KOTTWITZ-RAPOPORT STRATA IN THE SIEGEL MODULI SPACES

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This is a write-up note of the talk where I gave in the conference on “Géométrie arithmétique, représentations galoisiennes et formes modulaire”, held on June 6 – 8, 2007, at Université Paris-Nord, Villetaneuse, Paris. The goals are

- Describe results on the Kottwitz-Rapoport (KR) stratification and provide examples; those are due to Kottwitz-Rapoport, de Jong, Ngô-Genestier, Haines, Görtz, Tilouine, and C.-F. Yu.
- Report some results in the case $g = 3$ (joint work with Ulrich Görtz).

1. Moduli spaces

Let $g \geq 1$ be an integer, p a rational prime, $N \geq 3$ an integer with $(p, N) = 1$. Choose $\zeta_N \in \overline{\mathbb{Q}} \subset \mathbb{C}$ a primitive Nth root of unity and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Put $I := \{0, 1, \ldots, g\}$. Let $\mathcal{A}_I$ be the moduli space over $\overline{\mathbb{F}}_p$ parametrizing equivalence classes of objects

$$(A_0 \xrightarrow{\alpha} A_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} A_g, \lambda_0, \lambda_g, \eta),$$

where

- A_i is a g-dimensional abelian variety,
- α is an isogeny of degree p,
- λ_0 and λ_g are principal polarizations on A_0 and A_g, respectively.
- η is a symplectic level-N structure on A_0 w.r.t. ζ_N.

Put $\eta_0 := \eta$, $\eta_i := \alpha_* \eta_{i-1}$ for $i = 1, \ldots, g$, and $\lambda_{i-1} := \alpha^* \lambda_i$ for $i = g, \ldots, 2$. Let $\underline{A}_i := (A_i, \lambda_i, \eta_i)$. Then $\mathcal{A}_I$ parametrizes equivalence classes of objects

$$(\underline{A}_0 \xrightarrow{\alpha} \underline{A}_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \underline{A}_g),$$

where $\underline{A}_0 \in \mathcal{A}_{g,1,N}$, and for $i \neq 0$,

$$\underline{A}_i \in \mathcal{A}'_{g,p^{g-i},N} := \{ \underline{A} \in \mathcal{A}_{g,p^{g-i},N} \mid \ker \lambda \subset A[p] \}.$$

For a non-empty subset $J = \{i_0, \ldots, i_r\} \subset J$, let $\mathcal{A}_J$ be the moduli space over $\overline{\mathbb{F}}_p$ parametrizing equivalence classes of objects

$$(\underline{A}_{i_0} \xrightarrow{\alpha} \underline{A}_{i_1} \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \underline{A}_{i_r}),$$

where $\underline{A}_{i_0} \in \mathcal{A}_{g,1,N}$ if $i_0 = 0$, and $\underline{A}_{i_j} \in \mathcal{A}'_{g,p^{g-i_j},N}$ for others.

For $J_1 \subset J_2$, let $\pi_{J_1,J_2} : \mathcal{A}_{J_2} \to \mathcal{A}_{J_1}$ be the natural projection. The map π_{J_1,J_2} is proper and dominant. We have

(i) $\mathcal{A}_J^{\mathrm{ord}} \subset \mathcal{A}_J$ is dense (Ngô-Genestier [11], C.-F. Yu [14]).
(ii) $\mathcal{A}_J$ is equi-dimensional of dimension $g(g+1)/2$ (Görtz [6], also follows from (i)).
(iii) $\mathcal{A}_J$ is irreducible if $|J| = 1$ (de Jong [3]), and for $|J| \geq 2$, $\mathcal{A}_J$ has $(k_1 + 1) \ldots (k_r + 1)$ irreducible components, where $k_j := i_j - i_{j-1}$ (C.-F. Yu [14]).

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When $g=2$, we have the diagram

$$\begin{tikzcd}
& \mathcal{A}_I \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{A}_{\{0,1\}} \arrow[d] \arrow[dr] & \mathcal{A}_{\{0,2\}} \arrow[dl] \arrow[dr] & \mathcal{A}_{\{1,2\}} \arrow[dl] \arrow[d] \\
\mathcal{A}_{\{0\}} & \mathcal{A}_{\{1\}} & \mathcal{A}_{\{2\}}
\end{tikzcd}$$

Note that we have an involution $\theta_{\mathcal{A}}:\mathcal{A}_I\to\mathcal{A}_I$ which sends

$$(A_0\to A_1\to\cdots\to A_g,\lambda_0,\lambda_g,\eta)\mapsto(A_g^t\to\cdots\to A_0^t,\lambda_g^{-1},\lambda_0^{-1},\lambda_{g*}\eta_g).$$

Therefore, one may ignore $\mathcal{A}_{\{1,2\}}$ and $\mathcal{A}_{\{2\}}$. We know that $\mathcal{A}_{\{1\}}=\mathcal{A}_{2,p,N}$ is a 3-dimensional, irreducible variety with isolated singularities. Let

$$\Lambda^*_{2,p,N}:=\{\underline{A}\in\mathcal{A}_{2,p,N}\,;\,\ker\lambda=\alpha_p\times\alpha_p\,\},$$

be the set of "indecomposable" polarized superspecial points. Using the crystalline theory, we show that

(1) $\mathcal{A}^{\rm sing}_{\{1\}}=\Lambda^*_{2,p,N}$.
(2) When $p>2$, if $x\in\Lambda^*_{2,p,N}$, then $\mathcal{A}^{\wedge}_{\{1\},x}\simeq k[[X_1,X_2,X_3,X_4]]/(X_2X_3-X_1X_4)$.

Note that the set $\Lambda^*_{2,p,N}$ is used by Katsura-Oort [9] to construct the supersingular locus $\mathcal{S}_{\{0\}}$ of $\mathcal{A}_{2,1,N}$. For each $\xi\in\Lambda^*_{2,p,N}$, let S_ξ parametrize the isogenies $(\varphi:\underline{A}_1\to\underline{A}_2)$ of degree p with $\underline{A}_1=\xi$. One has $S_\xi\simeq\mathbf{P}^1$ and has a projection map $\mathrm{pr}_2:S_\xi\to\mathcal{S}_{\{0\}}$ via $(\varphi:\underline{A}_1\to\underline{A}_2)\mapsto\underline{A}_2$. One shows that

- The map $\coprod_{x\in\Lambda^*_{2,p,N}}S_\xi\to\mathcal{S}_{\{0\}}$ is surjective, and there are $p+1$ branches passing through each superspecial point.
- It induces an isomorphism $\coprod_{x\in\Lambda^*_{2,p,N}}S_\xi\simeq\widetilde{\mathcal{S}}_{\{0\}}$, where $\widetilde{\mathcal{S}}_{\{0\}}$ is the normalization of $\mathcal{S}_{\{0\}}$.

In fact, if one considers the supersingular locus $\mathcal{S}_{\{0,1\}}$ of $\mathcal{A}_{\{0,1\}}$, then the picture is clearer. We has

$$\mathcal{S}_{\{0,1\}}=\coprod_{\xi\in\Lambda^*_{2,p,N}}S'_\xi\coprod_{\gamma\in\Lambda_{2,1,N}}S'_\gamma,$$

where

$$S'_\xi=\{\varphi:\underline{A}_0\to\underline{A}_1;\underline{A}_1=\xi\,\}\simeq\mathbf{P}^1,$$

$$S'_\gamma=\{\varphi:\underline{A}_0\to\underline{A}_1;\underline{A}_0=\gamma\,\}\simeq\mathbf{P}^1.$$

If one has an isogeny $\varphi:\underline{A}_0\to\underline{A}_1$ of supersingular abelian surfaces, then either $\underline{A}_0\in\Lambda_{2,1,N}$ or $\underline{A}_1\in\Lambda^*_{2,p,N}$. We have natural projections

$$\mathcal{S}_{\{0\}}\xleftarrow{\mathrm{pr}_0}\mathcal{S}_{\{0,1\}}\xrightarrow{\mathrm{pr}_1}\mathcal{S}_{\{1\}}.$$

Using another projection pr_1, we describe the supersingular locus $\mathcal{S}_{\{1\}}$.

