An infinitesimal Liouville–Arnold theorem as criterion of reducibility for variational Hamiltonian equations

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Nr.	Name des Hauses * = siehe Anzeige	Inhaber	Straße	Ruf Vorwahl (0 26 44)	Ruhetag	P = Parkplatz	MA = Mittag und Abendtisch K = Kegelbahn	Saal von bis Personen	WP = Weinprobe	Tanz f. Gesell.
	Restaurants und Cafés									
1	**Hotel Weinstock** *	Rolf Geller	Linzhausenstraße 38	24 59	Mo	P	MA	10–400		n.V.
2	**Hotel Palm** *	Illa Schlang	Vor dem Leetor 13	25 32		P	MA, K	50–500	WP	n.V.
3	**Hotel-Restaurant Burg Ockenfels** *	E. Harms	Ockenfels	20 71/72		P	MA	40 u. 80		n.V.
4	**Hotel-Restaurant Gut Frühscheid**	Familie Wiedemeyer	Am Ronigerweg	70 15	Di	P	MA	10–50		n.V.
5	**Café Weiß**	Emil Weiß	Mittelstraße 7–11	22 95						
6	**Alt Linz**	Christel Klein	Mittelstraße 15	24 83			MA			
7	**Weinhaus Mohr**	Josef Tewes	Grabenstraße 3	33 16	Mi		MA	30		n.V.
8	**Hotel Wald**	Heinz Kloos	Linzhausenstraße 80	25 80		P	MA	35		
9	**Linzer Hof**	Geschw. Schaßberger	Am Halborn 10	27 65			MA	30–50		n.V.
10	**Burgklause**	Roland Grumptmann	Burgplatz 11	24 66	Mo		MA			
11	**Weinhaus Rebenhof** *	Erich Thür	Vor dem Leetor 2	26 80		P	MA	30–100	WP	n.V.
12	**Zum Köbes**	Jakob Lück	Buttermarkt 9	26 53	Mi					
13	**Weinhaus zur Linde**	Familie Hesse	Hospitalstraße 12	23 92						
14	**Brückenschänke**	Anneliese Seinsche	Linzhausenstraße 72	31 23	Do	P				
16	**Ristorante Pizzeria Franco**	Franco Soravia	Klosterstraße 5	22 00						
17	**Restaurant Burg Linz** *	Robert J. Schwaab	Burgplatz 4	70 21		P	MA	bis 200		n.V.
18	**Winzerhaus am Kaiserberg**	Käthe Oelpenich	Zum Winzerhaus 12	24 35		P	MA			n.V.
19	**Zur Alten Post** *	Rudi Stein	Rheinstraße 5	25 17			MA, K	20–90		n.V.
20	**Zur Bunten Stadt**	Günter Wagner	Neustraße 23	33 07	Do		MA	40–180		n.V.
21	**Bacchus-Keller** *	Ingeborg Probst	Marktplatz 17	24 80	Mo	P		50–80	WP	n.V.
22	**La Campagnola**	Fam. Di Lauro	Strohgasse 6	78 00	Di		MA			
23	**Sirtaki**	N. Gionarkias	Strohgasse 8	78 40	Mo		MA	30		
24	**Rheintor**	Christine Klinkusch	Burgplatz 10	27 38	Mi		MA			
25	**Em Stropp**	Klaus Staby	Neustraße 13	26 24	Mo					
26	**Zur Mühle** *	Tina Paffhausen	Mühlengasse 17	32 24	Di		MA	30		
27	**Schmeck'Lecker** *	Flesche-Almola	Burgplatz 13	79 37			MA			
28	**Zum alten Stern**	Toni Euskirchen	Asbacher Straße 139	26 59	Do	P	MA	20–100		n.V.
29	**„Go-In"**	Helene Nietzard	Asbacher Straße 17	29 94	Sa/So					Disco
30	**Em Pözje**	Jürgen Kryll	Neustraße 17	17 10	Di		MA	40		
31	**Zum Neutor**	Chr. Holberg	Neustraße 27	57 97	Mi					
32	**Zum Hammer**	Rita Schmahl	Am Hammer	25 60	Mo					
33	**Burgdiskothek Castello**	EPW, GmbH	Burgplatz 4	49 21		P		300		Disco
34	**Weinhaus Winzen**	Willi Winzen	Asbacher Straße 74	32 55	Di				WP	
35	**Zum Markus**	Salvatore Giobbe	Klosterstraße 8 a	32 85	Di		MA			
36	**Nibelungen im Rheinischen Hof**	Heinrich Klein	Rheinstraße 13	25 78			MA			
37	**Café Leber**	Geschw. Frings	Burgplatz 2	23 39	Do					
38	**Linzer Café-Restaurant**	Dietmar Brietzke	Kanzlerstraße	16 82			M	90		n.V.
39	**Café Schneider**	P. Schneider	Neustraße 15	25 02						
40	**Peppers Café, Galerie**	Klaus Pepper	Asbacher Straße 25	47 07				F.-Wohnung u. Kleinkunst		
41	**Martins Stuben**	Martin Scherer	Am Gestade 2	52 53	Mi	P				
42	**Em Stüffje**	Jürgen Kryll	Buttermarkt 10	34 70	Mo					
43	**Jägerklause**	Kurt Kurz	Marktplatz 27	69 18	Di					
44	**Grill am Markt**	Hartmut Rüdiger	Strohgasse 18	12 70			MA			
45	**Ital. Eiscafé**	Almola-Soravia	Burgplatz 12	41 00						
46	**Eis-Salon Dolomiten**	Julio Corazza	Rheinstraße 15	57 35						
47	**Ital. Eisdiele am Markt**	Elisabeth Sagui	Marktplatz 22	43 08		P				
48	**Eis-Salon Cordella**	Modesto Cordella	Klosterstraße 12	36 89						
49	**Bahnhofsgaststätte**	Katharina Schmidt	Bahnhofsplatz 1	34 28		P	MA			n.V.
50	**Kaktus**	Jean-Regis Dibus	Commenderiestraße 2	14 74	Mo		MA			

