VERTEX ALGEBRAS AND THE FORMAL LOOP SPACE

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1. Introduction

One of the salient mathematical features of string theory is the importance of vertex algebras. Their role in the theory can be compared to that of Lie algebras in the "ordinary" physics of point particles.

Mathematically, the approach of string theory can be cast in terms of analysis on the space of free loops, i.e., smooth maps $S^1 \to X$ where X is a given "spacetime" manifold. Accordingly, one has the folklore principle that constructions involving the space of free loops lead to vertex algebras. One class of such constructions is provided by the spaces of highest weight representations of loop groups. Another is Ω_X^{ch}, the chiral de Rham complex of an algebraic variety X, introduced by Malikov, Schectman and Vaintrob [MSV]. Heuristically, this complex should be interpreted in terms of LX, the space of free loops and its subvariety L^0X consisting of loops extending holomorphically into the unit disk. More precisely, Ω_X^{ch} can be thought of as the semiinfinite de Rham complex with coefficients in the space of distributions on LX supported on L^0X. This is not, however, the way Ω_X^{ch} has been defined mathematically. The definition given in [MSV] is of more computational nature and proceeds by constructing the action of the group of diffeomorphisms on the irreducible module over the Heisenberg algebra. In that approach it seems miraculous that such an action exists at all.

The aim of this paper is twofold. First, to give a precise mathematical theorem underlying the above folklore principle about vertex algebras. For this, we introduce an algebro-geometric version of the free loop space $\mathcal{L}(X)$ for any scheme X of finite type over a field. This is an ind-scheme containing $\mathcal{L}^0(X)$, the scheme of formal germs of curves on X studied in [DL]. We prove that both $\mathcal{L}(X)$ and $\mathcal{L}^0(X)$ themselves possess a non-linear version of the vertex algebra structure (which makes it clear that any natural linear construction applied to them should give a vertex algebra in the usual sense). More precisely, we use the geometric approach

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to vertex algebras developed by Beilinson and Drinfeld [BD1] and based on the concepts of chiral algebras and factorization algebras. The latter concept has a natural nonlinear version, that of a factorization semigroup. What we prove is that natural "global"versions of $\mathcal{L}(X)$, $\mathcal{L}^0(X)$ have natural structures of factorization semigroups. An earlier known example of a factorization semigroup is given by the affine Grassmannian [G], and this explains why the spaces of representations of loop groups are vertex algebras. Our construction is similar in spirit.

To give a good definition of the algebraic analog of the full loop space LX one has to overcome a certain subtlety. Namely, a natural approach would be to try to (ind-)represent a functor which to any commutative ring R associates the set of $R((t))$-points of X. (This is exactly how one defines the scheme $\mathcal{L}^0(X)$, with $R[[t]]$ instead of $R((t))$.) If X is affine, this indeed gives a good ind-scheme which we denote $\tilde{\mathcal{L}}(X)$. But when X is, say, projective, then (for R a field) there is no difference between $R[[t]]$-points and $R((t))$-points of X (valuative criterion of properness), so it may seem that nothing is gained by allowing Laurent series. To state this phenomenon differently, the ind-schemes $\tilde{\mathcal{L}}(U)$ for affine $U \subset X$ do not glue together well. This is in fact understandable on general grounds: the loop space LX is not the union of the LU since a loop need not spend all its time in any given U.

To get around this difficulty we adopt the following strategy. For an affine X we consider $\mathcal{L}(X)$, the formal neighborhood of $\mathcal{L}^0(X)$ in $\tilde{\mathcal{L}}(X)$. So we are dealing with formal loops which are "infinitesimal in the Laurent direction".Then, we prove that the $\mathcal{L}(U), U \subset X$, do indeed possess the right gluing properties. This is due to the infinitesimal nature of our loops.

Our second goal is to give a direct geometric construction of Ω_X^{ch} (for smooth X) in terms of our model for the loop space. By the above, this construction explains also the fact that Ω_X^{ch} is a sheaf of vertex algebras. In order to achieve this, we represent $\mathcal{L}(X)$ as an ind-pro-object in the category of schemes of finite type and then show that the shifted de Rham complexes of the terms of this ind-pro-system arrange naturally into a double inductive system whose inductive limit is identified with Ω_X^{ch}.

As with the study of formal arcs and motivic integration [DL], one can view our considerations as algebro-geometric analogs of the basic constructions of p-adic analysis. The difference between our ind-scheme $\mathcal{L}(X)$ and the more familiar scheme $\mathcal{L}^0(X)$ is similar to the difference between $\mathbb{Q}_p$ and $\mathbb{Z}_p$: while the latter is a pro-object in the category of schemes of finite type (resp. finite sets), the former is an ind-pro-object. Further, our approach to Ω_X^{ch} is similar to the construction of the space of locally constant functions with compact support on $\mathbb{Q}_p = \varinjlim_i \varprojlim_j p^{-i}\mathbb{Z}_p/p^j\mathbb{Z}_p$ as the double inductive limit of the spaces of functions on the finite sets $p^{-i}\mathbb{Z}_p/p^j\mathbb{Z}_p$, cf [P]. Notice that the reason that these spaces of functions indeed form a double inductive system (with respect to the maps of inverse image in the j-direction and direct image in the i-direction) is that the commutative squares in the ind-pro-system $p^{-i}\mathbb{Z}_p/p^j\mathbb{Z}_p$ are Cartesian (so that we have base change). This is an algebraic counterpart of the property of the local compactness of $\mathbb{Q}_p$, see [Kat]. In our situation it is equally important that the ind-scheme $\mathcal{L}(X)$ satisfies a certain formal analog of local compactness.

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2. Construction of the formal loop space

(2.1) Generalities on schemes and ind-schemes. If $\mathbf{C}$ is a category, then we denote by $\mathbf{Ind}(\mathbf{C})$ and $\mathbf{Pro}(\mathbf{C})$ the categories of ind- and pro-objects of $\mathbf{C}$, see [AM] [GV] for background. Thus, objects of $\mathbf{Ind}(\mathbf{C})$ (resp. $\mathbf{Pro}(\mathbf{C})$) are symbols "$\varinjlim_i$"C_i (resp. "$\varprojlim_i$"C_i) where (C_i) is a filtering inductive (resp. projective) system over $\mathbf{C}$, with morphisms defined as in *loc. cit.* Recall that $\mathbf{Ind}(\mathbf{C})$ can be considered as a full subcategory in $\mathbf{Fun}^\circ(\mathbf{C}, \mathbf{Sets})$, the category of contravariant functors.

Throughout the paper we fix a field k of characteristic 0. We denote by $\mathbf{Sch} \subset \mathbf{Lrs}$ the categories of schemes and locally ringed spaces over k. If R is a commutative ring, we will write $\underline{\mathrm{Spec}}\, R$ for the topological space (the set of prime ideals with the Zariski topology) underlying the affine scheme Spec R which is thus the ringed space $(\underline{\mathrm{Spec}}\, R, \mathcal{O}_{\mathrm{Spec}\ R})$.

By an ind-scheme we will mean in this paper an ind-object of $\mathbf{Sch}$ represented by an inductive system of closed embeddings. The category of ind-schemes will be denoted by $\mathbf{Isch}$. Let us make the category $\mathbf{Sch}$ into a Grothendieck site by using Zariski open coverings and let $\mathbf{Shf}$ be the category of sheaves of sets on $\mathbf{Sch}$. For any ind-scheme Y the functor η_Y on schemes represented by Y is then a sheaf, so we have the embeddings

$$\mathbf{Sch} \subset \mathbf{Isch} \subset \mathbf{Shf} \subset \mathbf{Fun}^\circ(\mathbf{Sch}, \mathbf{Sets}). \tag{2.1.1}$$

Since the category $\mathbf{Sch}$ has finite inverse limits, so do all the categories in (2.1.1) and the embeddings preserve them. On the contrary, finite direct limits, such as cokernels (when they exist) are preserved by the first two of the embeddings but not by the third one: cokernels in the category of sheaves are not the same as in the category of all functors (presheaves).

We denote by **Alg** the category of k-algebras and by $\mathbf{Aff} \subset \mathbf{Sch}$ the dual category of affine schemes. Note that **Isch** can be as well realized as a full subcategory in $\mathbf{Fun}^\circ(\mathbf{Aff}, \mathbf{Sets}) = \mathbf{Fun}(\mathbf{Alg}, \mathbf{Sets})$.

Given two contravariant functors $\phi, \phi' : \mathbf{Sch} \to \mathbf{Sets}$, and a morphism $F : \phi' \to \phi$, we will say that F is formally smooth (resp. formally étale), if for any nilpotent extension of affine schemes $S \subset S'$ the natural map

$$\phi'(S') \to \phi(S') \times_{\phi(S)} \phi'(S)$$

is surjective (resp. bijective).

We will say that F is an open embedding, if for any scheme S and any $u \in \phi(S)$ (which is the same as a morphism $\eta_S \to \phi$) the fiber product functor $\eta_S \times_\phi \phi'$ is representable by a scheme S' whose natural morphism to S is an open embedding.

We define formal smoothness and openness for morphisms of ind-schemes by considering their representable functors.

For a scheme Z we denote by $Z_{\mathrm{red}} \subset Z$ the corresponding reduced subscheme. We extend this notation to ind-schemes by applying it term by term in inductive systems.

Let X be a k-scheme of finite type. We denote by $\mathbf{Aff}_X$ (resp. $\mathbf{Aff}_X^{\mathrm{ft}}$) the category of schemes affine over X (resp. affine of finite type over X). For future use let us quote the following fact [EGAIV, Corollary 8.13.2].

(2.1.2) Proposition. *The category $\mathbf{Pro}(\mathbf{Aff}_X^{\mathrm{ft}})$ is equivalent to $\mathbf{Aff}_X$ via the functor* "$\varprojlim_n$"$S_n \mapsto \varprojlim_n S_n$.

(2.2) The scheme of germs of arcs.

Let X be a scheme. We denote by $\mathcal{L}^0(X)$ the scheme of germs of arcs on X, see [DL] [BLR, Theorem 7.6.4]. It represents the following covariant functor λ_X^0 on the category **Alg**:

$$\lambda_X^0 : R \mapsto \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\ R[[t]], X).$$

Here are some of the well-known properties of $\mathcal{L}^0(X)$. Note that if R is a local ring with maximal ideal M, then $R[[t]]$ is a local ring with maximal ideal $M[[t]] + tR[[t]]$. Thus for any scheme S the pair $(S, \mathcal{O}_S[[t]])$ is a locally ringed space.

(2.2.1) Proposition. *(a) For any scheme S we have*

$$\mathrm{Hom}_{\mathrm{Sch}}(S, \mathcal{L}^0(X)) = \mathrm{Hom}_{\mathbf{Lrs}}((S, \mathcal{O}_S[[t]]), X).$$

(b) The scheme $\mathcal{L}^0(X)$ is the projective limit of the schemes $\mathcal{L}^0_n(X)$, $n \in \mathbb{N}$, representing the functors

$$\lambda^0_{n,X} : R \mapsto \mathrm{Hom}_{\mathbf{Sch}}\big(\mathrm{Spec}\ R[t]/t^{n+1},\, X\big).$$

If X is of finite type, than so is each $\mathcal{L}^0_n(X)$.

(c) Denote $\pi_n : \mathcal{L}^0_n(X) \to X$, $\pi : \mathcal{L}^0(X) \to X$ the natural projections. They are affine morphisms. For an open subset $U \subset X$ we have $\pi^{-1}(U) = \mathcal{L}^0(U)$ and $\pi_n^{-1}(U) = \mathcal{L}^0_n(U)$.

(d) If X is smooth, then so is $\mathcal{L}^0_n(X)$ and $\mathcal{L}^0(X)$ is formally smooth.

Let us give the proof of (a), as it will serve as a template for a similar statement about the space of formal loops. It is enough to assume $S = \mathrm{Spec}\ R$ affine. The embedding of constant series and the evaluation at 0 give ring homomorphisms $R \overset{\alpha}{\to} R[[t]] \overset{\beta}{\to} R$ and hence morphisms of topological spaces

$$\underline{\mathrm{Spec}}\, R \overset{p}{\longleftarrow} \underline{\mathrm{Spec}}\, R[[t]] \overset{i}{\longleftarrow} \underline{\mathrm{Spec}}\, R. \tag{2.2.2}$$

(2.2.3) Lemma. *We have*

$$i^{-1}\mathcal{O}_{\mathrm{Spec}\ R[[t]]} = \mathcal{O}_{\mathrm{Spec}\ R}[[t]] = p_*\mathcal{O}_{\mathrm{Spec}\ R[[t]]}.$$

Proof: Let us prove the first equality. If $\mathfrak{p} \in \underline{\mathrm{Spec}}\, R$ is a prime ideal in R, then the stalk of $i^{-1}\mathcal{O}_{\mathrm{Spec}\ R[[t]]}$ at $\mathfrak{p}$ is the localization of $R[[t]]$ with respect to the multiplicative subset $\beta^{-1}(R-\mathfrak{p})$ while the stalk of $\mathcal{O}_{\mathrm{Spec}\ R}[[t]]$ is $R[[t]] \otimes_R R_{\mathfrak{p}}$ where $R_{\mathfrak{p}}$ denotes, as usual, the localization of R with respect to $R - \mathfrak{p}$. Now, to see that the two rings are the same, it suffices to use the following obvious property of formal power series rings: if A is a commutative ring and $f(t) \in A[[t]]$ is such that $f(0)$ is invertible in A, then $f(t)$ is invertible in $A[[t]]$.

The second equality is obvious: the stalk of $p_*\mathcal{O}_{\mathrm{Spec}\ R[[t]]}$ at $\mathfrak{p}$ is immediately seen to coincide with $R[[t]] \otimes_R R_{\mathfrak{p}}$. □

Now, composing with i defines a map of sets

$$\phi : \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\ R[[t]], X) \to \mathrm{Hom}_{\mathbf{Lrs}}\big((\mathrm{Spec}\ R, \mathcal{O}_{\mathrm{Spec}\ R}[[t]]), X\big).$$

A map ψ in the other direction comes from the second equality in (2.3.3). One verifies easily that ϕ and ψ are mutually inverse.

We will also need the following generalization of Proposition 2.2.1(c).

(2.2.4) Proposition. *If $\phi : X \to Y$ is an étale morphism, then the natural squares*

$$\begin{array}{ccc} \mathcal{L}^0_n(X) & \xrightarrow{\pi_n} & X \\ {\scriptstyle \mathcal{L}^0_n(\phi)} \downarrow & & \downarrow \phi \\ \mathcal{L}^0_n(Y) & \xrightarrow{\pi_n} & Y \end{array}, \qquad n \geq 0$$

as well as the analogous square for $\mathcal{L}^0(Y)$, are Cartesian.

Proof: We can assume that $X = \mathrm{Spec}\,(A), Y = \mathrm{Spec}\,(B)$ are affine. It is enough to prove that for each $n \geq 0$ the natural morphism

$$\alpha : \mathcal{L}^0_n(X) \to \mathcal{L}^0_n(Y) \times_Y X$$

is an isomorphism. Let us construct the inverse morphism β. Let R be a k-algebra and f be a morphism $S \to \mathcal{L}^0_n(Y) \times_Y X$. Thus, f corresponds to a pair of ring homomorphisms forming the horizontal arrows of the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{u} & R[t]/t^{n+1} \\ \phi^* \downarrow & & \downarrow \pi \\ A & \xrightarrow{v} & R. \end{array}$$

Here π is natural projection. Since ϕ is étale and π is nilpotent, there is a unique homomorphism $w : A \to R[t]/t^{n+1}$ such that both resulting triangles are commutative. Let g be the morphism $S \to \mathcal{L}^0_n(X)$ represented by w. Then we set $\beta(f) = g$. The verifications are obvious and left to the reader.

(2.3) Nil-Laurent series. Let R be a commutative ring. A nil-Laurent series is, by definition, a Laurent series $a(t) = \sum_{i \gg -\infty} a_i t^i \in R((t))$ such that all the a_i with $i < 0$ are nilpotent. The set of such series will be denoted $R((t))^{\surd}$.

It is clear that $R((t))^{\surd}$ is a subring in $R((t))$, the ring of all Laurent series. Indeed, let $\sqrt{R}$ be the radical of R (the set of all nilpotent elements) and set $R_{\mathrm{red}} = R/\sqrt{R}$. Consider the homomorphism $\rho : R((t)) \to R_{\mathrm{red}}((t))$ induced by the projection $R \to R_{\mathrm{red}}$. Then $R((t))^{\surd} = \rho^{-1}(R_{\mathrm{red}}[[t]])$.

(2.3.1) Proposition. *For $a = \sum a_i t^i \in R((t))^{\surd}$ the following are equivalent :*

(i) a is invertible in $R((t))^{\surd}$.

(ii) $\rho(a)$ is invertible in $R_{\mathrm{red}}[[t]]$.

(iii) The element a_0 is invertible in R.

Proof: (i)$\Rightarrow$(ii) is obvious. To see that (ii)$\Leftrightarrow$(iii), note that the invertibility of $\rho(a)$ implies the invertibility of the image of a_0 in R_{red} which certainly implies the invertibility of a_0 and the converse is equally obvious. Let us prove that (ii)$\Rightarrow$(i). Indeed, if $\rho(a)$ is invertible in $R_{\mathrm{red}}[[t]]$, we have $ab = 1 + c$ for some $b \in R((t))^{\surd}$, $c \in \mathrm{Ker}\,\rho = \sqrt{R}((t))$. But every element $c \in \sqrt{R}((t))$ is topologically nilpotent (setting $c = c_- + c_+$ with $c_- \in t^{-1}\sqrt{R}[t^{-1}]$ and $c_+ \in \sqrt{R}[[t]]$, we have that c_- is nilpotent while c_+ is topologically nilpotent), thus ab is invertible in $R((t))^{\surd}$, and a is invertible, too. $\square$

(2.3.2) Corollary. *If R is a local ring with maximal ideal M, then $R((t))^{\surd}$ is a local ring with maximal ideal $\rho^{-1}\big(M_{\rm red}[[t]] + tR_{\rm red}[[t]]\big)$.*

(2.4) The formal loop space. We now describe our main construction. Let X be a scheme of finite type over k. Define a covariant functor λ_X from **Alg** to sets as follows:

$$\lambda_X(R) = \mathrm{Hom}_{\mathbf{Sch}}\big(\mathrm{Spec}\ R((t))^{\surd}, X\big). \tag{2.4.1}$$

(2.4.2) Theorem. *(a) The functor λ_X is represented by an ind-scheme $\mathcal{L}(X)$, containing $\mathcal{L}^0(X)$.*

(b) $\mathcal{L}(X)$ is an inductive limit of nilpotent extensions of $\mathcal{L}^0(X)$. In particular, for any open set $Y \subset \mathcal{L}^0(X)$ there is a well-defined ind-scheme $\mathcal{L}(X)|_Y$.

(c) If U is an open subset in X, then the ind-scheme $\mathcal{L}(U)$ is identified with $\mathcal{L}(X)|_{p^{-1}(U)}$.

(d) If X is smooth, then $\mathcal{L}(X)$ is formally smooth.

The proof will be finished in the next subsection.

Let us now define a contravariant functor λ'_X on the category **Sch** by

$$\lambda'_X(S) = \mathrm{Hom}_{\mathbf{Lrs}}\big((S, \mathcal{O}_S((t))^{\surd}), X\big). \tag{2.4.3}$$

(2.4.4) Proposition. *For an affine scheme $S = \mathrm{Spec}(R)$ we have $\lambda'_X(S) = \lambda_X(R)$.*

Proof: Since $R((t))^{\surd}$ is a nilpotent extension of $R[[t]]$, we have $\underline{\mathrm{Spec}}\, R((t))^{\surd} = \underline{\mathrm{Spec}}\, R[[t]]$, so Spec $R((t))^{\surd}$ is the ringed space formed by $\underline{\mathrm{Spec}}\, R[[t]]$ and a certain sheaf of rings $\mathcal{O}_{\mathrm{Spec}\ R((t))^{\surd}}$ on it.

(2.4.5) Lemma. *With respect to the maps in (2.2.2) we have*

$$i^{-1}\mathcal{O}_{\mathrm{Spec}\ R((t))^{\surd}} = \mathcal{O}_{\mathrm{Spec}\ R}((t))^{\surd} = p_*\mathcal{O}_{\mathrm{Spec}\ R((t))^{\surd}}.$$

The proof is analogous to that of Lemma 2.2.3. Instead of the property of $A[[t]]$ quoted there, we use Proposition 2.3.1. □

Having Lemma 2.4.5, we prove Proposition 2.4.4 similarly to Proposition 2.2.1(a).

In virtue of Proposition 2.4.4, it suffices to show that the functor λ'_X on **Sch** is ind-representable. We start by estabilishing some of its properties.

(2.4.6) Proposition. *(a) For every scheme X the functor λ'_X is a sheaf on* **Sch**.

(b) If $U \subset X$ is an open subset, then the induced morphism of functors $\lambda'_U \to \lambda'_X$ is open.