2. Local model diagram and KR stratification

Let $V = \mathbb{Q}_p^{2g}, L_0 = \mathbb{Z}_p^{2g}$, ψ the standard alternating pairing, and $e_1, \dots, e_{2g}$ the standard basis. One has

$$\psi = \begin{pmatrix} 0 & \widetilde{I}_g \\ -\widetilde{I}_g & 0 \end{pmatrix}, \quad \widetilde{I}_g = \text{anti-diag}(1, \dots, 1).$$

Put $\Lambda_{-i} = \mathbb{Z}_p^{2g}$. Let $\psi_0 := \psi$ on $\Lambda_0 = L_0$. Define, for each $1 \le i \le 2g$, a map $\alpha : \Lambda_{-2g+i-1} \to \Lambda_{-2g+i}$ by $\alpha(e_i) = pe_i$ and $\alpha(e_j) = e_j$ if $j \neq i$. Let ψ_g be $\frac{1}{p}\cdot$ the pull-back of ψ_0 on Λ_{-g}; it is a perfect pairing. We get a lattice chain

$$\Lambda_I : \Lambda_{-g} \xrightarrow{\alpha} \dots \xrightarrow{\alpha} \Lambda_{-1} \xrightarrow{\alpha} \Lambda_0.$$

Denote by $\mathbf{M}_I^{\text{loc}}$ the local model associated to the lattice chain Λ_I. It is a projective scheme over $\mathbb{Z}_p$ which parametrizes the objects $(\mathcal{F}_{-i})$, where

- each $\mathcal{F}_{-i} \subset \Lambda_{-i} \otimes \mathcal{O}_S$ is a local free $\mathcal{O}_S$-submodule of rank g, locally a direct summand.
- $\mathcal{F}_0$ and $\mathcal{F}_{-g}$ are isotropic w.r.t. the pairings ψ_0 and ψ_{-g}, respectively.
- $\alpha(\mathcal{F}_{-i}) \subset \mathcal{F}_{-i+1}$.

We have the local model diagram [12]:

$$\begin{tikzcd} & \widetilde{\mathcal{A}}_I \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathcal{A}_I & & \mathbf{M}^{\text{loc}}_{I, \overline{\mathbb{F}}_p}, \end{tikzcd}$$

where

- $\widetilde{\mathcal{A}}_I$ is the moduli space over $\overline{\mathbb{F}}_p$ parametrizing equivalence classes of objects $(\underline{A}_\bullet, \xi)$, where $\underline{A}_\bullet \in \mathcal{A}_I$ and $\xi : H^1_{\text{DR}}(A_\bullet/S) \simeq \Lambda_I \otimes \mathcal{O}_S$ is an isomorphism of chains which is compatible with α and preserves the polarizations up to scalars.
- Let $\mathcal{G}_I$ be the group scheme over $\mathbb{Z}_p$ representing the functor $S \mapsto \text{Aut}(\Lambda_I \otimes \mathcal{O}_S, [\psi_0], [\psi_{-g}])$. This group acts on $\widetilde{\mathcal{A}}_I$ and $\mathbf{M}_I^{\text{loc}}$ from the left.
- The morphism ψ sends $(\underline{A}_\bullet, \xi) \mapsto \xi(\omega_\bullet)$, where $\omega_\bullet \subset H^1_{\text{DR}}(A_\bullet)$ is the Hodge filtration. The map ψ is $\mathcal{G}_I$-equivalent, surjective and smooth.
- $\widetilde{\mathcal{A}}_I \to \mathcal{A}_I$ is a $\mathcal{G}_I$-torsor.

We can also define the local model $\mathbf{M}_I^{\text{loc}}$ for each non-empty subset $J \subset I$, and have the local model diagram between $\mathcal{A}_J$, $\widetilde{\mathcal{A}}_J$ and $\mathbf{M}^{\text{loc}}_{J, \overline{\mathbb{F}}_p}$.

Consider the decomposition into $\mathcal{G}_I$-orbits:

$$\mathbf{M}^{\text{loc}}_{I, \overline{\mathbb{F}}_p} = \coprod_x \mathbf{M}^{\text{loc}}_{I,x}, \quad \widetilde{\mathcal{A}}_I = \coprod_x \widetilde{\mathcal{A}}_{I,x}.$$

Since φ is a $\mathcal{G}_I$-torsor, the stratification on $\widetilde{\mathcal{A}}_I$ descends to a stratification,

$$\mathcal{A}_I = \coprod_{x \in \text{Adm}_I(\mu)} \mathcal{A}_{I,x}.$$

This is called the Kottwitz-Rapoport (KR) stratification. The index set $\text{Adm}_I(\mu)$ is a finite subset of $\widetilde{W}$, the extended Weyl group for GSp_{2g}, and $\mu = (1, \dots, 1, 0, \dots, 0)$ (with $|\mu| = g$).

One has

$$\widetilde{W} = X_*(T) \rtimes W \subset \mathbf{A}(\mathbb{R}^{2g}),$$

where $T \subset \mathrm{GSp}_{2g}$ is the diagonal subgroup, $W = W(\mathrm{GSp}_{2g})$ the linear Weyl group, and $\mathbf{A}(\mathbb{R}^{2g})$ is the group of affine transformations on $\mathbb{R}^{2g}$. Let $\theta = (1, 2g)(2, 2g-1) \ldots (g, g+1)$. Then

$$W \simeq \{\sigma \in S_{2g} = W(\mathrm{GL}_{2g})\,;\theta\sigma = \sigma\theta\,\}.$$

By definition,

$$\mathrm{Adm}_I(\mu) = \{x \in \widetilde{W}\,; x \leq t_{w(\mu)} \text{ for some } w \in W\,\}.$$

$$\mathrm{Perm}_I(\mu) = \{x \in \widetilde{W} \subset \mathbf{A}(\mathbb{R}^{2g})\,; \mathbf{0} \leq x(w_i') - w_i' \leq \mathbf{1},\ \forall\, 1 \leq i \leq 2g\,\},$$

where $w_i' = (0, \ldots, 0, 1, \ldots, 1)$ with $|w_i'| = i$. Kottwitz and Rapoport [10] have shown that $\mathrm{Adm}_I(\mu) = \mathrm{Perm}_I(\mu)$.

In fact, $\mathrm{Adm}_I(\mu) \subset W_a\tau$.

- τ is the element that is less than μ and fixes the base alcove

$$\mathbf{a} = \{u \in \mathbb{R}^{2g}; u_1 + u_{2g} = \ldots u_g + u_{g+1},\ 1 + u_1 > u_{2g} > \cdots > u_{g+1} > u_g\,\}.$$

- W_a is the affine Weyl group, which is $< s_0, s_1, \ldots, s_g >$.

We have

$$s_i = (i, i+1)(2g+1-i, 2g-i), \quad i = 1, \ldots, g-1,$$
$$s_g = (g, g+1), \quad s_0 = (-1, 0, \ldots, 0, 1), (1, 2g),$$
$$\tau = (0, \ldots, 0, 1, \ldots, 1), (1, g+1)(2, g+2) \ldots (g, 2g).$$

We also have the following results

- Each stratum $\mathcal{A}_{I,x}$ is smooth of pure dimension $\ell(x)$.
- (Ngô-Genestier [11]) The p-rank function is constant on each KR stratum. Furthermore, one has

$$p\text{−rank}(x) = \frac{1}{2}\#\mathrm{Fix}(w),$$

  where we write $w = (\nu, w)$ and $\mathrm{Fix}(w) := \{i; w(i) = i\}$.

