Trace Functionals of the Kontsevich Quantization

Alexander Golubev
Max-Planck-Institute for Mathematics
Vivatsgasse 7, 53111 Bonn, Germany
e-mail address: golubev@mpim-bonn.mpg.de

January 11, 2000

Abstract: We generalize the notion of trace to the Kontsevich quantization algebra and show that for all symplectically realizable Poisson manifolds with a nilpotent gauge group the trace is given by integration with respect to a unimodular volume form.

1 Introduction

The gist of deformation quantization of a Poisson manifold lies in modifying the usual (commutative) product of functions to obtain a new operation (star product) in such a way that the first approximation is given by the Poisson bracket. This approach was first proposed in [BFFLS], and is at the moment one of the most developed methods to relate "classical" data represented by a Poisson manifold to a "quantum" associative algebra. With the advent of the Kontsevich star product in 1997 (see [Ko]), when the problem of existence of these algebras for arbitrary Poisson structures has been solved in full generality, it became abundantly clear that the utility of deformation quantization hinges on whether the resulting algebras are amenable to the standard operator theory tools, notwithstanding the fact that no actual operators are present. If this universal version of quantization is to be useful, some "spectral" properties must appear. Sure enough, on the infinitesimal level there is a Poisson trace, whose existence is intimately linked with the modular class (or, in mundane terms, with the existence of a unimodular measure - the one invariant with respect to all Hamiltonian flows) as demonstrated by Weinstein in [W2]. Another bit of information supporting the idea of the Kontsevich quantization algebras having spectral properties is the fact that on $\mathbb{R}^d$ any constant coefficient Poisson structure induces a Kontsevich quantized algebra with a trace given by integration.

On the other hand, considering all possible star products on symplectic manifolds, Connes et al. [CFS] were able to identify obstructions to the existence of a trace on a fixed star-product algebra. These turned out to be certain cocycles in the cyclic cohomology of the Hochshild complex. Moreover, the authors constructed a simple example where integration does not yield a trace functional. Combined with the fact that the Liouville measure on a symplectic manifold is unimodular *par excellence*, the results of Connes et al. permit one to conclude that unimodularity alone is not sufficient even in the nicest possible (symplectic) setting. Although later Fedosov [F] showed that on an arbitrary symplectic manifold there is a star-product quantization equipped with a trace given by integration, the following question remains: Is there a trace functional on the Kontsevich quantization algebra of an arbitrary Poisson manifold? The present paper aims to single out a class of Poisson manifolds that possess a trace functional on the Kontsevich quantization algebra. In the process we first briefly recall the essentials in Section 2, then study the Kontsevich quantization of the dual of nilpotent Lie algebras endowed with the standard Lie-Poisson structure (Section 3), go on to describe the properties of Morita equivalent Poisson manifolds in Section 4, and finally get everything together in Section 5.
Aknowledgments. We would like to thank Joseph Donin for many valuable suggestions, and generosity in sharing his insights.

2 Star Products

Given a smooth ($\mathcal{C}^\infty$) Poisson manifold (P, π), the set of smooth functions $\mathcal{C}^\infty(P)$ can be viewed as a commutative algebra. The star product on $\mathcal{C}^\infty(P)$ (c.f. [BFFLS]) is an associative $\mathbb{R}[[\hbar]]$-linear product on $\mathcal{C}^\infty(P)[[\hbar]] = \mathbb{Q}(P)$ expressed by the following formula for $f, g \in \mathcal{C}^\infty(P) \subset \mathbb{Q}(P)$:

$$(f,g) \mapsto f \star g = fg + \hbar\pi(f,g) + \hbar^2 B^2(f,g) + ... \in \mathbb{Q}(P),$$

where $\hbar$ is a formal variable, and B^i are bidifferential operators (bilinear maps $\mathbb{Q}(P) \times \mathbb{Q}(P) \longrightarrow \mathbb{Q}(P)$ which are differential operators of globally bounded order with respect to each argument). The product of arbitrary elements of $\mathbb{Q}(P)$ is defined by the condition of linearity over $\mathbb{R}$ and $\hbar$-adic continuity:

$$(\sum_{n\geq 0} \hbar^n f_n) \star (\sum_{n\geq 0} \hbar^n g_n) \; := \; \sum_{k,l\geq 0} \hbar^{k+l} f_k g_l \; + \sum_{k,l\geq 0,\; m\geq 1} \hbar^{k+l+m} B^m(f_k, g_l).$$

Now we proceed to give a brief account of the universal deformation quantization of $\mathbb{R}^d$ or a domain thereof due to Kontsevich [Ko]. From this point on, the symbol $\star$ refers to the canonical Kontsevich star product. In order to describe the terms proportional to $\hbar^n$ for any integer $n \geq 1$, Kontsevich introduced a special class G_n of oriented labeled graphs called *admissible graphs*.