1. Introduction.

The subject of investigation is a Hamiltonian vector–field H_f on a 2N–dimensional symplectic manifold (M,ω), which is integrable on some invariant symplectic submanifold $\mathcal{T} \subset M$, $\dim \mathcal{T} = 2n < 2N$. So $\mathcal{T}$ is foliated into invariant tori T^n_p depending on an n–dimensional parameter $p \in P \subset\subset \mathbb{R}^n$, and the flow on every torus T^n_p is of the form $\dot{q} = \nabla_p f_0(p)$ (f_0 is a restriction of the hamiltonian f to $\mathcal{T}$). Let $(T\mathcal{T})^{\perp} \subset T_{\mathcal{T}}M \equiv \cup_{m\in\mathcal{T}} T_m$ be the (skew–)normal bundle of $\mathcal{T}$. If S_t is a flow of H_f, then the normal bundle $(T\mathcal{T})^{\perp}$ is invariant for the tangent flow S_{t*}. We call the restriction of S_{t*} on $(T\mathcal{T})^{\perp}$ "the flow of the normal variational equation (NVE) of H_f along $\mathcal{T}$", and study the question: under what conditions is this flow reducible to the flow of a linear equation with coefficients independent of the point $q \in T^n_p$ (so–called reducibility problem; see e.g. Johnson, Sell (1981)). If such reducibility occur then in the "nondegenerate case" $\mathcal{T}$ is "KAM–stable". That is most of the tori T^n_p, $p \in P$, survive after a small hamiltonian perturbation of the system (this results from a perturbation theorem for lower–dimensional invariant tori of a linear system, see Eliasson (1988), Kuksin (1989), Pöschel (1989)).

It is known that if no additional conditions are imposed then the NVE may be non–reducible (see Johnson (1979), Herman (1983)). On the other hand, if in a neighborhood of $\mathcal{T}$ the conditions of the "degenerate Liouville–Arnold theorem" are fulfilled, then the vector–field H_f is integrable in the vicinity of $\mathcal{T}$ and NVE is trivially reducible (for the degenerate Liouville–Arnold theorem see Eliasson (1988) and its bibliography).

Our aim in this paper is to obtain some criterion of reducibility of the NVE, which is a rather straightforward infinitesimal version of the Liouville–Arnold theorem.

In the important case of codimension 1 $(N = n+1)$ this criterium gives as a test for reducibility some zero–curvature equation.

We are most interested in elliptic invariant submanifolds $\mathcal{T}$. For such a $\mathcal{T}$ with reducible flow of the NVE we give a definition of a spectrum of the flow and formulate the nondegeneracy condition sufficient for KAM–stability of $\mathcal{T}$ in terms of this spectrum.

§ 2. Criterion of reducibility.

We shall formulate the results in analytic case. So all the manifolds and the mappings are supposed to be analytic. Let the symplectic manifold (M,ω) be provided with Riemann metric dm and the submanifold $\mathcal{T}$ is symplectomorphic to $(\mathbb{T}^n \times P, dp \wedge dq)$, $\mathbb{T}^n = \{q\}$, $P = \{p\}$. That is, $\mathcal{T} = \Sigma_0(\mathbb{T}^n \times P)$ for an (analytic) map

$$\Sigma_0 : \mathbb{T}^n \times P \longrightarrow M , \quad \Sigma_0^* \omega = dp \wedge dq .$$

Below we identify $\mathcal{T}$ with $\mathbb{T}^n \times P$.

If S_t is a flow of hamiltonian vector field H_f, then the subbundles $T_{\mathcal{T}}M = \cup_{m\in\mathcal{T}} T_m M \subset TM$, $T\mathcal{T} \subset T_{\mathcal{T}}M$ and $(T\mathcal{T})^{\perp} \subset T_{\mathcal{T}}M$ (the skew–normal bundle to $T\mathcal{T}$ in $T_{\mathcal{T}}M$) are invariant for the tangent flow S_{t*}.

Definition 1. The flow S_{t*} of the NVE of the vector–field H_f along $\mathcal{T}$ (together with the underlying normal bundle $(T\mathcal{T})^{\perp}$ is called reducible if

1) there exist a symplectic trivialisation of the bundle $(T\mathcal{T})^{\perp}$,

$$\begin{array}{ccc} \mathbb{T}^n \times P \times Y & \xrightarrow{\phi} & (T\,\mathcal{T})^{\perp} \\ & \searrow \quad \swarrow & \\ & \mathcal{T} & \end{array} \tag{2.1}$$

where the fiber $Y = \mathbb{R}^{2m}_{y} = \mathbb{R}^{m}_{y_+} \times \mathbb{R}^{m}_{y_-}$, $m = N-n$, has the usual symplectic structure with the form $dy_+ \wedge dy_-$.

2) There exists an analytic symmetric $2m \times 2m$ – matrix $A(p)$ such that under this trivialisation the flow S_{t*} on $(T\,\mathcal{T})^{\perp}$ corresponds on $\mathbb{T}^n \times P \times Y$ to the flow of the equation

$$\dot{q} = \nabla f_0(p)\ , \quad \dot{p} = 0\ , \quad \dot{y} = JA(p)y \tag{2.2}$$

where $J(y_+,y_-) = (-y_-,y_+)$ (we use the same notation for operators and their matrices).

In the situation of the Definition 1, we will say (with some abuse of language) that the NVE is reducible.

Definition 2. The flow S_{t*} is called complex–reducible if its complexification in the bundle $(T\,\mathcal{T})^{\perp} \otimes_{\mathbb{R}} \mathbb{C}$ is reducible in the category of complex symplectic bundles, with some symmetric complex matrix $A(p)$.

Proposition 1. If the bundle $(T\,\mathcal{T})^{\perp}$ can be trivialized (i.e. if there exists an isomorphism Φ as in (2.1)), then some neighborhood of $\mathcal{T}$ in M is symplectomorphic to a neighborhood 0 of $\mathcal{T}_0 = \mathbb{T}^n \times P \times \{0\}$ in $\mathbb{T}^n \times P \times Y$ with the 2–form $dp \wedge dq + dy_+ \wedge dy_-$.

Proof. Let us consider the restriction on $(T\,\mathcal{S})^{\perp}$ of the geodesic flow on TM and take its M–projection:

$$\Xi : (T\,\mathcal{S})^{\perp} \longrightarrow M\ , \quad (x,\xi) \longmapsto \exp_x \xi\ ,$$

for $x \in M$, $\xi \in (T\,\mathcal{S})^{\perp}_x$. Let $(T\,\mathcal{S})^{\perp}_0$ be the zero–section of $(T\,\mathcal{S})^{\perp}$. Then for arbitrary $(x,0) \in (T\,\mathcal{S})^{\perp}_0$ the tangent map

$$\Xi_*(x,0) : T_{(x,0)}(T\,\mathcal{S})^{\perp} \simeq T_x\,\mathcal{S} \oplus (T_x\,\mathcal{S})^{\perp} \xrightarrow[\sim]{} T_xM \tag{2.3}$$