Number of μ-admissible elements.

We find the following formula in Haines [7, p.1272]:

$$N_g := \#\mathrm{Adm}_I(\mu, g) = \sum_{d=0}^{g} N_g^{g-d},$$

where N_g^{g-d} is the number of x with p-rank$=g-d$:

$$N_g^{g-d} = \binom{g}{d} 2^{g-d} \sum_{k=0}^{d} \binom{d}{k} 2^k a_k.$$

Here $a_0 = 1$ and for $n \geq 1$, $a_n := \#\{\sigma \in S_n\,; \sigma(i) \neq i\ \forall\, i\,\}$. One also has the formula $1 + \sum_{k=1}^{n} \binom{n}{k} a_k = n!$. From these, we get

n	0	1	2	3	4
a_n	1	0	1	2	9

$g=2$

p-rank	0	1	2	total
#	5	4	4	13

$g=3$

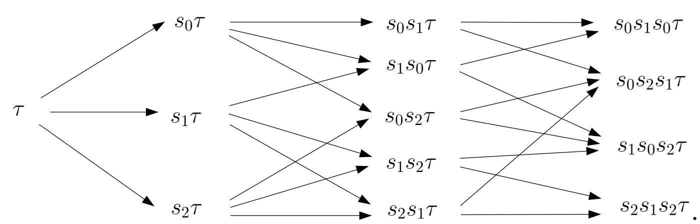
p-rank	0	1	2	3	total
#	29	30	12	8	79

$g=4$

p-rank	0	1	2	3	4	total
#	233	232	120	32	16	633

3. Example: $g=2$

The following are KR-types



Put $\mathrm{Adm}^i(\mu) := \{x \in \mathrm{Adm}(\mu);\, p\text{-rank}(x)=i\,\}$. We have

$$\begin{aligned}\mathrm{Adm}^2(\mu) &= \{s_0s_1s_0\tau, s_1s_0s_2\tau, s_2s_1s_2\tau, s_0s_2s_1\tau\},\\ \mathrm{Adm}^1(\mu) &= \{s_0s_1\tau, , s_1s_2\tau, s_2s_1\tau, s_1s_0\tau\},\\ \mathrm{Adm}^0(\mu) &= \{\tau, s_1\tau, s_0\tau, s_2\tau, s_0s_2\tau\}.\end{aligned} \tag{3.1}$$

We conclude [16]

- $\mathcal{A}_I^1 \subset \mathcal{A}_I^{\le 1}$ is not dense. This implies that p-rank strata do not form a stratification on $\mathcal{A}_I$.
- The supersingular locus $\mathcal{S}_I \subset \mathcal{A}_I$ consists of one-dimensional components and two-dimensional components. This rules out the possibility of equi-dimensionality of p-rank strata.
- The morphism $\mathcal{S}_I \to \mathcal{S}_{\{0\}}$ is not finite. This limits the method of p-adic monodromy to conclude an irreducibility result for p-rank strata in $\mathcal{A}_I$; see [16].

Geometric characterization for KR strata.

Let $a = (\underline{A}_0 \to \underline{A}_1 \to \underline{A}_2) \in \mathcal{A}_I(k)$. One wants to determine $KR(a)$ in $\mathrm{Adm}(\mu)$. Let $(\overline{M}_2 \to \overline{M}_1 \to \overline{M}_0)$ be the chain of de Rham cohomology groups, and let $\omega_i \subset \overline{M}_i$ be the

Hodge filtration. Put

$$G_0 := \ker(A_0 \to A_1), \quad G_1 := \ker(A_1 \to A_2).$$

From Dieudonné theory, we have

$$\omega_i/\alpha(\omega_{i-1}) = \operatorname{Lie} {G_i}^*, \quad \overline{M}_i/\omega_i + \alpha(\overline{M}_{i-1}) = \operatorname{Lie}(G_i^D).$$

Define

$$\sigma_i(a) := \dim \omega_i/\alpha(\omega_{i-1}), \quad \sigma_i'(a) := \dim \overline{M}_i/\omega_i + \alpha(\overline{M}_{i-1}).$$

Clearly, the invariant (σ_i, σ_i') characterizes the KR-types in $\operatorname{Adm}^1(\mu) \cup \operatorname{Adm}^2(\mu)$, as (G_0, G_1) is either (α_p, μ_p), $(\alpha_p, \mathbb{Z}/p)$, their switch, or $(*, *)$ with $* = \mathbb{Z}/p$ or μ_p.

Here is the correspondence:

p-rank(a)	2	2	2	2	1	1	1	1
$(\sigma_0(a), \sigma_0'(a))$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,1)$
$(\sigma_1(a), \sigma_1'(a))$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$
$KR(a)$	$s_0s_1s_0\tau$	$s_0s_2s_1\tau$	$s_1s_0s_2\tau$	$s_2s_1s_2\tau$	$s_0s_1\tau$	$s_1s_2\tau$	$s_2s_1\tau$	$s_1s_0\tau$

Note that $(\sigma_i(a), \sigma_i'(a)) = (1,1)$ for a supersingular. Introduce a new invariant:

$$\sigma_{02}(a) := \omega_0/\alpha^2(\omega_2), \quad \sigma_{02}'(a) := \dim \overline{M}_0/\omega_0 + \alpha^2(\overline{M}_2),$$

where $\alpha^2 : \overline{M}_2 \to \overline{M}_0$ is the composition. We get

p-rank(a)	0	0	0	0
$(\sigma_0(a), \sigma_0'(a))$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(\sigma_1(a), \sigma_1'(a))$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(\sigma_{02}(a), \sigma_{02}'(a))$	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$KR(a)$	$s_0s_2\tau$	$s_0\tau$	$s_2\tau$	$s_1\tau,\ \tau$

- The invariant $(\sigma_{02}, \sigma_{02}')$ does not determine the isomorphism class of finite subgroups $\ker(A_0 \to A_1)$; the latter has finer information than KR-types.
- It remains to distinguish $s_1\tau$ and τ. For this, we study the supersingular locus $\mathcal{S}_I$ of $\mathcal{A}_I$.

Suppose that $a = (\underline{A}_0 \to \underline{A}_1 \to \underline{A}_2) \in \overline{\mathcal{A}_{s_1\tau}}$, that is, $(\sigma_{02}(a), \sigma_{02}'(a)) = (2,2)$. Then from the description of $\mathcal{S}_I$, one has

$$a \in \mathcal{A}_\tau \iff \underline{A}_1 \in \Lambda_{2,p,N}^*.$$

Let $\underline{A}_0$ be any superspecial point, $A_0 \to A_1$ an isogeny of degree p, and $M_1 \subset M_1$ their Dieudonné modules. Then we have

$$\underline{A}_1 \in \Lambda_{2,p,N}^* \iff \text{In } \overline{M}_0 = M_0/pM_0,\ \langle \overline{M}_1, V\overline{M}_1 \rangle = 0.$$

Translating this property we have

- Let $a = (\underline{A}_\bullet) \in \overline{\mathcal{A}_{s_1\tau}}$ and $\overline{M}_\bullet$ the chain of de Rham cohomologies. Then $KR(a) = \tau \iff \langle \alpha(\overline{M}_1), \alpha(\omega_1) \rangle_0 = 0$.

This completes the geometric characterization of KR strata.

KR strata under the transition maps[*].