Definition 1 *An oriented graph Γ is admissible ($\Gamma \in G_n$, $(n \geq 1)$) if:*

1. *Γ has $n+2$ vertices labeled $\{1, 2, ...n, L, R\}$ where L and R stand for Left and Right respectively, and Γ has $2n$ oriented edges labeled $\{i_1, j_1, i_2, j_2, ..., i_n, j_n\}$;*
2. *The pair of edges $\{i_m, j_m\}, 1 \leq m \leq n$ starts at vertex m;*
3. *Γ has no loops (edges starting at some vertex and ending at that same vertex) and no parallel multiple edges (edges sharing the same starting and ending vertices).*

The class G_n is finite. For $n \geq 1$ the first edge i_k starting at vertex k has $n+1$ possible ending vertices since there are no loops, while the second edge j_k has only n possible "landing sites" since there is no parallel multiple edges. Thus there are $n(n+1)$ ways to draw the pair of edges starting at some vertex and therefore G_n has $n(n+1)^n$ elements. For $n = 0$, G_0 has only one element: the graph having $\{L, R\}$ as set of vertices and no edges. Of all admissible graphs, there is an important particular subclass, which we call *the class of unions of subgraphs*. This notion will be used in the sequel. Here is how we define the elements of this subclass:

Definition 2 *A graph $\Gamma \in G_r$ is the union of two subgraphs Γ_1 and Γ_2 with $r_1 + r_2 = r$, if the subset $(1, ..., r)$ of the set of vertices of Γ can be split into two parts $(a_1, ..., a_{r_1})$ and $(b_1, ..., b_{r_2})$ such that there is no edge between these two subsets of vertices.*

A bidifferential operator $(f, g) \mapsto B_\Gamma(f, g)$, $f, g \in \mathcal{C}^\infty(\mathbb{R}^d)$ is associated to each graph $\Gamma \in G_n$, $n \geq 1$. To each vertex k, $1 \geq k \geq n$ one associates the components $\pi_{i_k j_k}$ of the Poisson structure, f is associated to the vertex L

and g to the vertex R. Each edge such as i_k acts by partial differentiation with respect to x_{i_k} on its ending vertex. For $n = 0$ we simply have the usual product of f and g. In what follows we use the words "graph" and "bidifferential operator encoded by the graph" interchangeably.

Now we go on to describe the weights w_Γ. Let $\mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ be the upper half-plane. $\mathcal{H}_n$ will denote the configuration space $\{z_1, ..., z_n \in \mathcal{H} | z_i \neq z_j \text{ for } i \neq j\}$. $\mathcal{H}_n$ is an open submanifold of $\mathbb{C}_n$. Let $\phi : \mathcal{H}_2 \to \mathbb{R}/2\pi\mathbb{Z}$ be the function:

$$\phi(z_1, z_2) \;=\; \frac{1}{2\sqrt{-1}} \mathrm{Log} \frac{(z_2 - z_1)(\bar{z}_2 - z_1)}{(z_2 - \bar{z}_1)(\bar{z}_2 - \bar{z}_1)}.$$

$\phi(z_1, z_2)$ is extended by continuity for $z_1,\ z_2 \in \mathbb{R},\ z_1 \neq z_2$.

For a graph $\Gamma \in G_n$, the vertex k, $1 \leq k \leq n$ is associated with the variable $z_k \in \mathcal{H}$, the vertex L with $0 \in \mathbb{R}$, and the vertex R with $1 \in \mathbb{R}$.

