: Let $\mathcal{E}$ be a complex vector bundle on a CW-complex X, $Th(\mathcal{E})$ be its Thom space and $\tau_{\mathcal{E}} \in H^{2n}(Th(\mathcal{E}), \mathbf{Z}/l)$ be its Thom class. It is sufficient to verify that $q_i(\tau_{\mathcal{E}}) = s_i(\mathcal{E})\tau$. Using the standard splitting principle we may assume that $\mathcal{E}$ is a sum of line bundles L_j, $j = 1, \ldots, m$. By Lemma 3.3 and Corollary 3.2 one can easily see that in this case one has $q_i(\tau_{\mathcal{E}}) = (\sum_{j=1}^m c_1(L_j)^{l^i-1})\tau_{\mathcal{E}}$ which proves our claim in view of the definition of s_i.

Remark: One can also show by the same method that $P^i(\tau)$ corresponds to the characteristic class which takes a sum of line bundles $L_1 \oplus \ldots \oplus L_m$ to the i-th elementary symmetric function of the classes $c_1^{l-1}(L_j) \in H^{2(l-1)}(-, \mathbf{Z}/l)$. In particular for $l = 2$ the element $P^i(\tau) = Sq^{2i}(\tau)$ corresponds to the i-th Chern class.

For an almost complex manifold X of complex dimension n denote by $s(X)$ the value of the characteristic class s_n of the normal bundle of X on the fundamental class of X. If $[X]$ is the class of X in MU_{2n} one can verify easily that $s_{2n}([X]) = s(X)$.

Definition 3.5 *An element a in MU_* is called a (ν_n, l)-element if it belongs to $MU_{2(l^n-1)}$ and satisfies the following two conditions*

1. *the image of a in $H_*(\mathbf{MU}, \mathbf{Z})$ is divisible by l*

2. *$s_{l^n-1}(a)$ is not zero modulo l^2.*

A complex algebraic variety X is called a (ν_n, l)-variety if the class of X in MU_* is a (ν_n, l)-element, i.e. if

1. *$dim(X) = l^n - 1$*

2. *all characteristic numbers of X are divisible by l*

3. *$s(X) \neq 0(mod\ l^2)$*

Consider Q_i as a morphism in the stable homotopy category of the form

$$Q_i : \mathbf{H}_{\mathbf{Z}/l} \to S^{2l^n-1} \wedge \mathbf{H}_{\mathbf{Z}/l}$$

where $\mathbf{H}_{\mathbf{Z}/l}$ is the Eilenberg-MacLane spectrum representing ordinary cohomology with $\mathbf{Z}/l$-coefficients and denote by $\mathbf{\Phi}_i^{top}$ the homotopy fiber of this map such that we have a distinguished triangle

$$S^{2(l^i-1)} \wedge \mathbf{H}_{\mathbf{Z}/l} \xrightarrow{u} \mathbf{\Phi}_i^{top} \xrightarrow{v} \mathbf{H}_{\mathbf{Z}/l} \xrightarrow{Q_i} S^{2l^i-1} \wedge \mathbf{H}_{\mathbf{Z}/l}.$$

We are ready now to formulate our topological lemma.

Lemma 3.6 *Let $\psi : \mathbf{MU} \to \mathbf{\Phi}_n^{top}$ be a morphism such that the diagram*

$$\begin{array}{ccc} \mathbf{MU} & \stackrel{\psi}{\to} & \mathbf{\Phi}_n^{top} \\ \tau \downarrow & & \downarrow v \\ \mathbf{H}_{\mathbf{Z}/l} & \stackrel{Id}{\to} & \mathbf{H}_{\mathbf{Z}/l} \end{array}$$

commutes. Then for any (ν_n, l)-element $a \in MU_{2(l^n-1)}$ one has $\psi(a) \neq 0$.

Proof: Let us fix a and let $\mathbf{E}$ be the cofiber of the map $a : S^{2(l^n-1)} \to \mathbf{MU}$. The Thom class on $\mathbf{MU}$ defines a cohomology class in $H^0(\mathbf{E}, \mathbf{Z}/l)$ which we denote by τ_a. One obseves easily that all we have to show is that $Q_n(\tau_a)$ is not zero in $H^{2l^n-1}(\mathbf{E}, \mathbf{Z}/l)$. Let us assume that $n > 0$. Then $Q_i(\tau_a) = \beta q_i(\tau_a)$ since $\beta(\tau_a) = 0$ for $n > 0$. Therefore all we have to show is that the cohomology class $q_i(\tau_a)$ in $H^{2(l^n-1)}(\mathbf{E}, \mathbf{Z}/l)$ can not be lifted to a class in $H^{2(l^n-1)}(\mathbf{E}, \mathbf{Z}/l^2)$. Since the restriction homomorphism $H^{2(l^n-1)}(\mathbf{E}, \mathbf{Z}/l) \to H^{2(l^n-1)}(\mathbf{MU}, \mathbf{Z}/l)$ is a monomorphism it is sufficient to show that for any lifting $\tilde{\psi}$ of the class $q_i(\tau)$ in $H^{2(l^n-1)}(\mathbf{MU}, \mathbf{Z}/l)$ to $H^{2(l^n-1)}(\mathbf{MU}, \mathbf{Z}/l^2)$ one has $\tilde{\psi}(a) \neq 0$. By Lemma 3.4 we have $\tilde{\psi} = s_{l^n-1} + lc$ where c is an integral characteristic class. By Definition 3.5 the number $c(a)$ is divisible by l. Therefore $\tilde{\psi}(a) = s_{l^n-1}(a)$ in $\mathbf{Z}/l^2$ which is nonzero by the definition of (ν_n, l)-classes.

3.2 *Homotopy categories of algebraic varieties*

Let $\Delta^{op}Shv_{Nis}(Sm/k)$ be the category of simplicial sheaves of sets in the Nisnevich topology on the category of smooth schemes over k. This category replaces in our theory the category of topological spaces. Note that the categories of smooth schemes, sheaves of sets, simplicial smooth schemes and simplicial sets naturally embed into $\Delta^{op}Shv_{Nis}(Sm/k)$.

For a smooth variety U over k and a point u of U let $\mathcal{O}^h_{U,u}$ be the henselisation of the local ring of u in U. Then $Spec(\mathcal{O}^h_{U,u})$ is a limit of smooth schemes U_α over k and for any object $\mathcal{X}$ of $\Delta^{op}Shv_{Nis}(Sm/k)$ one defines a simplicial set $\mathcal{X}(Spec(\mathcal{O}^h_{U,u}))$ as the corresponding colimit of simplicial sets $\mathcal{X}(U_\alpha)$.

Definition 3.7 *A morphism of simplicial sheaves $f : \mathcal{X} \to \mathcal{Y}$ is called a simplicial weak equivalence if for any smooth scheme U over k and any point u of U the corresponding map of simplicial sets*

$$\mathcal{X}(Spec(\mathcal{O}^h_{U,u})) \to \mathcal{Y}(Spec(\mathcal{O}^h_{U,u}))$$

is a weak equivalence.

A class of important examples of a simplicial weak equivalences can be obtained as follows. Recall, that for a smooth variety X over we denote by $\check{C}(X)$ the simplicial scheme such that $\check{C}(X)_n = X^{n+1}$ and faces and degeneracy morphisms are given by partial projections and diagonals respectively. We have a canonical morphism $\check{C}(X) \to Spec(k)$.

Lemma 3.8 *Let X, Y be smooth schemes over k such that*

$$Hom(Y, X) \neq \emptyset.$$

Then the projection $\check{C}(X) \times Y \to Y$ is a simplicial weak equivalence.

Proof: Let U be a smooth scheme over k. Then the simplicial set $\check{C}(X)(U)$ is the simplex generated by the set $Hom(U, X)$ which is contractible if and only if $Hom(U, X) \neq \emptyset$. It implies immediately that for any U the map of simplicial sets

$$(\check{C}(X) \times Y)(U) \to Y(U) = Hom(Y, U)$$

is a weak equivalence.

Remark: One can strengthen the statement of Lemma 3.8 by showing that the map $\check{C}(X) \times Y \to Y$ is a simplicial weak equivalence if and only if X has a rational point over the residue field of any point of Y.

Let $\mathcal{H}_s(\Delta^{op}Shv_{Nis}(Sm/k))$ be the localization of $\Delta^{op}Shv_{Nis}(Sm/k)$ with respect to simplicial weak equivalences.

Definition 3.9 *An object $\mathcal{X}$ of $\Delta^{op}Shv_{Nis}(Sm/k)$ is called $\mathbf{A}^1$-local if for any other object $\mathcal{Y}$ the map*

$$Hom_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X}) \to Hom_{\mathcal{H}_s}(\mathcal{Y} \times \mathbf{A}^1, \mathcal{X})$$

induced by the projection $\mathcal{Y} \times \mathbf{A}^1 \to \mathcal{Y}$ is a bijection.

A morphism $f : \mathcal{Y} \to \mathcal{Y}'$ is called an $\mathbf{A}^1$-weak equivalence if for any $\mathbf{A}^1$-local object $\mathcal{X}$ the corresponding map

$$Hom_{\mathcal{H}_s}(\mathcal{Y}', \mathcal{X}) \to Hom_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X})$$

is a bijection.