[*]part of this subsection is incooperated with U. Görtz

Recall that we have

$$\mathcal{A}_I = \coprod_{x\in \mathrm{Adm}_I(\mu)} \mathcal{A}_{I,x}, \quad \mathrm{Adm}_I(\mu)\subset W_a\tau, \quad W_a=<s_0,\dots,s_g>,$$

$$\mathcal{A}_J = \coprod_{x\in \mathrm{Adm}_J(\mu)} \mathcal{A}_{J,x}, \quad \mathrm{Adm}_J(\mu)\subset W_J\backslash \widetilde{W}/W_J,$$

where $\mathrm{Adm}_J(\mu)$ is the image of $\mathrm{Adm}_I(\mu)$ in $W_J\backslash W_a\tau/W_J\subset W_J\backslash\widetilde{W}/W_J$ and $W_J=<s_i\,|\,i\notin J>$, a finite group. For $g=2$, we consider $J=I,\{0,1\},\{0,2\},\{1\},\{0\}$. For $x\in\mathrm{Adm}_I(\mu)$, let

$$[x]_J=\{y\in \mathrm{Adm}_I(\mu)\,|\,[y]=[x] \text{ in } W_J\backslash W_a\tau/W_J\}.$$

Let $\mathcal{A}_{[x]_J}$ be the corresponding KR stratum in $\mathcal{A}_J$, regarding $[x]_J$ as an element in $W_J\backslash\widetilde{W}/W_J$.

(1) $J=\{0,1\}$ and $W_J=<s_2>$. Using $\tau s_2=s_0\tau$, we compute

$$\begin{array}{ll} [\tau]_J=\{\tau, s_2\tau, s_0\tau, s_{02}\tau\}, & \text{dim=1},\\ [s_1\tau]_J=\{s_1\tau, s_{10}\tau, s_{21}\tau\}, & \text{dim=2},\\ [s_{12}\tau]_J=\{s_{12}\tau, s_{120}\tau, s_{212}\tau\}, & \text{dim=3},\\ [s_{01}\tau]_J=\{s_{01}\tau, s_{010}\tau, s_{201}\tau\}, & \text{dim=3}. \end{array}$$

We have

- (i) There are 2 ordinary irreducible components; they are (properly) contained in $\mathcal{A}_{[s_{01}\tau]_J}$ and $\mathcal{A}_{[s_{12}\tau]_J}$ respectively.
- (ii) There are 3 p-rank one irreducible components; they are (properly) contained in $\mathcal{A}_{[s_1\tau]_J}$, $\mathcal{A}_{[s_{01}\tau]_J}$ and $\mathcal{A}_{[s_{12}\tau]_J}$ respectively.
- (iii) The closure $\overline{\mathcal{A}_{[s_1\tau]_J}}$ is a smooth surface, which is the intersection of $\overline{\mathcal{A}_{[s_{01}\tau]_J}}$ and $\overline{\mathcal{A}_{[s_{12}\tau]_J}}$.
- (iv) $\mathcal{A}_{[\tau]_J}$ consists of "horizontal" components of the supersingular locus $\mathcal{S}_J$.
- (v) $\mathcal{S}_J\cap\mathcal{A}_{[s_1\tau]_J}$ consists of open "vertical" components of $\mathcal{S}_J$.
- (vi) $\mathcal{A}_{[s_{01}\tau]_J}\cup\mathcal{A}_{[s_{12}\tau]_J}$ is the smooth locus of $\mathcal{A}_J$.

Question: Is $\pi_{\{0\},J}:\overline{\mathcal{A}_{[s_1\tau]_J}}\to\mathcal{A}_{\{0\}}^{\mathrm{non-ord}}$ the blow-up of $\mathcal{A}_{\{0\}}^{\mathrm{non-ord}}$ at the singular (superspecial) points? We expect the answer to be YES.

(2) $J=\{0,2\}$ and $W_J=<s_1>$. Using $\tau s_1=s_1\tau$, we compute

$$\begin{array}{lll} [\tau]_J=\{\tau, s_1\tau\}, & \text{dim=0}, & H_2=\alpha_p\times\alpha_p,\\ [s_2\tau]_J=\{s_2\tau, s_{12}\tau, s_{21}\tau\}, & \text{dim=2}, & H_2(\eta)=\mu_p\times\alpha_p,\\ [s_0\tau]_J=\{s_0\tau, s_{10}\tau, s_{01}\tau\}, & \text{dim=2}, & H_2(\eta)=\mathbb{Z}/p\times\alpha_p,\\ [s_{02}\tau]_J=\{s_{02}\tau, s_{201}\tau, s_{120}\tau\}, & \text{dim=3}, & H_2(\eta)=\mathbb{Z}/p\times\mu_p,\\ [s_{212}\tau]_J=\{s_{212}\tau\}, & \text{dim=3}, & H_2=\mu_p\times\mu_p,\\ [s_{010}\tau]_J=\{s_{010}\}, & \text{dim=3}, & H_2=\mathbb{Z}/p\times\mathbb{Z}/p. \end{array}$$

Here $H_2(\eta)$ means $\ker(A_{0,\eta}\to A_{2,\eta})$ for a generic point η of this KR stratum. We have

- (i) There are 3 ordinary irreducible components. Two are $\mathcal{A}_{[s_{212}\tau]_J}$ and $\mathcal{A}_{[s_{010}\tau]_J}$, and the other is contained in the stratum $\mathcal{A}_{[s_{02}\tau]_J}$.
- (ii) There are 2 p-rank one irreducible components. They are contained in $\mathcal{A}_{[s_0\tau]_J}$ and $\mathcal{A}_{[s_2\tau]_J}$, respectively.
- (iii) The supersingular locus $\mathcal{S}_J$ has pure dimension 2. It is contained in $\overline{\mathcal{A}_{[s_{02}\tau]_J}}$.

(iv) The zero dimensional stratum $\mathcal{A}_{[\tau]_J}$ consists of points $(\underline{A}_0 \xrightarrow{F} \underline{A}_0^{(p)})$, where A_0 is superspecial.
(v) The union $\mathcal{A}_{[s_{212}\tau]_J} \cup \mathcal{A}_{[s_{010}\tau]_J} \cup \mathcal{A}_{[s_{02}\tau]_J}$ is the smooth locus.

In fact, in the module space $\mathcal{A}_I$ with Iwahori level structure, we have

$$\mathcal{S}_I = \overline{\mathcal{A}_{s_{021}\tau}} \cap \overline{\mathcal{A}_{s_{102}\tau}}.$$

(3) $J = \{1\}$ and $W_J =< s_0, s_2 >$. Using $\tau s_0 = s_2\tau$ and $\tau s_2 = s_0\tau$, we compute

$$\begin{array}{ll} [\tau]_J = \{\tau, s_0\tau, s_2\tau, s_{02}\tau\}, & \text{dim=0,} \\ [s_1\tau]_J = \{ \text{ the rest } \}, & \text{dim=3.} \end{array}$$

We have

(i) There is 1 ordinary irreducible component.
(ii) There is 1 p-rank one irreducible component.
(iii) The supersingular locus has pure dimension 1. Each component is isomorphic to $\mathbf{P}^1$. The intersection $S_J \cap \mathcal{A}_{[s_1\tau]_J}$ is the smooth locus of S_J.
(iv) The zero dimensional stratum $\mathcal{A}_{[\tau]_J}$ is the singular locus of $\mathcal{A}_J$, also the singular locus of $\mathcal{S}_J$, which is equal to the set $\Lambda^*_{2,p,N}$.
(v) The stratum $\mathcal{A}_{[s_1\tau]_J}$ is the smooth locus.

(4) $J = \{0\}$ and $W_J =< s_1, s_2 >$. We compute that $[\tau]_J$ is everything. The moduli space $\mathcal{A}_{\{0\}}$ is itself a KR stratum.

4. Some aspects for higher dimensional cases (joint with Ulrich Görtz)

For simplicity, we will restrict ourselves to the Iwahori level case. The results in this section can be extended to any parahoric level.

Numerical characterization.

