A remark on the connection between affine Lie algebras and soliton equations

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Abstract

The Lax equations of Drinfeld-Sokolov are derived in the framework of the Fock representation of the Clifford algebras. The derivation is based on the bilinear identities for τ-functions.

1. In a series of papers [1-8], the Fock representations of the Clifford algebras and the so-called boson-fermion correspondence (= the realization of the Fock representations) are employed to study soliton equations. Infinite dimensional Lie algebras and corresponding groups in the Clifford algebras are identified with the transformations of solutions of soliton equations. Representation theoretic aspects of these studies are summarized and developed further in [9]. Key notions are the τ-functions and the bilinear identities for them. They afford us a unification of two major methods in studying soliton equations: the linearization and the bilinearization.

On the other hand, Drinfeld and Sokolov [10] present a method to generate soliton equations. They construct the Lax equations starting from the realizations of the affine Lie algebras as Lie algebras over Laurent polynomials. They focus mainly on the hamiltonian structures of these soliton equations. Subsequently Wilson [11] and Imbens [12] consider the τ-functions for such equations.

In this note we show that the Lax equations of Drinfeld-Sokolov for the generalized modified KdV equations and also for the generalized KdV equations naturally arise in the framework of [9]. In other words, for affine Lie algebras $A_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$, we can derive the Lax equations of [10] on the basis of the Fock representation of the Clifford algebras.

In section 2, we summarize some of the results in [9]. In section 3, the bilinear identities are rewritten in terms of pseudo differential operators as in [6,8]. In section 4, we extract the Lax equations of [10] from the identities in section 2.

2. We recall here some of the results of [9] relevant to this note. For details and notations, readers are referred to [9].

In [9], the infinite dimensional Lie algebras A_∞, B_∞, B'_∞, C_∞, D_∞ and D'_∞ are realized as appropriate totalities of quadratic expressions of the free fermions (= linear combinations of generators of the Clifford algebras). The algebras B_∞, C_∞ and D_∞ are realized as subalgebras of A_∞ consisting of fixed points of suitable involutions of the Clifford algebra of charged free fermions. The affine Lie algebras $A_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$ are obtained as subalgebras of these algebras by imposing certain periodicity conditions on the coefficients of linear combinations of generators of A_∞, B_∞, C_∞ and D_∞ (the reduction).

The Fock representations of the Clifford algebras induce highest weight representations of these Lie algebras. Through the so-called boson-fermion correspondence these representations are realized on the polynomial algebras of infinitely many variables. Elements of the group orbits of highest weight vectors in these realizations are called the τ-functions.

We are interested in the equations satisfied by the τ-functions. Below we quote them from [9].

For the A_∞ case, we have a sequence of τ-functions $\{\tau_n(x)\}_{n\in\mathbb{Z}}$, $x = (x_1, x_2, \ldots)$. Each $\tau_n(x)$ corresponds to the fundamental weight Λ_n of A_∞ .

They satisfy the identity of the following form

$$0 = \oint \frac{dk}{2\pi i} e^{\xi(x-x',k)} k^{n-n'} \tau_n(x-\epsilon(k^{-1}))\tau_{n'}(x'+\epsilon(k^{-1})) \qquad (1)$$

for any x, x', $n \geq n'$, where the integration is taken over a small circuit around $k = \infty$ so that $\oint \frac{dk}{2\pi ik} = 1$. Here $\xi(x,k) = \sum_{j\geq 1} x_j k^j$ and $\epsilon(k^{-1}) = (\frac{1}{k}, \frac{1}{2k^2}, \frac{1}{3k^3}, \ldots)$. This is one way of writing the Plücker relations for the infinite dimensional Grassmann manifold. We refer to identities of this type as the bilinear identities.

For B_∞ (resp. C_∞) τ-functions are those for A_∞ subject to extra condition

$$\tau_{1-n}(x) = \tau_n(\tilde{x}), \quad \tilde{x} = (x_1, -x_2, x_3, \ldots), \quad n \in \mathbb{Z}$$

$$(\text{resp. } \tau_{-n}(x) = \tau_n(\tilde{x})).$$

This condition is a reflection of the involution which singles out B_∞ (resp. C_∞) as a subalgebra of A_∞ .