is a linear symplectomorphism and its restriction on $(T_x\,\mathcal{S})^{\perp}$ is the identical map. Spo by inverse function theorem the restriction of the map $\Xi \circ \Phi$ on some neighborhood 0^1 of $\mathcal{S}_0$ in $\mathbb{T}^n \times P \times \mathbb{R}^n$ defines an isomorphism and

$$(\Xi \circ \Phi)^* \omega|_{\mathcal{S}_0} = dp \wedge dq + dy_+ \wedge dy_-$$

Now by the relative Darboux theorem (see Arnold, Givental (1985), Weinstein (1977)) in a neighborhood 0 of $\mathbf{T}^n \times P \times \{0\}$ there exists a change of coordinates V such that

$$V_*|_{T\,\mathcal{S}(M)} = \mathrm{id} \tag{2.4}$$

and $(\Xi \circ \Phi \circ V)^* \alpha = dp \wedge dq + dy_+ \wedge dy_-$. ∎

Proposition 2. If the NVE for H_f along $\mathcal{S}$ is reducible, then in the symplectic cordinates (q,p,y) from Proposition 1

$$f(q,p,y) = f_0(p) + \frac{1}{2} <A(p)y,y> + O(|y|^3) \ . \tag{2.5}$$

Proof. Let us write f(q,p,y) as a series in y :

$$f = f^0(q,p) + f^1(q,p)\circ y + \frac{1}{2} <f^2(q,p)y,y> + O(|y|^3) \ . \tag{2.6}$$

Here f^1 is a vector in $\mathbb{R}^{2m}$ and f^2 is a symmetric linear operator. As the manifold $\mathcal{T} = \{y = 0\}$ is invariant for the vector–field H_f, we have $f^1 \equiv 0$; as the restriction of H_f on $\mathcal{T}$ is the Hamiltonian system with hamiltonian $f_0(p)$, we also have $f^0 = f_0(p)$. The flow of a NVE along $\mathcal{T}_0$ for the system with hamiltonian (2.5) is the one of equations

$$\dot{q} = \nabla f_0(p) \ , \ \ \dot{p} = = \ , \ \ \dot{y} = J f^2(q,p)y \ . \tag{2.7}$$

As $\Xi_*(x,0)|_{(T_x\mathcal{T})^\perp}$ is identical map $\forall x \in \mathcal{T}$ and $V_*(m)$ is identical $\forall x \in \mathcal{T}$, then the map Φ transforms solutions of the system (2.7) into trajectories of the flow $S_{t*}|_{(T\mathcal{T})^\perp}$. Sy by the item 2) of Definition 1 the set of solutions of equations (2.7) is equal to the one of the equation (2.2). Thus $f^2(q,p) \equiv a(p)$. ∎

In what follows for an analytic function g on M we write $g(m) = o(\mathrm{dist}(m,\mathcal{T}))^p$, $p \in \mathbb{Z}$, $p \geq 0$, if in every local chart on M with coordinates $(x_1,\dots,x_{2N})$ we have:

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} g(x) \right| = 0(\mathrm{dist}(x, \mathcal{T} \cap Q)^{p-|\alpha|}) \ \forall\, \alpha \in \mathbb{Z}^{2N} , \quad |\alpha| \le p .$$

Theorem 1. Let $f_1,\dots,f_n$ be analytic functions in some neighborhood of $\mathcal{T}$ such that $f_1 = f$ and

a) $[f_j,f_k](m) = o(\mathrm{dist}(m,\mathcal{T}))^2$, (2.8)

b) $\forall\, \tilde{q} \in \mathbf{T}^n$, $\tilde{p} \in P$ the vectors $H_{f_1}(\tilde{q},\tilde{p}) ,\dots,H_{f_n}(\tilde{q},\tilde{p})$ are linearly independent and are tangent to $T^n_{\tilde{p}} = \{(q,p) \in \mathcal{T} \mid p = \tilde{p}\}$.

Then $\forall\, p_0 \in P$ there exists a neighborhood P_0 of p_0 such that the NVE for H_f along $\mathcal{T}_0 = \mathbb{T}^n \times P_0$ is complex–reducible.

Remark 1. The assumption b) of the theorem results from a) and the following three assumptions:

i) $f_j(m) = o(\mathrm{dist}(m,\mathcal{T}))$,

ii) Hess $f(p) \not\equiv 0$,

iii) $\forall\, \tilde{q},\tilde{p}$ the vectors $H_{f_1}(\tilde{q},\tilde{p}),\dots,H_{f_n}(\tilde{q},\tilde{p})$ are linearly independent.

Indeed, by (i) the submanifold $\mathcal{T}$ is invariant for the flows S^j_t of H_{f_j} for all j and these flows commute on $\mathcal{T}^j$ by a) (see Lemma 1 below). So a set $M_{\tilde{q},\tilde{p}} = U_{t_1,\dots,t_n} S^1_{t_1} \circ \dots \circ S^n_{t_n}(\tilde{q},\tilde{p})$ is invariant for S_t for all $(\tilde{q},\tilde{p})$. This set is n–dimensional by iii) and contains a closure of the trajectory of H_f starting from $(\tilde{q},\tilde{p})$. By ii) the last is equal to $T^n_{\tilde{p}}$ for almst all $\tilde{p}$. So $M_{\tilde{q},\tilde{p}} = T^n_{\tilde{p}} \ \forall\, \tilde{q},\tilde{p}$ and the vector–fields $H_{f_1},\dots,H_{f_n}$ are tangent to T^n_p .

Proof of the theorem. Let S_t^j (j=1,...,n) be the flows of H_{f_j} and S_{t*}^j be the tangent flows on TM. By the assumption b) of the theorem the manifold $(T\mathcal{F})^\perp$ is invariant for S_{t*}^j $\forall j$.

Lemma 1. Restrictions of the flows S_{t*}^j on $(T\mathcal{F})^\perp$, j = 1,2,...,n, commute. In particular, the flows $S_t^j|_{(T\mathcal{F})^\perp}$ commute.