Let $a = (\underline{A}_0 \to \cdots \to A_g) \in \mathcal{A}_I(k)$. Let

$$M_\bullet : \ M_{-g} \to M_{-g+1} \to \cdots \to M_0, \quad VM_\bullet : VM_{-g} \to VM_{-g+1} \to \cdots \to VM_0.$$

be the associated chain of Dieudonné modules. Then we have

$$KR(a) = \mathrm{inv}(M_\bullet, VM_\bullet) \in \mathrm{Iw}\backslash \mathrm{GSp}_{2g}(L)/\mathrm{Iw} \simeq \widetilde{W},$$

where $L = \mathrm{Frac}\, W(k)$, Iw the standard Iwahori compact subgroup ($\equiv B_\triangle \bmod p$), $\widetilde{W}$ the extended Weyl group of GSp_{2g}.

Another way to think about KR types is as follows. Let

$$\overline{M}_\bullet : \ \overline{M}_{-g} \to \overline{M}_{-g+1} \to \cdots \to \overline{M}_0$$

be the chain of de Rham cohomologies, together with Hodge filtrations. Forget the F and V structure, just look at isomorphism classes of chains of vector spaces over k, together with

Hodge filtration as subspaces. Then the isomorphism classes give rise to the KR types.

Just as flag varieties, on one hand, we have a *group-theoretic description* for the cell decomposition (coming from the Bruhat decomposition). On the other hand, we use the *incidence relation* to construct these Schubert cells. The latter description is useful in the intersection theory.

Definition. Let $a = (\underline{A}_0 \to \cdots \to A_g) \in \mathcal{A}_I(k)$ and let $\overline{M}_{-g} \to \overline{M}_{-g+1} \to \cdots \to \overline{M}_0$ be the chain of de Rham cohomologies with Hodge filtration $\omega_{-i} \subset \overline{M}_{-i}$. Let $\alpha_{i,j} : \overline{M}_{-j} \to \overline{M}_{-i}$ be the composition for $0 \le i < j \le g$. Define

$$\sigma_{ij}(a) := \dim \omega_{-i}/\alpha_{ij}(\omega_{-j}), \quad \sigma'_{ij}(a) := \dim \overline{M}_{-i}/\omega_{-i} + \alpha_{ij}(\overline{M}_{-j}).$$

For $0 \le i, j \le g$, define

$$d_{ij}(a) := \dim \alpha_{0i}(\omega_{-i}) + \alpha_{0j}(\overline{M}_{-j})^{\perp}.$$

Clearly, the function

$$\underline{\sigma} : a \mapsto (\sigma_{ij}(a), \sigma'_{ij}(a), d_{ij}(a))$$

is constant on each KR stratum. This particularly implies that the function

$$p - \mathrm{rank}(a) = \sum_{i=0}^{g-1} 2 - \sigma_{i,i+1}(a) - s'_{i,i+1}(a)$$

is constant on each KR stratum. Conversely, we prove

Theorem 4.1. *KR strata are distinguished by the invariant $\underline{\sigma}$. That is, if $x \neq x' \in \mathrm{Adm}(\mu)$, then $\underline{\sigma}(\mathcal{A}_{I,x}) \neq \underline{\sigma}(\mathcal{A}_{I,x'})$.*

Shuffle construction.

The goal is to reduce geometric problems on KR strata $\mathcal{A}_x$ to those on *p-rank zero* KR strata and KR strata of the moduli spaces of *lower genus g*.

Observation: an Iwahori level structure on (A, λ) is a flag of finite group schemes

$$H_\bullet : \ 0 \subset H_1 \subset \cdots \subset H_g \subset A[p]$$

satisfying certain conditions. This structure is defined through the p-torsion subgroup $(A[p], \lambda)$ with polarization.

Let $\mathrm{BT}^1_{h,I}$ be the set of isomorphism classes of $(G, \lambda, H_\bullet)$ over k, where

- (G, λ) is a principally polarized BT^1 of height $2h$,
- $H_\bullet : \ H_1 \subset \cdots \subset H_h \subset G$ a flag of finite flat group schemes such that $< \lambda(H_h), H_h >= 0$ (Note that $\lambda : G \to G^D$).

We may formulate $\mathrm{BT}^1_{h,I}$ as a category of groupoids with objects as above. But let us regard it simply as a set for simplicity. Clearly, we have a surjective map

$$\mathrm{BT}^1_{h,I} \xrightarrow{KR} \mathrm{Adm}_I(\mu).$$

For $s \geq 1, t \geq 1$ with $s+t=g$, denote by $Sh(s,t)$ the set of maps

$$\varphi : \{0,1,\dots,g\} \to \{0,1,\dots,s\}$$

such that

$$\varphi(0)=0,\ \varphi(g)=s,\ \text{and}\ \ \varphi(i) \leq \varphi(i+1) \leq \varphi(i)+1,\ \forall\, i=0,\dots,g-1.$$

It is called the set of shuffle maps of s letters and t letters.

For example, let $\varphi \in Sh(4,3)$, we use φ to shuffle **123** into 1234 as follows. Suppose

$$\varphi\ :\ 0\ 1\ \underline{1}\ 2\ \underline{3}\ 3\ 4\ \underline{4}.$$

We underline the repeated numbers, remove them, and replace by **123**:

$$\varphi\ :\ 0\ 1\ \mathbf{1}\ 2\ \mathbf{2}\ 3\ 4\ \mathbf{3}.$$

For $\varphi \in sh(s,t)$, define $\varphi' : \{0,1,\dots,g\} \to \{0,1,\dots,t\}$, called the complement of φ, as follows.

$$\varphi'(0)=0, \quad \varphi'(i+1)+\varphi(i+1)=\varphi'(i)+\varphi(i)+1, \quad \forall\, i=0,\dots,g-1.$$

With information above, we construct a map

$$\mathrm{sh}_\varphi : \mathrm{BT}^1_{s,I} \times \mathrm{BT}^1_{t,I} \to \mathrm{BT}^1_{g,I}$$

by

$$\big((G,\lambda,H_\bullet),(G'\lambda' H'_\bullet)\big) \mapsto (G\times G', \lambda\times\lambda', \varphi(H_\bullet,H'_\bullet)),$$

where

$$\varphi(H_\bullet,H'_\bullet):\ K_1 \subset K_2 \subset \cdots \subset K_g \subset G\times G', \quad K_i = H_{\varphi(i)} \times H_{\varphi'(i)}.$$

The shuffle map sh_φ descends to the set $\mathrm{Adm}_I(\mu)$:

$$\begin{array}{ccc} \mathrm{BT}^1_{s,I}\times \mathrm{BT}^1_{t,I} & \xrightarrow{\mathrm{sh}_\varphi} & \mathrm{BT}^1_{g,I} \\ \big\downarrow{\scriptstyle (KR,KR)} & & \big\downarrow{\scriptstyle KR} \\ \mathrm{Adm}_I(\mu,s)\times \mathrm{Adm}_I(\mu,t) & \xrightarrow{\mathrm{sh}_\varphi} & \mathrm{Adm}_I(\mu,g). \end{array}$$

In general, the map sh_φ is not injective. But we have

- The restriction $\mathrm{sh}_\varphi : \mathrm{Adm}^0_I(\mu,g-f)\times \mathrm{Adm}^f_I(\mu,f) \to \mathrm{Adm}^f_I(\mu,g)$ is injective.
- $$\mathrm{Adm}^f_I(\mu,g) = \coprod_{\varphi\in Sh(g-f,f)} \mathrm{sh}_\varphi(\mathrm{Adm}^0_I(\mu,g-f)\times \mathrm{Adm}^f_I(\mu,f)).$$

These follow easily from the canonical decomposition $G = G^{\mathrm{et,m}} \oplus G^{\mathrm{loc,loc}}$.