As was shown in [14] there is a proper closed model structure on the category $\Delta^{op}Shv_{Nis}(Sm/k)$ where all monomorphisms are cofibrations and $\mathbf{A}^1$-weak equivalences are weak equivalences. This closed model structure is called the $\mathbf{A}^1$-closed model structure (as opposed to the simplicial closed model structure where weak equivalences are given by Definition 3.7) and the corresponding homotopy category $\mathcal{H}(k)$ is called the *homotopy category of schemes over $Spec(k)$*.

Let $\Delta^{op}_{\bullet}Shv_{Nis}(Sm/k)$ be the category of pointed simplicial sheaves. For a simplicial sheaf $\mathcal{X}$ we denote by $\mathcal{X}_+$ the simplicial sheaf $\mathcal{X} \coprod Spec(k)$ pointed by the canonical embedding $Spec(k) \to \mathcal{X} \coprod Spec(k)$. Define the smash product $\mathcal{X} \wedge \mathcal{Y}$ as the sheaf associated with the presheaf of the form $U \mapsto \mathcal{X}(U) \wedge \mathcal{Y}(U)$. One observes easily that this provides us with a symmetric monoidal structure on $\Delta^{op}_{\bullet}Shv_{Nis}(Sm/k)$ and on $\mathcal{H}_{\bullet}(k)$ with the unit object being $S^0 = Spec(k)_+$.

Denote by T the pointed simplicial sheaf given by the cocartesian square

$$\begin{array}{ccc} \mathbf{A}^1 - \{0\} & \to & \mathbf{A}^1 \\ \downarrow & & \downarrow \\ Spec(k) & \to & T \end{array}$$

This object plays the role of S^2 in our theory (note that there is a canonical isomorphism in $\mathcal{H}_{\bullet}(k)$ of the form $T \cong (\mathbf{P}^1, 0)$). We also have two different "circles"

S^1_s - the simplicial circle considered as an object of $\Delta^{op}_{\bullet}Shv_{Nis}(Sm/k)$

S^1_t - the scheme $\mathbf{A}^1 - \{0\}$ pointed by 1 considered as an object of the same category.

A T-spectrum over k is a sequence $\mathbf{E} = (E_i, e_i : T \wedge E_i \to E_{i+1})$. T-spectra and the obvious morphisms between them form a category which we denote

$Spect_T(k)$. Using the $\mathbf{A}^1$-closed model structure on $\Delta^{op}_{\bullet}Shv_{Nis}(Sm/k)$ one defines in the usual way ([4]) the strict and stable closed model structures on $Spect_T(k)$. The homotopy category $\mathcal{SH}(k)$ corresponding to the stable closed model structure is called the *stable homotopy category of schemes* over $Spec(k)$. For any pointed simplicial sheaf $\mathcal{X}$ the sequence

$$\Sigma^\infty_T \mathcal{X} = (T^{\wedge i} \wedge \mathcal{X}, Id)$$

is a T-spectrum called the suspension spectrum of $\mathcal{X}$. This construction gives us a functor $\mathcal{H}_\bullet(k) \to \mathcal{SH}(k)$. By abuse of notations we will usually write S^1_s, S^1_t ant T instead of $\Sigma^\infty_T S^1_s$, $\Sigma^\infty_T S^1_t$ and $\Sigma^\infty_T T$.

For any morphism $f : \mathcal{X} \to \mathcal{Y}$ of pointed simplicial sheaves define $cone(f)$ to be the sheaf associated with the presheaf of the form $U \mapsto cone(\mathcal{X}(U) \to \mathcal{Y}(U))$. We have a canonical morphism $\mathcal{Y} \to cone(f)$ which extends in the usual way to a sequence

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \to cone(f) \to S^1_s \wedge \mathcal{X}$$

called the cofibration sequence associated to f.

Theorem 3.10 *There exists a structure of a tensor triangulated category on $\mathcal{HS}(k)$ with the following properties*

1. *the shift functor $\mathbf{E} \to \mathbf{E}[1]$ is given by $\mathbf{E}[1] = S^1_s \wedge \mathbf{E}$*
2. *the functor Σ^∞_T takes cofibration sequences to distinguished triangles*
3. *the functor Σ^∞_T is a symmetric monoidal functor from $(\mathcal{H}_\bullet(k), \wedge)$ to $(\mathcal{SH}_\bullet(k), \wedge)$*
4. *the object T of $\mathcal{SH}(k)$ is invertible*

Let $\mathbf{E} = (E_i, e_i)$ be a spectrum. The morphisms $e_i : \Sigma_T E_i \to E_{i+1}$ define morphisms in the stable category of the form

$$\tilde{e}_i : \Sigma^\infty_T E_i[-i] \to \Sigma^\infty_T E_{i+1}[-(i+1)]$$

and one can describe the spectrum $\mathbf{E}$ in terms of the spectra $\Sigma^\infty_T E_i$ and morphisms $\tilde{e}_i$ as follows.

Lemma 3.11 *For any spectrum* $\mathbf{E} = (E_i, e_i)$ *there is a canonical distinguished triangle in* $\mathcal{SH}$ *of the form*

$$\oplus_{i\geq 0}\Sigma^\infty_T E_i[-i] \stackrel{\oplus(\tilde{e}_i - Id)}{\to} \oplus_{i\geq 0}\Sigma^\infty_T E_i[-i] \to \mathbf{E} \to \oplus_{i\geq 0}\Sigma^\infty_T E_i[-i+1]$$

There is a canonical isomorphism in $\mathcal{H}_\bullet$ of the form $S^1_s \wedge S^1_t = T$. Using this relation and Theorem 3.10 we conclude that both S^1_s and S^1_t are invertible in $\mathcal{SH}(k)$. Let us fix objects S^{-1}_s and S^{-1}_t and isomorphisms

$$S^1_s \wedge S^{-1}_s \cong S^0$$
$$S^1_t \wedge S^{-1}_t \cong S^0$$

Let further

$$S^n_s = \begin{cases} (S^1_s)^{\wedge n} & \text{for } n \geq 0 \\ (S^{-1}_s)^{\wedge -n} & \text{for } n \leq 0 \end{cases} \quad S^n_t = \begin{cases} (S^1_t)^{\wedge n} & \text{for } n \geq 0 \\ (S^{-1}_t)^{\wedge -n} & \text{for } n \leq 0 \end{cases}$$

and

$$S^{p,q} = S^q_t \wedge S^{p-q}_s$$

For any object $\mathbf{E}$ in $\mathcal{SH}(k)$ denote by $\mathbf{E}(q)[p]$ the object $S^{p,q} \wedge \mathbf{E}$. Define the *cohomology theory* $\tilde{E}^{*,*}$ associated to $\mathbf{E}$ by

$$\tilde{E}^{p,q}(X) = Hom_{\mathcal{SH}}(X, \mathbf{E}(p)[q])$$

and the *homology theory* $E_{*,*}$ associated to $\mathbf{E}$ by

$$\tilde{E}_{p,q}(X) = Hom_{\mathcal{SH}}(S^{p,q}, \mathbf{E} \wedge X).$$

These theories take distinguished triangles to long exact sequences and have suspension isomorphisms with respect to smashing with S^1_s, S^1_t and T. For a simplicial smooth scheme $\mathcal{X}$ we set

$$E^{*,*}(\mathcal{X}) = \tilde{E}^{*,*}(\Sigma^\infty_T \mathcal{X}_+)$$
$$E_{*,*}(\mathcal{X}) = \tilde{E}_{*,*}(\Sigma^\infty_T \mathcal{X}_+).$$

Remark: The obvious functor from the category of pointed simplicial sets to $\Delta^{op}_\bullet(Shv_{Nis}(Sm/k))$ extends to a tesor triangulated functor from the topological stable homotopy category to $\mathcal{SH}(k)$ which takes S^1 to S^1_s. This implies in particular that the permutation isomorphism on $S^1_s \wedge S^1_s$ equals -1. It is not true however that the permutation isomorphism on $S^1_t \wedge S^1_t$ is -1. Instead it equals $-\epsilon$ where ϵ is a certain element in $End(S^0)$. This element equals 1 if the base field k contains a square root of -1 but in general it is nontrivial.

3.3 *Eilenberg-MacLane spectra and motivic cohomology.*

Consider the quotient sheaf of abelian groups $L(\mathbf{A}^n)/L(\mathbf{A}^n - \{0\})$ as a sheaf of sets pointed by zero and denote it by $K(\mathbf{Z}(n), 2n)$.

One has a canonical morphism of pointed sheaves

$$e_n : T \wedge K(\mathbf{Z}(n), 2n) \to K(\mathbf{Z}(n+1), 2n+2)$$

and we define the *Eilenberg-MacLane spectrum* $\mathbf{H_Z}$ as the sequence $(K(\mathbf{Z}(n), 2n), e_n)$. Similarly starting with the sheaves $(L(\mathbf{A}^n)/L(\mathbf{A}^n - \{0\})) \otimes \mathbf{Z}/n$ one defines Eilenberg-MacLane spectra $\mathbf{H}_{\mathbf{Z}/n}$. The external product of cycles give us morphisms of pointed sheaves $K(\mathbf{Z}(n), 2n) \wedge K(\mathbf{Z}(m), 2m) \to K(\mathbf{Z}(n+m), 2(n+m))$ which fit together and produce the multiplication morphism

$$m : \mathbf{H_Z} \wedge \mathbf{H_Z} \to \mathbf{H_Z}$$

in $\mathcal{SH}(k)$ which satisfies the usual associativity and commutativity condition.