Proof. We shall prove that prove that the restrictions of the flows $(S_t^j)_*$ on $T_{\mathcal{F}}M$ commute. The statement is local and it is enough to prove it in a local chart Q on M with coordinates $(x_1,...,x_{2N})$. Let in this chart

$$H_{f_j} = V = (V^1,...,V^{2N}),\ H_{f_K} = W = (W^1,...,W^{2N})$$

for some $1 \le j,\ k \le n$, and $TV : TM \longrightarrow T(TM)$ be a vector–field of a variational equation for V. Let $(x_1,...,x_{2N}, \xi_1,...,\xi_{2N})$ be coordinates on TQ. Then $TV(x,\xi) = (V(x), \sum \frac{\partial}{\partial x_e} V(x)\ \xi_e)$ and the commutator [TV,tW] of the vector–fields TV, TW is equal to

$$[TV,TW] = (\sum (W^k \frac{\partial}{\partial x_k} - V^k \frac{\partial W}{\partial x_k}) ,$$

$$\sum (\frac{\partial^2 V}{\partial x_j \partial x_k} \xi_j W^k + \frac{\partial V}{\partial x_j} \frac{\partial W^j}{\partial x_k} \xi_k - \frac{\partial^2 W}{\partial x_j \partial x_k} \xi_j V^k - \frac{\partial W}{\partial x_j} \frac{\partial V^j}{\partial x_k} \xi_k)) .$$

The r.h.s. of the last equality is equal to T[V,W]. So

$$[TV,TW] = T[V,W] = TH_{[f_j,f_k]}$$

and

$$[TV|_{T\mathcal{T}M}, TW|_{T\mathcal{T}M}] = TH_{[f_j,f_k]}|_{T\mathcal{T}M}$$

because the commutation of vector–fields is a natural operation with respect to imbedding. By the assumption (2.8) $H_{[f_j,f_k]}(m) = o(\mathrm{dist}(m, \mathcal{T}))$. So the r.h.s. in the last equality is equal to zero, the restrictions of vector–fields TV, TW on $T_{\mathcal{T}}M$ commute and the lemma is proved. ■

Let us fix a point $q_0 \in \mathbb{Z}^n$, $q_0 = 0 \bmod 2\pi\,\mathbb{Z}^n$, and fix some analytic trivialization of the restriction of $(T\,\mathcal{T})^\perp$ on $q_0 \times P$,

$$(T\,\mathcal{T})^\perp|_{q_0\times P} \simeq P \times E \tag{2.9}$$

For $p \in P$, let $(T\,\mathcal{T}_p)^\perp$ be the restriction of $(T\,\mathcal{T})^\perp$ on the torus T^n_P . To prove the theorem, it is enough to trivialize the symplectic bundle $(T\,\mathcal{T}_p)^\perp$ by a map which depends on p in an analytic way, and to check that the restriction of the flow S_{t*} on $(T\,\mathcal{T}_p)^\perp$ is of the form (2.2).

Let $(e_1,\dots,e_n)$ be the usual basis of $\mathbb{Z}^n$ and $\mathfrak{S}^j_t(q,p) = (q + t\,e_j,p)$. By Lemma 1 and assumption b) of the theorem we can see that there exists a nondegenerate analytic matrix $D_{ij}(p)$ such that

$$\prod_{\ell=1}^{n} S_t^{\ell} D_{j\ell}\Big|_{(T\,\mathcal{T}_p)^{\perp}} = \mathfrak{G}_t^j \quad \forall j \tag{2.10}$$

(this is the first step from the classical proof of Liouville–Arnold theorem, see Arnold (1974), Moser, Zehnder (1980)). Let us denote by $\mathfrak{G}^j_{t*}(p)$ the flow on $(T\,\mathcal{T}_p)^{\perp}$,

$$\mathfrak{G}^j_{t*}(p) = \prod_{\ell=1}^{n} S_t^{\ell} D_{je*}, \quad j=1,\dots,n \ .$$

These flows are well–defined by Lemma 1. By (2.9) the monodromy operators $\mathfrak{G}^j_{2\pi*}$, $j=1,\dots,n$, define linear symplectomoprhisms of $(T\,\mathcal{T})^{\perp}_{(q_0,p)} \simeq E$. By Lemma A1 (see Appendix)

$$\mathfrak{G}^j_{2\pi\tau}(p) = e^{2\pi\, B^j(p)} \tag{2.11}$$

Here B^j are some analytic on p linear Hamiltonian operators in the complexification $E^c = E \otimes_{\mathbb{R}} \mathbb{C}$ of E (that is the matrix of B^j in $E^c = \mathbb{C}^n_{y_+} \times \mathbb{C}^n_{y_-}$ is of a form $B^j = J\, B^{j(s)}$; here J is the matrix of the operator $J(y_+,y_-) = (-y_-,y_+)$ and $B^{j(s)}$ is a symmetric matrix). As the operators $\mathfrak{G}^j_{2\pi*}$, $j=1,\dots,n$, commute, their logarithms $B^j(p)$ commute as well (these results, for example, from the representation (A3) for $B^j(p)$). Now we can trivialize the bundle $(T\,\mathcal{T}_p)^{\perp} \otimes \mathbb{C}$ with the help of a map

$$\begin{gathered} \mathbb{T}^n \times \{p\} \times E^c \longrightarrow (T\,\mathcal{T}_p)^{\perp} \otimes \mathbb{C}\ , \\ (q,p,\xi) \longmapsto \prod_{j=1}^{n} \mathfrak{G}^j_{q_j*}(p)(0,p,\prod_{j=1}^{n} e^{-q_j B^j(p)}\xi) \end{gathered} \tag{2.12}$$

The definition of this map is correct because the image does not change if the vector $(q_1,\dots,q_n)$ is replaced by $(q_1,\dots,q_j \pm 2\pi,\dots,q_n)$. It is symplectic because every map $\exp \tau B_j(p) : E^c \longrightarrow E^c$ and every flow $\mathfrak{G}^j_{t*}$ are symplectic. The map (2.12) depends on p in an analytic way because matrices $B^j(p)$ are analytic. Let us define a map Φ in (2.1) in such a way that $\Phi|_{\mathbb{T}^n\times\{p\}\times E^c}$ is equal to the map (2.12).

Let us write for brevity

$$q \circ \vec{\mathfrak{G}}_*(p) = \prod \mathfrak{G}^j_{q_j*}(p)\,, \quad q \circ \vec{B}(p) = \sum q_j B^j(p) \;.$$

From (2.10) we see that

$$S^1_{t*} = t\, D^-_1 \circ \vec{\mathfrak{G}}_* \tag{2.13}$$

here D^-_1 is the first row of the inverse matrix D^{-1}_{ij}. So if under the trivialization (2.12) $(T\,\mathcal{T}_p)^\perp \otimes \mathbb{C} \ni \chi \simeq (q,p,\xi)$ and $S^1_{t*}\,\chi \simeq (q_1,\xi_1)$, i.e. if

$$\begin{array}{ccc} \chi & \xleftarrow{\;\Phi\;} & (q,p,\xi) \\ \downarrow & & \\ S^1_{t*}\,\chi & \xleftarrow{\;\Phi\;} & (q_1,p,\xi_1) \end{array}\;,$$

then $q_1 = q + t\, D^-_1$ and

$$\xi_1 = e^{(q+t\,D^-_1)\cdot\vec{B}} \prod\nolimits_Y \circ ((-t\,D^-_1 - q)\cdot\vec{\mathfrak{G}}_*) \circ (t\,D^-_1\cdot\vec{\mathfrak{G}}_*) \circ$$

$$\circ\,(q \cdot \vec{\mathfrak{G}}_*(0,p,e^{-q\circ \vec{B}}\xi)) = (0,p,e^{t\, D_1^- \cdot \vec{B}}\xi)$$

(here $\prod_Y$ is a projection of $(T\,\mathcal{T}_p)^{\perp} \otimes \mathbb{C})_{q_0} \simeq \mathbb{T}^n \times Y^c$ on Y^c).