For any $x_1 \in \mathrm{Adm}_I(\mu,t)$, $x_2 \in \mathrm{Adm}_I(\mu,t)$ and $\varphi \in Sh(s,t)$, we get a shuffle morphism

$$\mathrm{sh}_\varphi : \mathcal{A}_{s,x_1} \times \mathcal{A}_{t,x_2} \to \mathcal{A}_{g,x},$$

where $x = \mathrm{sh}_\varphi(x_1,x_2)$. This produces various subvarieties in a KR stratum $\mathcal{A}_{g,x}$ which may give enough information about what we want to know on $\mathcal{A}_{g,x}$. For example, let x be any element say in $\mathrm{Adm}^f_I(\mu,g)$. Then there exist a unique $x_1 \in \mathrm{Adm}^0_I(\mu,g-f)$, $x_2 \in \mathrm{Adm}^f_I(\mu,f)$, and $\varphi \in Sh(g-f,f)$ such that $x = \mathrm{sh}_\varphi(x_1,x_2)$. So we have a morphism

$$\mathrm{sh}_\varphi : \mathcal{A}_{g-f,x_1} \times \mathcal{A}_{f,x_2} \to \mathcal{A}_{g,x}.$$

Geometric information on $\mathcal{A}_{g,x}$, for example possible Newton polygons, can be read from those on $\mathcal{A}_{g-f,x}$.

Admissible elements: g=3 and p-rank zero

We list all 29 μ-admissible elements with p-rank zero in the extended Weyl group $\widetilde{W} = X_*(T) \rtimes W(\mathrm{GSp}_6)$. Below

$$\tau = (0,0,0,1,1,1), (14)(25)(34), \quad s_0 = (-1,0,0,0,0,1), (16),$$

$$s_1 = (12)(56), \quad s_1 = (23)(45) \quad \text{and} \quad s_3 = (34).$$

Write $s_{i_1 i_2 \dots i_r}$ for the element $s_{i_1} s_{i_2} \cdots s_{i_r}$ in the affine Weyl group W_a.

KR	$(\nu, w) \in X_*(T) \rtimes W$	KR	$(\nu, w) \in X_*(T) \rtimes W$
(1) τ	(0,0,0,1,1,1), (14)(25)(34)	(16) $s_{310}\tau$	(0,0,1,0,1,1), (132645)
(2) $s_0\tau$	(0,0,0,1,1,1), (1463)(25)	(17) $s_{120}\tau$	(0,0,0,1,1,1), (16)(2453)
(3) $s_1\tau$	(0,0,0,1,1,1), (142635)	(18) $s_{320}\tau$	(0,0,1,0,1,1), (154623)
(4) $s_2\tau$	(0,0,0,1,1,1), (153624)	(19) $s_{230}\tau$	(0,1,0,1,0,1), (124653)
(5) $s_3\tau$	(0,0,1,0,1,1), (1364)(25)	(20) $s_{201}\tau$	(0,0,0,1,1,1), (1562)(34)
(6) $s_{10}\tau$	(0,0,0,1,1,1), (145)(263)	(21) $s_{301}\tau$	(0,0,1,0,1,1), (135)(642)
(7) $s_{20}\tau$	(0,0,0,1,1,1), (153)(246)	(22) $s_{121}\tau$	(0,1,0,1,0,1), (16)(25)(34)
(8) $s_{30}\tau$	(0,0,1,0,1,1), (13)(25)(46)	(23) $s_{231}\tau$	(0,,1,0,1,0,1), (1265)(34)
(9) $s_{01}\tau$	(0,0,0,1,1,1), (142)(356)	(24) $s_{312}\tau$	(0,0,1,0,1,1), (16)(2354)
(10) $s_{21}\tau$	(0,0,0,1,1,1), (15)(26)(34)	(25) $s_{323}\tau$	(0,1,1,0,0,1), (123654)
(11) $s_{31}\tau$	(0,0,1,0,1,1), (135)(264)	(26) $s_{3010}\tau$	(0,0,1,0,1,1), (132)(456)
(12) $s_{12}\tau$	(0,0,0,1,1,1), (16)(24)(35)	(27) $s_{3120}\tau$	(0,0,1,0,1,1), (16)(23)(45)
(13) $s_{32}\tau$	(0,0,1,0,1,1), (154)(236)	(28) $s_{3230}\tau$	(0,1,1,0,0,1), (123)(465)
(14) $s_{23}\tau$	(0,1,0,1,0,1), (124)(356)	(29) $s_{2301}\tau$	(0,1,0,1,0,1), (12)(34)(56)
(15) $s_{010}\tau$	(0,0,0,1,1,1), (145632)		

The partial (Bruhat) order on this finite set is expressed as follows. Two element x, y has relation $x < y$ in the Bruhat order if and only if there is a chain with $x = x_0 \to x_1 \to \cdots \to x_n = y$.

(1) $\tau \to s_0\tau,\ s_1\tau,\ s_2\tau,\ s_3\tau$
(2) $s_0\tau \to s_{10}\tau,\ s_{20}\tau,\ s_{30}\tau,\ s_{01}\tau$
(3) $s_1\tau \to s_{10}\tau,\ s_{01}\tau,\ s_{21}\tau,\ s_{31}\tau,\ s_{12}\tau$
(4) $s_2\tau \to s_{20}\tau,\ s_{21}\tau,\ s_{12}\tau,\ s_{32}\tau,\ s_{23}\tau$
(5) $s_3\tau \to s_{30}\tau,\ s_{31}\tau,\ s_{32}\tau,\ s_{23}\tau$
(6) $s_{10}\tau \to s_{010}\tau,\ s_{310}\tau,\ s_{120}\tau$
(7) $s_{20}\tau \to s_{120}\tau,\ s_{320}\tau,\ s_{230}\tau,\ s_{201}\tau$
(8) $s_{30}\tau \to s_{310}\tau,\ s_{320}\tau,\ s_{230}\tau,\ s_{301}\tau$
(9) $s_{01}\tau \to s_{010}\tau,\ s_{201}\tau,\ s_{301}\tau$
(10) $s_{21}\tau \to s_{201}\tau,\ s_{121}\tau,\ s_{231}\tau$
(11) $s_{31}\tau \to s_{310}\tau,\ s_{301}\tau,\ s_{231}\tau,\ s_{312}\tau$
(12) $s_{12}\tau \to s_{120}\tau,\ s_{121}\tau,\ s_{312}\tau$
(13) $s_{32}\tau \to s_{320}\tau,\ s_{312}\tau,\ s_{323}\tau$
(14) $s_{23}\tau \to s_{230}\tau,\ s_{231}\tau,\ s_{323}\tau$
(15) $s_{010}\tau \to s_{3010}\tau$
(16) $s_{310}\tau \to s_{3010}\tau,\ s_{3120}\tau$
(17) $s_{120}\tau \to s_{3120}\tau$
(18) $s_{320}\tau \to s_{3120}\tau,\ s_{3230}\tau$
(19) $s_{230}\tau \to s_{3230}\tau,\ s_{2301}\tau$
(20) $s_{201}\tau \to s_{2301}\tau$
(21) $s_{301}\tau \to s_{3010}\tau,\ s_{2301}\tau$
(22) $s_{121}\tau$ (max.)
(23) $s_{231}\tau \to s_{2301}\tau$
(24) $s_{312}\tau \to s_{3120}\tau$
(25) $s_{323}\tau \to s_{3230}\tau$
(26) $s_{3010}\tau$ (max.)
(27) $s_{3120}\tau$ (max.)
(28) $s_{3230}\tau$ (max.)
(29) $s_{2301}\tau$ (max.)

The following table indicates the possible Newton polygons occurring in each KR stratum. The symbol A represents the supersingular Newton polygon; the symbol B represents the Newton polygon with slopes $\frac{1}{3}$ and $\frac{2}{3}$. Let NP denote the set of the Newton polygons of points in the KR stratum.