For D_∞ we have τ-functions $\{\tau_{n_1,n_2,n}(x^{(1)}, x^{(2)})\}_{n_1,n_2,n\in\mathbb{Z}}$, $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots)$, $i = 1,2$. They satisfy the following bilinear identity

$$0 = (-1)^{n_2+n_2'} \oint \frac{dk}{2\pi ik} e^{\xi(x^{(1)}-x^{(1)'},k)} k^{n_1-n_1'+1} \times$$

$$\tau_{n_1,n_2,n}(x^{(1)}-\epsilon(k^{-1}),x^{(2)})\tau_{n_1',n_2',n'}(x^{(1)'}+\epsilon(k^{-1}),x^{(2)'}) \qquad (2)$$

$$+ \oint \frac{dk}{2\pi ik} e^{\xi(x^{(1)}-x^{(1)'},k)} k^{n_1-n_1'+1} \times$$

$$\tau_{n_1+1,n_2-1,n-1}(x^{(1)},x^{(2)}-\epsilon(k^{-1}))\tau_{n_1'-1,n_2'+1,n'+1}(x^{(1)'},x^{(2)'}+\epsilon(k^{-1}))$$

for any $n_1 - n_1' \geq n' - n \geq n_2' - n_2$, $x^{(i)},x^{(i)'}$, $i = 1,2$, and the symmetries

$$\tilde{\tau}_{1-n_1,1-n_2,-n}(x^{(1)},x^{(2)}) = (-)^{n(n_1+n_2)} \tau_{n_1,n_2,n}(x^{(1)},x^{(2)}), n_1,n_2,n \in \mathbb{Z} .$$

The algebras B_∞' and D_∞' are realized in terms of neutral free fermions and are suitable for spin representations of these algebras. As for the bilinear identities we refer to [9,5].

As a consequence of the reduction, τ-functions of the affine Lie algebras are subject to further constraints. The table of them together with the one for the corresponding highest weight are found in Tables 2,3 of [9].

3. In this section we rewrite bilinear identities in terms of pseudo differential operators of order 0. They are introduced through wave functions as follows.

Recall the definition of the wave functions for A_∞

$$w_n(x,k) = e^{\xi(x,k)} \frac{\tau_n(x-\epsilon(k^{-1}))}{\tau_n(x)}$$

$$w_n^*(x,k) = e^{-\xi(x,k)} \frac{\tau_n(x+\epsilon(k^{-1}))}{\tau_n(x)} .$$

We define pseudo differential operators P_n, Q_n by

$$w_n(x,k) = P_n(x,\partial)e^{\xi(x,k)}, \quad \partial = \partial/\partial x_1$$
$$w_n^*(x,k) = Q_n(x,-\partial)e^{-\xi(x,k)} . \tag{3}$$

Namely P_n and Q_n are defined through the expansions of w_n, w_n^* :

$$P_n(x,\partial) = \sum_{\alpha\geq 0} p_{n\alpha}(x)\partial^{-\alpha}, \quad w_n(x,k) = e^{\xi(x,k)}\Big(\sum_{\alpha\geq 0} p_{n\alpha}(x)k^{-\alpha}\Big) ,$$

$$Q_n(x,\partial) = \sum_{\alpha\geq 0} q_{n\alpha}(x)(-\partial)^{-\alpha}, \quad w_n^*(x,k) = e^{-\xi(x,k)}\Big(\sum_{\alpha\geq 0} q_{n\alpha}(x)k^{-\alpha}\Big) .$$

Similarly we set

$$w^{(i)}_{n_1,n_2,n}(x^{(1)},x^{(2)},k) = e^{\xi(x^{(i)},k)}\frac{\tau_{n_1,n_2,n}(x^{(1)}-\delta_{i1}\epsilon(k^{-1}),x^{(2)}-\delta_{i2}\epsilon(k^{-1}))}{\tau_{n_1,n_2,n}(x^{(1)},x^{(2)})},$$

$$= P^{(i)}_{n_1,n_2,n}(x^{(1)},x^{(2)},\partial^{(i)})e^{\xi(x^{(i)},k)} ,$$

$$w^{(i)*}_{n_1,n_2,n}(x^{(1)},x^{(2)},k) = e^{-\xi(x^{(i)},k)}\frac{\tau_{n_1,n_2,n}(x^{(1)}+\delta_{i1}\epsilon(k^{-1}),x^{(2)}+\delta_{i2}\epsilon(k^{-1}))}{\tau_{n_1,n_2,n}(x^{(1)},x^{(2)})} ;$$

$$= Q^{(i)}_{n_1,n_2,n}(x^{(1)},x^{(2)},-\partial^{(i)})e^{-\xi(x^{(i)},k)}, \quad \partial^{(i)} = \partial/\partial x_1^{(i)}, \ i = 1,2 .$$