So (2.2) holds with $A(p) = D_1^-(p) \cdot \vec{B}(p)$ and the theorem is proved. ■

An "almost inverse" to Theorem 1 statement easily results from Proposition 2:

Proposition 3. If the NVE for H_f along $\mathcal{T}$ is reducible, $p \in P$ and P_0 is a small enough neighborhood of p in P , then in a neighborhood of $\mathcal{T}_0 = \mathbf{T}^n \times P_0$. There are n analytic functions with the properties a), b).

To prove the statement it is enough to write the hamiltonian f in a form (2.5) and to choose $f_1 = f$, and for $j \geq 2$ $f_j(q,p,y) = f_j(p)$, where the vectors $\nabla f_0(p_0)$, $\nabla f_2(p_0),\dots,\nabla f_n(p_0)$ are linearly independent. ■

For the last proposition a natural question is whether the reduction of Theorem 1 can be done in the category of real bundles. This is true if in (2.11) the logarithms $B_j(p)$ of the monodromy operators can be constructed as real matrices. For Lemmas A1, A2 this is true if

$$\sigma(\mathfrak{G}^j_{2\pi *}) \cap (-\infty,0] = \emptyset \;\; \forall\, j \qquad (2.14)$$

(σ = spectrum) or if $\mathfrak{G}^j_{2\pi *}$, $j=1,\dots,n$, are replaced by their squares. The last takes place if the tori T^n_p are replaced by their 2^n–sheets covering

$$T^n_p \longrightarrow T^n_p, \quad q \longmapsto 2q$$

This covering induces a bundle $(T\,\mathcal{T})^{\perp}_{ind}$ with the induced flow $(S_{t*})_{ind}$ in it.

Corollary 1. Under the assumptions of Theorem 1, the bundle $(T\,\mathcal{T})^{\perp}_{ind}$ can be trivialized as a real bundle. For this trivialization the flow $(S_{t*})_{ind}$ is of a form (2.2).

To realize the first possibility let us mention that (2.13) holds if

$$\sigma(\mathfrak{G}^j_{2\pi*}) \in i\,\mathbb{R} \quad \forall j\ . \tag{2.15}$$

Definition 3. An invariant manifold $\mathcal{T}$ is called linearly stable for a vector–field H_f, if all the Liapunov exponents of every solution of H_f on $\mathcal{T}$ are equal to zero.

Lemma 2. Under the conditions of Theorem 1 the assumption (2.15) holds if and only if the invariant manifold $\mathcal{T}$ is linearly stable for every vector–field H_{f_j} $(j=1,\ldots,n)$.

Proof. Let us suppose that $\mathcal{T}$ is linearly stable $\forall\, H_{f_j}$, $j=1,\ldots,n$. Then by the definition of the flows $\mathfrak{G}^j_{t*}(p)$ for every $\epsilon > 0$ there exists C_ϵ such that

$$\|\mathfrak{G}^j_{\pm 2\pi n}(p)\| \le C_\epsilon e^{\epsilon n} \tag{2.16}$$

and so (2.15) is true.

Let us suppose that (2.15) holds. Then (2.16) is true $\forall\, \epsilon > 0$ with some C_ϵ . By

(2.13), (2.16) we see that $\|S^1_{t*}\| \le C^1_\epsilon e^{\epsilon n}$ and the same is true for all S^j_{t*} . So $\mathcal{T}$ is linearly stable for all H_{f_j} . ■

Theorem 2. Suppose the invariant manifold $\mathcal{T}$ is linearly stable for H_f and for some $p_0 \in P$ $H_f(p_0) \neq 0$. Then the NVE of H_f along $\mathcal{T}_0 = \mathbf{T}^n \times P_0$ (P_0 is a small enough neighborhood of p_0 in P) is reducible if and only if there are analytic functions $f_1,...,f_n$ such that $f_1 = f$ and the assumptions a), b) of Theorem 1 are fulfilled for $\mathcal{T} = \mathcal{T}_0$, together with

c) $\mathcal{T}_0$ is linearly stable for all H_{f_j} , $j = 1,...,n$.

In such a case the spectrum of the operator J A8p) is pure imaginary.

Proof. If the NVE is reducible then we can construct the functions $f_q,...,f_n$ as in Proposition 3. The manifold $\mathcal{T}_0$ is linearly stable for all H_{f_j} trivially.

Suppose now that the assumptions a)–c) are fulfilled. Then by Lemma 2 the assumption (2.15) holds and by Lemma A1 the matrices $B_j(p)$ (and, so, the trivialization Φ) can be real choosen. The last statement of the theorem is trivial because a system of the form (2.2) is linearly stable if and only if the spectrum of J A(p) is pure imaginary. ■

Remarks. 2) Propositions 1, 2 and Theorems 1, 2 have direct smooth versions with the same proofs.

3) Our proof of Theorems 1, 2 (but not of Lemmas A1, A2) does not use the finite dimensionality of the fibers of the bundle $(T\,\mathcal{T})^\perp$. If in (2.9) dim A = ∞ and we have sufficient spectral information on the flows S^j_{t*} and can construct "regular" logarithms $B_j(p)$ of the monodromy operators $\mathfrak{S}^j_{2\pi*}$ (see (2.11)), then our proof is valid.