One defines *motivic cohomology* as the cohomology theory $\tilde{H}^{p,q}(-, \mathbf{Z})$ represented by the spectrum $\mathbf{H_Z}$.

Remark: One can easily see that if k is a field of characteristic zero and X is a smooth variety over k the sheaf $L(X)$ considered as an object of $\mathcal{H}_\bullet(k)$ is the group completion of the abelian monoid $\coprod_{i \geq 0} SP^i X$ where $SP^i X$ is the symmetric product of X. One can also observe that $\mathbf{A}^n/(\mathbf{A}^n - \{0\})$ is canonically isomorphic in $\mathcal{H}_\bullet(k)$ to the sphere $S^{2n,n}$ and therefore our definition of $K(\mathbf{Z}(n), 2n)$ is a version of the Dold-Thom equivalence

$$(\coprod_{i \geq 0} SP^i(S^{2n}))^+ \cong K(\mathbf{Z}, 2n)$$

in topology.

Theorem 3.12 *Let k be a field which admits resolution of singularities and X be a simplicial smooth scheme over k. Then for all $p, q \in \mathbf{Z}$ there are canonical isomorphisms*

$$H^{p,q}(X, \mathbf{Z}) = H_B^{p,q}(X, \mathbf{Z})$$

and the same holds for finite coefficients.

Proof: See [14,].

Starting from this point we will denote the Beilinson motivic cohomology groups simply by $H^{*,*}(-,-)$.

Define the *motivic Steenrod algebra* as the algebra of endomorphisms of $\mathbf{H}_{\mathbf{Z}/l}$ in $\mathcal{SH}(k)$

$$\mathcal{A}^{p,q}(k,\mathbf{Z}/l) = Hom_{\mathcal{SH}}(\mathbf{H}_{\mathbf{Z}/l},\mathbf{H}_{\mathbf{Z}/l}(q)[p]) = H^p(\mathbf{H}_{\mathbf{Z}/l},\mathbf{Z}/l(q))$$

The algebra $\mathcal{A}^{p,q}(k,\mathbf{Z}/l)$ is analized in details in [26]. For our present purposes we will only need a few results of this paper explained below. Denote by $H^{*,*}$ the motivic cohomology ring of $Spec(k)$ with $\mathbf{Z}/l$-coefficients. Using Corollaries 2.2-2.4 one immediately gets the following result.

Proposition 3.13 *For any field k with resolution of singularities one has:*

1. $H^{p,q} = 0$ *in the following cases*
   (a) *if* $p > q$
   (b) *if* $q < 0$
   (c) *if* $q = 0$ *and* $p \neq 0$
   (d) *if* $q = 1$ *and* $p \neq 0, 1$
2. $H^{p,p} = K_p^M(k)/l$
3. $H^{0,1} = \mu_l(k)$ *where* μ_l *is the sheaf of l-th roots of unit.*

We denote by τ the generator of $H^{0,1} = \mu_2 = \mathbf{Z}/2$ for $l = 2$ and we assume $\tau = 0$ for $l \neq 2$. We also denote by ρ the class of -1 in $H^{1,1} = k^*/(k^*)^l$ (which is also zero for $l \neq 2$).

Theorem 3.14 *For a field k of characteristic zero the group $\mathcal{A}^{p,q}(k,l)$ is zero for $q < 0$ and the group $\mathcal{A}^{0,0}(k,l)$ is the cyclic group of order $\mathbf{Z}/l$ generated by the identity morphism $Id_{\mathbf{H}_{\mathbf{Z}/l}}$.*

Theorem 3.15 *The Kunnet homomorphism*

$$\tilde{H}^{*,*}(\mathbf{H}_{\mathbf{Z}/l},\mathbf{Z}/l)\otimes_{H^{*,*}}\tilde{H}^{*,*}(\mathbf{H}_{\mathbf{Z}/l},\mathbf{Z}/l)\to\tilde{H}^{*,*}(\mathbf{H}_{\mathbf{Z}/l}\wedge\mathbf{H}_{\mathbf{Z}/l},\mathbf{Z}/l)$$

is an isomorphism.

Composing the inverse to Kunnet homomorphism with the map

$$\tilde{H}^{*,*}(\mathbf{H}_{\mathbf{Z}/l}, \mathbf{Z}/l) \to \tilde{H}^{*,*}(\mathbf{H}_{\mathbf{Z}/l} \wedge \mathbf{H}_{\mathbf{Z}/l}, \mathbf{Z}/l)$$

given by the multiplication morphism $\mathbf{H}_{\mathbf{Z}/l} \wedge \mathbf{H}_{\mathbf{Z}/l} \to \mathbf{H}_{\mathbf{Z}/l}$ we get a homomorphism

$$\Delta : \tilde{H}^{*,*}(\mathbf{H}_{\mathbf{Z}/l}, \mathbf{Z}/l) \to \tilde{H}^{*,*}(\mathbf{H}_{\mathbf{Z}/l}, \mathbf{Z}/l(*)) \otimes_{H^{*,*}} H^{*}(\mathbf{H}_{\mathbf{Z}/l}, \mathbf{Z}/l).$$

In the following theorems $\beta \in \mathcal{A}^{1,0}(k,l)$ denotes the Bockstein homomorphism given by the distinguished triangle

$$\mathbf{H}_{\mathbf{Z}/l} \to \mathbf{H}_{\mathbf{Z}/l^2} \to \mathbf{H}_{\mathbf{Z}/l} \stackrel{\beta}{\to} S^1_s \wedge \mathbf{H}_{\mathbf{Z}/l}.$$

Theorem 3.16 *Let k be a field of characteristic zero. Then there exists a unique sequence of cohomological operations $P^i \in \mathcal{A}^{2i(l-1),l-1}$, $i \geq 0$ with the following properties*

1. $P^0 = Id$
2. *for any simplicial smooth scheme X and any $u \in H^n(X, \mathbf{Z}/l(i))$ one has $P^i(u) = 0$ for $n < 2i$ and $P^i(u) = u^l$ for $n = 2i$*
3. $\Delta(P^i) = \sum_{a+b=i} P^a \otimes P^b + \tau \sum_{a+b=i-2} \beta P^a \otimes \beta P^b$

Define operations $Q_i \in \mathcal{A}^{2l^i-1,l^i-1}$ inductively setting

$$Q_0 = \beta$$

$$Q_{i+1} = [Q_i, P^{l^i}].$$

Theorem 3.17 *Let k be a field of characteristic zero. Then operations Q_i have the following properties:*

1. *Q_i's generate an exterior subalgebra in $\mathcal{A}^{*,*}(l,k)$ i.e. $Q_iQ_j + Q_jQ_i = 0$ and $Q_i^2 = 0$*
2. *$\Delta(Q_i) = 1 \otimes Q_i + Q_i \otimes 1 + \sum \rho^{n_j} \phi_j \otimes \psi_j$ where $n_j > 0$ and ϕ_i, ψ_i are some operations of bidegrees (p,q) with $p > 2q$.*
3. *for any $i > 0$ there exist operations q_i such that $Q_i = [\beta, q_i]$*

Just as in the topological situation we can use operations Q_i to define Margolis motivic cohomology $\tilde{H}M_i^{*,*}(\mathcal{X})$ as cohomology groups of the complex

$$\tilde{H}^{*-2l^i+1,*-l^i+1}(\mathcal{X}, \mathbf{Z}/l) \stackrel{Q_i}{\to} \tilde{H}^{*,*}(\mathcal{X}, \mathbf{Z}/l) \stackrel{Q_i}{\to} \tilde{H}^{*+2l^i-1,*+l^i-1}(\mathcal{X}, \mathbf{Z}/l)$$

3.4 *Topological realization functor.*

In this section we assume that our base field can be embedded into the fields $\mathbf{C}$ of complex numbers. Any such embedding $i : k \subset \mathbf{C}$ gives us a functor from Sm/k to the category of simplicial sets given by $X \mapsto Sing(X(\mathbf{C}))$ where $Sing$ is the singular simplicial set functor and $X(\mathbf{C})$ is the complex manifold of $\mathbf{C}$-points of X with respect to the embedding i. As was shown in [14,] there is an endofunctor rr on $\Delta^{op}Shv_{Nis}(Sm/k)$ and a natural transformation $rr \to Id$ such that for any simplicial sheaf $\mathcal{X}$ the terma $rr(\mathcal{X})_n$ are direct sums of representable sheaves and the morphism $rr(\mathcal{X}) \to Id$ is a (simplicial) weak equivalence. Define a functor $t_{\mathbf{C}}$ from $\Delta^{op}Shv_{Nis}(Sm/k)$ to the category of simplicial sets setting $t_{\mathbf{C}}(\mathcal{X})$ to be the diagonal simplicial set of the bisimplicial set $Sing(rr(\mathcal{X})(\mathbf{C}))$. One observes easily that this functor takes $\mathbf{A}^1$-weak equivalences to weak equivalences and therefore defines the *topological realization functor* $t_{\mathbf{C}} : \mathcal{H}(k) \to \mathcal{H}$ (here $\mathcal{H}$ is the usual homotopy category) such that for any smooth scheme X the object $t_{\mathbf{C}}(X)$ is canonically isomorphisc in $\mathcal{H}$ to the object $X(\mathbf{C})$.