(c) Let $(U_\alpha)_{\alpha\in A}$ be an open covering of X. Then λ'_X is equal to the cokernel, in the category **Shf**, *of the pair of morphisms*

$$\coprod_{\alpha,\beta} \lambda_{U_\alpha\cap U_\beta} \rightrightarrows \coprod_\alpha \lambda_{U_\alpha},$$

Proof: The proposition follows from simple properties of representable functors on the category **Lrs**. If $\mathcal{T}=(T,\mathcal{O}_\mathcal{T})$ is a locally ringed space, we will call an open part of $\mathcal{T}$ a ringed space of the form $(U,\mathcal{O}_\mathcal{T}|_U)$ where $U\subset T$ is an open subset in the usual sense. An open embedding is, by definition, a morphism isomorphic to the inclusion of an open part. Accordingly, we have the concept of an open covering of $\mathcal{T}$. This makes **Lrs** into a Grothendieck site. For $\mathcal{X}\in$ **Lrs** let $\eta_\mathcal{X}$ be the corresponding representable functor on **Lrs**. As in (2.1), a morphism $F:\phi'\to\phi$ of functors **Lrs** $\to$ **Sets** will be called open, if for any $\mathcal{S}\in$ **Lrs** and any element $u\in\phi(\mathcal{S})$ (i.e., a morphism $\eta_\mathcal{S}\to\phi$) the fiber product functor $\eta_\mathcal{S}\times_\phi\phi'$ is representable by a locally ringed space $\mathcal{S}'$ whose natural morphism to $\mathcal{S}$ is an open embedding. Let us recall the following basic facts.

(2.4.7) Lemma. *(a) For any $\mathcal{X}\in$ **Lrs** the representable functor $\eta_\mathcal{X}$ is a sheaf on* **Lrs**.

(b) If $\mathcal{U}\subset\mathcal{X}$ is an open embedding in **Lrs**, *then $\eta_\mathcal{U}\to\eta_\mathcal{X}$ is an open morphism of functors.*

*(c) Let $\mathcal{X}\in$ **Lrs** and $(\mathcal{U}_\alpha)_{\alpha\in A}$ be an open covering of $\mathcal{X}$. Then in the category of sheaves of sets on* **Lrs** *we have the equality*

$$\eta_\mathcal{X}=\operatorname{Coker}\Big\{\coprod_{\alpha,\beta}\eta_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta}\rightrightarrows\coprod_\alpha\eta_{\mathcal{U}_\alpha}\Big\} \tag{2.4.8}$$

*or, explicitly, for any $\mathcal{S}\in$ **Lrs**,*

$$\eta_\mathcal{X}(\mathcal{S})=\varinjlim_{\mathcal{S}_\alpha}\operatorname{Ker}\Big\{\prod_\alpha\eta_{\mathcal{U}_\alpha}(\mathcal{S}_\alpha)\rightrightarrows\prod_{\alpha\beta}\eta_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta}(\mathcal{S}_\alpha\cap\mathcal{S}_\beta)\Big\}, \tag{2.4.9}$$

where the limit is taken over the set of open coverings $(\mathcal{S}_\alpha)_{\alpha\in A}$ of $\mathcal{S}$ (the indexing set A being fixed) ordered by simultaneous inclusion.

Proposition 2.4.6 follows from Lemma 2.4.7 applied to $\mathcal{X}=X$ (scheme considered as a ringed space) and the ringed spaces of the form $\mathcal{S}=(S,\mathcal{O}_S((t))^{\sqrt{\ }})$ where S is a scheme.

□

Proof of (2.4.7) : Part (a) is obvious, and Part (b) is proved in [EGA0, (4.5.2)]. Let us prove Part (c). For any $\phi\in\eta_\mathcal{X}(\mathcal{S})$ and any α, consider the ringed space

$\mathcal{S}_\alpha = (\phi^{-1}(\mathcal{U}_\alpha), \mathcal{O}_\mathcal{S}|_{\phi^{-1}(\mathcal{U}_\alpha)})$. Clearly, $(\mathcal{S}_\alpha)$ is an open covering of $\mathcal{S}$. Let ϕ_α be the restriction of ϕ to $\mathcal{S}_\alpha$. Then, (ϕ_α) is an element of $\prod_\alpha \eta_{\mathcal{U}_\alpha}(\mathcal{S}_\alpha)$ lying in the kernel (2.4.9). Thus we have constructed a map from $\eta_\mathcal{X}(\mathcal{S})$ to the right hand side of (2.4.9). On the other hand, assume that $(\phi_\alpha) \in \prod_\alpha \eta_{\mathcal{U}_\alpha}(\mathcal{S}_\alpha)$ is such that $\phi_\alpha(\mathcal{S}_\alpha \cap \mathcal{S}_\beta) \subset \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $\phi_\alpha|_{\mathcal{S}_\alpha\cap\mathcal{S}_\beta} = \phi_\beta|_{\mathcal{S}_\alpha\cap\mathcal{S}_\beta}$. The corresponding maps $\mathcal{S}_{\alpha,\mathrm{red}} \to \mathcal{U}_{\alpha,\mathrm{red}}$ glue together, giving a continuous map $\phi_\mathrm{red} : \mathcal{S}_\mathrm{red} \to \mathcal{X}_\mathrm{red}$. Moreover, there is an obvious sheaf homomorphism $\phi^* : \phi_\mathrm{red}^{-1}\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{S}$: the restriction of ϕ^* to $\mathcal{S}_\alpha$ is the composition of the chain of maps

$$(\phi_\mathrm{red}^{-1}\mathcal{O}_\mathcal{X})|_{\mathcal{S}_\alpha} = (\phi_{\alpha,\mathrm{red}}^{-1}\mathcal{O}_{\mathcal{U}_\alpha})|_{\mathcal{S}_\alpha} \xrightarrow{\phi_\alpha^*} \mathcal{O}_{\mathcal{S}_\alpha} = \mathcal{O}_\mathcal{S}|_{\mathcal{S}_\alpha}.$$

This establishes (2.4.9).

Let C be the cokernel as in (2.4.8) but taken in the category of presheaves. Then the cokernel in the category of sheaves is just the sheaf $\overline{C}$ associated to the presheaf C. By definition, for $\mathcal{S} \in \mathbf{Lrs}$ the set $C(\mathcal{S})$ consists of pairs $(\alpha, \phi : S \to \mathcal{U}_\alpha)$ taken modulo the identifications coming from morphisms of $\mathcal{S}$ into $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Now, by the definition of the sheaf associated to the presheaf,

$$\overline{C} = \varinjlim_{\mathcal{S}=\bigcup_{\gamma\in\Gamma}\mathcal{S}_\gamma} \mathrm{Ker}\left\{ \prod_{\gamma\in\Gamma} C(\mathcal{S}_\gamma) \to \prod_{\gamma,\gamma'\in\Gamma} C(\mathcal{S}_\gamma \cap \mathcal{S}_{\gamma'}) \right\},$$

where the limit is over all possible open coverings of $\mathcal{S}$ (with arbitrary indexing sets Γ) ordered by refinement. Notice now that specifying a section of C over $\mathcal{S}_\gamma$ includes specifying an index α from the set A indexing the covering $\{\mathcal{U}_\alpha\}$, so we get a map $p : \Gamma \to A$. Denoting $\mathcal{S}_\alpha = \bigcup_{p(\gamma)=\alpha} \mathcal{S}_\gamma$, we get an element of the right hand side of (2.4.9). This establishes the equivalence of (2.4.8) and (2.4.9). □

Note that Proposition 2.4.6(c) implies the following.

(2.4.10) Corollary. *We have* $\lambda_X = \varinjlim_{U\subset X \text{ affine}} \lambda_U$, *the limit being taken in the category* **Shf**.

(2.5) Proof of Theorem 2.4.2. We first assume that $X = \operatorname{Spec} A$ is an affine scheme of finite type. Consider the larger functor $\tilde{\lambda}_X$ on **Alg** defined by

$$\tilde{\lambda}_X(R) = \mathrm{Hom}_{\mathbf{Alg}}\big(A, R((t))\big).$$

(2.5.1) Proposition. *(a) The functor* $\tilde{\lambda}_X$ *is represented by an ind-scheme* $\tilde{\mathcal{L}}(X)$, *which is an inductive limit of affine schemes of infinite type.*

(b) If X *is smooth, then* $\tilde{\mathcal{L}}(X)$ *is formally smooth.*

Proof: (a) Consider the k-ind-scheme

$$\underline{k((t))} = \varinjlim_N \operatorname{Spec} k[a_l; -N \le l].$$

We can think of the a_i as the coefficients of an indeterminate Laurent series $\sum a_i t^i$. It is clear that $\underline{k((t))}$ represents the functor $\tilde{\lambda}_{\mathbb{A}^1}$. Since $\mathbb{A}^1$ is a k-algebra object in the category of schemes, multiplication of Laurent series makes $\underline{k((t))}$ into a k-algebra object in the category of ind-schemes. In particular, each polynomial $f \in k[x_1, ..., x_d]$ defines a morphism of ind-schemes $\tilde{\mathcal{L}}(f) : \underline{k((t))}^d \to \underline{k((t))}$. Hereafter we will write $k[x]$ instead of $k[x_1, ..., x_d]$ to simplify.

Suppose now that X is given in $\mathbb{A}^d$ by a system of algebraic equations, say

$$f_m(x_1, ..., x_d) = 0, \qquad m = 1, 2, ..., l.$$

The ind-scheme $\tilde{\mathcal{L}}(X)$ is realized as the closed sub-ind-scheme in $\underline{k((t))}^d$ defined as the intersection of the preimages of 0 under the $\tilde{\mathcal{L}}(f_m)$. More explicitly, replacing x_i by $x_i(t) = \sum_{j \geq -N} a_j^{(i)} t^j$ in the equations above, we get, for each N, a system of algebraic equations in $k[a_l^{(i)}; -N \leq l]$ for each N which defines a subscheme in $(\operatorname{Spec} k[a_l; -N \leq l])^d$. Our desired ind-scheme $\tilde{\mathcal{L}}(X)$ is the direct limit of these schemes as $N \to \infty$.

(b) The infinitesimal lifting condition for $\tilde{\lambda}_X$ is formulated for affine schemes $S = \operatorname{Spec}(R)$. We need to prove that for any surjection of k-algebras $R' \to R$ whose kernel I satisfies $I^n = 0$ for some n, the map of sets $\tilde{\lambda}_X(R') \to \tilde{\lambda}_X(R)$ is surjective. But the kernel of $R'((t)) \to R((t))$ is $I((t))$ which is also nilpotent of order n. So the smoothness of A implies that any morphism $A \to R((t))$ can be lifted to a morphism $A \to R'((t))$. $\square$

(2.5.2) Corollary. *The functor λ_X is represented by an ind-scheme $\mathcal{L}(X)$ which is the inductive limit of the formal neighborhoods of $\mathcal{L}^0(X)$ in the schemes forming an inductive system for $\tilde{\mathcal{L}}(X)$. If X is smooth, then $\mathcal{L}(X)$ is formally smooth.*

This proves parts (a), (b) and (d) of Theorem 2.4.2 for the case of affine X. Part (c) of the theorem follows from Proposition 2.4.6(c).

To prove Theorem 2.4.2 for general X, it is enough to establish the existence of the limit $\varinjlim_{U \subset X \text{ affine}} \mathcal{L}(U)$ in the category of ind-schemes. In fact, it is enough to take the limit over a finite set of U consisting of elements of some finite covering and their intersections. Indeed, given this, all the other properties follow from the affine case and from Proposition 2.4.6. But using Proposition 2.4.6(b) again, we see that for any pair $U' \subset U$ of affine open subsets in X the corresponding morphism of ind-schemes $\mathcal{L}(U') \to \mathcal{L}(U)$ is an open embedding. Further, the ind-scheme $\mathcal{L}(U)_{\text{red}}$ is actually a scheme, namely $\mathcal{L}^0(U)$. So our statement follows from the next general fact.

(2.5.3) Lemma. *Let $(Z_i)_{i \in I}$ be a finite diagram of ind-schemes in which all the arrows are open embeddings. Assume that the inductive limit of $Z_{i,\text{red}}$ exists in the category of ind-schemes. Then so does the inductive limit of the Z_i.*

Theorem 2.4.2 is proved. Let us also note the following fact.

(2.5.4) Proposition. *If $\phi : X \to Y$ is an étale morphism of schemes of finite type, then $\mathcal{L}(\phi) : \mathcal{L}(X) \to \mathcal{L}(Y)$ is formally étale.*

Proof: By the above we can assume that $X = \mathrm{Spec}\,(A)$, $Y = \mathrm{Spec}\,(B)$ are affine. If S is a scheme and we have two compatible maps

$$\alpha : S_{\mathrm{red}} \to \mathcal{L}(X), \qquad \beta : S \to \mathcal{L}(Y),$$

we must construct the unique map $S \to \mathcal{L}(X)$. If $S = \mathrm{Spec}\, R$, then we have a diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha^*} & R_{\mathrm{red}}((t))^{\surd} & = & R_{\mathrm{red}}[[t]] \\ \phi^* \uparrow & & \uparrow & & \\ B & \xrightarrow{\beta^*} & R((t))^{\surd}, & & \end{array}$$

where the right arrow, which is the projection ρ in (2.3), has nilpotent kernel. Therefore there is a unique homomorphism of rings $A \to R((t))^{\surd}$ making the diagram commute. $\square$

(2.6) The formal loop space as an ind-object. For future purposes we construct a certain class of local realizations of $\mathcal{L}(X)$ as an ind-object in the category of schemes.

Let $x_1, x_2, ..., x_d$ be the coordinates on the affine space $\mathbb{A}^d$. For any k-algebra A and any morphism $\phi : \mathrm{Spec}\, A \to \mathbb{A}^d$ let ϕ^* be the corresponding map $k[x_1, ..., x_d] \to A$.

(2.6.1) Proposition. *Assume that $X = \mathrm{Spec}\, A$ is an affine scheme and that $\phi : X \to \mathbb{A}^d$ is an étale morphism of schemes.*

(a) There is a unique morphism $\theta_\phi : \mathcal{L}(X) \to X$ such that $\theta_\phi(f)(\phi^* x_i) = f(\phi^* x_i)_0$ for any $f \in \lambda_X(R)$. The restriction of θ_ϕ to $\mathcal{L}_0(X)$ is equal to the projection p from (2.2.1)(b).

(b) We have $\mathcal{L}(\mathbb{A}^d) \times_{\mathbb{A}^d} X \simeq \mathcal{L}(X)$.

Proof: Consider the morphism of functors $\tilde{\lambda}_{\mathbb{A}^d} \to \eta_{\mathbb{A}^d}$ which maps a homomorphism $f \in \mathrm{Hom}_{\mathbf{Alg}}\big(k[x], R((t))\big)$ to the unique morphism in $\mathrm{Hom}_{\mathbf{Alg}}(k[x], R)$ such that $x_i \mapsto f(x_i)_0$. This morphism of functors can be seen as a morphism of ind-schemes $\tilde{\theta}_{\mathbb{A}^d} : \tilde{\mathcal{L}}(\mathbb{A}^d) \to \mathbb{A}^d$. It is clear that its restriction to $\mathcal{L}^0(\mathbb{A}^d)$ coincides with the projection p from (2.2.1)(b). Let $\theta_{\mathbb{A}^d}$ be the restriction of $\tilde{\theta}_{\mathbb{A}^d}$ to the ind-subscheme $\mathcal{L}(\mathbb{A}^d) \subset \tilde{\mathcal{L}}(\mathbb{A}^d)$. Since $\mathcal{L}(X)$ is an inductive limit of nilpotent extensions of the scheme $\mathcal{L}^0(X)$ and ϕ is formally étale, there is a map $\theta_\phi : \mathcal{L}(X) \to X$ such that the diagram

$$\begin{array}{ccc} \mathcal{L}^0(X) & \to & X \\ \downarrow & & \downarrow \phi \\ \mathcal{L}(X) & \to & \mathbb{A}^d \end{array}$$

commutes. Here, the lower horizontal arrow is the composition of maps

$$\theta_{\mathbb{A}^d} \circ \mathcal{L}(\phi) : \mathcal{L}(X) \to \mathcal{L}(\mathbb{A}^d) \to \mathbb{A}^d.$$

Let ψ be the resulting map $\mathcal{L}(X) \to \mathcal{L}(\mathbb{A}^d) \times_{\mathbb{A}^d} X$. We claim that ψ is an isomorphism. For this, we construct a map $\chi : \mathcal{L}(\mathbb{A}^d) \times_{\mathbb{A}^d} X \to \mathcal{L}(X)$ inverse to ψ. Let $S = \operatorname{Spec} R$ be a scheme. A morphism $S \to \mathcal{L}(\mathbb{A}^d) \times_{\mathbb{A}^d} X$ is a compatible pair

$$\big(\alpha : S \to \mathcal{L}(\mathbb{A}^d),\, \beta : S \to X\big).$$

We need to construct a map $\chi(\alpha,\beta) : S \to \mathcal{L}(X)$. First, $\alpha(S_{\rm red}) \subset \mathcal{L}^0(\mathbb{A}^d)$, and $\beta(S_{\rm red}) \subset X$. Thus, by Proposition 2.2.2, we have a map $\gamma : S_{\rm red} \to \mathcal{L}^0(X)$. The composition $\tilde\gamma$ of γ and the embedding $\mathcal{L}^0(X) \subset \mathcal{L}(X)$ gives a commutative diagram

$$\begin{array}{ccc} S_{\rm red} & \stackrel{\tilde\gamma}{\to} & \mathcal{L}(X) \\ \downarrow & & \downarrow {\scriptstyle \mathcal{L}(\phi)} \\ S & \stackrel{\alpha}{\to} & \mathcal{L}(\mathbb{A}^d). \end{array} \tag{2.6.3}$$

To complete the construction, we notice that $\mathcal{L}(\phi)$ is formally étale by Proposition 2.5.4 and so we have a map $S \to \mathcal{L}(X)$ splitting (2.6.2) into two commutative triangles. We take this map to be $\chi(\alpha,\beta)$. The verification that χ is inverse to ψ is straightforward. Proposition 2.6.1 is proved. □

Let $\mathbb{E}$ be the set of $\varepsilon = (\varepsilon_{-1}, \varepsilon_{-2}, ...)$, $\varepsilon_j \in \mathbb{Z}_+$ such that $\varepsilon_j = 0$ for almost all j. It is equipped with the natural partial order: $\varepsilon \le \varepsilon'$ if $\varepsilon_j \le \varepsilon'_j$ for all j. In the remainder of this section we assume that the k-scheme X is smooth. Thus X can be covered by affine open sets $U = \operatorname{Spec} A$ possessing étale maps $\phi : U \to \mathbb{A}^d$. For every such U, ϕ we consider the functor

$$\lambda^\varepsilon_\phi : R \mapsto \big\{ f \in \lambda_U(R) \,|\, f(\phi^* x_i)_j^{1+\varepsilon_j} = 0,\, \forall i \in [1,d],\, \forall j < 0 \big\}.$$

(2.6.3) Proposition. *The functor λ^ε_ϕ is representable by a closed subscheme $\mathcal{L}^\varepsilon(\phi) \subset \mathcal{L}(U)$, such that $\mathcal{L}(U)$ is the inductive limit of the schemes $\mathcal{L}^\varepsilon(\phi)$.*

Proof: We first consider the case $U = \mathbb{A}^d, \phi = \mathrm{Id}$. Let N be such that $\epsilon_j = 0$ for $j < N$. Define the scheme $\mathcal{L}^\varepsilon(\mathbb{A}^d)$ by

$$\mathcal{L}^\varepsilon(\mathbb{A}^d) = \operatorname{Spec}\big(k[a_l^{(i)}; -N \le l] / \big((a_l^{(i)})^{1+\epsilon_l}\big)\big). \tag{2.6.4}$$

It is immediate that this scheme represents the functor $\lambda^\varepsilon_{\rm Id}$, and that $\lambda^\varepsilon_\phi = \lambda^\varepsilon_{\rm Id} \times_{\lambda_{\mathbb{A}^d}} \lambda_U$. The proof of Proposition 2.6.1 implies that that the map

$$\lambda^\varepsilon_\phi \to \lambda^\varepsilon_{\rm Id} \times_{\eta_{\mathbb{A}^d}} \eta_U, \quad f \mapsto \big(f \circ \phi^*, \theta_\phi(f)\big),$$

is an isomorphism of functors. Thus, the closed subscheme

$$\mathcal{L}^\varepsilon(\mathbb{A}^d) \times_{\mathbb{A}^d} U \subset \mathcal{L}(\mathbb{A}^d) \times_{\mathbb{A}^d} U = \mathcal{L}(U), \tag{2.6.5}$$

see Proposition 2.6.1.(b), represents the functor λ^ε_ϕ. It is clear by the definition that $\lambda_U = \varinjlim_\varepsilon \lambda^\varepsilon_\phi$ in the category of functors $\mathbf{Alg} \to \mathbf{Sets}$ and thus we have that $\mathcal{L}(U) = \varinjlim_\varepsilon \mathcal{L}^\varepsilon(\phi)$ in the category of ind-schemes. □

Let us note the following reformulation of this fact, to be used later.

(2.6.6) Corollary. *If ϕ, ψ are two étale maps $U \to \mathbb{A}^d$, then the ind-objects "$\varinjlim_\varepsilon$"$\mathcal{L}^\varepsilon(\phi)$, "$\varinjlim_\varepsilon$"$\mathcal{L}^\varepsilon(\psi)$ are isomorphic, i.e., for any ε there is ε' such that $\mathcal{L}^\varepsilon(\phi) \subset \mathcal{L}^{\varepsilon'}(\psi)$ and vice versa.*

Proof: Given ψ, any map $S \to \mathcal{L}(U)$ with S an affine scheme, factors through some $\mathcal{L}^{\varepsilon'}(\psi)$. Now take $S = \mathcal{L}^\varepsilon(\phi)$ which we know to be an affine scheme by (2.6.4-5). □

(2.7) The formal loop space as an ind-pro-object. We keep the notations of (2.6). Thus $U = \operatorname{Spec}(A)$ is affine and $\phi : U \to \mathbb{A}^d$ is étale. The schemes $\mathcal{L}^\varepsilon(\phi)$ are of infinite type. In fact, each of these schemes is a projective limit of schemes of finite type, so $\mathcal{L}(U)$ can be viewed as an ind-pro-object in the category of schemes of finite type over k. In this subsection we construct explicit ind-pro-systems for $\mathcal{L}(U)$.