KR	NP	KR	NP	KR	NP
(1) τ	A	(11) $s_{31}\tau$	B	(21) $s_{301}\tau$	A,B
(2) $s_0\tau$	A	(12) $s_{12}\tau$	A	(22) $s_{121}\tau$	A
(3) $s_1\tau$	A	(13) $s_{32}\tau$	B	(23) $s_{231}\tau$	A,B
(4) $s_2\tau$	A	(14) $s_{23}\tau$	B	(24) $s_{312}\tau$	A,B
(5) $s_3\tau$	A	(15) $s_{010}\tau$	A,B	(25) $s_{323}\tau$	A,B
(6) $s_{10}\tau$	B	(16) $s_{310}\tau$	A,B	(26) $s_{3010}\tau$	A,B
(7) $s_{20}\tau$	B	(17) $s_{120}\tau$	A,B	(27) $s_{3120}\tau$	A,B
(8) $s_{30}\tau$	A	(18) $s_{320}\tau$	A,B	(28) $s_{3230}\tau$	A,B
(9) $s_{01}\tau$	B	(19) $s_{230}\tau$	A,B	(29) $s_{2301}\tau$	A,B
(10) $s_{21}\tau$	A	(20) $s_{201}\tau$	A,B		

Numerical invariants for $g = 3$.

The following is the result of computation of the invariants $(\sigma_{ij}, \sigma'_{ij})$ and d_{ij}. Recall these invariants. Let $s = (\underline{A}_0 \to \cdots \to \underline{A}_g)$ be a point in $\mathcal{A}_I(k)$. Let $(\overline{M}_{-g} \xrightarrow{\alpha} \overline{M}_{-g+1} \ldots, \xrightarrow{\alpha} \overline{M}_0)$ be the associated chain of de Rham cohomologies. For $0 \le i < j \le g$, write $\alpha_{ij} : \overline{M}_{-j} \to \overline{M}_{-i}$ for the composition. Define

$$\sigma_{ij}(s) := \dim \omega_{-i}/\alpha_{ij}(\omega_{-j}), \quad \sigma'_{ij}(s) := \dim \overline{M}_{-i}/(\omega_{-i} + \alpha_{ij}(\overline{M}_{-j})).$$

For $1 \le i, j \le g-1$, define

$$d_{ij}(s) = \dim \alpha_{0i}(\omega_{-i}) + \alpha_{0j}(\overline{M}_{-j})^{\perp}.$$

Given an element $x \in \mathrm{Adm}(\mu)$, we use the expression $x = (\nu, w)$ to compute the lattice $(\mathcal{L}'_\bullet)$ with $t\Lambda'_{-i} \subset \mathcal{L}_{-i} \subset \Lambda'_{-i}$. Then we use this lattice to compute the invariants $(\sigma_{ij}, \sigma'_{ij})$ and d_{ij}. We first compute the invariants $(\sigma_{ij}, \sigma'_{ij})$ for each (p-rank zero μ-admissible) element x.

KR	$(\sigma_{02}, \sigma'_{02})$	$(\sigma_{13}, \sigma'_{13})$	$(\sigma_{03}, \sigma'_{03})$	KR	$(\sigma_{02}, \sigma'_{02})$	$(\sigma_{13}, \sigma'_{13})$	$(\sigma_{03}, \sigma'_{03})$
(1) τ	(2,2)	(2,2)	(3,3)	(16) $s_{310}\tau$	(2,2)	(1,2)	(2,2)
(2) $s_0\tau$	(2,2)	(2,2)	(2,3)	(17) $s_{120}\tau$	(2,2)	(1,2)	(2,3)
(3) $s_1\tau$	(2,2)	(2,2)	(3,3)	(18) $s_{320}\tau$	(2,2)	(2,1)	(2,2)
(4) $s_2\tau$	(2,2)	(2,2)	(3,3)	(19) $s_{230}\tau$	(2,1)	(2,2)	(2,2)
(5) $s_3\tau$	(2,2)	(2,2)	(3,2)	(20) $s_{201}\tau$	(1,2)	(2,2)	(2,3)
(6) $s_{10}\tau$	(2,2)	(1,2)	(2,3)	(21) $s_{301}\tau$	(1,2)	(2,2)	(2,2)
(7) $s_{20}\tau$	(2,2)	(2,2)	(2,3)	(22) $s_{121}\tau$	(2,2)	(2,2)	(3,3)
(8) $s_{30}\tau$	(2,2)	(2,2)	(2,2)	(23) $s_{231}\tau$	(2,1)	(2,2)	(3,2)
(9) $s_{01}\tau$	(1,2)	(2,2)	(2,3)	(24) $s_{312}\tau$	(2,2)	(2,1)	(3,2)
(10) $s_{21}\tau$	(2,2)	(2,2)	(3,3)	(25) $s_{323}\tau$	(2,1)	(2,1)	(3,1)
(11) $s_{31}\tau$	(2,2)	(2,2)	(3,2)	(26) $s_{3010}\tau$	(1,2)	(1,2)	(1,2)
(12) $s_{12}\tau$	(2,2)	(2,2)	(3,3)	(27) $s_{3120}\tau$	(2,2)	(1,1)	(2,2)
(13) $s_{32}\tau$	(2,2)	(2,1)	(3,2)	(28) $s_{3230}\tau$	(2,1)	(2,1)	(2,1)
(14) $s_{23}\tau$	(2,1)	(2,2)	(3,2)	(29) $s_{2301}\tau$	(1,1)	(2,2)	(2,2)
(15) $s_{010}\tau$	(1,2)	(1,2)	(1,3)				

In the following two tables some KR strata are already distinguished by the invariants $(\sigma_{ij}, \sigma'_{ij})$.

$(\sigma_{03}, \sigma'_{03})$	(1,2)	(2,1)	(2,2)	(2,2)	(1,3)	(3,1)
$(\sigma_{02}, \sigma'_{02})$	(1,2)	(2,1)	(2,2)	(1,1)	(1,2)	(2,1)
$(\sigma_{13}, \sigma'_{13})$	(1,2)	(2,1)	(1,1)	(2,2)	(1,2)	(2,1)
KR	(26) $s_{3010}\tau$	(28) $s_{3230}\tau$	(27) $s_{3120}\tau$	(29) $s_{2301}\tau$	(15) $s_{010}\tau$	(25) $s_{323}\tau$

$(\sigma_{03}, \sigma'_{03})$	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)
$(\sigma_{02}, \sigma'_{02})$	(1,2)	(2,1)	(2,2)	(2,2)	(2,2)
$(\sigma_{13}, \sigma'_{13})$	(2,2)	(2,2)	(2,1)	(1,2)	(2,2)
KR	(21) $s_{301}\tau$	(19) $s_{230}\tau$	(18) $s_{320}\tau$	(16) $s_{310}\tau$	(8) $s_{30}\tau$

The following two tables are given by the invariants $(\sigma_{03}, \sigma'_{03}) = (2,3)$ and $(\sigma_{03}, \sigma'_{03}) = (3,2)$, respectively. There are two classes in the each set of classes with invariants $(\sigma_{ij}, \sigma'_{ij})$ constant. They are distinguished by the invariant d_{12} in the first table (resp. by the invariant d_{21} in the second table). Notice that each pair of classes has the inclusion relation. In the first table, every smaller element is obtained by dropping s_2 from the bigger element. In the second table, every smaller element is obtained by dropping s_1 from the bigger element.