Similarly, one can define the topological realization functor $t_{\mathbf{C}}$ from $\mathcal{SH}(k)$ to the usual stable homotopy category SH. This is a tensor triangulated functor and one has canonical isomorphisms $t_{\mathbf{C}}(S^1_s) = S^1$ and $t_{\mathbf{C}}(S^1_t) = S^1$.

For an object $\mathbf{E}$ in $\mathcal{SH}(k)$ denote by E^p_{top} the cohomology theory given by

$$E^p_{top}(X) = Hom_{\mathcal{SH}}(t_{\mathbf{C}}(X), S^p \wedge t_{\mathbf{C}}(\mathbf{E}))$$

and by E^{top}_p the homology theory given by

$$E^{top}_p(X) = Hom_{\mathcal{SH}}(S^p, t_{\mathbf{C}}(X) \wedge t_{\mathbf{C}}(\mathbf{E}))$$

such that for a smooth variety X we have

$$E^p_{top}(X) = (t_{\mathbf{C}}(\mathbf{E})^p(X(\mathbf{C}))$$

$$E^{top}_p(X) = (t_{\mathbf{C}}(\mathbf{E})_p(X(\mathbf{C})).$$

Then we have canonical morphisms $E^{p,q}(X) \to E^p_{top}(X)$, $E_{p,q}(X) \to E^{top}_p(X)$.

Remark: One can easily see that the theory E^p_{top} is represented by an object $\mathbf{E}_{top}$ of $\mathcal{SH}(k)$ which is $(0,1)$-periodic i.e. such that there is a canonical isomorphism $\mathbf{E}_{top} \cong \mathbf{E}_{top}(1)$. Form the point of view of the homotopy theory of algebraic varieties spectra of the form $\mathbf{E}_{top}$ are usually completely misterious.

The following result is a corollary of our definition of motivic cohomology and the Dold-Thom theorem.

Proposition 3.18 *For any embedding $i : k \subset \mathbf{C}$ there is a canonical isomorphism*

$$t_{\mathbf{C}}(\mathbf{H_Z}) = \mathbf{H_Z}$$

where $\mathbf{H_Z}$ is the usual topological Eilenberg-Maclane spectrum. The same holds for finite coefficients.

For an embedding $k \subset \mathbf{C}$ let $t_{\mathbf{C}} : \mathcal{A}^{*,*}(l,k) \to \mathcal{A}(l)$ be the homomorphism induced by Proposition 3.18 by the topological realization functor $t_{\mathbf{C}}$. One can choose an algebraic model $B_{et}\mathbf{Z}/l$ for the classifying space $B\mathbf{Z}/l$ such that the generator $a \in H^1(B\mathbf{Z}/l, \mathbf{Z}/l)$ comes from a motivic cohomology class $a \in H_B^{1,1}(B_{et}\mathbf{Z}/l, \mathbf{Z}/l)$. Thus properties of operations P^i given in Theorem 3.16 together with the standard topological results on the uniqueness of reduced power operations imply immediately that for any embedding $k \subset \mathbf{C}$ one has $t_{\mathbf{C}}(P^i) = P^i$ and in particular one has:

Lemma 3.19 *For any embedding $k \subset \mathbf{C}$ one has $t_{\mathbf{C}}(Q_i) = Q_i$.*

3.5 *Algebraic cobordisms.*

In this section we define a spectrum **MGL** called the algebraic cobordisms spectrum and prove some of its very basic properties. To do it we need to introduce Thom spaces of vector bundles on smooth schemes over k. Let $\mathcal{E} \to X$ be such a bundle and let $\mathcal{E} - s(X)$ be the complement to the zero section. Then the Thom space $Th(\mathcal{E})$ of $\mathcal{E}$ is the pointed sheaf given by the cocartesian square

$$\begin{array}{ccc} \mathcal{E} - s(X) & \to & \mathcal{E} \\ \downarrow & & \downarrow \\ Spec(k) & \to & Th(\mathcal{E}) \end{array}$$

One observes easily that there is a canonical isomorphism of pointed sheaves of the form $Th(\mathcal{E} \oplus \mathcal{O}) = T \wedge Th(\mathcal{E})$ and therefore considering universal bundles on the Grassmannians $G(m,n)$ one gets in the usual way a spectrum which we denote **MGL** and call the algebraic cobordisms spectrum.

Lemma 3.20 *Let X be a connected smooth scheme over a field k which admits resolution of singularities and $\mathcal{E}$ be a vector bundle over X of dimension d. Then there are canonical isomorphisms*

$$\tilde{H}^{p,q}(Th(\mathcal{E}),\mathbf{Z}) = H^{p-2d,q-d}(X,\mathbf{Z})$$

and the same holds for finite coefficients. In particular there is a canonical isomorphism $\tilde{H}^{2d,d}(Th(\mathcal{E}),\mathbf{Z}) = \mathbf{Z}$.

Proof: By [14,] we have a canonical isomorphism in the homotopy category $\mathcal{H}_{\bullet}(k)$ of the form

$$Th(\mathcal{E}) \cong \mathbf{P}(\mathcal{E}\oplus\mathcal{O})/\mathbf{P}(\mathcal{E}).$$

Together with the projective bundle theorem in DM ([25, Prop. 3.5.1]) this implies the statement of the lemma.

The generator of the group $\tilde{H}^{2d,d}(Th(\mathcal{E}),\mathbf{Z})$ is called the Thom class of $\mathcal{E}$. Combining Lemma 3.20 with 3.11 and 2.3(2) we get the following result.

Theorem 3.21 *Let k be a field which admits resolution of singularities. Then one has $\tilde{H}^{p,q}(\mathbf{MGL},\mathbf{Z}) = 0$ for $p > 2q$ and $\tilde{H}^{0,0}(\mathbf{MGL},\mathbf{Z}) = \mathbf{Z}$. The same holds for finite coefficients.*

Denote by $\tau : \mathbf{MGL} \to \mathbf{H}_{\mathbf{Z}/l}$ the canonical generator of $\tilde{H}^{0,0}(\mathbf{MGL},\mathbf{Z}/l)$. Note that the first part of Theorem 3.21 implies immediately that if α is an element of the motivic Steenrod algebra of bidegree (p,q) with $p > 2q$ then $\alpha(\tau) = 0$. Let $k \subset \mathbf{C}$ be an embedding of our base field into the field of complex numbers. Then there is a canonical isomorphism in the stable homotopy category of the form $t_{\mathbf{C}}(\mathbf{MGL}) = \mathbf{MU}$. In particular for any smooth scheme X over k we get natural homomorphisms $MGL_{p,q}(X) \to MU_p(X(\mathbf{C}))$. The following theorem is the only nontrivial result on algebraic cobordisms which we will use in this paper.

Theorem 3.22 *Let X be a smooth projective variety of dimension d over a subfield k of $\mathbf{C}$. Then there exists an element $\phi_X \in MGL_{2d,d}(X)$ such that the image $t_{\mathbf{C}}(\phi_X)$ of ϕ_X in $MU_{2d}(X(\mathbf{C}))$ is the fundamental class of $X(\mathbf{C})$ in MU-homology.*

3.6 *Main theorem.*

We are ready now to prove the main theorem of this section (Theorem 3.25), which is essentially the only result of the general theory described here which we use in the second part of the paper. We start with some preliminary results. Consider the operation Q_i as a morphism in the category $\mathcal{SH}$ of the form $\mathbf{H_{Z/\mathit{l}}} \to S^1_s \wedge T^{l^i-1} \wedge \mathbf{H_{Z/\mathit{l}}}$ and let $\mathbf{\Phi}_i$ be its homotopy fiber such that we have a distinguished triangle of the form

$$T^{l^i-1} \wedge \mathbf{H}_{\mathbf{Z}/l} \xrightarrow{u} \mathbf{\Phi}_i \xrightarrow{v} \mathbf{H}_{\mathbf{Z}/l} \xrightarrow{Q_i} S^1_s \wedge T^{l^i-1} \wedge \mathbf{H}_{\mathbf{Z}/l}.$$

Denote by $\tilde{\tau}$ the composition

$$\mathbf{MGL} \wedge \mathbf{H}_{\mathbf{Z}/l} \xrightarrow{\tau\wedge Id} \mathbf{H}_{\mathbf{Z}/l} \wedge \mathbf{H}_{\mathbf{Z}/l} \xrightarrow{m} \mathbf{H}_{\mathbf{Z}/l}$$

where τ is the Thom class described in Section 3.5 and m is the multiplication homomorphism.

Lemma 3.23 *There exists a morphism $\tilde{\phi}_n : \mathbf{MGL} \wedge \mathbf{\Phi}_n \to \mathbf{\Phi}_n$ such that the diagram*

$$\begin{array}{ccc}
\mathbf{MGL} \wedge T^{l^n-1} \wedge \mathbf{H}_{\mathbf{Z}/l} & \xrightarrow{\tilde{\tau}} & T^{l^n-1} \wedge \mathbf{H}_{\mathbf{Z}/l} \\
\downarrow & & \downarrow u \\
\mathbf{MGL} \wedge \mathbf{\Phi}_n & \xrightarrow{\tilde{\phi}_n} & \mathbf{\Phi}_n \\
\downarrow & & \downarrow v \\
\mathbf{MGL} \wedge \mathbf{H}_{\mathbf{Z}/l} & \xrightarrow{\tilde{\tau}} & \mathbf{H}_{\mathbf{Z}/l}
\end{array}$$

commutes.