4) The reducibility of the NVE along $\mathcal{T}_0$ was proved via its reducibility along the tori $\{(q,p) \in \mathcal{T} \mid p = \text{const}\}$. So the proof can be used for proving the reducibility of a linear Hamiltonian equation

$$\dot{q} = \omega , \quad \dot{y} = J\, A(q)y \quad (q \in \mathbb{T}^n, y \in Y)$$

to a constant–coefficient Hamiltonian equation $\dot{y} = J\, \bar{A}\, y$ by means of symplectic transformation $y = C(q)y$. This reduction is possible if in the phase space $\mathbb{T}^n \times \mathbb{R}^n \times Y$ there are functions $f_j(q,p,y)$ $(j = 1,2,\dots,n)$ of the form $f_j = \omega_j \cdot p + \frac{1}{2} <A_j(q)y,y>$ such that $\omega_1 = \omega$, $A_1 = A$, $\det(\omega_1^t,\omega_2^t,\dots,\omega_n^t) \neq 0$ and $\forall\, j,k$

$$\tfrac{1}{2}(\omega_j \cdot \nabla)A_k(q) - \tfrac{1}{2}(\omega_k \cdot \nabla)A_j(q) + A_k(q)J\, A_j(q) - A_j(q)J\, A_k(q) \equiv 0 \ .$$

5) In the special case $n=1$ we need no "infinitesimal integrals" other than $f_1 = f$, and the assumptions a), b) of Theorem 1 are fulfilled in a trivial way. For $n = 1$ Theorem 1 + Corollary 1 coincide with the Floquet theorem (see Arnold, Givental (1985)). For a less trivial example, see § 4 below.

3. Elliptic case.

Definition 4. The invariant manifold $\mathcal{T}$ is called weakly elliptic if the NVE of H_f along $\mathcal{T}$ is reducible and operator $J\, A(p)$ in (2.2) has pure imaginary spectrum $\{\pm\, i\, \lambda_j(p)\}$. $\mathcal{T}$ is called elliptic if it is weakly elliptic and operator $J\, A(p)$ is semi-simple (i.e. is diagonal in some complex symplectic basis) $\forall p \in P$.

One can treat Theorem 2 as a weak ellipticity criterion.

Clearly, submanifold $\mathcal{T}$ is elliptic if it is weakly elliptic and $\lambda_j(p) \neq \lambda_k(p)$ for $j \neq k$.

Remark 6. Finite–dimensional elliptic invariant submanifold of infinite codimension appear in the study of nonlinear partial differential equations which are integrable in terms of theta–functions. See Kuksin (1989), § 4.

For an elliptic invariant submanifold $\mathcal{T}$ the spectrum $\{\pm i\, \lambda_j(p)\}$ is not defined in an unique way:

Proposition 4. Let the submanifold $\mathcal{T}$ is elliptic and the flow $S_t|_{\mathcal{T}}$ is nondegenerate:

$$\det \partial\omega | \partial p \not\equiv 0\ , \quad \omega(p) = \nabla f_0(p)\ . \tag{3.1}$$

Let us consider some another trivialisation of S_{t*} with Φ' and A' in (2.1), (2.2) instead of Φ and A. Let $\sigma(J\, A'(p)) = \{\pm i\, \mu_j(p)\}$. Then for every j there exist $k = k(j)$, $s = s(j) \in \mathbb{Z}^n$ such that

$$\mu_j(p) = \lambda_k(p) + s \cdot \omega(p) \quad \forall\, p\ . \tag{3.2}$$

Moreover, every n numbers of the form $\mu_j(p) = \lambda_j(p) + s_j(p) \cdot \omega(p)$, $s_j \in \mathbb{Z}^n$, may be achieved as a spectrum of a Hamiltonian operator $J\, A'(p)$ for some trivialisation Φ'.

Proof. Let $\{\varphi_j^{\pm}(p)\}$, $\varphi_j^{-}(p) = \overline{\varphi_j^{+}}(p)$, be symplectic basic of Y^c,

$J\,A(p)\,\varphi_j^{\pm} = \pm\, i\,\lambda_j(p)\,\varphi_j^{\pm}$. Then the mapping $\Phi' : \mathbf{T}^n \times P \times Y \longrightarrow (T\,\mathcal{J})^{\perp}$, which maps $(q,p,\varphi_j^{\pm})$ to $\exp(\mp\, s_j, a)\,\varphi_j^{\pm}$, transforms the flow S_{t*} into a flow of an equation (2.2) with an operator $A'(p)$ such that

$$J\,A'(p)\,\varphi_j^{\pm} = \pm\, i(S_j \cdot \omega(p) + \lambda_j(p))\varphi_j^{\pm} \;.$$

Thus the second statement is proved.

To prove the first one let us mention that $\Phi^{-1} \circ \Phi'(q+\omega t, p, e^{i\mu_j t}\varphi_j'^{+})$ is a solution of (2.2) (here $J\,A'\,\varphi_j'^{\pm} = \pm\, i\,\mu_j \varphi_j'^{\pm}$) . Let $\Phi^{-1} \circ \Phi'(q,p,\varphi_j'^{\pm}) = \sum x_k^{\pm}(q,p)\varphi_k^{\pm}$. Then the solution may be rewritten as $\exp(i\,\mu_j t)\sum x_k^{\pm}(q+\omega t,p)\varphi_k^{\pm}$. So

$$i\,\mu_j\, x_k^{\pm} + \frac{\partial}{\partial \omega} x_k^{\pm} = \pm\, \lambda_k\, x_k^{\pm} \quad \forall\, k \;. \tag{3.3}$$

Among the functions $x_k^{\pm}$ there are nonzero ones. Let us suppose that $x_{k_0}^{+}(q,p) \not\equiv 0$. For (3.1) the components of the vector $(\omega_1,\dots,\omega_n)$ are rationally independent for almost all $p \in P$. Then by (3.3) $x_{k_0}^{+} = C(p)\exp i\, S \cdot q$ for some $s \in \mathbb{Z}^n$ and $\mu_j = \lambda_{k_0} - s \cdot \omega$. Thus the first assertion is proved, too.

Let us consider a family of subgroups of additive groupe $\mathbb{Z}$ of a form $\omega(p) \cdot \mathbb{Z}^n$, $p \in P$, and corresponding factor groups $G(p) = \mathbb{Z}/\omega(p) \cdot \mathbb{Z}^n$. For a weakly elliptoc submanifold $\mathcal{J}$ let us define elements $\Lambda_1(p),\dots,\Lambda_n(p)$ of $G(p)$ as follows:

$$\Lambda_j(p) = \lambda_j(p) + \omega(p) \cdot \mathbb{Z}^n \in G(p) \;. \tag{3.4}$$

The following definition is motivated by Proposition 4:

Definition 5. If $\mathcal{T}$ is an weakly elliptic invariant submanifold, then the depending on $p \in P$ set

$$\Lambda(p) = \{\Lambda_1(p),\dots,\Lambda_n(p)\} \subset G(p)$$

is called spectrum of $\mathcal{T}$.