$(\sigma_{03},\sigma'_{03})$	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)
$(\sigma_{02},\sigma'_{02})$	(1,2)	(1,2)	(2,2)	(2,2)	(2,2)	(2,2)
$(\sigma_{13},\sigma'_{13})$	(2,2)	(2,2)	(1,2)	(1,2)	(2,2)	(2,2)
d_{12}	2	3	2	3	2	3
KR	(9)$s_{01}\tau$	(20) $s_{201}\tau$	(6) $s_{10}\tau$	(17) $s_{120}\tau$	(2) $s_0\tau$	(7) $s_{20}\tau$

$(\sigma_{03},\sigma'_{03})$	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)
$(\sigma_{02},\sigma'_{02})$	(2,1)	(2,1)	(2,2)	(2,2)	(2,2)	(2,2)
$(\sigma_{13},\sigma'_{13})$	(2,2)	(2,2)	(2,1)	(2,1)	(2,2)	(2,2)
d_{21}	1	2	1	2	1	2
KR	(14) $s_{23}\tau$	(23) $s_{231}\tau$	(13) $s_{32}\tau$	(24) $s_{312}\tau$	(5) $s_3\tau$	(11) $s_{31}\tau$

The following is the table for superspecial KR strata. Note that $(\sigma_{03},\sigma'_{03})=(3,3)$ implies $(\sigma_{02},\sigma'_{02})=(2,2)$ and $(\sigma_{13},\sigma'_{13})=(2,2)$.

$(\sigma_{03},\sigma'_{03})$	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)
d_{12}	2	2	3	3	3	3
d_{21}	1	2	1	2	2	2
d_{11}				2	3	3
d_{22}				3	2	3
KR	(1) τ	(3) $s_1\tau$	(4) $s_2\tau$	(10) $s_{21}\tau$	(12) $s_{12}\tau$	(22) $s_{121}\tau$

Supersingular KR strata.

Call a KR stratum $\mathcal{A}_x$ *supersingular* if it is contained in the supersingular locus $\mathcal{S}_I$. The following are supersingular KR-types:

$$\{\tau, s_1\tau, s_2\tau, s_{12}\tau, s_{21}\tau, s_{121}\tau, s_0\tau, s_3\tau, s_{03}\tau\} = W_{\{0,3\}}\tau \cup W_{\{1,2\}}\tau.$$

Note that their union is properly contained in the supersingular locus $\mathcal{S}_I$.

Theorem 4.2.
(a) (Case: $x \in W_{\{0,3\}}\tau$) *Let $\Lambda_{3,1,N}$ be the set of superspecial principally polarized abelian 3-folds with a level-N structure over $\overline{\mathbb{F}}_p$. Then*

(1) *The closure $\overline{\mathcal{A}_{s_{121}\tau}}$ has $|\Lambda_{3,1,N}|$ irreducible components, and each irreducible component is isomorphic to*
$$\mathrm{GL}_3/B_\triangle = \{(\underline{a},\underline{b}) \in \mathbf{P}^2\times\mathbf{P}^2 \,|\, \underline{a}\cdot\underline{b}=0\} =: X \subset \mathbf{P}^2\times\mathbf{P}^2.$$
(2) *The closure $\overline{\mathcal{A}_{s_{21}\tau}}$ has $|\Lambda_{3,1,N}|$ irreducible components, and each irreducible component is isomorphic to $\{(\underline{a},\underline{b}) \in X \,|\, \underline{b}\cdot\underline{b}^{(p)}=0\}$.*
(3) *The closure $\overline{\mathcal{A}_{s_{12}\tau}}$ has $|\Lambda_{3,1,N}|$ irreducible components, and each irreducible component is isomorphic to $\{(\underline{a},\underline{b}) \in X \,|\, \underline{a}\cdot\underline{a}^{(p)}=0\}$.*
(4) *The closure $\overline{\mathcal{A}_{s_1\tau}}$ has $|\Lambda_{3,1,N}|$ irreducible components, and each irreducible component is isomorphic to*
$$F_{\mathbf{P}^2}\cap X = \{(a, a^{(p)}) \,|\, a\cdot\underline{a}^{(p)}=0\}.$$

(5) *The closure* $\overline{\mathcal{A}_{s_2\tau}}$ *has* $|\Lambda_{3,1,N}|$ *irreducible components, and each irreducible component is isomorphic to*

$$V_{\mathbf{P}^2} \cap X = \{\, (b^{(p)}, b) \,|\, b \cdot \underline{b}^{(p)} = 0 \,\}.$$

(6) $|\mathcal{A}_\tau| = |\Lambda_{3,1,N}| \cdot |U(3)(\mathbb{F}_p)/B_0(\mathbb{F}_p)|$, *where* B_0 *is a Borel subgroup over* $\mathbb{F}_p$.

(b) (Case: $x \in W_{\{1,2\}}\tau$) *Let* $J = \{1, 2\}$, *and* $\Lambda_J := \pi_{J,I}(\mathcal{A}_\tau)$, *where* $\pi_{J,I} : \mathcal{A}_I \to \mathcal{A}_J$ *is the natural projection.*

(1) *The closure* $\overline{\mathcal{A}_{s_{30}\tau}}$ *has* $|\Lambda_J|$ *irreducible components, and each irreducible component is isomorphic to* $\mathbf{P}^1 \times \mathbf{P}^1$.

(2) *The closure* $\overline{\mathcal{A}_{s_3\tau}}$ *has* $|\Lambda_J|$ *irreducible components, and each irreducible component is isomorphic to* $\mathbf{P}^1$.

(3) *The closure* $\overline{\mathcal{A}_{s_0\tau}}$ *has* $|\Lambda_J|$ *irreducible components, and each irreducible component is isomorphic to* $\mathbf{P}^1$.

(4) $|\mathcal{A}_\tau| = |\Lambda_J| \cdot (p^2 + 1)$.

References

[1] C.-L. Chai and P. Norman, Bad reduction of the Siegel moduli scheme of genus two with $\Gamma_0(p)$-level structure. *Amer. J. Math.* **112** (1990), 1003–1071.

[2] C.-L. Chai and P. Norman, Singularities of the $\Gamma_0(p)$-level structure. *J. Algebraic Geom.* **1** (1992), 251–178.

[3] A.J. de Jong, The moduli spaces of polarized abelian varieties. *Math. Ann.* **295** (1993), 485–503.

[4] A.J. de Jong, The moduli spaces of principally polarized abelian varieties with $\Gamma_0(p)$-level structure. *J. Algebraic Geom.* **2** (1993), 667–688.

[5] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 22. Springer-Verlag, Berlin, 1990. xii+316 pp.

[6] U. Görtz, On the flatness of local models for the symplectic group. *Adv. Math.* **176** (2003), 89–115.

[7] T. Haines, The combinatorics of Bernstein functions. *Trans. Amer. Math. Sci.* **353** (2001), 1251–1278.

[8] T. Haines, Introduction to Shimura varieties with bad reduction of parahoric type. *Harmonic analysis, the trace formula, and Shimura varieties*, 583–642, Clay Math. Proc., 4, Amer. Math. Soc., 2005.

[9] T. Katsura and F. Oort, Families of supersingular abelian surfaces, *Compositio Math.* **62** (1987), 107–167.

[10] R. E. Kottwitz and M. Rapoport, Minuscule alcoves for GL_n and GSp_{2n}. *Manuscripta Math.* **102** (2000), 403–428.

[11] B.C. Ngô and A. Genestier, Alcôves et p-rang des variétés abéliennes. *Ann. Inst. Fourier (Grenoble)* **52** (2002), 1665–1680.

[12] M. Rapoport and Th. Zink, *Period Spaces for p-divisible groups*. Ann. Math. Studies 141, Princeton Univ. Press, 1996.

[13] J. Tilouine, Siegel Varieties and p-Adic Siegel Modular Forms. *Doc. Math.* Extra Volume: John H. Coates' Sixtieth Birthday (2006) 781–817

[14] C.-F. Yu, Irreducibility of the Siegel moduli spaces with parahoric level structure. *Int. Math. Res. Not.* **2004**, No. 48, 2593–2597.

[15] C.-F. Yu, The supersingular loci and mass formulas on Siegel modular varieties. *Doc. Math.* **11** (2006) 449–468.

[16] C.-F. Yu, Irreducibility and p-adic monodromies on the Siegel moduli spaces. math.NT/0612839, 29 pp.

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