Proof: Since both columns are parts of distinguished triangles it is sufficient to show that the diagram

$$\begin{array}{ccc}
\mathbf{MGL} \wedge \mathbf{H}_{\mathbf{Z}/l} & \xrightarrow{\tilde{\tau}} & \mathbf{H}_{\mathbf{Z}/l} \\
{\scriptstyle Id\wedge Q_i}\downarrow & & \downarrow Q_n \\
\mathbf{MGL} \wedge S^1_s \wedge T^{l^i-1} \wedge \mathbf{H}_{\mathbf{Z}/l} & \xrightarrow{\tilde{\tau}} & S^1_s \wedge T^{l^i-1} \wedge \mathbf{H}_{\mathbf{Z}/l}
\end{array}$$

commutes. This follows from the definition of $\tilde{\tau}$, the formula 3.17(2) for $\Delta(Q_n)$ and the fact that $\tilde{H}^{p,q}(\mathbf{MGL}, \mathbf{Z}/l) = 0$ for $p > 2q$ (Theorem 3.21).

Let $\mathcal{Y}$ be an object of $\mathcal{SH}$ and $\mathcal{Y} \to S^0$ be a morphism. Let further $\tilde{\phi}_n$ be a morphism satisfying the condition of Lemma 3.23 and $\phi_{\mathcal{Y}}$ be an element of $MGL_{2d,d}(\mathcal{Y})$. For any object $\mathcal{X}$ of $\mathcal{SH}$ consider the homomorphism

$$p_{\phi_{\mathcal{Y}},\phi_n} : \tilde{\Phi}_n^{*,*}(\mathcal{Y} \wedge \mathcal{X}) \to \tilde{\Phi}_n^{*-2d,*-d}(\mathcal{X})$$

which takes an element α in $\tilde{\Phi}_n^{*,*}(\mathcal{Y} \wedge \mathcal{X})$ given by a morphism

$$\alpha : \mathcal{Y} \wedge \mathcal{X} \to \mathbf{\Phi}_n(*)[*]$$

to the element given by the composition

$$T^d \wedge \mathcal{X} \stackrel{\phi_{\mathcal{Y}} \wedge Id}{\to} \mathbf{MGL} \wedge \mathcal{Y} \wedge \mathcal{X} \stackrel{Id \wedge \alpha}{\to} \mathbf{MGL} \wedge \mathbf{\Phi}_n(*)[*] \stackrel{\tilde{\phi}_n(*)[*]}{\to} \mathbf{\Phi}_n(*)[*]$$

Proposition 3.24 *Let $\mathcal{Y}$, $\mathcal{X}$ be objects of $\mathcal{SH}$, $p : \mathcal{Y} \to S^0$ be a morphism and $\phi_{\mathcal{Y}}$ be an element in $\tilde{MGL}_{2(l^n-1),l^n-1}(\mathcal{Y})$ such that the image $t_{\mathbf{C}}(p(\phi_{\mathcal{Y}}))$ of $\phi_{\mathcal{Y}}$ in $MU_*(pt)$ is a (ν_n, l)-element. Then for any choice of a morphism ϕ_n satisfying the conditions of Lemma 3.23 the following diagram commutes up to multiplication by a nonzero element in $\mathbf{Z}/l$*

$$\begin{array}{ccc}
\tilde{\Phi}_n^{*,*}(\mathcal{X}) & \stackrel{p^*}{\to} & \tilde{\Phi}_n^{*,*}(\mathcal{Y} \wedge \mathcal{X}) \\
v \downarrow & & \downarrow p_{\phi_{\mathcal{Y}},\phi_n} \\
\tilde{H}^{*,*}(\mathcal{X}, \mathbf{Z}/l) & \stackrel{u}{\to} & \tilde{\Phi}_n^{*-2(l^n-1),*-(l^n-1)}(\mathcal{X})
\end{array}$$

Proof: It follows from the construction of $p_{\phi_{\mathcal{Y}},\phi_n}$ that the composition $p_{\phi_{\mathcal{Y}},\phi_n} \circ p^*$ is given by the morphism

$$f : T^{l^n-1} \wedge \mathbf{\Phi}_n \stackrel{p_*(\phi_{\mathcal{Y}}) \wedge Id}{\longrightarrow} \mathbf{MGL} \wedge \mathbf{\Phi}_n \stackrel{\phi_n}{\to} \mathbf{\Phi}_n$$

By Theorem 3.14 and the definition of $\mathbf{\Phi}_n$ we have

$$Hom_{\mathcal{SH}}(T^{l^n-1} \wedge \mathbf{\Phi}_n, \mathbf{\Phi}_n) = \mathbf{Z}/l$$

The generator of this group corresponds to the composition of the left vertical and the lower horizontal arrows of our diagram. Thus to prove the proposition it is sufficient to show that the morphism f is not zero. Denote the (ν_n, l)-element $t_{\mathbf{C}}(p_*(\phi_{\mathcal{Y}}))$ by a. By Proposition 3.18 and Lemma 3.19 we have $t_{\mathbf{C}}(\mathbf{\Phi}_n) = \mathbf{\Phi}_n^{top}$. Thus applying the topological realization functor we reduce the problem to showing that the composition

$$t_{\mathbf{C}}(f) : S^{2(l^n-1)} \wedge \mathbf{\Phi}_n^{top} \stackrel{a\wedge Id}{\to} \mathbf{MU} \wedge \mathbf{\Phi}_n^{top} \stackrel{t_{\mathbf{C}}(\phi_n)}{\to} \mathbf{\Phi}_n^{top}$$

is not zero. This follows immediately from Lemma 3.6.

Let k be a subfield in $\mathbf{C}$ and X be a variety over k. Denote by I_X the image of the homomorphism

$$\oplus_{i\ge 0} MGL_{2i,i}(X) \to \oplus_{i\ge 0} MGL_{2i,i}(Spec(k)) \stackrel{t_{\mathbf{C}}}{\to} MU_*(pt).$$

Note that Theorem 3.22 implies that I_X contains classes of complex varieties $Y(\mathbf{C})$ for all smooth projective varieties Y over k such that $Hom(Y,X) \neq \emptyset$.

Theorem 3.25 *Let k be a subfield in $\mathbf{C}$ and X be a smooth variety over k such that I_X contains a (ν_n, l)-element. Denote by $\tilde{\mathcal{X}}$ the object of $\mathcal{SH}$ given by the distinguished triangle*

$$\tilde{\mathcal{X}} \to \Sigma_T^\infty \check{C}(X)_+ \to S^0 \to \tilde{\mathcal{X}}[1]$$

Then $\tilde{H}M_n^{p,q}(\tilde{\mathcal{X}}) = 0$ for all $p, q \in \mathbf{Z}$.

Proof: Since $Hom(X,X) \neq \emptyset$, Lemma 3.8 implies that the morphism

$$\Sigma_T^\infty \check{C}(X)_+ \wedge \Sigma_T^\infty X_+ \to \Sigma_T^\infty X_+$$

is an isomorphism in $\mathcal{SH}$ and therefore $\Sigma_T^\infty X_+ \wedge \tilde{\mathcal{X}} = 0$. Proposition 3.24 implies now that the homomorphisms

$$\tilde{\Phi}_n^{p,q}(\tilde{\mathcal{X}}) \to \tilde{H}^{p,q}(\tilde{\mathcal{X}}, \mathbf{Z}/l) \to \tilde{\Phi}_n^{p-2(l^n-1),q-(l^n-1)}(\tilde{\mathcal{X}})$$

are zero for all $p, q \in \mathbf{Z}$ which is tautologically equivalent to the statement of the theorem.

Remark: One can easily see that for any X the subset I_X is an ideal in MU_*. Using a little more of the general theory one can show that this ideal is in fact invariant under the action of the Landweber-Novikov operations on MU_*. In particular it is true that if I_X contains a (ν_n, l)-element for some n and l then it also contains (ν_m, l)-elements for all $m \le n$. It is also true that if I_X contains an element satisfying the second condition of Definition 3.5 for siome n and l then it contains a (ν_n, l)-element.