The important reason to prove the reducibility of the NVE is Proposition 2 which provide a hamiltonian $f(q,p,y)$ with the useful normal form (2.5). For nondegenerate hamiltonians of the form (2.5) (see the condition (3.5) below) one can prove that the family $\mathcal{T} = \Sigma_0(\mathbb{T}^n \times P)$ of invariant tori $\Sigma_0(\mathbb{T}^n \times \{p\})$, $p \in P$, is KAM–stable in the following sense:

Definition 6. A family of invariant tori $\mathcal{T} = \Sigma_0(\mathbb{T}^n \times P)$ of the Hamiltonian vector–field H_f is called KAM–stable if for an arbitrary analytic function Υ and for ϵ small enough, the vector–field $H_{f+\epsilon\Upsilon}$ has an invariant set $\mathcal{T}_\epsilon = \Sigma_\epsilon(\mathbb{T}^n \times P_\epsilon)$. Here

1) P_ϵ is a Cantor–set in P and

$$\mathrm{mes}(P\setminus P_\epsilon) \longrightarrow 0 \quad (\epsilon \longrightarrow 0) \ ,$$

2) the map $\Sigma_\epsilon : \mathbb{T}^n \times P_\epsilon \longrightarrow M$ is Lipschitz and it is ϵ–close to $\Sigma_0|_{\mathbb{T}^n \times P_\epsilon}$;

3) the tori $\Sigma_\epsilon(\mathbb{T}^n \times \{p\})$, $p \in P_\epsilon$, are invariant for the vector–field $H_{f+\epsilon\Upsilon}$.

To prove the KAM–stability one has to apply a theorem on perturbation of a linear system (see Eliasson (1988), Kuksin (1989), Pöschel (1989)) to the vector–field H_f with f in the form (2.5) after a simple space–dilation (see Kuksin (1989), § 1). In such a way we get the following result:

Theorem 3. Suppose the invariant manifold $\mathcal{T}$ is weakly elliptic for the NVE of H_f and for the spectrum $\{\Lambda_j(p) \mid j=1,\dots,n\}$ of NVE we have:

$$\Lambda_j(p) \not\equiv 0 \ \ \forall j, \quad \Lambda_j(p) \not\equiv \Lambda_k(p) \ \ \forall j \neq k \tag{3.5}$$

Then Λ is KAM–stable.

Remark 7. There is a natural smooth version of Theorem 3. In order to prove it one has to write down a smooth version of perturbation theorem for lower–dimensional invariant tori using usual smoothing techniques of J. Moser. Clearly it is possible but this work still has not been done.

Remark 8. In order to prove KAM–stability of $\mathcal{T}$ via a smooth version of the arguments (see Remark 6) it is enough to prove "KAM–reducibility" of NVE. That is, for every $\delta > 0$ we must be able to find a smooth trivialisation (2.1) such that inthe equation (2.2) the matrix A8p) does not depend on q if p lies out of some Cantor set of measure δ.

Remark 9. In Johnson, Sell (1981) the hyperbolic situation was considered. It was proved that if normal bundle $(T\,\mathcal{T})^{\perp}$ is trivial and the flow S_{t*} has full Sacker–Sell spectrum then NVE is KAM–reducible and $\mathcal{T}$ as a family of "doubled tori" is KAM–stable.

4. Example.

Let $N = n+1$ and suppose the symplectic Riemann manifold M is polarizable. Then the bundles TM and $T\mathcal{T}$ are trivial and so the bundle $(T\mathcal{T})^{\perp}$ is also trivial. This results from the fact that the symplectic bundles TM, $T\mathcal{T}$, $(T\mathcal{T})^{\perp}$ can be given complex structures (see Arnold (1978), Arnold, Givental (1985)) and that a one–dimensional complex bundle which is a factor–bundle of a trivial complex bundle is trivial (see Hirzebruch (1966)). So by the Proposition 1, in a neighborhood of $\mathcal{T}$ in M there are symplectic coordinates (q,p,y) $(q \in \mathbb{T}^n, p \in P \subset \mathbb{R}^n, y = (y_+,y_-) \in 0 \subset \mathbb{R}^2)$ and $\mathcal{T} = \{y = 0\}$. In this coordinates the hamiltonians $f_1,\dots,f_n$ we are looking for, can be written in the form

$$f_j(q,p,y) = f_j^0(p) + \tfrac{1}{2} \langle A_j(q,p)y,y\rangle + O(|y|^3) .$$

So $[f_j,f_k] = \langle \mathfrak{A}_{jk}(q,p)y,y\rangle + O(|y|^3)$,

$$\mathfrak{A}_{jk} = \tfrac{1}{2} (\nabla_p f_j^0 \cdot \nabla_q A_k - \nabla_p f_k^0 \cdot \nabla_q A_j) + A_k J A_j - A_j J A_k .$$

Let us denote

$$\nabla_p f_j^0 = \omega_j(p) = (\omega_{j,1},\dots,\omega_{j,n}) \in \mathbb{R}^n ,$$

$$\mathfrak{A}_j(p) = J A_j(p) \in s\ell(2) = s\ell(2,\mathbb{R}) .$$

The assumptions of Theorem 1 are fulfilled if we can construct the functions

$f_1,\dots,f_n$ and the matrices $\mathfrak{A}_1,\dots,\mathfrak{A}_n \in s\ell(2)$ in such a way that $f_1 = f_0$ and $\mathfrak{A}_1 = J\,A_0$ ($f_0,\mathfrak{A}_0$ are given), for every $p \in P$ the vectors $\omega_1(p),\dots,\omega_n(p)$ span $\mathbb{R}^n$, and

$$J\,\mathfrak{A}_{jk} \equiv (\omega_j\cdot\nabla_q)\mathfrak{A}_k - (\omega_k\cdot\nabla_q)\mathfrak{A}_j + [\mathfrak{A}_k,\mathfrak{A}_j] \equiv 0 \ . \tag{4.1}$$

$$\forall\, j,k = 1,\dots,n \ .$$

Let us suppose that Hess $f_0(p_c) \neq 0$ and, so, near p_0 the map $p \longmapsto \omega = \nabla f_0(p)$ is invertible. Then to prove KAM–reducibility of NVE (see Remark 8), for Remark 4 we have to construct smooth vectors $\omega_j = I_j(i)$ and smooth matrices $\mathfrak{A}_j(q,\omega)$ $(j = 2,\dots,n)$ which solve equations (4.1) with $\omega_1 = \omega$, $\mathfrak{A}_1 = \mathfrak{A}_1(q,p)$, $\nabla f_0(p) = \omega$, for ω out of a set of small measure δ in such a way that

$$\det(\omega_1^t,\dots,\omega_n^t) \neq 0 \ . \tag{4.2}$$

In particular, if $n=2$, then we have to find a vector ω_2 and a matrix $\mathfrak{A}_2(q) \in s\ell(2)$ such that

$$\begin{gathered}(\omega_1\cdot\nabla_q)\mathfrak{A}_2-(\omega_2\cdot\nabla_q)\mathfrak{A}_1 + [\mathfrak{A}_2,\mathfrak{A}_1] = 0 \ , \\ \det(\omega_1^t,\omega_2^t) \neq 0\end{gathered} \tag{4.3}$$

(the last relation excludes the trivial solution $\mathfrak{A}_2 = \lambda\mathfrak{A}_1$, $\omega_2 = \lambda\omega_1$).