The weight w_Γ is defined by integrating an $2n$-form over $\mathcal{H}_n$:

$$w_\Gamma \;=\; \frac{1}{n!(2\pi)^{2n}} \int_{\mathcal{H}_n} \bigwedge_{1 \leq k \leq n} (d\phi(z_k,\ I_k) \wedge d\phi(z_k,\ J_k)),$$

where I_k (J_k) denotes the variable or real number associated with the ending vertex of the edge i_k (j_k). It is showed in [Ko] that this integral is absolutely convergent. As one can see, the weights do not depend on the Poisson structure or the dimension of the underlying manifold. Combining these constructions Kontsevich proved

Background Theorem 1 (Kontsevich) *Let π be a Poisson structure in a domain of $\mathbb{R}^d$. The formula*

$$f \,\star\, g := \sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma,\pi}(f, g)$$

defines an associative product up to gauge equivalence.

Furthermore, all the machinery above as well as Background Theorem 1 generalize to arbitrary Poisson manifolds using localization. There are no cohomological obstructions.

Finally, we deal with trace functionals on star-product algebras. Consider the linear map Tr on the compactly supported Kontsevich quantization algebra $\mathbb{Q}_c(P)$ with values in formal Laurent power series $\hbar^{-[d/2]}\mathbb{R}[[\hbar]]$:

$$\mathsf{Tr}_{P,\mu}(F) \;=\; \hbar^{-[d/2]} \int_P F\mu,$$

where [ ] denotes the integral part of a number and d is the dimension of P. When P is symplectic this functional (up to a contant) coincides with the usual one as defined in [CFS], provided μ is the Liouville volume form. In general, we may have many meaningful choices for the volume form, which is emphasized by the subscript. Following Fedosov [F] we declare that the functional Tr satisfies the trace property with respect to the Kontsevich star product if

$$\mathsf{Tr}(F \star G) \ = \ \mathsf{Tr}(G \star F).$$

Consequently, the problem of defining a trace functional amounts to proving that a linear functional Tr satisfies the trace property. Formally, we have

Definition 3 *The linear functional* Tr *on* $\mathbb{Q}_c(P)$ *is called a trace functional if it satisfies the trace property.*

3 Quantization of $\mathfrak{g}^*$

Given a finite-dimensional Lie group G, we denote its Lie algebra by $\mathfrak{g}$, the dual of $\mathfrak{g}$ by $\mathfrak{g}^*$. The structure constants of $\mathfrak{g}$ are determined by the set of relations with respect to a basis of $\mathfrak{g}$:

$$[X_i, X_j] \ = \ X_k C_{ij}^k, \ \ X_i \in \mathfrak{g} \ \ \forall i \in \{1, ..., d\}.$$

Now we let $(x_1, ..., x_d)$ denote linear coordinates on $\mathfrak{g}^*$. Then the natural Lie-Poisson structure

$$\pi \ = \sum_{i,j} \pi_{ij} \partial_i \wedge \partial_j,$$

is expressed in terms of coordinates via

$$\pi_{ij} \ = \ x_k C_{ij}^k.$$

We begin by looking at the role of unimodularity in vanishing of the Poisson trace.

Proposition 1 *For any* $f, \ g \in \mathcal{C}_c^\infty(\mathfrak{g}^*)$,

$$\int_{\mathbb{R}^d} \pi(f, g) dx_1 \wedge ... \wedge dx_d \ = \ 0$$

if and only if

$$\sum_i C_{ij}^i \ = \ 0 \ \ \forall j \in \{1, ..., d\}.$$

Proof. We proceed by making a simple observation: $C^k_{ij}\partial_i\partial_j f$ adds up to zero by virtue of C^k_{ij} being antisymmetric. Now straightforward integration by parts of a single component $\pi_{ij}\partial_i f\partial_j g$ with subsequent summation does the trick. Q.E.D.
Remark. Proposition 1 is basically a rephrasing of the statement made in [W2], according to which "on the dual of a Lie algebra $\mathfrak{g}$, with its Lie-Poisson structure, the modular vector field with respect to any translation-invariant density is the constant vector field with value $\mathsf{Tr}\ ad$, the trace of the adjoint representation".

Now we are in a position to state the main theorem of this section.

Theorem 1 *For all $F, G \in \mathbb{Q}_c(\mathfrak{g}^*)$ the equality*

$$\int_{\mathbb{R}^d} F\star G dx_1\wedge ...\wedge dx_d \;=\; \int_{\mathbb{R}^d} G\star F dx_1\wedge ...\wedge dx_d$$

holds for $\mathfrak{g}$ nilpotent.