Remark: There is another way to prove Theorem 3.25 which does not use algebraic cobordisms and the topological realization functor but it is less enlightening than the one we give here. In order to get an alternative proof replace the condition that I_X contains a (ν_n, l)-element by the condition that there exists a (ν_n, l)-variety Y such that $\check{C}(X)\times Y \to Y$ is a weak equivalence. Consider the Thom spectrum $Th(N_X)$ of the normal bundle to X which comes together with a canonical stable map $T^{\wedge(l^n-1)} \to Th(N_X)$. Let *cone* be the cone of this map in $\mathcal{SH}$. Then we have $Th(N_X)\wedge\tilde{\mathcal{X}} \cong 0$ and therefore the morphism $cone[-1]\wedge\tilde{\mathcal{X}} \to T^{\wedge(l^n-1)}\wedge\tilde{\mathcal{X}}$ is a weak equivalence. The Thom class $Th(N_X) \to H_{\mathbf{Z}}$ gives a canonical morphism $cone[-1] \to H_{\mathbf{Z}}[-1]$ which, after smashing with $\tilde{\mathcal{X}}$, gives a map $S^1_s \wedge T^{\wedge(l^n-1)} \wedge \tilde{\mathcal{X}} \to \tilde{\mathcal{X}} \wedge H_{\mathbf{Z}}$. This map gives a homomorphism of graded abelian groups

$$\phi : H^{*,*}(\tilde{\mathcal{X}}, \mathbf{Z}/l) \to H^{*-2l^n+1,*-l^n+1}(\tilde{\mathcal{X}}, \mathbf{Z}/l).$$

Using the realations between the action of the Steenrod operations on the Thom class and characteristic classes (which can be proven in the algebro-geometrical context using the splitting principle in exactly the same way as in topology) one can show that for any x in $H^{*,*}(\tilde{\mathcal{X}}, \mathbf{Z}/l)$ one has

$$x = Q_n(\phi(x)) \pm \phi(Q_n(x)).$$

Therefore, ϕ is a contracting homotopy for the complex $(H^{*,*}(\tilde{\mathcal{X}}, \mathbf{Z}/l), Q_n)$ which shows that $\tilde{H}M_n^{*,*} = 0$.

4 Pfister quadrics and their motives.

In this section we prove the following theorem by constructing varieties satisfying the conditions of Theorem 2.25 for $l = 2$ and any n.

Theorem 4.1 *Let k be a field of characteristic zero. Then for any $q \in \mathbf{Z}$ one has $H_L^{q+1,q}(Spec(k), \mathbf{Z}_{(2)}) = 0$.*

We will deduce Conjecture 1.1 from Theorem 4.1 in Section 5.

Let k be a field of characteristic not equal to 2. For elements $a_1, \ldots, a_n$ in k^* let $< a_1, \ldots, a_n >$ be the quadratic form $\sum a_i x_i^2$. Let further

$$<< a_1, \ldots, a_n >>=< 1, -a_1 > \otimes \ldots \otimes < 1, -a_n > .$$

Denote by $Q_{\underline{a}} = Q_{a_1,\ldots,a_n}$ the projective quadric of dimension $2^{n-1} - 1$ given by the equation $<< a_1, \ldots, a_{n-1} >>= a_n t^2$. This quadric is called the norm quadric associated with the sequence $(a_1, \ldots, a_n)$. The following result is well known but we decided to include the proof since it is crucial for our main theorem.

Proposition 4.2 *The quadric $Q_{\underline{a}}$ is a 2-splitting variety for $\underline{a}$.*

Proof: We are going to show now that if $Q_{\underline{a}}$ has a rational point over k then $\underline{a}$ is divisible by 2 in $K_n^M(k)$. In particular it implies that $\underline{a}$ is divisible by 2 in the generic point of $Q_{\underline{a}}$. We need first the following result.

Lemma 4.3 *For any $\underline{a} = (a_1, \ldots, a_n)$ the following two conditions are equivalent*

1. *$Q_{\underline{a}}$ has a k-rational point (i.e. the form*

$$<< a_1, \ldots, a_{n-1} >> \oplus < -a_n >$$

   represents zero over k).

2. *The form $<< a_1, \ldots, a_n >>$ represents zero over k.*

Proof: Since $<< a_1, \ldots, a_{n-1} >> \oplus < -a_n >$ is a subform in $<< a_1, \ldots, a_n >>$ the first condition implies the second one for obvious reasons. Let us show that if $<< a_1, \ldots, a_n >>$ represents zero over k then so does $<< a_1, \ldots, a_{n-1} >> \oplus < -a_n >$. Denote the quadric given by the equation $<< a_1, \ldots, a_n >>= 0$ by $P_{\underline{a}}$. By [6, Corollary 1.6] we may assume that $<< a_1, \ldots, a_n >>$ is hyperbolic. The quadric $Q_{\underline{a}}$ is a section of $P_{\underline{a}}$ by a linear subspace L of codimension $2^{n-1} - 1$ in $\mathbf{P}^{2^n-1}$. On the other hand since $P_{\underline{a}}$ is hyperbolic for any rational point p of $P_{\underline{a}}$ there exists a linear subspace H of dimension $dim(P_{\underline{a}})/2 = 2^{n-1} - 1$ which lies on $P_{\underline{a}}$ and passes through p. The intersection of H and L is a point on $Q_{\underline{a}}$.

To rpove Proposition 4.2 we proceed by induction on n. Consider first the case $n = 2$. Then $Q_{\underline{a}}$ is given by the equation $x^2 - a_1 y^2 = a_2 z^2$. We may assume that it has a point of the form $(x_0, y_0, 1)$ (otherwise a_1 is a square root in k and the statement is obvious). Then a_2 is the norm of the element $x_0 + a_1^{1/2} y_0$ from $k(a_1^{1/2})$ and thus the symbol (a_1, a_2) is divisible by 2.

Suppose that the lemma is proven for weights smaller than n. The quadric $Q_{\underline{a}}$ is given by the equation $<< a_1, \ldots, a_{n-1} >>= a_n t^2$. The form $<< a_1, \ldots, a_{n-1} >>$ is of the form $< 1 > \oplus \mu_{\underline{a}}$. By induction we may assume that our point $q \in Q_{\underline{a}}(k)$ belongs to the affine part $t \neq 0$. Consider the plane L generated by points $(1, 0, \ldots, 0)$ and q. The restriction of the quadratic form $<< a_1, \ldots, a_{n-1} >>$ to L is of the form $<< b >>$ for some b (the idea is that L is a "subfield" in the vector space where $<< a_1, \ldots, a_{n-1} >>$ lives). Consider the field extension $k(b^{1/2})$. The form $<< b >>$ and therefore the form $<< a_1, \ldots, a_{n-1} >>$ represents zero over $k(b^{1/2})$ and thus by the inductive assumption $(a_1, \ldots, a_{n-1}) = 0$ in $K^M_{n-1}(k(b^{1/2}))/2$. On the other hand by the construction $<< b >>$ represents a_n and therefore we have $a_n \in Im N_{k(b^{1/2})/k} \subset k^*$ which proves the proposition.

The fact that $Q_{\underline{a}}$ is rational over its function field follows from the observation that any quadric with a rational point is rational and that any variety has a rational point over its function field. Therefore to finish the proof of Theorem 4.1 it remains to show that $H^{n+1,n}(\check{C}(Q_{\underline{a}}), \mathbf{Z}_{(2)}) = 0$.

Denote the simplicial scheme $\check{C}(Q_{\underline{a}})$ by $\mathcal{X}_{\underline{a}}$. Our proof goes in two steps. First we use two results due to Markus Rost to show that

$$H^{2^n-1, 2^{n-1}}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}) = 0.$$

Then we use Theorem 3.25 to show that the group $H^{n+1,n}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)})$ embeds into $H^{2^n-1, 2^{n-1}}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)})$ and therefore is zero as well.

Theorem 4.4 *For any $n > 1$ there exists a distinguished triangle in DM_-^{eff} of the form*

$$M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}}) \to M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-1]$$

such that $M_{\underline{a}}$ is a direct summand of $M(Q_{\underline{a}})$.

The proof of Theorem 4.4 is based on the following important result which is due to M. Rost (see [19])[2].

[2] There is another approach to the proof of Theorem 4.4 which does not require Theorem 4.5 but uses more of the theory developed in Section 3.

Theorem 4.5 (M. Rost) *There exists a direct summand $M_{\underline{a}}$ of the motive $M(Q_{\underline{a}})$ together with two morphisms*

$$\psi^* : \mathbf{Z}(2^{n-1}-1)[2^n-2] \to M_{\underline{a}}$$

$$\psi_* : M_{\underline{a}} \to \mathbf{Z}$$

such that for any field F over k where $Q_{\underline{a}}$ has a point the sequence

$$\mathbf{Z}(2^{n-1}-1)[2^n-2] \to (M_{\underline{a}})_F \to \mathbf{Z}$$

is a splitting distinguished triangle in $DM^{eff}(F)$.

The compositions

$$\mathbf{Z}(2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(Q_{\underline{a}})$$

$$M(Q_{\underline{a}}) \to M_{\underline{a}} \to \mathbf{Z}$$

are equal to the fundamental class and the canonical morphism $M(Q_{\underline{a}}) \to \mathbf{Z} = M(Spec(k)$ respectively.

Remark: Theorem 4.5 is formulated in [19] in the context of Chow motives. It implies the same result in the category $DM^{eff}(k)$ by [25].

Theorem 4.4 is, in fact, a rather formal corollary of Theorem 4.5. We start with the following lemmas.

Lemma 4.6 *The sequence of morphisms*

$$M(Q_{\underline{a}})(2^{n-1}-1)[2^n-2] \stackrel{Id\otimes\psi^*}{\to} M(Q_{\underline{a}}) \otimes M_{\underline{a}} \stackrel{Id\otimes\psi_*}{\to} M(Q_{\underline{a}})$$

is a splitting distinguished triangle.