The equation (4.3) is the equation of zero curvature (see Faddeev, Takhtajan (1987)) with non–standard periodicity conditions (that is, the periodicity is not with respect to the directions ω_1, ω_2, but to some other directions). Well–known gauge transformations

$$\mathfrak{A}_2 \longrightarrow (\omega_2 \cdot \nabla_q) G\, G^{-1} + G\, \mathfrak{A}_2 G^{-1}$$

$$\mathfrak{A}_1 \longrightarrow (\omega_1 \cdot \nabla_q) G\, G^{-1} + G\, \mathfrak{A}_1 G^{-1}$$

($G = G(q)$ is an analytic symplectic matrix) transforms solutions of (4.3) into new ones. It provides a means to construct new solutions of (4.3) from the trivial ones.

Clearly, the equation (4.3) cannot be solved for arbitrary analytic $\mathfrak{A}_1(q) \in sl(2)$ for all $\omega_1 \in \nabla f_1^0(P)$ because some NVE with $N = n+1$ are not reducible (at least for some Liouvilleau frequencies ω_1, see Johnson (1979), Herman (1983)). Nevertheless we have the following

Conjecture. If the matrix $\mathfrak{A}_1(q)$ is analytic then $\forall\, \delta > 0$ there exist a smooth matrix $\mathfrak{A}_2$ and a smooth vector ω_2 which solve (4.3) fro ω_1 out of some set of measure δ.

If it is true then NVE is KAM–reducible and $\mathcal{T}$ is KAM–stable (see Remark 8).

Appendix. On logarithms of analytic symplectic matrices.

Let C_p, $p \in P$, be a symplectic matrix of order $2m$ analytic on p. Let for some $p_0 \in P$ the matrix C_{p_0} be reversible.

Lemma A1. There exist a neighborhood P_0 of p_0 and an analytic complex matrix B_p , $p \in P_0$, which is a branch of $L_n C_p$:

$$\exp B_p = C_p \quad . \tag{A1}$$

The matrix is Hamiltonian:

$$(JB_p) = (JB_p)^t \tag{A2}$$

and may be choosen real if the spectrum $\sigma(C_{p_0})$ contains no real negative points.

Proof. As $\sigma(C_{p_0}) \not\ni 0$, there exists a contour $\Gamma \subset \mathbb{C}$ such that $\sigma(C_{p_0})$ lies inside Γ and 0 lies outside Γ . The same is true for $\sigma(C_p)$, $p \in P_0$ if P_0 is small enough. For $\lambda \in \Gamma$ let us fix some branch $\ell n\, \lambda$ of $\mathrm{Ln}\, \lambda$ and set

$$B_p = \frac{1}{2\pi i} \oint_\Gamma \frac{\ell n\, \lambda}{C_p - \lambda}\, d\lambda \quad . \tag{A3}$$

Then $\exp B_p = C_p$ (see Dunford, Schwartz (1958), Ch. VII) and so (A1) is proved. To prove (A2) let us mention that

1) the operator B_p in (A3) depends on C_p in a continuous way;

2) single–spectrum symplectic matrices are dense among symplectic matrices;

3) single–spectrum symplectic matrix is diagonal in some symplectic basis and for it (A2) is evident.

So it remains to prove the last statement. It is well–known (Arnold (1974), Arnold, Givental (1985)) that for an invertible symplectic matrix C , the spectrum $\sigma(C)$ consists of pairs of points λ, λ^{-1} $(\lambda \in \mathbb{R})$; pairs of points λ, λ^{-} $(|\lambda| = 1)$ and quadruples $\lambda, \bar\lambda, \lambda^{-1}, \bar\lambda^{1}$ $(\lambda \in \mathbb{C}\backslash\mathbb{R}, |\lambda| \neq 1)$. So in the present situation $\sigma(C_{p_0}) = S_1 \cup (S_2 \cup \bar S_2)$, where $S_1 = \{\lambda_j\} \subset \mathbb{R}_+$, $S_2 = \{\mu_j\} \subset \{\lambda | \operatorname{Im} \lambda > 0\}$. Let us take a (nonconnected) contour Γ_0 of the form $\Gamma_0 = \bigcup_{\lambda_j} \Gamma(\lambda_j) \bigcup_{\mu_j} (\Gamma(\mu_0) \cup -\bar\Gamma(\mu_j))$. Here $\Gamma(\lambda_j)$, $\Gamma(\mu_j)$ are small circles centered at λ_j , μ_j (thus $\Gamma(\lambda_j) = -\bar\Gamma(\lambda_j) \ \forall \lambda_j$) . We can do it in such a way that $\Gamma_0 \cap (-\infty,0] = \phi$, and so we can take for $\ln z$ a branch of Ln z which is real for $\lambda \in \mathbb{R}_+$. With such a choice of Γ in (A3) one can see in a trivial way that $B_p = \overline{B_p}$. ∎

Lemma A2. Under the assumptions of Lemma A1 there exists an analytic real Hamiltonian matrix $\tilde B_p$, $p \in P_0$ such that $\exp \tilde B_p = C_p^2$, $p \in P_c$.

Proof. Let $\tilde\Gamma \in \mathbb{C}$ be a contour containing all negative eigenvalues of C_p and no other eigenvalues. Then for $p \in P_0$ (P_0 is small enough there exists a smooth splitting $\mathbb{R}^{2m}$ into two invariant for C_p symplectic subspaces, $\mathbb{R}^{2m} = E_1 \oplus E_2$, such taht the spectrum of $C_p|_{E_1}$ is negative and lies inside $\tilde\Gamma$ and the spectrum of $C_p|_{E_2}$ lies outside $\tilde\Gamma$ and out of $(-\infty,0]$. Then by Lemma A1 $C_p|_{E_2} = \exp B_p^2$ for some real Hamiltonian operator $B_p^{(2)}$, and $(C_p|_{E_2} = \exp B_p^{(2)}$ for some real Hamiltonian operator $B_p^{(2)}$, and $(C_p|_{E_1})^2 = \exp B_p^{(1)}$. This operator has all the properties we need.

∎

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