Consider the functor $\lambda^\varepsilon_{n\phi} : R \mapsto \lambda^\varepsilon_\phi(R)/\sim_{n\phi}$, where

$$f \sim_{n\phi} g \iff f(\phi^* x_i) - g(\phi^* x_i) \in t^{n+1}R[[t]], \quad \forall i.$$

(2.7.1) Proposition. *(a) The functor $\lambda^\varepsilon_{n\phi}$ is representable by a scheme $\mathcal{L}^\varepsilon_n(\phi)$ of finite type which is a nilpotent extension of the scheme $\mathcal{L}^0_n(U) = \mathcal{L}^0_n(\phi)$. Moreover, $\mathcal{L}^\varepsilon(\phi) = \varprojlim_n \mathcal{L}^\varepsilon_n(\phi)$.*

(b) The schemes $\mathcal{L}^\varepsilon_n(\phi)$ form an ind-pro-system with Cartesian squares ($n' \geq n$, $\varepsilon' \leq \varepsilon$)

$$\begin{array}{ccc} \mathcal{L}^\varepsilon_{n'}(\phi) & \hookrightarrow & \mathcal{L}^{\varepsilon'}_{n'}(\phi) \\ \downarrow & & \downarrow \\ \mathcal{L}^\varepsilon_n(\phi) & \hookrightarrow & \mathcal{L}^{\varepsilon'}_n(\phi) \end{array}$$

where the vertical arrows are smooth affine morphisms.

(c) The ind-pro-object "$\varinjlim_\varepsilon$""$\varprojlim_n$"$\mathcal{L}^\varepsilon_n(\phi)$ is independent, up to isomorphism, on ϕ.

Proof: Claim (a) is entirely similar to Proposition 2.6.3. We first consider the case $U = \mathbb{A}^d, \phi = \mathrm{Id}$ and define the scheme $\mathcal{L}^\varepsilon_n(\mathbb{A}^d)$ by

$$\mathcal{L}^\varepsilon_n(\mathbb{A}^d) = \operatorname{Spec}\big(k[a^{(i)}_l; -N \leq l \leq n]/\big((a^{(i)}_l)^{1+\varepsilon_l}\big)\big), \quad N \gg 0. \tag{2.7.2}$$

It represents $\lambda^{\varepsilon}_{n\mathrm{Id}}$. As before, we see that for an arbitrary étale $\phi : U \to \mathbb{A}^d$ the fiber product scheme

$$\mathcal{L}^{\varepsilon}_n(\phi) := \mathcal{L}^{\varepsilon}_n(\mathbb{A}^d) \times_{\mathbb{A}^d} X \tag{2.7.3}$$

represents the functor $\lambda^{\varepsilon}_{n\phi}$.

Claim (b) is obvious in the case $\phi = \mathrm{Id}$. The general case follows from (2.7.3) since the since the base change of a smooth affine morphism is still smooth affine, and the base change of a Cartesian square is Cartesian.

Claim (c) follows from Corollary 2.6.6 and Proposition 2.3.2. □

Passing to the limit in ϵ we get the ind-scheme $\mathcal{L}_n(\phi) = \varinjlim_{\epsilon} \mathcal{L}^{\epsilon}_n(\phi)$. It represents the functor $R \mapsto \lambda_U(R)/\sim_{n\phi}$. As in (2.7.2) we get

$$\mathcal{L}_n(\mathbb{A}^d) = \varinjlim_{N} \operatorname{Spf} k[a^{(i)}_l, 0 \leq l \leq n][[a^{(i)}_l, -N \leq l < 0]], \tag{2.7.4}$$

and we see that $\mathcal{L}_n(\mathbb{A}^d)$ is formally smooth. Further, as in (2.7.3), we see that

$$\mathcal{L}_n(\phi) = \mathcal{L}_n(\mathbb{A}^d) \times_{\mathbb{A}^d} X, \tag{2.7.5}$$

from which we see that $\mathcal{L}_n(\phi)$ is also formally smooth.

3. The global loop space

(3.1) Localization on a curve. Consider the Lie algebra Der $k[[t]]$ and the group scheme Aut $k[[t]]$ over k. They form a Harish-Chandra pair [BB] [F, §6.1]. By construction, we have the action of this pair on the scheme $\mathcal{L}^0(X)$ and on the ind-scheme $\mathcal{L}(X)$ constructed in §2.

Let C be a smooth curve over k. The well known procedure of localization, see *loc. cit.* and [GKF], gives then a scheme $\mathcal{L}^0(X)_C$ and an ind-scheme $\mathcal{L}(X)_C$ over C defined as follows. Let $C^\wedge \to C$ be the scheme of pairs (c, t_c) where $c \in C$ and t_c is a formal coordinate near c. Then $C^\wedge$ has a natural (Der $k[[t]]$, Aut $k[[t]]$)-structure, i.e. a simply transitive Der $k[[t]]$-action extending the fiberwise Aut $k[[t]]$-action. We define

$$\mathcal{L}^0(X)_C = C^\wedge \times_{\mathrm{Aut}\ k[[t]]} \mathcal{L}^0(X), \quad \mathcal{L}(X)_C = C^\wedge \times_{\mathrm{Aut}\ k[[t]]} \mathcal{L}(X). \tag{3.1.1}$$

If X is affine, we define, in a similar way, the ind-scheme $\tilde{\mathcal{L}}(X)_C$ starting from $\tilde{\mathcal{L}}(X)$. Because of the simple transitivity of the Der $k[[t]]$-action on $C^\wedge$, the (ind-)schemes thus constructued possess natural connections along C which are compatible with the embeddings $\mathcal{L}^0(X)_C \subset \mathcal{L}(X)_C$ and, for X affine, $\mathcal{L}^0(X)_C \subset \tilde{\mathcal{L}}(X)_C$.

Note that $\mathcal{L}^0(X)_C$ is nothing but the scheme of infinite jets of morphisms $C \to X$, so it is easy to describe explicitly the functor represented by $\mathcal{L}^0(X)_C$ (and also by $\mathcal{L}(X)_C$, $\tilde{\mathcal{L}}(X)_C$). Let us introduce some notations. For a scheme S and a morphism $f : S \to C$ we denote by $\Gamma(f) \subset S \times C$ the graph of f. Let $f_{\mathrm{red}} : S_{\mathrm{red}} \to C$ be the restriction of f to S_{red}. We have then the following sheaves of k-algebras on $S \times C$ supported on $\Gamma(f)$:

- $\mathcal{O}_f^\wedge$, the completion of $\mathcal{O}_{S\times C}$ along $\Gamma(f)$, i.e. the sheaf of functions on the formal neighborhood of $\Gamma(f)$.
- $\mathcal{K}_f^\wedge$, the sheaf of functions on the punctured formal neighborhood of $\Gamma(f)$. Thus $\mathcal{K}_f^\wedge$ is obtained from $\mathcal{O}_f^\wedge$ by inverting the local equations of the divisor $\Gamma(f) \subset S \times C$.
- $\mathcal{K}_f^{\sqrt{\ }} \subset \mathcal{K}_f^\wedge$, the subsheaf of sections whose restriction to $S_{\mathrm{red}} \times C$ lies in $\mathcal{O}_{f_{\mathrm{red}}}^\wedge \subset \mathcal{K}_{f_{\mathrm{red}}}^\wedge$.

The following is straightforward and left to the reader, part (a) being proved similarly to Proposition 2.1.1.

(3.1.2) Proposition. *(a) The sheaves $\mathcal{O}_f^\wedge$ and $\mathcal{K}_f^{\sqrt{\ }}$ are sheaves of local rings.*

*(b) The scheme $\mathcal{L}^0(X)_C$ represents the functor $\lambda^0_{X,C}$: **Sch** $\to$ **Sets** which associates to a scheme S the set of pairs (f, ρ) where $f : S \to C$ is a morphism of schemes and $\rho : (\Gamma(f), \mathcal{O}_f^\wedge) \to X$ is a morphism of ringed spaces.*

(c) The ind-scheme $\mathcal{L}(X)_C$ ind-represents the functor $\lambda_{X,C}$ defined as in (b) but with $\mathcal{O}_f^\wedge$ replaced by $\mathcal{K}_f^{\sqrt{\ }}$.

(d) Similarly, when X is affine, the ind-scheme $\tilde{\mathcal{L}}(X)_C$ ind-represents the functor $\tilde{\lambda}_{X,C}$ defined as in (b) but with $\mathcal{O}_f^\wedge$ replaced by $\mathcal{K}_f^\wedge$.

(3.2) Factorization semigroups. Let C be a smooth curve, as before. For any surjection $J \twoheadrightarrow I$ of finite sets and $i \in I$ we denote by J_i the preimage of i. To such a surjection one associates, in a standard way, the "diagonal" embedding $\Delta^{(J/I)} : C^I \hookrightarrow C^J$. Let $U^{(J/I)} \subset C^J$ be the locus of $(c_j)_{j\in J}$ such that $c_j \neq c_{j'}$ whenever the images of j and j' in I are different. We denote by $j^{(J/I)} : U^{(J/I)} \hookrightarrow C^J$ the embedding.

(3.2.1) Definition. *Let Y be an ind-scheme over C. A factorization semigroup structure on Y is a collection of ind-schemes Y_I over C^I for each nonempty finite set I such that $Y_{\{1\}} = Y$, equipped with the following data:*

(a) An identification $\Delta^{(J/I)*}Y_J \xrightarrow{\sim} Y_I$ for every surjection $J \twoheadrightarrow I$, compatible with compositions of surjections.

(b) An isomorphism of $U^{(J/I)}$-ind-schemes

$$\kappa^{(J/I)} \,:\, j^{(J/I)*}(\textstyle\prod_{i\in I} Y_{J_i}) \xrightarrow{\sim} j^{(J/I)*}Y_J$$

for every $J \twoheadrightarrow I$, such that for $K \twoheadrightarrow J$ the isomorphism $\kappa^{(K/I)}$ coincides with the composition $\kappa^{(J/I)} \circ (\prod_{i\in I} \kappa^{(K_i/J_i)})$.

(3.2.2) Remarks. (a) This is a nonlinear counterpart of the concept of a factorization algebra due to Beilinson and Drinfeld [BD1] [G], see also (6.1) below. Factorization semigroups can be used to construct factorization algebras by applying "natural" linear constructions.

(b) More generally, we can speak about a factorization semigroup structure on any functor $\mathcal{Y}$: **Sch** $\to$ **Sets** (not necessarily representable by an ind-scheme) which is equipped with a morphism to C (i.e. to the representable functor η_C).

(3.2.3) Example. Let G be an affine algebraic group over k. Then $\mathcal{L}^0(G)$ is a group scheme and $\tilde{\mathcal{L}}(G)$ is a group ind-scheme over C. The quotient ind-scheme $\mathfrak{Gr}_G = \tilde{\mathcal{L}}(G)/\mathcal{L}^0(G)$ is known as the affine Grassmannian associated to G. The natural family of such Grassmannians over C, i.e. the ind-scheme $\mathfrak{Gr}_G(C) = \tilde{\mathcal{L}}(G)_C/\mathcal{L}^0(G)_C$ is known to have a structure of a factorization semigroup, see [G, §5.2.2].

Now, the main result of this section is the following.

(3.2.4) Theorem. *Let X be a scheme of finite type. Then the C-scheme $\mathcal{L}^0(X)_C$ and the C-ind-scheme $\mathcal{L}(X)_C$ possess natural structures of factorization semigroups so that the embedding $\mathcal{L}^0(X)_C \subset \mathcal{L}(X)_C$ is a factorization homomorphism. Similarly, if X is affine, then the C-ind-scheme $\tilde{\mathcal{L}}(X)_C$ has a natural structure of a* *factorization semigroup so that the embedding $\mathcal{L}^0(X)_C \subset \tilde{\mathcal{L}}(X)_C$ is a factorization homomorphism.*

(3.3) Factorization semigroup structure on the functors represented by $\mathcal{L}^0(X)_C, \tilde{\mathcal{L}}(X)_C$ and $\mathcal{L}(X)_C$. Let S be a scheme and f_I be I-uple of morphisms $f_i : S \to C$, $i \in I$. We denote by $\Gamma(f_I) \subset S \times C$ the union of the graphs of the f_i's and by $f_{I,\mathrm{red}} \subset S_{\mathrm{red}} \times C$ the union of the graphs of the $f_{i,\mathrm{red}}$. Let us introduce, similarly to (3.1), the following sheaves of rings on $S \times C$ with support in $\Gamma(f_I)$:

- $\mathcal{O}^{\wedge}_{f_I}$, the completion of $\mathcal{O}_{S\times C}$ along $\Gamma(f_I)$, i.e. the sheaf of functions on the formal neighborhood of $\Gamma(f_I)$.
- $\mathcal{K}^{\wedge}_{f_I}$, the sheaf of functions on the punctured formal neighborhood of $\Gamma(f_I)$.
- $\mathcal{K}^{\sqrt{\ }}_{f_I} \subset \mathcal{K}^{\wedge}_{f_I}$, the subsheaf of sections whose restriction to $S_{\mathrm{red}} \times C$ lies in $\mathcal{O}^{\wedge}_{f_{I,\mathrm{red}}} \subset \mathcal{K}^{\wedge}_{f_{I,\mathrm{red}}}$.

(3.3.1) Proposition. *The sheaves $\mathcal{O}^{\wedge}_{f_I}$ and $\mathcal{K}^{\sqrt{\ }}_{f_I}$ are sheaves of local rings.*

The proof is analog to the proof of Proposition 2.1.1, and is left to the reader.

(3.3.2) Definition. *Let I be a nonempty finite set.*

(a) We define the contravariant functors λ^0_{X,C^I}, λ_{X,C^I} from **Sch** *to* **Sets** *as follows. For a scheme S the set $\lambda^0_{X,C^I}(S)$ consists of pairs (f_I, ρ) such that*

$$f_I \in \mathrm{Hom}_{\mathbf{Sch}_k}(S, C^I) \quad \textit{and} \quad \rho \in \mathrm{Hom}_{\mathbf{Lrs}_k}\big((\Gamma(f_I), \mathcal{O}^{\wedge}_{f_I}),\, X\big).$$

The contravariant functor λ_{X,C^I} is defined similarly but with $\mathcal{O}^{\wedge}_{f_I}$ replaced by $\mathcal{K}^{\sqrt{\ }}_{f_I}$.

(b) If X is affine, the functor $\tilde{\lambda}_{X,C^I}$ from **Alg** *to* **Sets** *is such that the set $\tilde{\lambda}_{X,C^I}(R)$ consists of pairs (f_I, ρ) with*

$$f_I \in \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}\, R, C^I) \quad \textit{and} \quad \rho \in \mathrm{Hom}_{\mathbf{Alg}_k}\big(k[X], \mathrm{H}^0(\Gamma(f_I), \mathcal{K}^{\wedge}_{f_I})\big).$$

The embeddings of sheaves of rings $\mathcal{O}^{\wedge}_{f_I} \hookrightarrow \mathcal{K}^{\sqrt{\ }}_{f_I} \hookrightarrow \mathcal{K}^{\wedge}_{f_I}$ induce embeddings of functors $\lambda^0_{X,C^I} \hookrightarrow \lambda_{X,C^I} \hookrightarrow \tilde{\lambda}_{X,C^I}$.

(3.3.3) Proposition. *Let X be a fixed scheme of finite type. The families of functors (λ^0_{X,C^I}), $(\tilde{\lambda}_{X,C^I})$, (λ_{X,C^I}) with I running over nonempty finite sets, form each a factorization semigroup in the category of functors.*

Proof: This is almost obvious by construction. Indeed, let $J \twoheadrightarrow I$ be a surjection of nonempty sets. Then $\Delta^{(J/I)*}$ of the Jth functor in any of the three families takes S into the set of (f_J, ρ) where f_J is a morphism $S \to C^J$ which in fact lie in the image of $\Delta^{(J/I)}$. Thus f_J comes from a uniquely defined I-tuple f_I. Now, $\Gamma(f_J) = \Gamma(f_I)$ and so each of the three sheaves $\mathcal{O}^{\wedge}, \mathcal{K}^{\wedge}, \mathcal{K}^{\sqrt{\ }}$ associated to them coincide. This gives the datum (a) of Definition 3.2.1. Similarly, $j^{(J/I)^*}$ applied to the Jth functor in any of the families, takes S into the set of (f_J, ρ) where $f_J : S \to U^{(J/I)}$. But this means that $\Gamma(f_J) = \coprod_{i\in I} \Gamma(f_{J_i})$ and hence on the level of set of morphisms of $\Gamma(f_J)$ equipped with any of the three sheaves of rings, we get a direct product. This gives the datum (b), i.e. the isomorphism $\kappa^{(J/I)}$. The associativity of these isomorphisms is obvious. $\square$

Therefore, to establish Theorem 3.2.4, we need only to prove the representability of the functors λ^0_{X,C^I}, λ_{X,C^I} and, when X is affine, of $\tilde{\lambda}_{X,C^I}$.

(3.4) The global space of germs of arcs.

(3.4.1) Proposition. *The functor λ^0_{X,C^I} is representable by a scheme $\mathcal{L}^0(X)_{C^I}$ of infinite type over C^I.*

Proof: For $n > 0$ let $C^{(n)}$ be the nth symmetric product of C. As C is a smooth curve, $C^{(n)}$ is identified with the Hilbert scheme $\mathrm{Hilb}^n(C)$ parametrizing subschemes in C of finite length n. Explicitly, to a point of $C^{(n)}$, i.e. an effective divisor D on C of degree n, there corresponds the subscheme $Z_D = \mathrm{Spec}\big(\mathcal{O}_C/\mathcal{O}_C(-D)\big)$. The following lemma is well-known.

(3.4.2) Lemma. *Let T be a k-scheme, and X, Z be any T-schemes. Assume that the morphism $Z \to T$ is finite, and that $X \to T$ is of finite type. The contravariant functor*

$$\mathbf{Sch} \to \mathbf{Sets}, \quad S \mapsto \mathrm{Hom}_T(S \times Z, X),$$

is represented by a T-scheme $\underline{Hom}_T(Z, X)$.

Let Z be the total space of the family of the schemes Z_D, $D \in C^{(n)}$. We set $Map^n(C, X) = \underline{Hom}_{C^{(n)}}(Z, X \times C^{(n)})$. Let u^n_I be the composition of the maps

$$C^I \to C^{I\times[1,n+1]} \to C^{((n+1)|I|)}$$

where the first map takes the I-uple $(c_i, i \in I)$ to the $I\times[1,n+1]$-uple $(c_i, c_i, ..., c_i, i \in I)$, each c_i counted $n+1$ times, and the second map is the projection from the Cartesian product to the symmetric product. Let $\mathcal{L}^0_n(X)_{C^I}$ be the fiber product

$$\mathcal{L}^0_n(X)_{C^I} = C^I \times_{C^{((n+1)|I|)}} Map^{(n+1)|I|}(C, X). \tag{3.4.3}$$

For any I-uple $c \in C^I$ we have inclusions of subschemes of C

$$u^0_I(c) \subset u^1_I(c) \subset u^2_I(c) \subset \cdots \tag{3.4.4}$$

Thus we get a projective system of schemes

$$\mathcal{L}^0_1(X)_{C^I} \leftarrow \mathcal{L}^0_2(X)_{C^I} \leftarrow \mathcal{L}^0_3(X)_{C^I} \leftarrow \cdots$$

The morphisms in this projective system are affine because the embeddings $u_I^n(c) \subset u_I^{n+1}(c)$ are purely nilpotent. Therefore we have the limit scheme

$$\mathcal{L}^0(X)_{C^I} = \lim_{\longleftarrow n} \mathcal{L}^0_n(X)_{C^I}.$$

We claim that the scheme $\mathcal{L}^0(X)_{C^I}$ represents the functor λ^0_{X,C^I}. Indeed a morphism from a k-scheme S to $\mathcal{L}^0_n(X)_{C^I}$ is a pair (f_I, ψ) where f_I is a morphism $S \to C^I$ and

$$\psi \, : \, (\Gamma(f_I), \mathcal{O}_{u_I^n(f_I)}) \to (X, \mathcal{O}_X)$$

is a morphism of ringed spaces. Here $u_I^n(f_I) \subset S \times C$ is the subscheme of relative length $n|I|$ over S corresponding to the S-point f_I of C^I via u_I^n. When we pass to the limit we get $\lim_{\longleftarrow n} \mathcal{O}_{u_I^n(f_I)} = \mathcal{O}^\wedge_{f_I}$. □

(3.4.5) Remark. When $X = \operatorname{Spec} A$ is affine, the scheme $\mathcal{L}^0(X)_C$, being the scheme of infinite jets of maps $C \to A$, is the spectrum of a commutative $\mathcal{O}_C$-algebra with connection along C (in fact, of the universal such algebra with connection generated by $\mathcal{O}_C \otimes_k A$). According to Beilinson and Drinfeld [BD1], [G], commutative $\mathcal{O}_C$-algebras with connection are particular case of chiral algebras which, in their turn, give factorization algebras. So our construction in this case is a particular case of theirs.

(3.4.6) Proposition. *If $\phi : X \to Y$ is an étale morphism of schemes of finite type and $\pi : C \to D$ is an étale morphism of smooth curves, then the square*

$$\begin{array}{ccc}
\mathcal{L}^0(X)_{C^I} & \xrightarrow{\mathcal{L}^0(\phi)_{\pi^I}} & \mathcal{L}^0(Y)_{D^I} \\
\downarrow & & \downarrow \\
\mathcal{L}^0_0(X)_{C^I} & \longrightarrow & \mathcal{L}^0_0(Y)_{D^I}
\end{array}$$

is Cartesian. Here $\mathcal{L}^0_0(X)_{C^I}$ is the same as in (3.4.3).