Proof: Let *cone* be a cone of the morphism $\psi^* : \mathbf{Z}(2^{n-1}-1)[2^n-2] \to M_{\underline{a}}$. The morphism $\psi_* : M_{\underline{a}} \to \mathbf{Z}$ factors through a well defined morphism $\phi : cone \to \mathbf{Z}$. By Theorem 4.5 ϕ is an isomorphism in the generic point of $Q_{\underline{a}}$. It implies immediately by general properties of category $DM^{eff}(k)$ that $\phi \otimes Id_{M(Q_{\underline{a}})}$ is an isomorphism. Thus the sequence

$$M(Q_{\underline{a}})(2^{n-1}-1)[2^n-2] \to M(Q_{\underline{a}}) \otimes M_{\underline{a}} \to M(Q_{\underline{a}})$$

can be extended to a distinguished triangle. This triangle splits since by duality ([25, Theorem 4.3.2]) we have

$$Hom(M(Q_{\underline{a}}), M(Q_{\underline{a}})(2^{n-1}-1)[2^n-1]) = H^{2^{n+1}-3, 2^n-2}(Q_{\underline{a}} \times Q_{\underline{a}}, \mathbf{Z})$$

and the right hand side group is zero by Corollary 2.3(2).

Lemma 4.7 *Consider the canonical morphism $M(\mathcal{X}_{\underline{a}}) \to \mathbf{Z}$. Let further N be an object of the thick*[3] *(resp. localizing) subcategory in $DM^{eff}(k)$ generated by objects of the form $M(Q_{\underline{a}}) \otimes L$. Then this morphism induces isomorphisms*

$$Hom(\mathbf{Z}, N) \to Hom(M(\mathcal{X}_{\underline{a}}), N)$$

(resp. isomorphisms $Hom(N, M(\mathcal{X}_{\underline{a}})) \to Hom(N, \mathbf{Z})$).

Proof: It is clearly sufficient to consider $N = M(Q_{\underline{a}}) \otimes L$ for some L in $DM_{-}^{eff}(k)$. In this case our result follows from duality since the morphism $M(\mathcal{X}_{\underline{a}}) \otimes M(Q_{\underline{a}}) \to M(Q_{\underline{a}})$ is an isomorphism by Lemma 3.8.

By Lemma 4.7 the morphism $\psi_* : M_{\underline{a}} \to \mathbf{Z}$ has a canonical lifting to a morphism $\tilde{\psi}_* : M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}})$. Together with the composition

$$\tilde{\psi}^* : M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \to \mathbf{Z}(2^{n-1}-1)[2^n-2] \xrightarrow{\psi^*} M_{\underline{a}}$$

it gives us a sequence of morphisms

$$M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \xrightarrow{\tilde{\psi}^*} M_{\underline{a}} \xrightarrow{\tilde{\psi}_*} M(\mathcal{X}_{\underline{a}})$$

such that $\tilde{\psi}_* \circ \tilde{\psi}^* = 0$. Let *cone* be a cone of $\tilde{\psi}^*$. Then $\tilde{\psi}_*$ factors through a unique morphism $\phi : cone \to M(\mathcal{X}_{\underline{a}})$ and we have to show that ϕ is an isomorphism. The category of objects N such that $\phi \otimes Id_N$ is an isomorphism is a localizing subcategory. Since $M(\mathcal{X}_{\underline{a}}) \otimes M(Q_{\underline{a}}) = M(Q_{\underline{a}})$ we conclude from Lemma 4.6 that it contains $M(Q_{\underline{a}})$ and therefore it contains $M(\mathcal{X}_{\underline{a}})$. On the other hand we have a commutative diagram

$$\begin{array}{ccc} cone \otimes M(\mathcal{X}_{\underline{a}}) & \to & M(\mathcal{X}_{\underline{a}}) \otimes M(\mathcal{X}_{\underline{a}}) \\ \downarrow & & \downarrow \\ cone & \to & M(\mathcal{X}_{\underline{a}}) \end{array}$$

[3] The thick (resp. localizing) subcategory in a triangulated category $\mathcal{D}$ generated by a class of objects A is the minimal triangulated subcategory in $\mathcal{D}$ which is closed under direct summands (resp. direct summands and arbitrary direct sums) and contains A.

with both vertical arrows are clearly isomorphisms. This finishes the proof of Theorem 4.4.

Theorem 4.8 (M. Rost) *For $\underline{a} = (a_1, \ldots, a_n)$ the natural homomorphism*

$$H^{2^n-1,2^{n-1}}(Q_{\underline{a}}, \mathbf{Z}) \to k^*$$

is a monomorphism.

Proof: Since the dimension of $Q_{\underline{a}}$ equals $2^{n-1} - 1$ the left hand side group is isomorphic to the group $H^{2^{n-1}-1}(Q_{\underline{a}}, \underline{K}^M_{2^{n-1}})$ by Corollary 2.4(1) and our result follows from [18].

Corollary 4.9 *For $\underline{a} = (a_1, \ldots, a_n)$ we have*

$$H^{2^n-1,2^{n-1}}(\mathcal{X}_{\underline{a}}, \mathbf{Z}) = 0.$$

Proof: By Theorem 4.4 we have an exact sequence

$$H^{0,1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}) \to H^{2^n-1,2^{n-1}}(\mathcal{X}_{\underline{a}}, \mathbf{Z}) \to H^{2^n-1,2^{n-1}}(M_{\underline{a}}, \mathbf{Z}) \to H^{1,1}(\mathcal{X}_{\underline{a}}, \mathbf{Z})$$

Since $M_{\underline{a}}$ is a direct summand of $M(Q_{\underline{a}})$ Theorem 4.8 implies that the motivic cohomology group $H^{2^n-1,2^{n-1}}(M_{\underline{a}}, \mathbf{Z})$ embeds to the group $H^{2^n-1,2^{n-1}}(\bar{M}_{\underline{a}}, \mathbf{Z})$ where $\bar{M}_{\underline{a}}$ is a lifting of $M_{\underline{a}}$ to the algebraic closure of k. Since

$$H^{2^n-1,2^{n-1}}(\mathcal{X}_{\underline{a}} \times Spec(\bar{k}), \mathbf{Z}) = H^{2^n-1,2^{n-1}}(Spec(\bar{k}), \mathbf{Z}) = 0$$

we conclude that the second arrow in this exact sequence is zero. On the other hand by $H90(1, 2)$, Corollary 2.13(1) and Proposition 2.7 we have

$$H^{0,1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}) = H^{0,1}(Spec(k), \mathbf{Z}) = 0$$

which finishes the proof of the corollary.

Proposition 4.10 *For any sequence $(a_1, \ldots, a_n)$ one has*

$$H^{n+1,n}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}) = 0.$$

Proof: Consider the object $\tilde{\mathcal{X}}_{\underline{a}}$ in $\mathcal{SH}(k)$ given by the distinguished triangle

$$\tilde{\mathcal{X}}_{\underline{a}} \to \Sigma_T^\infty(\mathcal{X}_{\underline{a}})_+ \to S^0 \to \tilde{\mathcal{X}}_{\underline{a}}[1]$$

Since $H^{p,q}(Speck(k), \mathbf{Z}) = 0$ for $p > q$ (2.3(1)) the canonical homomorphisms $\tilde{H}^{p,q}(\mathcal{X}_{\underline{a}}, \mathbf{Z}) \to H^{p,q}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z})$ are isomorphisms for $p > q$. Thus it is sufficient to verify that $\tilde{H}^{n+1,n}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}_{(2)}) = 0$. Since there exists an extension of degree two E/k such that $Q_{\underline{a}}(Spec(E)) \neq \emptyset$ Lemma 3.8 together with the standard transfer argument implies that all the motivic cohomology groups of $\tilde{\mathcal{X}}_{\underline{a}}$ have exponent at most 2. Thus it is sufficient to show that the image of $\tilde{H}^{n+1,n}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}_{(2)})$ in $\tilde{H}^{n+1,n}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}/2)$ is zero. Let u be an element of this image. Consider the composition of cohomological operations $Q_{n-2}Q_{n-3}\dots Q_1$. It maps u to an element of $\tilde{H}^{2^n-1,2^{n-1}}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}/2)$ and Theorem 3.17(3) implies that this element belongs to the image of the corresponding integral motivic cohomology group. Thus by Corollary 4.9 it is zero. It remains to verify that the composition $Q_{n-2}Q_{n-3}\dots Q_1$ is a monomorphism.