Proof. First of all we note that this Theorem will follow if we manage to show the equality above for $f, g \in \mathcal{C}^\infty_c(\mathfrak{g}^*)$. Next we invoke the formula (due to Cattaneo and Felder [CaFe]) relating the weight of $B^n(f,g)$ to that of $B^n(g,f)$. Namely, $w_{B^n(f,g)} = (-1)^n w_{B^n(g,f)}$. This signifies that for all even n the corresponding integrals do not affect the (presumed) trace property. From now on we work only with the graphs involving odd n. To manipulate the operators effectively, we introduce the following notation: $B^n_k(f,g)$ stands for a bidifferential operator that has order n in the star product formula, and, in addition, has exactly k edges acting summarily on f AND g. Nothing else is specified, but below we take one particular operator at a time so that this apparent ambiguity does not matter. Now, by the linearity of π, it follows that $n \le k \le 2n$. An important fact used later is that any graph denoted by $B^n_n(f,g)$ necessarily includes at least one loop (an expression of the form $\partial_{i_m}\pi_{i_1j_1}\partial_{i_1}\pi_{i_2j_2}...\partial_{i_{m-1}}\pi_{i_mj_m}$ after possible reshuffling). As Kathotia [Ka] showed, such graphs vanish for nilpotent Lie algebras due to a theorem of Varadarajan [V]:

Background Theorem 2 (Varadarajan, Theorem 3.5.4.) *Let $\mathfrak{g}$ be a d-dimensional nilpotent Lie algebra over a field of characteristic zero. Then there is a basis $\{X_1, ..., X_d\}$ of $\mathfrak{g}$ such that the structure constants C^k_{ij} defined by $[X_i, X_j] = X_k C^k_{ij}$ satisfy*

$$C^r_{ij} \;=\; 0,\;\; \forall r \ge min(i,j).$$

An easy corollary to the above theorem assures that any cyclic product of the form $C^{j_m}_{i_1 j_1} C^{j_1}_{i_2 j_2} ... C^{j_{m-1}}_{i_m j_m}$ $(m > 1)$ vanishes and this, in turn, yields the vanishing of loop graphs.

Now since we have seen that in the nilpotent setting there are no loop graphs, the idea is to rewrite the integrals of B^n_k as combinations of those of B^n_n. This is accomplished via integration by parts. Once again using the linearity of the Poisson structure, we have a vertex such that one outgoing edge acts on f, whereas the other acts on g (otherwise $k = n$ and we are done). We use one of these edges to perform integration by parts. As a result we obtain a few integrals of admissible graphs of the form B^n_{k-1}. To see that we first note that all nilpotent groups are unimodular. Indeed, if there is a $C^i_{ij} \neq 0$ with respect to any basis, we would have an X_i such that $[X_i, X_j] = X_i C^i_{ij}$, and iterating this l times we would violate the condition $\mathfrak{g}^l = 0$. So the admissibility of the resulting graphs as well as the decreasing of the number of edges acting on f and g follows from Proposition 1 and its proof. Inductively we get rid of all antisymmetric (n odd) graphs. Q.E.D.
Remark. This result is in some sense the best attainable. Consider a unimodular Lie algebra $\mathfrak{u}$ such that the Kontsevich quantization of $\mathfrak{u}^*$ does contain loop graphs. Then by choosing f, g supported in a small neighborhood of the origin in $\mathfrak{u}^*$, we can assure that the contribution of the graphs with n odd, $k = n$, is not eliminated by some freaky cancellation. Thus the statement of Theorem 1 is not true for arbitrary unimodular Lie algebras.