First let us examine the consequences of the bilinear identities for A_∞ . Hereafter for a pseudo differential operator $P(x,\partial) = \sum_j p_j(x)\partial^j$, we denote its formal adjoint by $P(x,\partial)^* = \sum_j (-\partial)^j p_j(x)$ and by P_- its negative part:

$P(x,\partial)_- = \sum_{j<0} p_j(x)\partial^j$. Then as in [4,8] we have

<u>Proposition 1</u>. The bilinear identity (1) implies

$$(P_n(x,\partial)\partial^{n-n'}Q_{n'}(x,\partial)^*)_- = 0 \ ,$$

i.e. $P_n(x,\partial)\partial^{n-n'}Q_{n'}(x,\partial)^*$ is a differential operator of order $n - n'$.

Similarly, we have

<u>Proposition 2</u>. As a consequence of the bilinear identity (2), the following relations are valid

$$(P^{(1)}_{n_1,n_2,n}(x,\partial)\partial^{n_1-n_1'}Q^{(1)}_{n_1',n_2',n'}(x,\partial)^*)_-$$

$$= (-)^{n_2+n_2'+1} \sum_{\alpha+\beta=n_2-n_2'-1} \tilde{p}^{(2)}_{n_1+1,n_2-1,n-1;\alpha}(x)\partial^{-1}\tilde{q}^{(2)}_{n_1'-1,n_2'+1,n'+1;\beta}(x) \ ,$$

where $\partial = \partial^{(1)}$, $x = (x^{(1)},x^{(2)})$, $P^{(i)}_{n_1,n_2,n}(x,\partial) = \sum_{\alpha\geq 0} p^{(i)}_{n_1,n_2,n;\alpha}(x)\partial^{-\alpha}$,

$Q^{(i)}_{n_1,n_2,n}(x,\partial) = \sum_{\alpha\geq 0} q^{(i)}_{n_1,n_2,n;\alpha}(x)\partial^{-\alpha}$, $\tilde{p}^{(2)}_{n_1,n_2,n;\alpha} = \dfrac{\tau_{n_1,n_2,n}}{\tau_{n_1-1,n_2+1,n+1}} p^{(2)}_{n_1,n_2,n;\alpha}$,

$\tilde{q}^{(2)}_{n_1,n_2,n;\alpha} = \dfrac{\tau_{n_1,n_2,n}}{\tau_{n_1+1,n_2-1,n-1}} q^{(2)}_{n_1,n_2,n;\alpha}$.

Under extra condition on τ-functions we have

<u>Proposition 3</u>. The constraint

$$\tau_m(\tilde{x}) = \tau_n(x)$$

implies the relations

$$w^*_m(\tilde{x},-k) = w_n(x,k) ,$$

$$P_m(\tilde{x},\partial) = Q_n(x,\partial) = P_n(x,\partial)^{*-1} .$$

4. In this section we consider exclusively wave functions corresponding to affine Lie algebras. We derive linear differential equations for them and see that they are of the same type as those introduced by Drinfeld and Sokolov [10].