Proof: Similarly to the proof of Proposition 2.2.2, this follows from the fact that all the embeddings in (3.4.4) are purely nilpotent. □

(3.5) The meromorphic loop space of an affine scheme. Here we prove the ind-representability of the functor $\tilde{\lambda}_{X,C^I}$ for X affine. We first treat the case when $X = \mathbb{A}^1$. Let I be fixed. To every point $(c_i)_{i \in I} \in C^I$ we associate the effective divisor $\sum c_i$ on C. For every $m, n \geq 0$ let $\mathcal{A}_{mn}$ be the vector bundle on C^I whose fiber over (c_i) is the vector space of global sections of the coherent 0-dimensional sheaf

$$\mathcal{O}_C\big(m{\textstyle\sum} c_i\big) / \mathcal{O}_C\big(-n{\textstyle\sum} c_i\big)$$

on C. Thus $\operatorname{rk}(\mathcal{A}_{mn}) = |I|(m+n)$. Let A_{mn} be the total space of the bundle $\mathcal{A}_{mn}$, considered as an algebraic variety over C^I. When m, n vary, these varieties form a double inductive-projective system and the following is then obvious.

(3.5.1) Proposition. *(a) For every $m>0$ the limit $\varprojlim_n A_{mn}$ exists as a scheme. The ind-scheme $\tilde{\mathcal{L}}(\mathbb{A}^1)_{C^I} = \varinjlim_m \varprojlim_n A_{mn}$ represents the functor $\tilde{\lambda}_{\mathbb{A}^1, C^I}$.*

(b) $\tilde{\mathcal{L}}(\mathbb{A}^1)_{C^I}$ has a natural structure of a k-algebra object in the category of ind-schemes over C^I.

Proof: A morphism of k-schemes $\operatorname{Spec} R \to A_{mn}$ is a pair of a morphism of k-schemes $f_I : S \to C^I$, and a morphism of sheaves of k-algebras $\mathcal{O}_{A_{mn}} \otimes_{\mathcal{O}_{C^I}} R \to R$ (here, the right hand side is identified with the constant sheaf on C^I). There is a canonical isomorphism

$$\varinjlim_m \varprojlim_n \operatorname{Hom}\big(\mathcal{O}_{A_{mn}} \otimes_{\mathcal{O}_{C^I}} R, R\big) \simeq \mathrm{H}^0\big(\Gamma(f_I), \mathcal{K}_{f_I}\big) \simeq \operatorname{Hom}_{\mathbf{Alg}}\big(k[x], \mathrm{H}^0(\Gamma(f_I), \mathcal{K}_{f_I})\big).$$

Claim (a) is proved. In part (b), the variety A_{mn} is not a ring scheme. However there is an obvious map $A_{mn} \times A_{m'n'} \to A_{m+m', n+n'}$ which induces a ring structure on the limit of the ind-pro system. $\square$

(3.5.2) Proposition. *For any affine X of finite type the functor $\tilde{\lambda}_{X,C^I}$ is representable by an ind-scheme $\tilde{\mathcal{L}}(X)_{C^I}$ over C^I.*

Proof: By part (b) of Proposition 3.5.1, for every $f \in k[x_1, ..., x_d]$ we have a morphism of ind-schemes

$$\tilde{\mathcal{L}}(f)_{C^I} : \big(\tilde{\mathcal{L}}(\mathbb{A}^1)_{C^I}\big)^d \to \tilde{\mathcal{L}}(\mathbb{A}^1)_{C^I}.$$

If now the scheme X is given in some $\mathbb{A}^d$ be equations $f_j(x_1, ..., x_d) = 0$, then $\tilde{\lambda}_{X,C^I}$ is represented by the ind-scheme $\tilde{\mathcal{L}}(X)_{C^I}$ which is the intersection, in $\big(\tilde{\mathcal{L}}(\mathbb{A}^1)_{C^I}\big)^d$, of the preimages of 0 under the $\tilde{\mathcal{L}}(f_j)_{C^I}$, i.e. the inverse limit of an obvious diagram in the category of ind-schemes. $\square$

(3.5.3) Corollary. *For an affine scheme X the functor λ_{X,C^I} is represented by an ind-scheme $\mathcal{L}(X)_{C^I}$ which is the inductive limit of the formal neighborhoods of $\mathcal{L}^0(X)_{C^I}$ in the schemes of an inductive system for $\tilde{\mathcal{L}}(X)_{C^I}$.*

(3.6) The global loop space of an arbitrary scheme. Let now X be an arbitrary scheme of finite type. The (ind-)representability of the functor λ_{X,C^I} follows from Corollary 3.5.3 for the affine case and from the general gluing properties of the functors summarized in the next proposition.

(3.6.1) Proposition. *(a) The functor λ_{X,C^I} is a sheaf on* **Sch**.

(b) If $U \subset X$ is an open subset, then the induced morphism of functors $\lambda_{U,C^I} \to \lambda_{X,C^I}$ is open.

(c) Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of X. Then λ_{X,C^I} is equal to the cokernel, in the category **Shf**, *of the pair of morphisms*

$$\prod_{\alpha,\beta} \lambda_{U_\alpha\cap U_\beta,C^I} \rightrightarrows \prod_\alpha \lambda_{U_\alpha,C^I}.$$

The proof is similar to that of Proposition 2.4.2, using Lemma 2.4.3 and the fact that $\mathcal{K}^{\sqrt{}}_{f_I}$ is a sheaf of local rings on $\Gamma(f_I)$. An immediate corollary of Proposition 3.6.1 and Corollary 3.5.3 is that the functor λ_{X,C^I} is representable by an ind-scheme $\mathcal{L}(X)_{C^I}$ over C^I. Then, Proposition 3.3.3 implies that the collections of schemes and ind-schemes $\mathcal{L}^0(X)_{C^I}$ and $\mathcal{L}(X)_{C^I}$ are factorization semigroups in the categories of schemes and ind-schemes. Theorem 3.2.4 is proved.

(3.6.2) Proposition. *If $\phi : X \to Y, \pi : C \to D$ are étale morphisms, then the induced morphism*

$$\mathcal{L}(\phi)_{\pi^I} : \mathcal{L}(X)_{C^I} \to \mathcal{L}(Y)_{D^I}$$

is formally étale.

The proof is similar to that of Proposition 2.5.4.

(3.7) Example : the cases $C = \mathbb{A}^1$ and $X = \mathbb{A}^d$. Let $C = \operatorname{Spec} k[t]$. If $S = \operatorname{Spec} R$, a morphism $f_I = (f_i) : S \to C^I$ is the same as a collection of elements $b_i = f_i^*(t) \in R$. Assume these have been fixed. Then the subscheme $u_I^n(f_I) \subset S\times C$ from the proof of Proposition 3.4.1 is described explicitly :

$$u_I^n(f_I) = \operatorname{Spec}\big(R[t]/\textstyle\prod_{i\in I}(t-b_i)^{n+1}\big). \tag{3.7.1}$$

This implies the following.

(3.7.2) Proposition. *(a) The ring*

$$H^0(\Gamma(f_I), \mathcal{O}^\wedge_{f_I}) = \varprojlim_n \big(R[t]/\textstyle\prod_i (t-b_i)^{n+1}\big)$$

is identified with the set of series $\sum_{l=0}^\infty a_l(t)\prod_i (t-b_i)^l$, where $a_l(t) \in R[t]$ are polynomials of degree less than $|I|$ (such polynomials form a set of representatives for $R[t]/\prod_i(t-b_i)$).

(b) The ring $H^0(\Gamma(f_I), \mathcal{K}_{f_I})$ is identified with the set of series $\sum_{l\gg-\infty} a_l(t)\prod_i(t-b_i)^l$, where $a_l(t)$ are as in (a).

(c) The subring $H^0(\Gamma(f_I), \mathcal{K}^{\sqrt{}}_{f_I})$ is identified with the series as in (b) but with the condition that all the coefficients of the polynomials $a_l(t), l < 0$, are nilpotent elements of R.

(3.7.3) Notation. For fixed $b_i \in R$ and $a \in H^0(\Gamma(f_I), \mathcal{K}_{f_I})$ we denote by $a_l(t) \in R[t]$ the lth coefficient of the series corresponding to a by Proposition 3.7.2(b) and by $a_{l\nu} \in R$, $\nu = 0, ..., |I|-1$, the νth coefficient of the polynomial $a_l(t)$.

Assume moreover that $X = \mathbb{A}^d$ with coordinates $x_1, ..., x_d$. In this case we can give a completely explicit description of the ind-schemes $\tilde{\mathcal{L}}(\mathbb{A}^d)_{\mathbb{A}^I}$ and $\mathcal{L}(\mathbb{A}^d)_{\mathbb{A}^I}$, using Proposition 3.7.2. Indeed,

$$\tilde{\lambda}_{\mathbb{A}^d, \mathbb{A}^I}(R) = \big\{ (b_i, \rho) \,\big|\, (b_i) \in R^I,\ \rho : k[x] \to H^0(\Gamma(f_I), \mathcal{K}_{f_I}) \big\}.$$

The algebra homomorphism ρ is uniquely determined by the choice of

$$\rho(x_j) = \sum_{l \gg -\infty} \sum_{\nu=0}^{|I|-1} a_{l\nu}^{(j)} t^\nu \prod_{i \in I} (t - b_i)^l, \quad a_{l\nu}^{(j)} \in R.$$

A choice of $\rho(x_j)$ is the same as a choice of elements $a_{l\nu}^{(j)} \in R$. Thus the universal case corresponds to $b_i, a_{l\nu}^{(j)}$, with $i \in I$, $j \in [1, d]$, $l \in \mathbb{Z}$ and $\nu \in [0, |I| - 1]$, being independent variables, i.e.

$$\tilde{\mathcal{L}}(\mathbb{A}^d)_{\mathbb{A}^I} = \varinjlim_N \operatorname{Spec} k\big[b_i, a_{l\nu}^{(j)}; l \geq -N\big]. \tag{3.7.4}$$

and

$$\mathcal{L}(\mathbb{A}^d)_{\mathbb{A}^I} = \varinjlim_N \operatorname{Spf} k[b_i, a_{l\nu}^{(j)}; l \geq 0][[a_{l\nu}^{(j)}; -N \leq l \leq -1]]. \tag{3.7.5}$$

Notice further that

$$\mathcal{L}_n^0(\mathbb{A}^d)_{\mathbb{A}^I} = \operatorname{Spec} k[b_i, a_{l\nu}^{(j)}; 0 \leq l \leq n].$$

In particular, we have a natural morphism

$$\theta_{\mathbb{A}^d, I} : \mathcal{L}(\mathbb{A}^d)_{\mathbb{A}^I} \to \mathcal{L}_0^0(\mathbb{A}^d)_{\mathbb{A}^I} \tag{3.7.6}$$

taking all $a_{l\nu}^{(j)}, l \neq 0$, to 0.

(3.8) The global loop space as an ind-object. We first consider the case $X = \mathbb{A}^d, C = \mathbb{A}^1$, employing the notations of (3.7). Let $\varepsilon = (\varepsilon_{-1}, \varepsilon_{-2}, ...) \in \mathbb{E}$ be as in (2.6). Define the scheme

$$\mathcal{L}^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I} = \operatorname{Spec} \big(k[b_i, a_{l\nu}^{(j)}] / (a_{l,\nu_1}^{(j)} \cdots a_{l,\nu_{1+\varepsilon_l}}^{(j)}) \big), \tag{3.8.1}$$

where $j \in [1, d]$, $l < 0$ and $\nu_1, ..., \nu_{1+\varepsilon_l} \in [0, |I| - 1]$ are arbitrary. It is clear that

$$\mathcal{L}(\mathbb{A}^d)_{\mathbb{A}^I} = \varinjlim_\varepsilon \mathcal{L}^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I}.$$

next, assume that X is an arbitrary smooth scheme of finite type and C is an arbitrary smooth curve. Then X can be covered by open $U = \operatorname{Spec} A$ possessing an étale map $\phi : U \to \mathbb{A}^d$ and similarly C can be covered by open V possessing an étale $\pi : V \to \mathbb{A}^1$. For each such (U, ϕ), (V, π) we get a formally étale map of ind-schemes

$$\mathcal{L}(\phi)_{\pi^I} : \mathcal{L}(U)_{V^I} \to \mathcal{L}(\mathbb{A}^d)_{\mathbb{A}^I}$$

and we define the subscheme $\mathcal{L}^\varepsilon(\phi)_{\pi^I} \subset \mathcal{L}(U)_{V^I}$ to be the pullback of $\mathcal{L}^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I}$.

(3.8.2) Proposition. *The ind-scheme $\mathcal{L}^\varepsilon(\phi)_{\pi^I}$ is in fact an affine scheme, fitting into a Cartesian diagram*

$$\begin{array}{ccc}
\mathcal{L}^\varepsilon(\phi)_{\pi^I} & \hookrightarrow & \mathcal{L}(U)_{V^I} \\
{\scriptstyle \theta_{\phi,\pi^I}}\downarrow & & \downarrow{\scriptstyle \mathcal{L}(\phi)_{\pi^I}} \\
\mathcal{L}_0^0(U)_{V^I} & \xrightarrow{\mathcal{L}_0^0(\phi)_{\pi^I}} & \mathcal{L}_0^0(\mathbb{A}^d)_{\mathbb{A}^I}.
\end{array}$$

The ind-object "$\varinjlim_\varepsilon$" $\mathcal{L}^\varepsilon(\phi)_{\pi^I}$ *in* **Sch** *is isomorphic to* $\mathcal{L}(U)_{V^I}$.

The proof (including the construction of θ_{ϕ,π^I}) is similar to that of Propositions 2.6.1 and 2.6.3 and is left to the reader.

(3.9) The global loop space as an ind-pro-object. We keep the notation of (3.8). For $\varepsilon \in \mathbb{E}, n \geq 1$ consider the scheme

$$\mathcal{L}_n^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I} = \operatorname{Spec}\big(k[b_i, a_{l\nu}^{(j)}; l \leq n]/(a_{l\nu_1}^{(j)} \cdots a_{l,\nu_{1+\varepsilon_l}}^{(j)})\big), \tag{3.9.1}$$

which we regard as a quotient scheme of $\mathcal{L}^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I}$, see (3.8.1). Denoting by $\mathcal{R}_n \rightrightarrows \mathcal{L}^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I}$ the corresponding equivalence relation, let $\mathcal{R}_{n\phi\pi} \rightrightarrows \mathcal{L}^\varepsilon(\phi)_{\pi^I}$ be its pullback under the map $\mathcal{L}^\varepsilon(\phi)_{\pi^I} \to \mathcal{L}^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I}$.

(3.9.2) Proposition. *(a) The quotient $\mathcal{L}_n^\varepsilon(\phi)_{\pi^I} = \mathcal{L}^\varepsilon(\phi)_{\pi^I}/\mathcal{R}_{n\phi\pi}$ exists and is a scheme of finite type fitting into a Cartesian square*

$$\begin{array}{ccc}
\mathcal{L}_n^\varepsilon(\phi)_{\pi^I} & \longrightarrow & \mathcal{L}_n^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I} \\
\downarrow & & \downarrow \\
\mathcal{L}_0^0(U)_{V^I} & \longrightarrow & \mathcal{L}_0^0(\mathbb{A}^d)_{\mathbb{A}^I}.
\end{array}$$

In particular, the natural map $\mathcal{L}_n^\varepsilon(\phi)_{\pi^I} \to \mathcal{L}_n^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I}$ is étale.

(b) The scheme $\mathcal{L}_n^\varepsilon(\phi)_{\pi^I}$ is a nilpotent extension of $\mathcal{L}_n^0(U)_{V^I}$.

(c) The schemes $\mathcal{L}_n^\varepsilon(\phi)_{\pi^I}$ form a double ind-pro-system with Cartesian squares ($n' \geq n,\ \varepsilon \leq \varepsilon'$)

$$\begin{array}{ccc}
\mathcal{L}_{n'}^\varepsilon(\phi)_{\pi^I} & \hookrightarrow & \mathcal{L}_{n'}^{\varepsilon'}(\phi)_{\pi^I} \\
\downarrow & & \downarrow \\
\mathcal{L}_n^\varepsilon(\phi)_{\pi^I} & \hookrightarrow & \mathcal{L}_n^{\varepsilon'}(\phi)_{\pi^I}
\end{array}$$

where the vertical arrows are smooth affine morphisms.

(d) The ind-pro-object "$\varinjlim_\varepsilon$" "$\varprojlim_n$" $\mathcal{L}_n^\varepsilon(\phi)_{\pi^I}$ *is independent on ϕ, π.*

Proof: The proof is similar to that of Proposition 2.7.1. The proof of the second part of Claim (a) is as follows. The natural morphism of schemes $\mathcal{L}_0^0(U)_{V^I} \to \mathcal{L}_0^0(\mathbb{A}^d)_{\mathbb{A}^I}$ being locally of finite presentation, by base change and [EGAIV, §17] it suffices to check that it is is formally étale. If S is a scheme and we have two compatible pairs

$$(f_I, \rho) \in \lambda^0_{0,\phi,\pi^I}(S_{\mathrm{red}}), \quad (f'_I, \rho') \in \lambda^0_{0,\mathbb{A}^d,\mathbb{A}^I}(S),$$

we must construct the unique pair $(\tilde{f}_I, \tilde{\rho}) \in \lambda^0_{0,\phi,\pi^I}(S)$. By definition, we have a commutative square

$$\begin{array}{ccc} S_{\mathrm{red}} & \xrightarrow{f_I} & C^I \\ \downarrow & & \downarrow \pi^I \\ S & \xrightarrow{f_I'} & \mathbb{A}^I. \end{array}$$

Thus, π being étale, there is a unique lift $\tilde{f}_I \in \mathrm{Hom}_{\mathbf{Sch}}(S, C^I)$. Then, $\Gamma(\tilde{f}_I)_{\mathrm{red}} = \Gamma(f_I)$. We have also a commutative diagram

$$\begin{array}{ccc} \Gamma(f_I) & \xrightarrow{\rho} & U \\ \downarrow & & \downarrow \phi \\ \Gamma(\tilde{f}_I) & \longrightarrow & \mathbb{A}^d \end{array}$$

where the lower arrow is the composition of the chain of maps

$$\Gamma(\tilde{f}_I) \xrightarrow{\mathrm{Id}\times\pi} \Gamma(f_I') \xrightarrow{\rho'} \mathbb{A}^d.$$

Thus, ϕ being étale, there is a unique lift $\tilde{\rho} \in \mathrm{Hom}_{\mathbf{Sch}}\big(\Gamma(\tilde{f}_I), U\big)$. We are done.

The smoothness of the vertical arrows in Claim (c) is proved as follows. Set

$$A_n^N = k[\mathbb{A}^I][a_\nu^{(j)}; -N \leq l \leq n].$$

Then, $\mathcal{L}_n^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I}$ is a closed subset of $\mathrm{Spec}\, A_n^N$ if $N >> 0$. We have a Cartesian square

$$\begin{array}{ccc} \mathrm{Spec}\, A_{n'}^N & \rightarrow & \mathrm{Spec}\, A_n^N \\ \uparrow & & \uparrow \\ \mathcal{L}_{n'}^\varepsilon(\phi)_{\pi^I} & \rightarrow & \mathcal{L}_n^\varepsilon(\phi)_{\pi^I} \end{array}$$

where the horizontal arrow is the obvious smooth affine projection, and the vertical arrows are the composition of the chain of maps

$$\mathcal{L}_{n'}^\varepsilon(\phi)_{\pi^I} \xrightarrow{\sim} \mathcal{L}_n^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I} \times_{\mathcal{L}_0^0(\mathbb{A}^d)_{\mathbb{A}^I}} \mathcal{L}_0^0(U)_{V^I} \rightarrow \mathcal{L}_n^\varepsilon(\mathbb{A}^d)_{\mathbb{A}^I} \hookrightarrow \mathrm{Spec}\, A_n^N.$$

Thus, the projection $\mathcal{L}_{n'}^\varepsilon(\phi)_{\pi^I} \rightarrow \mathcal{L}_n^\varepsilon(\phi)_{\pi^I}$ is smooth and affine, by base change. $\square$

4. $\mathcal{D}$-MODULES OVER IND-SCHEMES

(4.1) Reminder on $\mathcal{D}$-modules. Let $\mathbf{Sch}^{\mathrm{ft}}$ denote the category of k-schemes of finite type. For any such scheme X let $\mathbf{O}_X$ be the category of all quasi-coherent $\mathcal{O}_X$-modules. For a morphism $f : X \to Y$ in $\mathbf{Sch}^{\mathrm{ft}}$ we denote by f_*, f^* the functors of the direct and inverse images on $\mathbf{O}_X$, $\mathbf{O}_Y$. If f is a closed embedding and $\mathcal{E} \in \mathbf{O}_Y$, let

$$f^!\mathcal{E} = f^{-1}\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{E}) \tag{4.1.1}$$

be the inverse image of the subsheaf of $\mathcal{E}$ consisting of sections supported scheme-theoretically on $f(X) \subset Y$.