Lemma 4.11 *The ideal $I_{Q_{\underline{a}}}$ contains $(\nu_i, 2)$-elements for all $i \leq n-1$.*

Proof: Fix an embedding $k \subset \mathbf{C}$. By Theorem 3.22 the ideal $I_{Q_{\underline{a}}}$ contains the cobordisms classes of all smooth subvarieties in$Q_{\underline{a}}$ which includes in particular all the plane sections. Thus it is sufficient to show that a quadric Q in $\mathbf{CP}^{2^i}$ satisfies the conditions of Definition 3.5 for $l = 2$ and $n = i$. Let $j : Q \to \mathbf{CP}^{2^i}$ be the embedding. Since Q is the set of zeros of a generic section of the line bundle $\mathcal{O}(2)$ on $\mathbf{CP}^{2^i}$ we have a short exact sequence

$$0 \to T_Q \to j^*(T_{\mathbf{CP}^{2^i}}) \to j * (\mathcal{O}(2)) \to 0$$

and therefore the class of the normal bundle to Q equals to $j^*(T_{\mathbf{CP}^{2^i}} - \mathcal{O}(2))$ in $K_0(Q)$. In particular any characteristic number of Q can be obtained by evaluating a cohomology class from $H^{2(2^i-1)}(\mathbf{CP}^{2^i}, \mathbf{Z})$ on the fundamental class of the quadric $[Q]$ in $H_{2(2^i-1)}(\mathbf{CP}^{2^i}, \mathbf{Z})$. The calss $[Q]$ is divisible by 2 which implies that all the chracteristic numbers of Q are divisible by 2. To compute $s(Q)$ we may use the fact that the characteristic classes $s_i(-)$ are additive. In particular we have:

$$s(Q) = -s_{2^i-1}(T_Q)[Q] = (s_{2^i-1}(T_{\mathbf{CP}^{2^i}}) - s_{2^i-1}(\mathcal{O}(2)))[Q]$$

where on the right hand side $[Q]$ denotes the fundamental class of Q in $H_{2(2^i-1)}(\mathbf{CP}^{2^i},\mathbf{Z})$. We have $T_{\mathbf{CP}^{2^i}}=(2^i+1)\mathcal{O}(1)$ in $K_0(\mathbf{CP}^{2^i})$ and therefore

$$-s(Q)=(s_{2^i-1}(T_{\mathbf{CP}^{2^i}})-s_{2^i-1}(\mathcal{O}(2)))[Q]=2((2^i+1)-2^{2^i-1})=2(mod\ 4).$$

By Theorem 3.25 and Lemma 4.11 we have $\tilde{H}M_i^{*,*}(\tilde{\mathcal{X}}_{\underline{a}})=0$ for all $i\leq n-1$. To show that the composition $Q_{n-2}\dots Q_1$ is a monomorphism it is sufficient to show that the operation Q_i acts monomorphically on the group $\tilde{H}^{n-i+2(2^{i-1}-1),n-i+2^{i-1}-1}(\tilde{\mathcal{X}}_{\underline{a}},\mathbf{Z}/2)$ for $i=1,\dots,n-2$. Since $\tilde{H}M_i^{*,*}(\tilde{\mathcal{X}}_{\underline{a}})=0$ the kernel of Q_i on this group is covered by the image of $\tilde{H}^{n-i-1,n-i}(\tilde{\mathcal{X}}_{\underline{a}},\mathbf{Z}/2)$. The later group is zero by the inductive assumption that H90(n-1,2) holds, Proposition 2.7 and Corollary 2.13(2).

5 Applications.

In this section we prove some corollaries of our main theorem 4.1. We start with the following result.

Theorem 5.1 (Milnor Conjecture) *For any field k of characteristic not equal to 2 and any $n\geq 0$, $m>0$ the norm residue homomorphisms*

$$K_n^M(k)/2^m\to H^n_{et}(k,\mu_{2^m}^{\otimes n})$$

are isomorphisms.

Proof: For fields k of characteristic zero this follows immediately from Theorem 4.1 and Corollary 2.13. To prove the theorem for fields of positive characteristic we need the following lemma.

Lemma 5.2 *Let n be an integer and l be a prime such that for any field k of characteristic zero the norm-residue homomorphisms*

$$K_n^M(k)/l\to H^n_{et}(k,\mu_l^{\otimes n})$$

are isomorphisms. Then the same holds for all fields k of characteristic different from l.

Proof: The standard transfer argument shows that it is sufficient to prove that the norm-residue homomorphism is an isomorphism for fields k which have no extensions of degree prime to l. In particular we may assume that k is a perfect field. Then the standard Witt ring construction gives us a complete discrete valuation ring R with the function field K of characteristic zero and the residue field k. Define the Milnor K-ring $K_*^M(R)$ as the quotient of the tensor algebra of R^* with respect to the ideal generated by elements of the form $x \otimes (1-x)$. The natural homomorphism $K_n^M(R)/l \to K_n^M(k)/l$ is an isomorphism since R is henselian. Choosing a generator $\pi \in K^*$ of K^* over R^* we may construct a commutative diagram of the form

$$\begin{array}{ccccc} K_n^M(k)/l & \stackrel{i_K}{\to} & K_{n+1}^M(K)/l & \stackrel{p_K}{\to} & K_n^M(k)/l \\ \downarrow & & \downarrow & & \downarrow \\ H_{et}^n(k, \mu_l^{\otimes n}) & \stackrel{i_H}{\to} & H_{et}^{n+1}(K, \mu_l^{\otimes(n+1)}) & \stackrel{p_H}{\to} & H_{et}^n(k, \mu_l^{\otimes n}) \end{array}$$

as follows. Take the vertical arrows to be the norm-residue homomorphisms. To get the left horizontal arrows observe that there are canonical homomorphisms

$$K_n^M(k)/l \cong K_M^n(R)/l \to K_n^M(K)/l$$

$$H_{et}^n(k, \mu_l^{\otimes n}) \cong H_{et}^n(Spec(R), \mu_l^{\otimes n}) \to H_{et}^n(K, \mu_l^{\otimes n})$$

and we define i_K and i_H as compositions of these homomorphisms with the multiplication by the class of π. We define p_K as the standard residue map which is determined by the property that its value on a symbol of the form $(\pi^m a_1, \dots, a_{n+1})$ where $a_i \in R^*$ equals $m(a_2, \dots, a_{n+1})$. To define p_H proceed as follows. Let $j : Spec(K) \to Spec(R)$ and $x : Spec(k) \to Spec(R)$ be the canonical open and closed embeddings. By [11, Lemma 1.4 p.183] we have $\mathbf{R}j_*(\mathbf{G}_m) = j_*(\mathbf{G}_m)$. We have a natural homomorphism $j_*(\mathbf{G}_m) \to i_*(\mathbf{Z})$ and therefore we get a morphism in the derived category

$$\mathbf{R}j_*(\mu_l^{\otimes n+1}) = (\mathbf{R}j_*(\mathbf{G}_m) \overset{L}{\otimes}_{\mathbf{Z}} \mathbf{Z}/l) \otimes_{\mathbf{Z}/l} \mu_l^{\otimes n} \to i_*(\mathbf{Z}/l) \otimes_{\mathbf{Z}/l} \mu_l^{\otimes n} = i_*(\mu_l^{\otimes n})$$

which gives the morphism p_H on the etale cohomolgy groups. One verifies easily that this diagram is indeed commutative and that the compositions of the horizontal arrows are identity homomorphisms. That finishes the proof of Lemma 5.2.

Corollary 5.3 *For any field k of characteristic not equal to 2 and any $n \geq 0$ the etale cohomology group $H^{n+1}_{et}(k, \mathbf{Z}_2(n))$ is torsion free.*

The following nice corollary of Theorem 5.1 is due to S. Bloch.

Corollary 5.4 *Let $\alpha \in H^i(X, \mathbf{Z})$ be a 2-torsion element in the integral cohomology of a complex algebraic variety X. Then there exists a divisor Z on X such that the restriction of α to $X - Z$ is zero.*

Proof: Since $2\alpha = 0$, α is the image of an element α' in $H^{i-1}(X, \mathbf{Z}/2)$ with respect to the Bockstein homomorphism $\beta : H^{i-1}(X, \mathbf{Z}/2) \to H^i(X, \mathbf{Z})$. By Theorem 5.1 there exists a dense open subset $U = X - Z$ of X such that α' on U is in the image of the canonical map $(\mathcal{O}^*(U))^{\otimes(i-1)} \to H^{i-1}(U, \mathbf{Z}/2)$. Since this map factors through the integral cohomology group we have $\beta(\alpha') = 0$ on U.

Denote by $B/m(n)$ the complex truncation at the cohomological level n of the complex of sheaves $\mathbf{R}\pi_*(\pi^*(\mu_m^{\otimes n}))$ where $\pi : (Sm/k)_{et} \to (Sm/k)_{Nis}$ is the canonical morphism of sites. Combining Theorem 4.1 with [22, Theorem 5.9] we get.

Theorem 5.5 *For a smooth variety X over a field of characteristic zero and any $m > 0$ there are canonical isomorphisms*

$$H^{p,q}(X, \mathbf{Z}/2^m) = \mathbf{H}^p_{Nis}(X, B/2^m(q))$$

In particular, for any X as above $H^{p,q}(X, \mathbf{Z}/2^m) = 0$ for $p < 0$.

In [3] S. Bloch and S. Lichtenbaum constructed a spectral sequence which starts from Higher Chow groups of a field and converges to its algebraic K-theory. Combining their result with Theorem 5.5 and the comparison result relating Higher Chow groups and motivic cohomology ([25, Prop. 4.2.9 and Th. 4.3.2]) we obtain the following.

Theorem 5.6 *Let k be a field of characteristic zero. Then there exists a spectral sequence, natural with respect to extensions of k, with the E_2-term of the form*

$$E_2^{p,q} = \begin{cases} H^{p-q}_{et}(k, \mathbf{Z}/2) & \text{for } p, q \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

which converges to $K_{-p-q}/2(k)$ (K-theory with $\mathbf{Z}/2$-coefficients).

Corollary 5.7 *Let X be a complex algebraic variety of dimension d. Then the canonical homomorphisms $K_i^{alg}/2(X) \to K_i^{top}/2(X)$ are isomorphisms for $i \geq d-1$ and monomorphisms for $i = d-2$.*

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