$A_n^{(1)}$ case. The τ-functions satisfy the relations

$$\tau_{j+n+1} = \tau_j, \quad \partial\tau_j/\partial x_{(n+1)\nu} = 0, \quad \nu = 1,2,\ldots .$$

The τ-function τ_j corresponds to the highest weight Λ_j of $A_n^{(1)}$, $0 \le j \le n$. By Propositions 1,3, we have

$$P_{j+1}(x,\partial)\partial = (\partial+v_j(x))P_j(x,\partial) ,$$

where

$$v_j(x) = (\partial \log)(\tau_j(x)/\tau_{j+1}(x)) .$$

Recalling the relation (3) between wave functions and pseudo differential operators, we get

$$(\partial+v_j(x))w_j(x,k) = k\ w_{j+1}(x,k) . \tag{4}$$

This system of linear differential equations of the first order is exactly of the same type as those in [10] used to describe the generalized modified KdV equations for the case of $A_n^{(1)}$. Adjoint wave functions $w_j(x,k)$ satisfy the adjoint equations. For a fixed j , the wave function $w_j(x,k)$ satisfies

$$(\partial+v_{n+j}(x))\dots(\partial+v_j(x))w_j(x,k) = k^{n+1}w_j(x,k) \ .$$

Expanding the operator in the right hand side and noting the relation $v_{n+j+1} = v_j$, we get an operator of the form

$$\partial^{n+1} + \sum_{\alpha=0}^{n-1} u_{j\alpha}(x)\partial^{\alpha} \ .$$

The operator of this type appears in [10] to define scalar Lax equations for $A_n^{(1)}$.

$D_{n+1}^{(2)}$ <u>case</u>. In this case we have $2(n+1)$ different τ-functions. Among them the relations

$$\tau_j(x) = \tau_{1-j}(\tilde{x}) = \tau_{j+2(n+1)}(x), \ \partial\tau_j/\partial x_{2(n+1)\nu} = 0, \ \nu = 1,2,\dots$$

hold. As in the $A_n^{(1)}$ case, we have (4) representing the linear differential equations for the modified equation.

By proposition 3 wave functions and adjoint wave functions are related by

$$w^*_{1-j}(\tilde{x},-k) = w_j(x,k) \ .$$

These correspond to the highest weight Λ_{j-1} of $D^{(2)}_{n+1}$ $2 \leq j \leq n$. Taking this fact into account, we decompose the differential equation for $w_j(x,k)$ into a pair of differential equations for $w_j(x,k)$ and $w^*_j(\tilde{x},-k)$.

$$L_1(x,\partial)w_j(x,k) = (\partial+v_{2n+2-j}(x))\ldots(\partial+v_j(x))w_j(x,k)$$
$$= k^{2(n-j)+3}w^*_j(\tilde{x},-k) ,$$
$$L_2(x,\partial)w^*_j(\tilde{x},-k) = (\partial+v_{j-1}(x))\ldots(\partial+v_{1-j}(x))w^*_j(\tilde{x},-k)$$
$$= k^{2j-1}w_j(x,k) .$$

The differential operators L_1, L_2 are "skew symmetric" in the sense

$$L_i(x,\partial)^* = -L_i(\tilde{x},\partial), \quad i = 1,2 .$$

Thus setting $\tilde{x} = x$, we obtain the pairs of scalar differential operators as in [10] for $2 \leq j \leq n$. Similar calculation using the bilinear identity for B'_∞ and extra bilinear identities in [6] gives us the scalar Lax equations of [10] corresponding to extreme vertices of the Dynkin diagram of $D^{(2)}_{n+1}$.

The cases $A^{(2)}_{2n}$ and $C^{(1)}_n$ are treated analogously.

$D^{(1)}_n$ case. The τ-functions satisfy the constraints

$$\tau_{j_1+2(n-1),j_2+2,j}(x) = (-)^{j(j_1+j_2)}\tau_{1-j_1,1-j_2,-j}(\tilde{x}) = \tau_{j_1,j_2,j}(x), x=(x^{(1)},x^{(2)}), \quad (5)$$

$$\partial\tau_{j_1,j_2,j}/\partial x^{(1)}_{2(n-1)\nu} + \partial\tau_{j_1,j_2,j}/\partial x^{(2)}_{2\nu} = 0, \quad \nu = 1,2,\ldots .$$