For $X \in \mathbf{Sch}^{\mathrm{ft}}$ let $\mathbf{D}_X$ be the category of right $\mathcal{D}$-modules on X. It is defined as follows, see [BD2, §7.10] or [G, §0.2.2]. If X is smooth, then we have the sheaf of rings $\mathcal{D}_X$ of differential operators from $\mathcal{O}_X$ to itself. An object of $\mathbf{D}_X$ is then a sheaf of right $\mathcal{D}_X$-modules quasicoherent over $\mathcal{O}_X$. Next, if X admits a closed embedding into a smooth scheme Y, one defines $\mathbf{D}_X$ as $\mathbf{D}_{Y,X}$, the full subcategory of $\mathbf{D}_Y$ consisting of modules supported (as sheaves) on X. This definition is independent on the choice of embedding: if X is embedded into two smooth schemes Y_1 and Y_2, then one has an equivalence $\mathbf{D}_{Y_1,X} \to \mathbf{D}_{Y_2,X}$ which is unique up to a unique isomorphism of functors. Now, given any X, an embedding into a smooth scheme always exists locally on X. Therefore we have an open covering $X = \bigcup U_\alpha$, the categories $\mathbf{D}_{U_\alpha}$, $\mathbf{D}_{U_{\alpha\beta}}$ etc. and the obvious restriction functors among them. One then defines an object $\mathcal{M}$ of $\mathbf{D}_X$ as a collection of objects $\mathcal{M}_\alpha \in \mathbf{D}_{U_\alpha}$ together with isomorphisms of their images in $\mathbf{D}_{U_{\alpha\beta}}$ whose images in $\mathrm{Mor}(\mathbf{D}_{U_{\alpha\beta\gamma}})$ satisfy the obvious compatibility conditions.

Given $X \in \mathbf{Sch}^{\mathrm{ft}}$ and $\mathcal{M} \in \mathbf{D}_X$, we define a sheaf $\mathcal{M}^{\mathcal{O}} \in \mathbf{O}_X$ as follows. If X admits a closed embedding $i : X \to Y$ with Y smooth and $\mathcal{M}$ is represented by a sheaf of right $\mathcal{D}_Y$-modules supported on X (which we also denote $\mathcal{M}$) then we set $\mathcal{M}^{\mathcal{O}} = i^!\mathcal{M}$. This definition is easily seen to be independent on the choice of Y. In the case of a general X one defines $\mathcal{M}^{\mathcal{O}}$ by gluing the sheaves given by above procedures on open parts of X admitting embeddings into scmooth schemes.

Any morphism $f : X \to Y$ in $\mathbf{Sch}^{\mathrm{ft}}$ induces functors $f_\bullet, f^\bullet$ of the direct and inverse image on $\mathbf{D}_X$, $\mathbf{D}_Y$. In the particular case when X, Y are smooth, we can view $\mathcal{M} \in \mathbf{D}_X$ as a sheaf on X and

$$f_\bullet(\mathcal{M}) = f_*(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

Here $\mathcal{D}_{X\to Y}$ is the sheaf of differential operators from $f^{-1}\mathcal{O}_Y$ to $\mathcal{O}_X$, see [BB] [G].

If f is a closed embedding (X and Y are enot assumed smooth), the functor $f_\bullet$ is exact, and we have a canonical embedding

$$\mathcal{M}^{\mathcal{O}} \simeq f^! f_\bullet(\mathcal{M}) \hookrightarrow f^* f_\bullet(\mathcal{M}). \tag{4.1.2}$$

For example, if X is non-reduced and $i : X_{\mathrm{red}} \hookrightarrow X$ is the reduced part, then $i_\bullet$ identifies $\mathbf{D}_{X_{\mathrm{red}}}$ with $\mathbf{D}_X$.

If f is a smooth morphism and $\omega_{X/Y}$ is its relative canonical bundle, then

$$f^\bullet(\mathcal{M}) = f^*(\mathcal{M}^{\mathcal{O}}) \otimes_{\mathcal{O}_X} \omega_{X/Y}.$$

We have then a canonical embedding

$$\mathcal{M}^{\mathcal{O}} \hookrightarrow f_*((f^\bullet\mathcal{M})^{\mathcal{O}} \otimes \omega_{X/Y}^{-1}). \tag{4.1.3}$$

We have the following base change property.

(4.1.4) Lemma. *Suppose that in a Cartesian diagram of schemes of finite type*

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \overset{j}{\hookrightarrow} & Y' \end{array}$$

the morphisms f, f' are smooth, i, j are closed embeddings. Then for any $\mathcal{M} \in \mathbf{D}_Y$, we have

$$i_\bullet f^\bullet \mathcal{M} \simeq f'^\bullet j_\bullet \mathcal{M}.$$

The projection formula holds for right $\mathcal{D}$-modules. More precisely, if f is an embedding of smooth k-schemes and $\mathcal{M} \in \mathbf{D}_X$, $\mathcal{N} \in \mathbf{D}_Y$, then there is a canonical isomorphism

$$f_\bullet\big(f^\bullet(\mathcal{N}) \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}\big) \simeq \mathcal{N} \otimes_{\mathcal{O}_Y} f_\bullet(\mathcal{M}) \otimes_{\mathcal{O}_Y} \omega_Y^{-1}.$$

(4.2) $\mathcal{D}$-modules over pro-schemes.

(4.2.1) Definition. *(cf.* [Kap].*) Let A be a filtering poset and $(\mathbf{C}_\alpha)_{\alpha \in A}$ be an inductive system of categories labelled by A. In other words, for each $\alpha \le \beta$ we have a functor $i_{\alpha\beta} : \mathbf{C}_\alpha \to \mathbf{C}_\beta$, for each $\alpha \le \beta \le \gamma$ a natural isomorphism $i_{\beta\gamma} \circ i_{\alpha\beta} \Rightarrow i_{\alpha\gamma}$ and these isomorphisms satisfy the obvious coherence conditions for any $\alpha \le \beta \le \gamma \le \delta$.*

The inductive limit $2\varinjlim_\alpha \mathbf{C}_\alpha$ is the category whose objects are pairs (α, x_α), $\alpha \in A, x_\alpha \in \mathrm{Ob}(\mathbf{C}_\alpha)$ and

$$\mathrm{Hom}(x_\alpha, y_\beta) = \varinjlim_{\gamma \ge \alpha,\beta} \mathrm{Hom}_{\mathbf{C}_\alpha}(i_{\alpha\gamma}(x_\alpha), i_{\beta\gamma}(y_\beta)).$$

(4.2.2) Definition. *A k-scheme X_∞ (possibly of infinite type) is called compact if it can be represented as $\lim\limits_{\longleftarrow\alpha} X_\alpha$ where $(X_\alpha)_{\alpha\in A}$ is a filtering projective system over $\mathbf{Sch}_k^{\mathrm{ft}}$ such that all the maps $p_{\alpha\beta} : X_\beta \to X_\alpha$, $\alpha \le \beta$, are affine morphisms.*

(4.2.3) Proposition. *The category of compact k-schemes can be identified with a full subcategory in $\mathbf{Pro}(\mathbf{Sch}_k^{\mathrm{ft}})$, via $X_\infty = \lim\limits_{\longleftarrow\alpha} X_\alpha \mapsto$ "$\lim\limits_{\longleftarrow\alpha}$"$X_\alpha$.*

(4.2.4) Definition. *A compact scheme X_∞ is called smooth if it can be represented as $\lim\limits_{\longleftarrow\alpha} X_\alpha$ for some (X_α) as in (4.2.2) with the extra property that each X_α is a smooth scheme and each $p_{\alpha\beta}$ is a smooth affine morphism.*

(4.2.5) Example. For any smooth $X \in \mathbf{Sch}_k^{\mathrm{ft}}$, the scheme $\mathcal{L}^0(X)$ is smooth and compact.

Let X_∞ be a compact scheme and (X_α) be as in Definition 4.2.2. We have two inductive systems of categories $(\mathbf{O}_{X_\alpha}, p^*_{\alpha\beta})$, $(\mathbf{D}_{X_\alpha}, p^\bullet_{\alpha\beta})$. We set $\mathbf{D}_{X_\infty} = 2\lim\limits_{\longrightarrow\alpha} \mathbf{D}_{X_\alpha}$, see also [KT, §1.9]. Informally, an object of $\mathbf{D}_{X_\infty}$ is a "$\mathcal{D}$-module pulled back from some X_α". The category $\mathbf{D}_{X_\infty}$ is independent, up to a canonical equivalence of categories, on the choice of (X_α) as in (4.2.2). Let $p_\alpha : X_\infty \to X_\alpha$ be the projection. Let $p^\bullet_\alpha : \mathbf{D}_{X_\alpha} \to \mathbf{D}_{X_\infty}$, $\mathcal{M}^\alpha \mapsto (\alpha, \mathcal{M}^\alpha)$ be the canonical functor. If the compact scheme X_∞ is smooth, there is the functor

$$\mathbf{D}_{X_\infty} \to 2\lim_{\longrightarrow\alpha} \mathbf{O}_{X_\alpha} \subset \mathbf{O}_{X_\infty}, \quad p^\bullet_\alpha \mathcal{M}_\alpha \mapsto (p^\bullet_\alpha \mathcal{M}_\alpha)^{\mathcal{O}} := (p^\bullet_{\alpha\beta}\mathcal{M}_\alpha \otimes \omega^{-1}_{X_\beta})_{\beta\ge\alpha}.$$

In particular, we can associate to $\mathcal{M}$ its space of global sections

$$\Gamma(X_\infty, \mathcal{M}) = \lim_{\longrightarrow\beta\ge\alpha} \Gamma\big(X_\beta, p^\bullet_{\alpha\beta}\mathcal{M}_\alpha \otimes \omega^{-1}_{X_\beta}\big). \tag{4.2.6}$$

(4.3) $\mathcal{D}$-modules over ind-schemes.

(4.3.1) Definition. *Let A be a filtering poset and $(\mathbf{C}_\alpha)_{\alpha\in A}$ be a projective system of categories labelled by A. In other words, for each $\alpha \le \beta$ we have a funtctor $j_{\alpha\beta} : \mathbf{C}_\beta \to \mathbf{C}_\alpha$ and for any $\alpha \le \beta \le \gamma$ a natural isomorphism $j_{\alpha\beta} \circ j_{\beta\gamma} \Rightarrow j_{\alpha\gamma}$ satisfying the obvious compatibility conditions.*

The projective limit $2\lim\limits_{\longleftarrow} \mathbf{C}_\alpha$ is the category whose objects are systems consisting of objects $x_\alpha \in \mathbf{C}_\alpha$ given for all $\alpha \in A$ and isomorphisms $j_{\alpha\beta}(x_\beta) \to x_\alpha$ given for each $\alpha \le \beta$ and satisfying the compatibility condition for each $\alpha \le \beta \le \gamma$. Morphisms are defined in the obvious way.

(4.3.2) Definition. *Let X^∞ be an ind-scheme. We say that X_∞ is discrete if it can be represented as $X^\infty =$ "$\varinjlim_\alpha$" X^α where $(X^\alpha)_{\alpha\in A}$ is a filtering inductive system over $\mathbf{Sch}_k^{\mathrm{ft}}$ such that each map $i_{\alpha\beta} : X^\alpha \to X^\beta$, $\alpha \leq \beta$, is a closed embedding.*

(4.3.3) Example. For any ϕ, U as in (2.7) the ind-scheme $\mathcal{L}_n(\phi)$ is discrete.

Let X^∞ be a discrete ind-scheme and (X^α) be as in Definition 4.3.2. We have then the projective system of categories $(\mathbf{O}_{X^\alpha}, i^!_{\alpha\beta})$. We define $\mathbf{O}_{X^\infty} = 2\varprojlim \mathbf{O}_{X^\alpha}$. If $(\mathcal{E}^\alpha, \gamma_{\alpha\beta} : \mathcal{E}^\alpha \to i^!_{\alpha\beta}\mathcal{E}^\beta)$ is an object of $\mathbf{O}_{X^\infty}$, then the spaces of global sections $\Gamma(X^\alpha, \mathcal{E}^\alpha)$ form an inductive system and we define

$$\Gamma(X^\infty, \mathcal{E}) = \varinjlim_\alpha \Gamma(X^\alpha, \mathcal{E}^\alpha). \tag{4.3.4}$$

We will also use the category $\hat{\mathbf{O}}_{X^\infty}$ which is the limit of the projective system of categories $(\mathbf{O}_{X^\alpha}, i^*_{\alpha\beta})$. There is a functor

$$\mathbf{O}_{X^\infty} \times \hat{\mathbf{O}}_{X^\infty} \to \mathbf{O}_{X^\infty}, \quad (\mathcal{E}, \hat{\mathcal{F}}) \mapsto \mathcal{E} \otimes \hat{\mathcal{F}} = (\mathcal{E}^\alpha \otimes_{\mathcal{O}_{X^\alpha}} \hat{\mathcal{F}}^\alpha).$$

See [BD2, §7.11.4] for more details on $\mathbf{O}_{X^\infty}$, $\hat{\mathbf{O}}_{X^\infty}$.

We set also $\mathbf{D}_{X^\infty} = 2\varinjlim_\alpha (\mathbf{D}_{X^\alpha}, i_{\alpha\beta\bullet})$. Let i_α be the embedding $X^\alpha \hookrightarrow X^\infty$. Let $i_{\alpha\bullet} : \mathbf{D}_{X^\alpha} \to \mathbf{D}_{X\infty}$, $\mathcal{M}^\alpha \mapsto (\alpha, \mathcal{M}^\alpha)$ be the canonical functor. It is exact. There is also the functor

$$\mathbf{D}_{X^\infty} \to \mathbf{O}_{X^\infty}, \quad i_{\alpha\bullet}\mathcal{M}^\alpha \mapsto (i_{\alpha\bullet}\mathcal{M}^\alpha)^{\mathcal{O}} := (i_{\alpha\beta\bullet}\mathcal{M}^\alpha)_{\beta\geq\alpha},$$

see (4.1.2). In particular, to any $\mathcal{M} \in \mathbf{D}_{X^\infty}$ we can associate its space of global sections: If $\mathcal{M}$ is represented by $\mathcal{M}^\alpha \in \mathbf{D}_{X^\alpha}$, then

$$\Gamma(X^\infty, \mathcal{M}) = \varinjlim_{\beta\geq\alpha} \Gamma(X^\beta, (i_{\alpha\beta\bullet}\mathcal{M}^\alpha)^{\mathcal{O}}). \tag{4.3.4}$$

(4.3.5) Remark. (a) We have an exact functor

$$2\varinjlim_\alpha \mathbf{O}_{X^\alpha} \to \mathbf{O}_{X^\infty}, \quad (\alpha, \mathcal{E}^\alpha) \mapsto (i_{\alpha\beta *}\mathcal{E}^\alpha)_{\beta\geq\alpha}.$$

The two categories are not equivalent in general.

(b) The category $\mathbf{O}_{X^\infty}$ is closed by inductive limits.

(c) If the ind-scheme X^∞ is not discrete anymore, the $\mathcal{O}_{X^\alpha}$-module $i^!_{\alpha\beta}\mathcal{E}^\beta$ may not be quasi-coherent. However the category $\mathbf{O}_{X^\infty}$ is still well-defined.

Let $X^\infty = \lim\limits_{\longrightarrow\alpha} X^\alpha$ be a formally smooth discrete ind-scheme. Following [BD2, Prop. 7.11.8 and 7.12.13] we define the tangent sheaf of X to be the object $\hat\Theta_{X^\infty} \in \hat{\mathbf{O}}_{X^\infty}$ such that

$$\mathcal{E} \otimes \hat\Theta_{X^\infty} = \lim_{\longrightarrow\alpha} \mathcal{H}om(\Omega^1_{X^\alpha}, \mathcal{E}^\alpha), \quad \forall \mathcal{E} \in \mathbf{O}_{X^\infty}. \tag{4.3.6}$$

Denoting $i^\alpha : X^\alpha \to X^\infty$ the canonical embedding, we have that $i^{\alpha *}\hat\Theta_{X^\infty}$ is, locally, a projective $\mathcal{O}_{X^\alpha}$-module (possibly of infinite rank).

(4.4) $\mathcal{D}$-modules over ind-pro-schemes. The following definition is inspired by the paper of K. Kato [Kat].

(4.4.1) Definition. *An ind-scheme X^∞_∞ is called locally compact if it can be represented as*

$$X^\infty_\infty = \lim_{\longrightarrow\alpha\in A} \lim_{\longleftarrow\beta\in B} X^\alpha_\beta$$

where (X^α_β) is a bi-filtering ind-pro-system over $\mathbf{Sch}^{\mathrm{ft}}$ with the following properties:

(1) For each $\beta \in B$ and $\alpha \leq \alpha' \in A$ the structure map $i^{\alpha\alpha'}_b : X^\alpha_\beta \to X^{a'}_b$ is a closed embedding.

(2) For each $\alpha \in A$ and $\beta \leq \beta' \in B$ the structure map $p^\alpha_{\beta\beta'} : X^\alpha_{b'} \to X^\alpha_\beta$ is an affine morphism.

(3) For each $\alpha \leq \alpha' \in A$ and $\beta \leq \beta' \in B$ the commutative square

$$\begin{array}{ccc} X^\alpha_{\beta'} & \hookrightarrow & X^{\alpha'}_{\beta'} \\ \downarrow & & \downarrow \\ X^\alpha_\beta & \hookrightarrow & X^{\alpha'}_\beta \end{array}$$

is Cartesian.

We denote by **Lc** $\subset$ **Isch** *the full subcategory formed by locally compact ind-schemes.*

Let (X^α_β) be an ind-pro-system as above. The maps $p^\alpha_{\beta\beta'}$ being affine, the projective limit $\lim\limits_{\longleftarrow\beta} X^\alpha_\beta$ is represented by a compact scheme (possibly of infinite type) denoted by X^α_∞. Similarly, we have the discrete ind-schemes $X^\infty_\beta = \lim\limits_{\longrightarrow\alpha} X^\alpha_\beta$. By definition $X^\infty_\infty = \lim\limits_{\longrightarrow\alpha} X^\alpha_\infty$.

(4.4.2) Proposition. *For an ind-pro-system satisfying the conditions of (4.4.1) we also have $X^\infty_\infty = \lim\limits_{\longleftarrow\beta} X^\infty_\beta = \lim\limits_{\longleftarrow\beta}\lim\limits_{\longrightarrow\alpha} X^\alpha_\beta$.*

Proof: By passing to the functors represented by our (ind-)schemes, we reduce the statement (a) to the following lemma whose proof we leave to the reader.

(4.4.3) Lemma. *Let $(S^\alpha_\beta)_{\alpha\in A,\beta\in B}$ be a bi-filtering ind-pro-system of sets. Then there is a canonical map*

$$c: \varinjlim_{\alpha} \varprojlim_{\beta} S^\alpha_\beta \to \varprojlim_{\beta} \varinjlim_{\alpha} S^\alpha_\beta.$$

If, moreover, all the squares in (S^α_β) are Cartesian, then c is an isomorphism.

(4.4.4) Definition. *We say that a locally compact ind-scheme X^∞_∞ is smooth if it admits a presentation as in (4.1.1) where:*

(1) All the $p^\alpha_{\beta\beta'}$ are smooth morphisms of relative dimension $d_{\beta\beta'}$ independent on α and at least one X^α_β is smooth.

(2) All the ind-schemes $X^\infty_\beta = \varinjlim_{\alpha} X^\alpha_\beta$ are formally smooth.

(4.4.5) Example. If X is a smooth affine variety admitting an etale map to $\mathbb{A}^d$, then the ind-scheme $\mathcal{L}(X)$ is locally compact and smooth.

Let X^∞_∞ be a locally compact smooth ind-scheme and (X^α_β) be an ind-pro-system as in (4.4.1), (4.4.4). By Lemma 4.1.4, we have then a double inductive system of categories $(\mathbf{D}_{X^\alpha_\beta}, i^{\alpha\alpha'}_{\beta\bullet}, p^{\alpha\bullet}_{\beta\beta'})$ and we define the category of (right) $\mathcal{D}$-modules on X^∞_∞ to be $\mathbf{D}_{X^\infty_\infty} = 2\varinjlim_{\alpha,\beta} \mathbf{D}_{X^\alpha_\beta}$.

(4.4.6) Remark. Although we have defined $\mathbf{D}_{X^\infty_\infty}$ as an abstract category, it is impossible, in general, to view its objects as sheaves in a more conventional sense. For example, it is impossible to associate to an object of $\mathbf{D}_{X^\infty_\infty}$ a vector space of global sections. Indeed, if such an object $\mathcal{M}$ is represented by some $\mathcal{M}^\alpha_\beta \in \mathbf{D}_{X^\alpha_\beta}$, then the spaces of global sections of the coherent sheaves

$$(p^{\alpha'\bullet}_{\beta\beta'} i^{\alpha\alpha'}_{b\bullet} \mathcal{M}^\alpha_\beta)^{\mathcal{O}} = p^{\alpha'*}_{\beta\beta'}(i^{\alpha\alpha'}_{\beta\bullet}\mathcal{M}^\alpha_\beta)^{\mathcal{O}} \otimes \omega_{X^{\alpha'}_{\beta'}/X^{\alpha'}_\beta}$$

do not form an inductive system because of the twist by the relative canonical class. For a compact smooth pro-scheme it is possible to get around this problem by untwisting by the absolute canonical classes of the terms of the projective system (4.2.6). To achieve the same effect in the ind-pro-case one would need to make sense of the canonical class of the ind-scheme X^α_∞, i.e., of the determinant of the (possibly infinite-dimensional) vector bundle $\hat{\Theta}_{X^\alpha_\infty}$. The impossibility of doing this ("the determinantal anomaly") is precisely the reason why there is no natural space $\Gamma(X^\infty_\infty, \mathcal{M})$.

5. De Rham complexes on ind-schemes

(5.1) Reminder on the De Rham complexes. Let X be a smooth algebraic variety over k and $\mathcal{M} \in \mathbf{D}_X$ be a right $\mathcal{D}_X$-module. Its de Rham complex $\mathcal{DR}(\mathcal{M})$ is given by

$$\mathcal{DR}^i(\mathcal{M}) = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^{-i}, \mathcal{M}) = \mathcal{M} \otimes_{\mathcal{O}_X} \bigwedge^{-i} \Theta_X, \quad i \leq 0. \tag{5.1.1}$$

If (x_j) is an etale coordinate system on an open part of X, then the differential is given by the formula $d = \sum \partial_{x_j} \otimes dx_j$ where $dx_j \in \Omega^1_X$ is considered as the contraction operator $\bigwedge^{-i} \Theta_X \to \bigwedge^{-i-1} \Theta_X$. We denote by $DR(\mathcal{M}) = \Gamma(X, \mathcal{DR}(\mathcal{M}))$ the complex of global sections.

Let $i : X \to Y$ be a closed embedding of smooth varieties and $\mathcal{M} \in \mathbf{D}_X$. The embedding (4.1.2) induces an embedding of the de Rham complexes $\mathcal{DR}(\mathcal{M}) \hookrightarrow i^*\mathcal{DR}(i_\bullet \mathcal{M})$ and therefore an embedding of the complexes of global sections:

$$DR(\mathcal{M}) \hookrightarrow DR(i_\bullet(\mathcal{M})). \tag{5.1.2}$$

Let $p : X \to Y$ be a smooth morphism of smooth varieties of relative dimension d and $\mathcal{M} \in \mathbf{D}_Y$. The embedding (4.1.3) induces an embedding of de Rham complexes which now involves a shift in the degrees:

$$p^*\mathcal{DR}(\mathcal{M}) \hookrightarrow \mathcal{DR}(p^\bullet(\mathcal{M})[d]).$$

It is induced by the map

$$p^* \bigwedge^{i} \Theta_Y \to \omega_{X/Y} \otimes \bigwedge^{i+d} \Theta_X.$$

In particular, we get an embedding of the complexes of global sections

$$DR(\mathcal{M}) \hookrightarrow DR(p^\bullet \mathcal{M})[d]. \tag{5.1.3}$$

Note that without passing to the de Rham complexes there is no embedding of the space of sections of $\mathcal{M}$ into that of $p^\bullet \mathcal{M}$. The map (5.1.3) can be seen as a $\mathcal{D}$-module manifestation of the fact that "fermions cancel the determinantal anomaly".

(5.1.4) Example. Let $Y = \mathbb{A}^d$ with coordinates $a_1, ..., a_d$ and $i : X \hookrightarrow Y$ be the embedding of the affine subspace $\{a_1 = ... = a_l = 0\}$, $l \leq d$. The algebra $\Gamma(Y, \mathcal{D}_Y)$ is just the Heisenberg (Weyl) algebra D_Y generated by $a_1, ..., a_d$ and $a_1^*, ..., a_d^*$ subject to the relations

$$[a_m, a_n] = [a_m^*, a_n] = 0, \quad [a_m^*, a_n] = \delta_{mn}.$$

The space of global sections $\Gamma(Y, i_\bullet\omega_X)$ is the right D_Y-module

$$\omega_{XY} = D_Y \Big/ (a_m^*, a_n,\, n \le l \le m) D_Y.$$

Let also C_Y be the Clifford algebra generated by odd elements $b_1, ..., b_d,\ b_1^*, ..., b_d^*$ subject to the relations

$$[b_m, b_n]_+ = [b_m^*, b_n^*]_+ = 0, \quad [b_m^*, b_n]_+ = \delta_{mn}.$$

Denote $CD_Y = C_Y \otimes_k D_Y$ the tensor product algebra. Then the global de Rham complex of $i_\bullet\omega_X$ is identified with the right CD_Y-module

$$DR_{XY} = CD_Y \Big/ \big(a_m^*, a_n, b_p,\, n \le l \le m,\, p = 1, ..., d\big) CD_Y.$$

(5.2) De Rham complexes for ind-schemes. Let $X^\infty = \varinjlim_{\alpha \in A} X^\alpha$ be a formally smooth discrete ind-scheme. Denote by $i^\alpha : X^\alpha \hookrightarrow X$ the canonical embedding. The considerations of (5.1) generalize easily to give the global de Rham complex of any $\mathcal{M} \in \mathbf{D}_{X^\infty}$. Explicitly, let $\mathcal{M}$ have the form $i_{\alpha\bullet}\mathcal{M}^\alpha$. Then the ith term of its de Rham complex is

$$DR^i(\mathcal{M}) = \varinjlim_{\beta \ge \alpha} \Gamma(X^\beta, (i_\bullet^{\alpha\beta}\mathcal{M}^\alpha)^{\mathcal{O}} \otimes_{\mathcal{O}_{X^\alpha}} i^{\alpha *} \bigwedge\nolimits^{-i} \hat{\Theta}_{X^\infty}). \tag{5.2.1}$$

(5.2.2) Proposition. *Let $f : X^\infty \to Y^\infty$ be a morphism of formally smooth discrete ind-schemes which can be included into a Cartesian square*

$$\begin{array}{ccc} X^\infty & \xrightarrow{p} & Y^\infty \\ \downarrow & & \downarrow \\ X^0 & \xrightarrow{p^0} & Y^0 \end{array}$$

where $p^0 : X^0 \to Y^0$ is a smooth morphism of schemes of finite type, of relative dimension d. Then for any $\mathcal{M} \in \mathbf{D}_Y$ we have an embedding of the shifted de Rham complexes:

$$DR(\mathcal{M}) \hookrightarrow DR(p^\bullet\mathcal{M})[d].$$

Proof: The conditions on p in the proposition are equivalent to the following: we can represent $X^\infty = \varinjlim_{\alpha \in A} X^\alpha$, $Y^\infty = \varinjlim_{\alpha \in A} Y^\alpha$ with the same filtering poset A and represent p by a morphism of inductive systems $(p^\alpha : X^\alpha \to Y^\alpha)$ such that for each $\alpha \le \beta$ the arising commutative square is Cartesian and each p^α is smooth of relative dimension d.

But in this situation the possibility of the embedding follows from (5.1.3) and from the base change for Cartesian squares (Lemma 2.1.4). □

(5.3) De Rham complexes for ind-pro-schemes. Let X^∞_∞ be a locally compact smooth ind-scheme and $\mathcal{M}$ be an object of $\mathbf{D}_{X^\infty_\infty}$. We fix an ind-pro-system (X^α_β) for X as in (4.4.1), (4.4.4). We have then the formally smooth discrete ind-schemes X^∞_β and projections $p_\beta : X^\infty_\infty \to X^\infty_\beta$. The category $\mathbf{D}_{X^\infty_\infty}$ being the double direct limit of $\mathbf{D}_{X^\alpha_\beta}$, we can think of $\mathcal{M}$ as being of the form $p^\bullet_\beta \mathcal{M}_b$ for some $\mathcal{M}_b \in \mathbf{D}_{X^\infty_\beta}$. Recall that $d_{\beta\beta'}$ denotes the relative dimension of the smooth morphism $p^\alpha_{\beta\beta'} : X^\alpha_{\beta'} \to X^\alpha_\beta$, $\beta \leq \beta'$. We choose numbers $d_\beta, \beta \in B$ such that $d_{\beta\beta'} = d_\beta - d_{\beta'}$ (this can be done uniquely up to an overall constant). Proposition 5.2.2 implies then that the shifted global de Rham complexes $DR(p^\bullet_{\beta\beta'}\mathcal{M}_\beta)[d_{\beta'}]$ form an inductive system of complexes of vector spaces and we define the de Rham complex of $\mathcal{M}$ to be

$$DR(\mathcal{M}) = \varinjlim_{\beta' \geq \beta} DR(p^\bullet_{\beta\beta'}\mathcal{M}_\beta)[d_{\beta'}]. \tag{5.3.1}$$

(5.4) The de Rham complexes on $\mathcal{L}(X)$. We now specialize to the particular case $X^\infty_\infty = \mathcal{L}(X)$ where X is a smooth affine algebraic variety over k admitting an etale map ϕ to $\mathbb{A}^d$. In this case $A = \mathbb{E}, B = \mathbb{N}$ with the terms of the ind-pro-systems being $\mathcal{L}^\varepsilon_n(\phi)$, see (2.6.7). Given an object $\mathcal{M} \in \mathbf{D}_{\mathcal{L}(X)}$ we associate to it its de Rham complex $DR(\mathcal{M})$ as in (5.3). Note that it is independent on the choice of an etale map to $\mathbb{A}^d$ because two such maps lead to isomorphic ind-pro-objects in the category $\mathbf{Sch}^{\mathrm{ft}}$ (Proposition 2.7.1c).

Let now X be an arbitrary smooth algebraic variety over k. and $\mathcal{M}$ be an object of $\mathbf{D}_{\mathcal{L}(X)}$. By covering X with affine open U admitting etale maps to $\mathbb{A}^d$, we get a complex of sheaves $U \mapsto DR(\mathcal{M}|_{\mathcal{L}(U)})$ which we denote $\mathcal{DR}(\mathcal{M})$. Recall that we have the diagram

$$X \xleftarrow{p} \mathcal{L}^0(X) \xhookrightarrow{i} \mathcal{L}(X). \tag{5.4.1}$$

Thus, every right $\mathcal{D}_X$-module $\mathcal{N}$ gives an object $i_\bullet p^\bullet \mathcal{N}$ of $\mathbf{D}_{\mathcal{L}(X)}$. We write $\mathcal{CDR}(\mathcal{N})$ for $\mathcal{DR}(i_\bullet p^\bullet \mathcal{N})$ and call it the chiral de Rham complex of $\mathcal{N}$. In particular, we write $\mathcal{CDR}_X$ for $\mathcal{DR}(i_\bullet p^\bullet \omega_X)$. More generally, denoting $p_n : \mathcal{L}^0(X) \to \mathcal{L}^0_n(X)$ the projection, we can start with any right $\mathcal{D}$-module $\mathcal{N}$ on the algebraic variety $\mathcal{L}^0_n(X)$: then $i_\bullet p^\bullet_n \mathcal{N}$ is an object of $\mathbf{D}_{\mathcal{L}(X)}$ and we can form its de Rham complex.

(5.4.2) Example. Let $X = \mathbb{A}^1$. Then the complex of global sections of $\mathcal{CDR}_X$, i.e., the complex $DR(i_\bullet p^\bullet \omega_{\mathbb{A}^1})$ can be found explicitly as follows.

Let V be the topological k-vector space $k((t))$ and $V^\star$ be its topological dual (over k). Denote by $= \langle l, v\rangle$ the canonical pairing of $l \in V^\star$ and $v \in V$.

Then $V^\star$ can be identified with $k((t))dt$, the space of 1-forms, the pairing between V and $V^\star$ being $(f, \omega) \to \mathrm{res}(f \cdot \omega)$. Let D be the Heisenberg algebra generated by $V^\star$ and V with $[l, v] = \langle lv\rangle$ and C be the Clifford algebra generated by $V^\star, V$ with $[l, v]_+ = \langle l, v\rangle$. Denote $CD = C \otimes_k D$. Explicitly, $CD = \varinjlim_{M\geq 0} \varprojlim_{N\geq 0} CD_M^N$, where CD_M^N is the algebra generated by symbols a_n, b_n for $-N \leq n \leq M$ and a_n^*, b_n^* for $-M \leq n \leq N$, subject to the relations

$$[a_m, a_n] = [a_m^*, a_n^*] = 0, \quad [a_m^*, a_n] = \delta_{m,-n}$$

$$[b_m, b_n]_+ = [b_m^*, b_n^*]_+ = 0, \quad [b_m^*, b_n]_+ = \delta_{m,-n},$$

$$[a_m, b_n] = [a_m^*, b_n] = [a_m, b_n^*] = [a_m^*, b_n^*] = 0.$$

As M, N vary, these algebras obviously form an ind-pro-system. Let $I \subset CD$ be the right ideal generated by (possibly infinite) linear combinations of $a_n, b_n, a_{n+1}^*, b_{n+1}^*$, $n < 0$.

(5.4.3) Proposition. *The de Rham complex $DR(i_\bullet p^\bullet \omega_{\mathbb{A}^1})$ is identified with the (irreducible) CD-module $Vac := CD/I$.*

Remark. Although this description is similar to (5.1.4), there is a difference: here the ideal is generated by b_{n+1}^*, b_n for $n \geq 0$ while in (5.1.4) all the b_n are in the ideal. This is because in our present situation we are dealing with a semiinfinite de Rham complex obtained as an inductive limit of usual de Rham complexes with respect to maps shifting the degrees.

Proof: To simplify, we write $\mathcal{L} = \mathcal{L}(\mathbb{A}^1)$, etc. We have

$$\mathcal{L}_M^\varepsilon = \operatorname{Spec}\big(k[a_l; -N_\varepsilon \leq l \leq M]/(a_l^{1+\varepsilon_l})\big),$$

where, for each $\varepsilon \in \mathbb{E}$, we set $N_\varepsilon = \max\{l; \varepsilon_{-l} \neq 0\}$. Hence

$$\mathcal{L}_M = \varinjlim_N \operatorname{Spf}\big(k[a_l; 0 \leq l \leq M][[a_l; -N \leq l < 0]]\big).$$

For any $M \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$, we put

$$Y_M^N = \operatorname{Spec}\big(k[a_l; -N \leq l \leq M]\big).$$

Then $\mathcal{L}_M$ is just the limit, over $N > 0$, of the formal completions of Y_M^N along Y_M^0. Since the de Rham complex with coefficients in the $\mathcal{D}$-module of distributions along a subvariety depends only on the completion along this subvariety, we can write

$$DR(i_\bullet p^\bullet \omega_{\mathbb{A}^1}) = \varinjlim_{M,N} DR(i_{M,N,\bullet}\omega_{Y_M^0})[M], \tag{5.4.4}$$

where $i_{M,N} : Y_M^0 \hookrightarrow Y_M^N$ is the embedding. Note that $i_{M,N}$ is just the embedding of an affine subspace, so we are in the situation of Example 5.1.4.

Let D_M^N be the subalgebra in CD_M^N generated by the allowable a_n, a_n^* only. It is identified with the algebra of polynomial differential operators on functions of $a_{-N}, ..., a_M$, with a_{-i}^* corresponding to $\partial/\partial a_i$. We see therefore that the (M, N)th term of the inductive system in (5.4.4) is identified with

$$Vac_M^N := CD_M^N \Big/ (a_i, i < 0,\, a_i^*, i \leq 0,\, b_i^*)CD_M^N. \tag{5.4.5}$$

Denote by 1_M^N the generator of this module. Then for $N \leq N'$ and any M the embedding takes 1_M^N to $1_M^{N'}$ while for $M \leq M'$ and any N it takes 1_M^N into $1_{M'}^N b_{M+1} ... b_{M'}$. This means that the inductive limit of algebras $CD = \varinjlim_M \varprojlim_N CD_M^N$ will act on the inductive limit of vector spaces (5.4.5) and the image of 1_0^0 will be annihilated by $a_{n+1}^*, b_{n+1}^*, a_n, b_n$, $n < 0$. It is equally clear (from comparison of the bases) that the limit is in fact an irreducible CD-module generated by the above vector. $\square$

6. Identification of the chiral de Rham complex

In this section we construct, in a geometric way, the structure of a vertex algebra on the chiral de Rham complex $\mathcal{CDR}_X$ and compare it with the construction of [MSV].

(6.1) Factorization algebras, and De Rham complexes on $\mathcal{L}(X)_{C^I}$. Let C be a smooth curve, as before. For any non-empty finite set I we set $U^{(I)} = U^{(I/I)}$, $\Delta^{(I)} = \Delta^{(I/\{1\})}$ and $j^{(I)} = j^{(I/I)}$ (see §3.2). Hereafter we write U, Δ, j instead of $U^{(I)}$, $\Delta^{(I)}$, $j^{(I)}$ if I has cardinal 2. We will need the following notation : for any (possibly empty) I let $\bar{I} = I \sqcup \heartsuit$ be the corresponding pointed set. Any surjection $\bar{I} \twoheadrightarrow \bar{J}$ maps $\heartsuit$ to $\heartsuit$. Let us recall the definition of a factorization algebra, see [BD1, §3.4].

(6.1.1) Definition. *(a) Let $\mathcal{E}$ be a quasi-coherent sheaf on C. A structure of a factorization algebra on $\mathcal{E}$ is a collection of quasi-coherent $\mathcal{O}_{C^I}$-modules $\mathcal{E}_I$ for each non-empty finite set I, such that $\mathcal{E}_I$ is flat along the diagonal strata, $\mathcal{E}_{\{1\}} = \mathcal{E}$, and*

– an isomorphism of $\mathcal{O}_{C^I}$-modules $\Delta^{(J/I)*}\mathcal{E}_J \xrightarrow{\sim} \mathcal{E}_I$ for every $J \twoheadrightarrow I$, compatible with the compositions of $J \twoheadrightarrow I$,

– an isomorphism of $\mathcal{O}_{U^{(J/I)}}$-modules

$$\kappa^{(J/I)} : j^{(J/I)*}(\boxtimes_I \mathcal{E}_{J_i}) \xrightarrow{\sim} j^{(J/I)*}\mathcal{E}_J$$

for every $J \twoheadrightarrow I$, compatible with the compositions of $J \twoheadrightarrow I$,

– a global section $1_\mathcal{E} \in \mathrm{H}^0(C, \mathcal{E})$ such that for every $f \in \mathcal{E}$ one has $1_\mathcal{E} \boxtimes f \in \mathcal{E}_{\{1,2\}} \subset j_* j^*(\mathcal{E} \boxtimes \mathcal{E})$ and $\Delta^*(1_\mathcal{E} \boxtimes f) = f$.

(b) A module over $\mathcal{E}$ is a quasi-coherent sheaf $\mathcal{M}$ on C with a collection of quasi-coherent $\mathcal{O}_{C^{\bar{I}}}$-modules $\mathcal{M}_{\bar{I}}$ for each non-empty finite set I, such that $\mathcal{M}_{\bar{I}}$ is flat along the diagonal strata, $\mathcal{M}_{\{\heartsuit\}} = \mathcal{M}$, and

– an isomorphism of $\mathcal{O}_{C^{\bar{I}}}$-modules $\Delta^{(\bar{J}/\bar{I})*}\mathcal{M}_{\bar{J}} \xrightarrow{\sim} \mathcal{M}_{\bar{I}}$ for every $\bar{J} \twoheadrightarrow \bar{I}$, compatible with the compositions of $\bar{J} \twoheadrightarrow \bar{I}$,

– an isomorphism of $\mathcal{O}_{U^{(\bar{J}/\bar{I})}}$-modules

$$\kappa^{(\bar{J}/\bar{I})} : j^{(\bar{J}/\bar{I})*}\big((\boxtimes_I \mathcal{E}_{J_i}) \boxtimes \mathcal{M}_{\bar{J}_\heartsuit}\big) \xrightarrow{\sim} j^{(\bar{J}/\bar{I})*}\mathcal{M}_{\bar{J}}$$

for every $\bar{J} \twoheadrightarrow \bar{I}$, compatible with the compositions of $\bar{J} \twoheadrightarrow \bar{I}$, and such that

– for any $f \in \mathcal{M}$ one has $1_\mathcal{E} \boxtimes f \in \mathcal{M}_{\{1,\heartsuit\}} \subset j_* j^*(\mathcal{E} \boxtimes \mathcal{M})$ and $\Delta^*(1_\mathcal{E} \boxtimes f) = f$.

We have the following immediate global counterpart of (5.4).

(6.1.2) Proposition. *For any right $\mathcal{D}_X$-module $\mathcal{M}$ there is a unique complex of $X \times C^I$-sheaves $\mathcal{DR}(\mathcal{M})_{C^I}$ such that:*

(a) the fiber of $\mathcal{DR}(\mathcal{M})_C$ at a point of the curve C is isomorphic to the complex $\mathcal{DR}(\mathcal{M})$,

(b) the collection $\big(\mathcal{DR}(\omega_X)_{C^I}\big)$ is a factorization algebra on the curve C. The collection $\big(\mathcal{DR}(\mathcal{M})_{C^I}\big)$ is a $\big(\mathcal{DR}(\omega_X)_{C^I}\big)$-module.

Proof: A choice of an element $i_0 \in I$ defines a morphism of schemes $p_{i_0} : \mathcal{L}^0(X)_{C^I} \to X$ as follows. Recall that $Lc^0(X)_{C^I}$ represents the functor λ^0_{X,C^I} which takes a scheme S into the set of pairs (f_I, ρ) where $f_I : S \to C^I$ is a morphism of schemes and ρ is a morphism of $\widehat{\Gamma(f_I)}$, the formal neighborhood of $\Gamma(f_I) \subset S \times C$, into X. Now, restricting ρ onto the graph of f_{i_0}, which is a subscheme in $\widehat{\Gamma(f_I)}$, isomorphic to S, we get a natural transformation from λ^0_{X,C^I} into the functor represented by X, so a morphism p_{i_0}.

Denote by $i : \mathcal{L}^0(X)_{C^I} \to \mathcal{L}(X)_{C^I}$ the embedding. For any $\mathcal{M} \in \mathbf{D}_X$, we form the object $i_\bullet p^\bullet_\heartsuit \mathcal{M} \in \mathbf{D}_{\mathcal{L}(X)_{C^{\bar{I}}}}$. The general construction of (5.3), applied to the restrictions of $i_\bullet p^\bullet_\heartsuit \mathcal{M}$ onto open subsets in $X \times C^{\bar{I}}$, gives then a complex of sheaves on $X \times C^{\bar{I}}$ which we denote $\mathcal{DR}(\mathcal{M})_{C^{\bar{I}}}$.