In this case the wave functions $w^{(1)}_{j,1,0}(x,k)$, $w^{(1)}_{n-1+j,2,0}(x,k)$, $1 \le j \le n-1$, $a_0(x,k) = k^{-1}\frac{\tau_{-1,1,1}(x)}{\tau_{0,0,0}(x)} w^{(1)}_{2n-3,3,1}(x,k)$, $a_n(x,k) = k^{-1}\frac{\tau_{n-2,2,1}(x)}{\tau_{n-1,1,0}(x)} w^{(1)}_{n-2,2,1}(x,k)$ satisfy a system of linear differential equations of the first order. By Proposition 2, we have

$$k\, w^{(1)}_{(i-1)(n-1)+j+1,i,0}(x,k) =$$
$$(\partial + v_{(i-1)(n-1)+j,i}(x))\, w^{(1)}_{(i-1)(n-1)+j,i,0}(x,k)\ ,\ i = 1,2,\ 1 \le j \le n-1\ ,$$
$$k\, w^{(1)}_{1,1,0}(x,k) =$$
$$(\partial + \partial\log\frac{\tau_{0,0,0}(x)}{\tau_{1,1,0}(x)})\, w^{(1)}_{2n-2,2,0}(x,k) + \frac{\tau_{2,0,-1}(x)}{\tau_{1,1,0}(x)}\, a_0(x,k)\ ,$$
$$\partial a_0(x,k) = \frac{\tau_{-1,1,1}(x)}{\tau_{0,0,0}(x)}\, w^{(1)}_{2n-2,2,0}(x,k)\ ,$$
$$k\, w^{(1)}_{n,2,0}(x,k) = (\partial + \partial\log\frac{\tau_{n-1,1,0}(x)}{\tau_{n,2,0}(x)})\, w^{(1)}_{n-1,1,0}(x,k) + \frac{\tau_{n+1,1,-1}(x)}{\tau_{n,2,0}(x)}\, a_n(x,k)\ ,$$
$$\partial a_n(x,k) = \frac{\tau_{n-2,2,1}(x)}{\tau_{n-1,1,0}(x)}\, w^{(1)}_{n-1,1,0}(x,k)\ .$$

Here we have used the relations like

$$P^{(1)}_{1,1,0}\partial = (\partial + \partial\log\frac{\tau_{0,0,0}}{\tau_{1,1,0}} + \frac{\tau_{2,0,-1}}{\tau_{1,1,0}}\,\partial^{-1}\,\frac{\tau_{-1,0,0}}{\tau_{0,0,0}})\, P^{(1)}_{0,0,0}$$
$$P^{(1)}_{-1,1,1}\partial^{-1} = \frac{\tau_{0,0,0}}{\tau_{-1,1,1}}\,\partial^{-1}\,\frac{\tau_{-1,1,1}}{\tau_{0,0,0}}\, P^{(1)}_{0,0,0}\ .$$

The above system is shown to be of the same type as that of [10] for the modified equation when restricted to $\tilde{x} = x$.

Further we have

$$w^{(1)}_{2n-1-j,2,0}(x,k) = w^{(1)*}_{j,1,0}(\tilde{x},-k)$$

by (5). These correspond to the highest weight Λ_j of $D_2^{(1)}$, $2 \le j \le n-2$. As in the case of $D_{n+1}^{(2)}$, we can derive equations among $w_{j,1,0}^{(1)}(x,k)$ and $w_{j,1,0}^{(1)*}(\tilde{x},-k)$. They take the following forms

$$L_1(x,\partial)w_{j,1,0}^{(1)}(x,k) = k^{2(n-j)-1}w_{j,1,0}^{(1)*}(\tilde{x},-k) \;,$$

$$L_2(x,\partial)w_{j,1,0}^{(1)*}(\tilde{x},-k) = k^{2j-1}w_{j,1,0}^{(1)}(x,k) \;,$$

$$L_1(x,\partial) = (\partial+v_{2n-2-j,2}) \cdots (\partial+v_{n,2}) \times$$

$$(\partial+\partial\log \frac{\tau_{n-1,1,0}}{\tau_{n,2,0}} + \frac{\tau_{n+1,1,-1}}{\tau_{n,2,0}}\partial^{-1}\frac{\tau_{n-2,2,1}}{\tau_{n-1,1,0}})(\partial+v_{n-2,1}) \cdots (\partial+v_{j,1})$$

and a similar expression for L_2. These pseudo differential operators satisfy

$$L_i(x,\partial)^* = -L_i(\tilde{x},\partial), \; i = 1,2 \;.$$

The operators corresponding to extreme vertices are derived by using D'_∞.

The case $A_{2n-1}^{(2)}$ is similar.

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