Notice that in the particular case where $\mathcal{M} = \omega_X$ the object $p^\bullet_{i_0}\mathcal{M}$ of the category $\mathbf{D}_{\mathcal{L}^0(X)_{C^I}}$ is independent (up to a unique isomorphism) of the choice of $i_0 \in I$. Indeed, objects of the latter category are, by definition, pairs $(n, \mathcal{N})$ where $\mathcal{N}$ is a right $\mathcal{D}$-module on $\mathcal{L}^0_n(X)_{C^I}$ and two such pairs $(n, \mathcal{N})$ and $(n', \mathcal{N}')$ are isomorphic, if the pullbacks of $\mathcal{N}$ and $\mathcal{N}'$ to $\mathcal{L}^0_m(X)_{C^I}$, $m \geq n, n'$ are isomorphic as right $\mathcal{D}$-modules. Since the pullback for right $\mathcal{D}$-modules is just the $\mathcal{O}$-module pullback tensored with the relative canonical class, $p^\bullet_{i_0}\omega_X$ is represented by $(n, \omega_{\mathcal{L}^0_n(X)_{C^I}})$ for any n, and thus is clearly independent on i_0.

Then, the general construction in (5.3) gives a complex of sheaves on $X \times C^I$, denoted by $\mathcal{DR}(\omega_X)_{C^I}$.

To prove Claim (a) it is sufficient to observe that the fiber of $\mathcal{L}(X)_C$ at a point $0 \in C$ is isomorphic to $\mathcal{L}(X)$. Recall that $\mathcal{L}(X)_C$ represents the contravariant functor $\lambda_{X,C} : \mathbf{Sch} \to \mathbf{Sets}$ such that $\lambda_{X,C}(S)$ is the set of pairs (f, ρ) such that

$$f \in \mathrm{Hom}_{\mathbf{Sch}}(S, C) \quad \text{and} \quad \rho \in \mathrm{Hom}_{\mathbf{Lrs}}\big((\Gamma(f), \mathcal{K}^{\surd}_f), X\big).$$

Thus the fiber at 0 represents the subfunctor

$$S \mapsto \big\{(f,\rho) \in \lambda_{X,C}(S) \,\big|\, f(S)_{\mathrm{red}} = \{0\}\big\}.$$

Let t be a local coordinate on C centered at 0. For any f as above we have $\big(\Gamma(f), \mathcal{K}^{\surd}_f\big) = \big(S, \mathcal{O}_S((t))^{\surd}\big)$. Note that the isomorphism of $\mathcal{L}(X)$ and the fiber of $\mathcal{L}(X)_C$ at 0 is compatible with the ind-pro-systems in (2.7), (3.9).

Claim (b) follows from the fact that $\mathcal{L}^0(X)_C$ and $\mathcal{L}(X)_C$ form factorization algebras, that the inclusion of the former into the latter is a morphism of factorization algebras and that passing to the de Rham complex takes Cartesian products of (ind-)schemes to tensor products of vector spaces. □

(6.2) Reminder on chiral and vertex algebras. Let us recall the basic facts on chiral and vertex algebras. See [BD1, §3], [K] for more details. Let C be a smooth curve, as before. For any right $\mathcal{D}_C$-module $\mathcal{M}$ let

$$\varepsilon_{\mathcal{M}} \,:\, j_\bullet j^\bullet(\omega_C \boxtimes \mathcal{M}) \to \Delta_\bullet \mathcal{M}$$

be the composition of the projection $j_\bullet j^\bullet(\omega_C \boxtimes \mathcal{M}) \to j_\bullet j^\bullet(\omega_C \boxtimes \mathcal{M})/(\omega_C \boxtimes \mathcal{M})$ and of the isomorphism $j_\bullet j^\bullet(\omega_C \boxtimes \mathcal{M})/(\omega_C \boxtimes \mathcal{M}) \xrightarrow{\sim} \Delta_\bullet\Delta^\bullet(\omega_C \boxtimes \mathcal{M}) \xrightarrow{\sim} \Delta_\bullet\mathcal{M}$.

(6.2.1) Definition. *(a) A chiral algebra over C is a right $\mathcal{D}_C$-module $\mathcal{A}$ with two maps $\mu_{\mathcal{A}} \in \mathrm{Hom}_{\mathbf{D}_{C^2}}\big(j_\bullet j^\bullet(\mathcal{A} \boxtimes \mathcal{A}), \Delta_\bullet\mathcal{A}\big)$ and $1_{\mathcal{A}} \in \mathrm{Hom}_{\mathbf{D}_C}(\omega_C, \mathcal{A})$ such that*

– *the map $\mu_{\mathcal{A}}(1_{\mathcal{A}}, id_{\mathcal{A}})$ coincides with $\varepsilon_{\mathcal{A}}$,*

– *the map $\mu_{\mathcal{A}}$ is antisymmetric, and it satisfies the Jacobi identity.*

(b) A module over a chiral algebra $\mathcal{A}$ over C is a right $\mathcal{D}_C$-module $\mathcal{M}_C$ with a map $\mu_{\mathcal{M}} \in \mathrm{Hom}_{\mathbf{D}_{C^2}}\big(j_\bullet j^\bullet(\mathcal{A} \boxtimes \mathcal{M}), \Delta_\bullet\mathcal{M}\big)$ such that

– *the map $\mu_{\mathcal{M}}(1_{\mathcal{A}}, id_{\mathcal{M}})$ coincides with $\varepsilon_{\mathcal{M}}$,*

– *the map $\mu_{\mathcal{M}}$ is compatible with $\mu_{\mathcal{A}}$.*

For any factorization algebra on $\mathcal{E}$, each sheaf $\mathcal{E}_I$ has a canonical left $\mathcal{D}_{C^I}$-module structure, compatible with the factorization structure, such that the section $1_{\mathcal{E}}$ is a horizontal (see [BD1, Proposition 3.4.8]). It is proved in [BD1, §3.4.9] that the right $\mathcal{D}_C$-module $\mathcal{A} = \mathcal{E} \otimes_{\mathcal{O}_C} \omega_C$ is a chiral algebra over C. The map $\mu_{\mathcal{A}}$ is the composition of the chain of maps

$$j_\bullet j^\bullet(\mathcal{A} \boxtimes \mathcal{A}) = \mathcal{E}_{\{1,2\}} \otimes_{\mathcal{O}_{C^2}} j_\bullet j^\bullet \omega_{C^2} \to \mathcal{E}_{\{1,2\}} \otimes_{\mathcal{O}_{C^2}} \Delta_\bullet\omega_C = \Delta_\bullet\mathcal{A}.$$

Here the 1-st equality is the 2-nd isomorphism in 6.1.1.(a), the second arrow is ε_{ω_C}, and the last equality results from the 1-st isomorphism in 6.1.1.(a) and the projection formula for Δ.

(6.2.2) Definition. *(a) A vertex algebra is a k-supervector space V with an even vector $1 \in V$, an even endomorphism $\partial \in End(V)$, and an even linear map $V \to End(V)[[z, z^{-1}]]$, $a \mapsto a(z) = \sum_n a_n z^{-n-1}$. These data satisfy the following axioms:*

– *$\partial(1_V) = 0$, $1_V(z) = id_V$, $a_n(1_V) = 0$ if $n \geq 0$, $a_{-1}(1_V) = a$,*

– *$[\partial, a(z)] = \partial_z a(z)$,*

– *we have $(z-w)^N[a(z), b(w)] = 0$ for $N >> 0$.*

(b) A module over a vertex algebra V is a k-supervector space W with an even endomorphism $\partial \in End(W)$, and an event linear map $V \to End(W)[[z, z^{-1}]]$, $a \mapsto a^W(z) = \sum_n a^W_n z^{-n-1}$. These data satisfy the following axioms :

– *$1^W_V(z) = id_W$, $L^W_{-1} = \partial$, $a^W_n(1_W) = 0$ if $n \geq 0$,*

– $[\partial, a^W(z)] = \partial_z a^W(z)$,
– *we have* $(z-w)^N[a^W(z), b^W(w)] = 0$ *for* $N >> 0$.

Let t be a formal coordinate on C centered at the point 0. Let $t_1 = t\otimes 1$, $t_2 = 1\otimes t$ be the corresponding coordinates on C^2. Let $\mathcal{A}_0$ be the fiber at the point $0 \in C$ of a chiral algebra $\mathcal{A}$. Let μ_0 be the fiber at 0 of the map $\Delta^\bullet(\mu_{\mathcal{A}}) : \Delta^\bullet j_\bullet j^\bullet(\mathcal{A}\boxtimes\mathcal{A}) \to \Delta^\bullet\Delta_\bullet(\mathcal{A}) \to \mathcal{A}$. We set

$$a(z)(b) = \sum_{n\in\mathbb{Z}} \mu_0\big((a\otimes b)(t_1-t_2)^{-n}\big)z^{-n-1}, \qquad \forall a, b \in \mathcal{A}_0. \tag{6.2.3}$$

Let $\mathcal{A}^\wedge$ be the space of section of $\mathcal{A}$ on the punctured formal neighborhood of C at the point 0. There is a canonical identification

$$\mathcal{A}_0 \simeq \mathcal{A}^\wedge/\mathcal{A}^\wedge \cdot \partial_t k((t)).$$

The operator ∂ on $\mathcal{A}_0$ is

$$\partial(a) = -a\cdot\partial_t, \qquad \forall a \in \mathcal{A}_0. \tag{6.2.4}$$

The following basic fact is due to Beilinson-Drinfeld (see [HL], [B] for a proof).

(6.2.5) Lemma. *The triple formed by* $\mathcal{A}_0$ *and the maps* (6.2.3), (6.2.4) *is a vertex algebra.*

Thus, Proposition 6.1.2 gives the following.

(6.2.6) Theorem. *The De Rham complex* $\mathcal{CDR}(\omega_X)$ *is a sheaf of vertex algebras on* X*. For any right* $\mathcal{D}_X$*-module* $\mathcal{M}$ *the De Rham complex* $\mathcal{CDR}(\mathcal{M})$ *is a sheaf of* $\mathcal{DR}(\omega_X)$*-modules.*

(6.3) The vacuum module and the chiral de Rham complex. Here we recall the original construction of the chiral de Rham complex of X as given in [MSV]. One first considers the case $X = \mathbb{A}^d$. Similarly to Example 5.4.2, let CD_M^N be the $\mathbb{Z}/2\mathbb{Z}$-graded k-algebra generated by even elements a_{in}, a^*_{in} and odd elements b_{in}, b^*_{in}, with $i = 1, ..., d$ and $-N \le n \le M$ for a_{in}, b_{in} and $-M \le n \le N$ for a^*_{in}, b^*_{in}, modulo the relations

$$a^*_{im}a_{jn} - a_{jn}a^*_{im} = \delta_{ij}\delta_{m,-n}, \qquad a_{im}a_{jn} - a_{jn}a_{im} = a^*_{im}a^*_{jn} - a^*_{jn}a^*_{im} = 0,$$

$$b^*_{im}b_{jn} + b_{jn}b^*_{im} = \delta_{ij}\delta_{m,-n}, \qquad b_{im}b_{jn} + b_{jn}b_{im} = b^*_{im}b^*_{jn} + b^*_{jn}b^*_{im} = 0.$$

We further require that the letters a and b commute in all cases. Let $CD = \varinjlim_M \varprojlim_N CD^N_M$. Let $CD^+ \subset CD$ be the right ideal generated by (possible infinite) linear combinations of $a^*_{i,n+1}, b^*_{i,n+1}, a_{in}, b_{in}$ with $n < 0$. The vacuum module

$Vac = CD/CD^+$ has a structure of a vertex algebra such that the generating series associated to the 0- and (-1)-modes of a_i, b_i are given by:

$$a^*_{i,-1}(z) = \sum_{n\in\mathbb{Z}} a^*_{in} z^{-n-1}, \quad a_{i0}(z) = \sum_{n\in\mathbb{Z}} a_{in} z^{-n},$$

and similarly with $b^*_{i,-1}(z)$, $b_{i0}(z)$. The generating series associated to other modes are obtained by differentiation, using the action of ∂ given by

$$\partial(a^*_{in}) = -n a^*_{i,n-1}, \quad \partial(a_{in}) = -(n-1)a_{i,n-1},$$

and idem for b^*_{in}, b_{in}. The map

$$\delta = \sum_{i,n} a^*_{in} b_{i,-n} \,:\, Vac \to Vac$$

is a derivation of vertex algebras with zero square.

Setting $x_i = a_{i0}$ makes Vac into a module over $k[x_1, ..., x_n]$, the coordinate ring of $\mathbb{A}^d$. We denote this ring shortly by $k[x]$. In [MSV] the authors consider the quasicoherent sheaf $\Omega^{ch}_{\mathbb{A}^d} := Vac \otimes_{k[x]} \mathcal{O}_{\mathbb{A}^d}$ corresponding to $k[x]$-module Vac and extend the vertex algebra structure to it. One also has a vertex algebra structure on

$$Vac^\wedge := Vac \otimes_{k[x_i]} k[[x_i]].$$

Now let X be a smooth algebraic variety, $U \subset X$ be an open subset and $\phi : U \to \mathbb{A}^d$ be an etale map. Let $0 \in X$ be a point such that $\phi(0) = 0$ and let $X^\wedge$ be the formal neighborhood of 0 ion X. Then $x'_i = \phi^* x_i$ are the coordinates on $X^\wedge$.

In [MSV] the authors construct a sheaf Ω^{ch}_X of differential vertex algebras on X as the unique such sheaf satisfying the following condition. For any ϕ as above, there is an isomorphism of vertex algebras

$$\phi_{ch} \,:\, \Omega^{ch}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X^\wedge} \to \phi^* \Omega^{ch}_{\mathbb{A}^d} \otimes_{\mathcal{O}_{\mathbb{A}^d}} \mathcal{O}_{\mathbb{A}^{d\wedge}} = Vac^\wedge$$

which coincides, for $U \subset \mathbb{A}^d$, with the automorphism of $Vac^\wedge$ introduced in [MSV, Theorem 3.7]. Our aim in the rest of this paper is to prove the following fact.

(6.3.1) Theorem. *There is an isomorphism of sheaves of differential vertex algebras* $\Omega^{ch}_X \simeq \mathcal{DR}(\omega_X)$.

(6.4) The factorization algebra associated to the vacuum module. As the first step in proving Theorem 6.3.1, let us describe the factorization algebra corresponding to $Vac = \Gamma(\mathbb{A}^d, \Omega^{ch}_{\mathbb{A}^d})$. In this, we follow [BD1] [G]: the constructions below are a particular instance of the general concept of the chiral enveloping algebra of a Lie*-algebra. For the convenience of the reader we give a self-contained presentation.

Fix a smooth projective curve C. Recall that ω_C is the sheaf of 1-forms on C (in the Zariski topology). Let I be a finite set. Consider the product $C^I \times C$ and its projections p, q to C^I and C. Let us specialize the notation of (3.3) to the case when $S = C^I$ and $f_I = \mathrm{Id} : C^I \to C^I$. We denote the subvariety $\Gamma(f_I)$ simply by

$$\Gamma_I = \big\{ \big((c_i), x\big) \in C^I \times C \big| x \in \{c_i\} \big\}.$$

Similarly we write $\mathcal{O}_I^\wedge, \mathcal{K}_I^\wedge$ for $\mathcal{O}_{f_I}^\wedge$, $\mathcal{K}_{f_I}^\wedge$. Set

$$[[\mathcal{O}]]_I = p_* \mathcal{O}_I^\wedge, \quad ((\mathcal{O}))_I = p_* \mathcal{K}_I^\wedge,$$
$$[[\omega]]_I = p_*(\mathcal{O}_I^\wedge \otimes q^* \omega_C), \quad ((\omega))_I = p_*(\mathcal{K}_I^\wedge \otimes q^* \omega_C).$$

These are (non-quasicoherent) sheaves of $\mathcal{O}$-modules on C^I. Informally, the "fiber" of, say, $((\omega))_I$ at a point $(c_i) \in C^I$ is the space of sections of ω_C on the punctured formal neighborhood of the set $\{c_i\}$, and similarly in the other cases. Note that the sum of residues defines a morphism

$$\mathrm{res} : ((\omega))_I \to \mathcal{O}_{C^I},$$

trivial on $[[\omega]]_I$.

Consider the super-vector space $k^{d|d}$ with basis consisting of even vectors $v_1, ..., v_d$ and odd vectors $v_{d+1}, ..., v_{2d}$. The sheaf

$$\mathfrak{h}_I^\wedge = \Big(\big(((\mathcal{O}))_I \oplus ((\omega))_I \big) \otimes k^{d|d} \Big) \ \oplus \mathcal{O}_{C^I} \cdot \gamma$$

is then a Lie superalgebra in the category of left $\mathcal{D}_{C^I}$-modules, with respect to the bracket

$$[\mathcal{O}_C^{d|d}, \mathcal{O}_C^{d|d}] = [\omega_C^{d|d}, \omega_C^{d|d}] = 0,$$

$$[v_i \otimes f, v_j \otimes \omega] = \delta_{ij} \mathrm{res}(f\omega) \cdot \gamma, \quad \forall f \in \mathcal{O}_C, \forall \omega \in \omega_C,$$

and with γ being a central element. Similarly, let $\mathfrak{h}_I^{\wedge,+} \subset \mathfrak{h}_I^\wedge$ be the super-Lie subalgebra $([[\mathcal{O}]]_I \oplus [[\omega]]_I) \otimes k^{d|d}$. For any surjective map $J \twoheadrightarrow I$ there are obvious isomorphisms

$$\Delta^{(J/I)*} \mathfrak{h}_J^\wedge \xrightarrow{\sim} \mathfrak{h}_I^\wedge, \qquad j^{(J/I)*}\big(\boxtimes_I \mathfrak{h}_{J_i}^\wedge\big) \xrightarrow{\sim} j^{(J/I)*} \mathfrak{h}_J^\wedge.$$

Let $\mathcal{U}_{C^I}^\wedge$ be the quotient of the associative enveloping algebra of $\mathfrak{h}_I^\wedge$, in the category of left $\mathcal{D}_{C^I}$-modules, by the right ideal generated by $\gamma - 1$. Consider the sheaf

$$\mathcal{V}ac_{C^I} = \mathcal{U}_{C^I}^\wedge / \mathcal{U}_{C^I}^{\wedge;+},$$

where $\mathcal{U}_{C^I}^{\wedge;+} \subset \mathcal{U}_{C^I}^\wedge$ is the right ideal generated by $\mathfrak{h}_I^{\wedge,+}$. The collection $(\mathcal{V}ac_{C^I})$ is clearly a factorization algebra. We denote $\mathcal{V}ac = \mathcal{V}ac_C \otimes \omega_C$ the chiral algebra associated to the factorization algebra $(\mathcal{V}ac_{C^I})$.

(6.4.1) Lemma. *Vac is isomorphic, as a vertex algebra, to the fiber of the chiral algebra $\mathcal{V}ac$ at any point of C.*

Proof: Fix a point $0 \in C$. Let t be a local coordinate on C centered at 0. Let $\mathcal{V}ac_0$ be the fiber of $\mathcal{V}ac$ at 0. We first identify $\mathcal{V}ac_0$ with Vac as a vector space and then compare the vertex algebra structures.

The space $\big(k((t)) \oplus k((t))dt\big) \otimes k^{d|d}$ is identified with the linear span of a_{im}, a^*_{im}, b_{im}, b^*_{im}, where $m \in \mathbb{Z}$, in such a way that

$$v_i t^m \mapsto a^*_{im}, \quad v_{i+d} t^m \mapsto b^*_{im}, \quad v_i t^{m-1} dt \mapsto a_{im}, \quad v_{i+d} t^{m-1} dt \mapsto b_{im}.$$

This isomorphism gives an isomorphism of k-vector spaces $\mathcal{V}ac_0 \simeq Vac$.

Next, the operator ∂ on $\mathcal{V}ac_0$ is the image of $\partial^{\mathcal{U}}$ modulo $\mathcal{U}_0^{\wedge,+}$. The vertex algebras $\mathcal{V}ac_0$ and Vac are completely determined by the operator ∂ and the formal series $a^*_{i,-1}(z)$, $b^*_{i,-1}(z)$, $a_{i0}(z)$, $b_{i0}(z)$. The operators ∂ coincide obviously.

The vertex algebra structure on $\mathcal{V}ac_0$ is the fiber at the point $(0,0)$ of the composition of the chain of maps

$$\mu \,:\, j_\bullet j^\bullet(\mathcal{V}ac \boxtimes \mathcal{V}ac) \xrightarrow{\sim} \mathcal{V}ac_{C^2} \otimes_{\mathcal{O}_{C^2}} j_\bullet j^\bullet \omega_{C^2} \xrightarrow{\varepsilon_{\omega_C}} \mathcal{V}ac_{C^2} \otimes_{\mathcal{O}_{C^2}} \Delta_\bullet \omega_C \xrightarrow{\sim} \Delta_\bullet \mathcal{V}ac.$$

Let $\mu_0 \,:\, \mathcal{V}ac_0 \otimes \mathcal{V}ac_0 \otimes k((t_1 - t_2)) \to \mathcal{V}ac_0$ be this fiber. There is a pair of compatible linear isomorphisms

$$\mathfrak{h}^\wedge_{\{1\},0} \simeq \big(k((t)) \oplus k((t))dt\big)^{d|d} \oplus k \cdot \gamma, \quad \mathfrak{h}^{\wedge,+}_{\{1\},0} \simeq \big(k[[t]] \oplus k[[t]]dt\big)^{d|d}.$$

Fix elements $a(t), b(t) \in k[t,t^{-1}]^{d|d}$. Recall that t_1, t_2 are the coordinates $t \otimes 1, 1 \otimes t$ on C^2. Let $a(t-t_1) \otimes b(t-t_2)$ be the corresponding element in the stalk of the sheaf $\mathcal{V}ac_C \boxtimes \mathcal{V}ac_C$ at the point $(0,0) \in C^2$. For any $f(t) \in k((t))$, the factorization morphism $j_\bullet j^\bullet(\mathcal{V}ac_C \boxtimes \mathcal{V}ac_C) \xrightarrow{\sim} j_\bullet j^\bullet(\mathcal{V}ac_{C^2})$ maps

$$a(t-t_1) \otimes b(t-t_2) \otimes f(t_1 - t_2) \quad \text{to} \quad a(t-t_1) \cdot b(t-t_2) \otimes f(t_1 - t_2),$$

where $\cdot$ is the product in the algebra $\mathcal{U}^\wedge_{C^2}$. Composing it with the map ε_{ω_C} in (6.2) we get

$$\mu\big(a(t-t_1)dt_1 \otimes b(t-t_2)dt_2 \otimes f(t_1-t_2)\big) =$$

$$= \sum_n a_n(t-t_2) \cdot b(t-t_2) \otimes \mathrm{res}_{t_1=t_2}\big((t_1-t_2)^n f(t_1-t_2) dt_1 dt_2\big),$$

where $a(t_2 - t_1) = \sum_n a_n(t_2) t_1^n$ is the expansion in $k[t_2, t_2^{-1}][[t_1]]^{d|d}$. In the particular case where $a(t) = at^{-1}$ with $a \in k^{d|d}$, and $f(t) = t^{-n}$ with $n \geq 0$, we have

$$\mu_0\big(at^{-1}dt_1 \otimes b(t)dt_2 \otimes (t_1-t_2)^{-n}\big) = (at^{-n}) \cdot b(t) \otimes dt.$$

In particular, this formula gives

$$\mu_0\big((a^*_{i,-1}\otimes b)(t_1-t_2)^{-n}\big)=a^*_{in}(b),\quad \mu_0\big((b^*_{i,-1}\otimes b)(t_1-t_2)^{-n}\big)=b^*_{in}(b).$$

A similar computation gives

$$\mu_0\big((a_{i0}\otimes b)(t_1-t_2)^{-n}\big)=a_{i,n+1}(b),\quad \mu_0\big((b_{i0}\otimes b)(t_1-t_2)^{-n}\big)=b_{i,n+1}(b).$$

We are done. □

(6.4.2) Corollary. *Theorem 6.3.1 is true for $X=\mathbb{A}^d$.*

Proof: The same argument as in Proposition 5.4.3 identifies the space of global sections of $\mathcal{CDR}_{\mathbb{A}^d}$ with Vac, and the latter is identified with $\Gamma(\mathbb{A}^d,\Omega^{ch}_{\mathbb{A}^d})$.

(6.5) The action of étale morphisms I. To prove Theorem 6.3.1 in full generality, it suffices to establish the following lemma.

(6.5.1) Lemma. *(a) For any affine open set $U\subset X$ and an étale map $\phi: U\to\mathbb{A}^d$, there is an isomorphism of differential vertex algebras $F_\phi : \phi^*\Omega^{ch}_{\mathbb{A}^d,\wedge}\xrightarrow{\sim}k[X^\wedge]\otimes_{\mathcal{O}_X}\mathcal{DR}(\omega_X)$.*

(b) If $U\subset\mathbb{A}^d$, then $F_\phi^{-1}\circ F_{\mathrm{Id}}=\phi_{ch}$ is the isomorphism constructed in [MSV].

The plan of the proof is as follows. We will construct an isomorpism (F_I) of factorization algebras and obtain F_ϕ as the fiber of $F_{\{1\}}$ at a point of C. It is enough to assume that $C=\mathbb{A}^1$. Set

$$CD^\wedge_I=\Gamma(\mathbb{A}^I,\mathcal{U}^\wedge_{C^I}),\quad CD^{\wedge,+}_I=\Gamma(\mathbb{A}^I,\mathcal{U}^{\wedge,+}_{C^I}),$$

$$DR(\omega_U)_I=\Gamma\big(\mathcal{L}(U)_{\mathbb{A}^I},\mathcal{DR}(\omega_U)_{\mathbb{A}^I}\big),\quad Vac_I=\Gamma(\mathbb{A}^I,\mathcal{V}ac_{C^I}).$$

Thus, $Vac_I=CD^\wedge_I/CD^{\wedge,+}_I$. We first construct a right action

$$DR(\omega_U)_I\otimes CD^\wedge_I\to DR(\omega_U)_I \tag{6.5.2}$$

commuting with the factorization maps. To prove that the factorization algebras $(\mathcal{V}ac_{\mathbb{A}^I})$, $(\mathcal{DR}(\omega_U)_{\mathbb{A}^I})$ are isomorphic, it is then sufficient to check that the right $CD^\wedge_I$-module $DR(\omega_U)_I$ has a cyclic vector whose annihilator is $CD^{\wedge,+}_I$. Observe that, since the map (6.5.2) depends on ϕ, the resulting isomorphism of sheaves of vertex algebras $\phi^*\Omega^{ch}_{\mathbb{A}^d,\wedge}\xrightarrow{\sim}k[X^\wedge]\otimes_{\mathcal{O}_X}\mathcal{DR}(\omega_X)$ will also depend on ϕ.

(6.5.3) The case $U=\mathbb{A}^1$. First we consider the particular case where $U=\mathbb{A}^d$ and $\phi=\mathrm{Id}$. What we do in this case is to provide an explicit identification of the right $CD^\wedge_I$-module $DR(\omega_{\mathbb{A}^d})_I$. To simplify we set $\mathcal{L}=\mathcal{L}(\mathbb{A}^d)_{\mathbb{A}^I}$, etc. We proceed as in Example 5.4.2. For any $n,N\in\mathbb{N}\cup\{\infty\}$ we set

$$B=k[\mathbb{A}^I],\quad A^N_n=B[a^{(j)}_{l\nu};-N\le l\le n],\quad Y^N_n=\operatorname{Spec}A^N_n, \tag{6.5.4}$$

where $a^{(j)}_{l\nu}$ are independent variables such that $j \in [1,d]$, $l \in \mathbb{Z}$, $\nu \in [0, |I|-1]$. Then

$$\mathcal{L}_n = \lim_{\longrightarrow N} \mathrm{Spf}\,\big(A^0_n[[a^{(j)}_{l\nu}; -N \leq l < 0]]\big), \quad \mathcal{L}^\varepsilon_n = \mathrm{Spec}\,\big(A^{N_\varepsilon}_n/(a^{(j)}_{l\nu_1}\cdots a^{(j)}_{l\nu_{1+\varepsilon_l}}; l<0)\big),$$

see (3.9.1), where we set $N_\varepsilon = \max\{l;\, \varepsilon_{-l} \neq 0\}$ for each $\varepsilon \in \mathbb{E}$. Note that

$$CD^\wedge_I = \lim_{\longrightarrow n}\lim_{\longleftarrow N} CD_{Y^N_n}.$$

To simplify again we set $Y^\varepsilon_n = Y^{N_\varepsilon}_n$. Thus Y^ε_N is an affine space of finite dimension. There are closed embeddings $\mathcal{L}^0_n \subset \mathcal{L}^\varepsilon_n \subset Y^\varepsilon_n$. Let $i_{n\varepsilon} : \mathcal{L}^0_n \hookrightarrow Y^\varepsilon_n$ is the composite embedding. We write $\omega_{\mathcal{L}^0_n, Y^\varepsilon_n}$ for the right $\mathcal{D}$-module $i_{n\varepsilon\bullet}\omega_{\mathcal{L}^0_n}$ on Y^ε_n.

Consider the right $\mathcal{D}_{\mathbb{A}^d}$-module $\mathcal{M} = \omega_{\mathbb{A}^d}$. Let $\mathcal{M}^\varepsilon_n \in \mathbf{D}_{\mathcal{L}^\varepsilon_n}$ be the subsheaf of $\omega_{\mathcal{L}^0_n, Y^\varepsilon_n}$ consisting of the sections supported (scheme-theoretically) on $\mathcal{L}^\varepsilon_n$. Similarly, let $\mathcal{DR}(\mathcal{M}^\varepsilon_n) \in \mathbf{O}_{\mathcal{L}^\varepsilon_n}$ be the subsheaf of $\mathcal{DR}(\omega_{\mathcal{L}^0_n, Y^\varepsilon_n})$ consisting of the sections supported on $\mathcal{L}^\varepsilon_n$. Let $DR(\omega_n)_I$ be the space of global sections of $\mathcal{DR}(\mathcal{M}_n)$. By definition,

$$\lim_{\longrightarrow \varepsilon} \Gamma\big(\mathcal{L}^\varepsilon_n, \mathcal{DR}(\mathcal{M}^\varepsilon_n)\big) = DR_{\mathcal{L}^0_n, Y^\infty_n}.$$

Let us denote this space by $DR(\omega_n)_I$.

We get a right $\lim_{\longleftarrow N} CD_{Y^N_n}$-action on $DR(\omega_n)_I$ such that $DR(\omega_n)_I$ is the quotient of $\lim_{\longleftarrow N} CD_{Y^N_n}$ by $\lim_{\longleftarrow N} CD^+_{Y^N_n}$. Since $DR(\omega_{\mathbb{A}^d})_I = \lim_{\longrightarrow n} DR(\omega_n)_I[nd]$, we get a right action of $CD^\wedge_I = \lim_{\longrightarrow n}\lim_{\longleftarrow N} CD_{Y^N_n}$ on $DR(\omega_{\mathbb{A}^d})_I$ such that

$$DR(\omega_{\mathbb{A}^d})_I \simeq CD^\wedge_I / CD^{\wedge,+}_I \simeq Vac_I,$$

thus achieving our goal in the case $X = \mathbb{A}^d$.

(6.6) Etale change of coordinates in Clifford algebras. In order to prove Lemma 6.5.1 for general $\phi : U \to \mathbb{A}^d$ we need some elementary observations about Clifford algebras.

If Y is a smooth algebraic variety, we denote by $\mathcal{CD}_Y$ the sheaf of differential operators in $\Omega^\bullet_Y$, the commutative superalgebra of differential forms.

In particular, if $Y = \mathbb{A}^l$ with coordinates $y_1, ..., y_l$, then y_i, dy_i are free generators of (the algebra of global sections of) $\Omega^\bullet_Y$ and we denote $\partial_i = \partial/\partial y_i$ and $\xi_i =$ "$\partial/\partial\, dy_i$" the corresponding derivations, which are thus global sections of $\mathcal{CD}_{\mathbb{A}^l}$.

Let now V be an affine open subset of a smooth variety Y, and let $\psi : V \to \mathbb{A}^l$ be an étale map. Fix a point $0 \in U$ such that $\phi(0) = 0$. Let $y'_i = \phi^* y_i$ be the coordinate on the formal neighborhood $Y^\wedge$ of 0 in Y. There are then uniquely determined derivations ∂'_i, ξ'_i of $\Omega_{Y^\wedge}$ such that

$$[\partial'_i, y'_j] = [\xi'_i, dy'_j] = \delta_{ij}, \quad [\partial'_i, dy'_j] = [\xi'_i, y'_j] = 0.$$

Set $k[[y]] = k[[y_1, y_2, ..., y_l]]$, and $CD_{Y^\wedge} = k[Y^\wedge] \otimes_{\mathcal{O}_Y} \mathcal{CD}_Y$. The étale map ψ gives an isomorphism of formal schemes $Y^\wedge \xrightarrow{\sim} \mathbb{A}^{l,\wedge}$. Set $\psi' = \psi^{-1}$.

(6.6.1) Lemma. *For any ψ as above, there is a unique $k[Y^\wedge]$-algebra isomorphism*
$$\psi_\sharp : CD_{Y^\wedge} \xrightarrow{\sim} \psi^* CD_{\mathbb{A}^l,\wedge}$$
such that
$$\psi_\sharp(dy'_i) = \sum_j \partial_j \phi_i(y) dy_j, \quad \phi_\sharp(\xi'_i) = \sum_j \partial'_i \phi'_j(\phi(y))\xi_j,$$
$$\phi_\sharp(\partial'_i) = \sum_j \partial'_i \phi'_j(\phi(y))\partial_j + \sum_{j,k,l} \partial'_i \partial'_k \phi'_j(\phi(y))\partial_l \phi_k(y) dy_k \xi_j,$$
where $\phi = (\phi_1, ..., \phi_l)$.

Proof: It suffices to observe that, since the map ψ is étale, $CD_{Y^\wedge}$ is a free $k[Y^\wedge]$-module with basis
$$(\partial'_1)^{r_1}(\partial'_2)^{r_2}\cdots(\partial'_l)^{r_l} \otimes (\xi'_1)^{m_1}\cdots(\xi'_d)^{m_l}(dy'_1)^{n_1}\cdots(dy'_d)^{n_l},$$
where $r_i, m_i, n_i \in \mathbb{N}$. Then use the coordinates change formulas, see [L, chap. II] for instance. □

(6.7) The action of étale morphisms II. Let now $\phi : U \to \mathbb{A}^d$ be a general étale morphism. By Proposition 3.9.2 we have an isomorphism of schemes
$$\mathcal{L}_n^\varepsilon(\phi)_{\mathbb{A}^I} \simeq \mathcal{L}_n^\varepsilon \times_{\mathcal{L}_0^0} \mathcal{L}_0^0(U)_{\mathbb{A}^I}.$$
There is an obvious map $Y_n^N \to \mathcal{L}_0^0$, since $\mathcal{L}_0^0 = Y_0^0 = \operatorname{Spec} A_0^0$, which restricts to the map $\mathcal{L}_n^\varepsilon \to \mathcal{L}_0^0$ when $N = N_\varepsilon$. For any N, consider the fiber product
$$Y_{n\phi}^N = Y_n^N \times_{\mathcal{L}_0^0} \mathcal{L}_0^0(U)_{\mathbb{A}^I}.$$
Denote by $\phi_n^N : Y_{n\phi}^N \to Y_n^N$ the projection to the first factor. Being a base change of an étale morphism $\mathcal{L}_0^0(\phi)_{\mathrm{Id}} : \mathcal{L}_0^0(U)_{\mathbb{A}^I} \to \mathcal{L}_0^0$, see Proposition 3.9.2.$(a)$, it is itself étale. Set $0 = \{a_{0\nu}^{(j)} = 0\} \in Y_0^0(B)$. Fix $0 \in Y_{0\phi}^0(B)$ mapping to 0 by ϕ_n^N. Let $Y_0^{0,\wedge}, Y_{0\phi}^{0,\wedge}$ be the formal neighborhoods of 0, and set
$$Y_n^{N,\wedge} = Y_0^{0,\wedge} \times_{Y_0^0} Y_n^N, \quad Y_\phi^{N,\wedge} = Y_{0\phi}^{0,\wedge} \times_{Y_0^0} Y_n^N.$$
By Lemma 6.6.1 applied to ϕ_n^N, we have a ring isomorphism
$$\phi_{n\sharp}^N : CD_{Y_{n\phi}^{N,\wedge}} \xrightarrow{\sim} \phi_n^{N*} CD_{Y_n^{N,\wedge}}.$$
Thus, by passing to the limits, we get a ring isomorphism
$$\phi_{\infty\sharp}^\infty : CD_I^\wedge = \varinjlim_n \varprojlim_N CD_{Y_n^{N\wedge}} \to \varinjlim_n \varprojlim_N CD_{Y_{n\phi}^{N\wedge}}. \tag{6.7.1}$$
Denote the algebra on the right by $CD_{I,\phi}^\wedge$. Note that
$$DR(\omega_U)_I = \varinjlim_{n,N} DR(\omega_{\mathcal{L}_n^0(U)_{\mathbb{A}^I}, Y_{n\phi}^N})[nd]$$
is a module over $CD_{I,\phi}^\wedge$. We obtain, therefore, a structure of a $CD_I^\wedge$-module on $DR(\omega_U)_I$.

(6.7.2) Lemma. *With respect to the above structure of a $CD^\wedge_I$-module, $DR(\omega_U)_I$ is isomorphic to Vac_I.*

Proof: Follows from the fact that each $DR(\omega_{\mathcal{L}^0_n(U),Y^N_{n\phi}})$ is isomorphic to the vacuum module over $CD^\wedge_{Y^N_{n\phi}}$.

Note that both $DR(\omega_U)_I$ and Vac_I have distinguished generators. Namely $1'_I \in DR(\omega_U)_I$ is the image of the following tensor on $\mathcal{L}^0_0(U)_{\mathbb{A}^I} = U \times \mathbb{A}^I$:

$$dx_1...dx_d dt_1...dt_{|I|} \otimes \frac{\partial}{\partial t_1} \wedge ... \wedge \frac{\partial}{\partial t_{|I|}} \in \omega_{U\times\mathbb{A}^I} \otimes \bigwedge^{|I|} \Theta_{U\times A^I}.$$

The generator $1''_I \in Vac_I$ is the image of $1 \in CD^\wedge_I$. We denote $F_I : Vac_I \to DR(\omega_U)_I$ the unique module isomorphism taking $1''_I$ to $1'_I$. Let $\mathcal{F}_I : \mathcal{V}ac_I \to \mathcal{DR}(\omega_U)_I$ be the corresponding moprhism of quasicoherent sheaves on $U \times \mathbb{A}^I$.

(6.7.3) Lemma. *The $(\mathcal{F}_I)$ commute with factorization maps and thus form an isomorphism of factorization algebras.*

Proof: Follows from the fact that $(1'_I)$ and $(1''_I)$ are compatible with factorization structures: in the notation of (6.1.1) we have $\kappa^{J/I}(\boxtimes 1'_{J_i}) = 1'_J$ and similarly for $1''_I$.

Let F_ϕ be the fiber of $F_{\{1\}}$ at the point $0 \in \mathbb{A}^1$. Lemma 6.7.3 implies that F_ϕ is a morphism of vertex algebras. This establishes part (a) of Lemma 6.5.1

We now prove Lemma 6.5.1(b). So we assume $U \subset \mathbb{A}^d$ and need to compare two automorphisms of the vertex algebra $\Gamma(U, \mathcal{V}ac)$, namely $F_\phi^{-1} \circ F_{\mathrm{Id}}$ and ϕ_{ch}. Notice that the vertex algebra Vac is strongly generated by the fields $a^*_{i,-1}(z)$, $a_{i0}(z)$, $b^*_{i,-1}(z)$, $b_{i0}(z)$, see [K], and $\Gamma(U, \mathcal{V}ac)$ is obtained by localization. Thus it is enough to compare the two automorpisms on the elements

$$a^*_{i,-1},\ a_{i0},\ b^*_{i,-1},\ b_{i0} \in Vac. \tag{6.7.4}$$

To simplify we set $Y = Y^\infty_\infty = \varinjlim_N \varprojlim_n Y^N_n$, $Y_\phi = Y^\infty_{\infty\phi} = \varinjlim_N \varprojlim_n Y^N_{n\phi}$, $\phi = \phi^\infty_\infty$. Note that the fiber at $0 \in \mathbb{A}^1$ of the B-algebras CD_Y, $CD_{Y^\wedge}$ are respectively

$$CD, \quad CD^\wedge := CD \otimes_{k[a_{i0}]} k[[a_{i0}]].$$

Following our simplified notation, we denote the isomorphism (6.7.1) simply by $\phi_\sharp$. Let $\phi_{\sharp 0}$ be the fiber of $\phi_\sharp$ at $0 \in \mathbb{A}^1$. We have $\phi_{\sharp 0}(CD^{+,\wedge}) \subset CD^{+,\wedge}$. In particular, $\phi_{\sharp 0}$ gives an automorphism of the $CD^\wedge$-module $Vac^\wedge$ which is compatible with the vertex algebra structure. To prove Claim (b) it suffices therefore to check that

$\phi_{\sharp 0} = \phi_{ch}$ on elements (6.7.4). Recall that $\mathcal{L}_0^0(X)_{\mathbb{A}^1} = X \times \mathbb{A}^1$ for any X. We get a commutative diagram

$$\begin{array}{ccccc} Y^N_{n\phi} & \twoheadrightarrow & Y^0_{0\phi} & = & U \times \mathbb{A}^1 \\ \phi^N_n \downarrow & & \downarrow & & \downarrow \phi \\ Y^N_n & \twoheadrightarrow & Y^0_0 & = & \mathbb{A}^d \times \mathbb{A}^1. \end{array}$$

This diagram induces a diagram of B-algebra homomorphisms

$$\begin{array}{ccc} CD_{Y^\wedge_\phi} & \hookleftarrow & CD_{Y^{0\wedge}_{0\phi}} \\ \phi_\sharp \downarrow & & \downarrow \phi^0_{0\sharp} \\ \phi^* CD_{Y^\wedge} & \hookleftarrow & \phi_0^{0*} CD_{Y_0^{0\wedge}}. \end{array}$$

We see that the images of the elements (6.7.4) by $\phi_\sharp$ and $\phi^0_{0\sharp}$ coincide, modulo the identification

$$a_{i0} \mapsto x_i, \quad b_{i0} \mapsto dx_i, \quad b^*_{i,-1} \mapsto \xi_i, \quad a^*_{i,-1} \mapsto \partial_i.$$

It suffices to observe that ϕ_{ch} and $\phi^0_{0\sharp}$ act similarly on the elements (5.4.9), see the formulas [MSV, (3.17)] for ϕ_{ch} and Lemma 6.6.1 for $\phi^0_{0\sharp}$. We are done.

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