4 Morita Equivalence

The essential sources used here are [W1], [X], and [GG]. Following [W1] we recall that a *full dual pair* $P_1 \overset{\rho_1}{\leftarrow} W \overset{\rho_2}{\rightarrow} P_2$ consists of two Poisson manifolds (P_1, π_1) and (P_2, π_2), a symplectic manifold W, and two submersions $\rho_1 : W \to P_1$ and $\rho_2 : W \to P_2$ such that ρ_1 is Poisson, ρ_2 is anti-Poisson, and the fibers of ρ_1 and ρ_2 are symplectic orthogonal to each other. A Poisson (or anti-Poisson) mapping is said to be *complete* if the pullback of a complete Hamiltonian flow under this mapping is complete. A full dual pair is called complete if both ρ_1 and ρ_2 are complete. The Poisson manifolds (P_1, π_1) and (P_2, π_2) are *Morita equivalent* if there exists a complete full dual pair $P_1 \overset{\rho_1}{\leftarrow} W \overset{\rho_2}{\rightarrow} P_2$ such that ρ_1 and ρ_2 both have connected and simply connected fibers. The notion of Morita equivalence of Poisson manifolds was introduced and studied by Xu [X], as a classical analogue of the Morita equivalence of C^*-algebras.

Theorem 2 *The property of being equipped with a trace functional on the Kontsevich quantization algebra is an invariant of Morita equivalence.*

Proof. By the hypothesis we have P_1, P_2 Morita equivalent, and $\mathbb{Q}_c(P_1)$ possesses a trace functional. Whence

$$\mathsf{Tr}_{P_1,\mu'}(F) \;=\; \hbar^{-[d/2]}\int_{P_1} F\mu',$$

where μ' is a unimodular volume form. Now we recall a result from [GG]. Here $\mathrm{mod}(P_i)$ denotes the modular class on respective manifolds.

Background Theorem 3 (Ginzburg, Golubev) *Let P_1 and P_2 be Morita equivalent and let, in addition, P_1 be locally unimodular. Then P_2 is also locally unimodular and* $\mathrm{mod}(\mathrm{P}_1)$ *goes to* $\mathrm{mod}(\mathrm{P}_2)$ *under the natural isomorphism of the first Poisson cohomology groups $E : H^1_\pi(P_1) \overset{\cong}{\to} H^1_\pi(P_2)$, i. e. $E(\mathrm{mod}(P_1)) = \mathrm{mod}(P_2)$.*

Using the above theorem we conclude P_2 is unimodular too. Furthermore, starting off with μ', via some constructions introduced in [GG] we arrive at a global unimodular volume form μ'' on P_2. At this point we invoke a theorem of Weinstein [W1] concerning transversal Poisson structures in full dual pairs:

Background Theorem 4 (Weinstein) *Let $P_1 \overset{\rho_1}{\leftarrow} W \overset{\rho_2}{\to} P_2$ be a full dual pair. For each $x \in W$, the transverse Poisson structures on P_1 and P_2 at $\rho_1(x)$ and $\rho_2(x)$ are anti-isomorphic. Consequently, if* $\dim(P_1) \leq \dim(P_2)$*, P_2 is locally anti-isomorphic to the product of P_1 with a symplectic manifold.*

Now we introduce a fixed open cover $\{U_\alpha\}_{\alpha\in J}$ of W such that $U'_\alpha = \rho_1(U_\alpha)$, $U''_\alpha = \rho_2(U_\alpha)$ and the latter ones are open covers of P_1 and P_2 respectively. They are so fine that there is a local anti-isomorphism of Background Theorem 4 in each U_α (by refining the cover of W we can always achieve that). From Background Theorem 4 we infer the relation between the local expressions for μ' and μ'', which we denote by μ'_α and μ''_α. Namely, either $\mu'_\alpha = \mu''_\alpha \wedge \kappa$ (this happens to be the case if $\dim(P_1) > \dim(P_2)$), or $\mu''_\alpha = \mu'_\alpha \wedge \kappa$ ($\dim(P_1) < \dim(P_2)$), or $\mu' = a\mu''$ ($\dim(P_1) = \dim(P_2)$), where κ is a Liouville volume form and a is a nonzero constant. By the standard manifold theory we may view U_α's as domains of $\mathbb{R}^d$. A crucial fact needed at this juncture is the following: the graphs in U'_α, U''_α are unions of subgraphs (c. f. Definition 2) of symplectic and transversal components. Moreover, the symplectic components necessarily have equal number of edges acting on f and g, so that up to a constant, the graphs can be thought of as those involving transversal components only. Now assembling the above volume forms and the graphs involving transversal components in the integral, we see that the vanishing of one on U'_α entails the vanishing of its counterpart on U''_α. Finally, using a partition of unity subordinate to the cover, we globalize the trace functional. Q.E.D.

5 The Main Theorem via Symplectic Realizations

We recall the appropriate definitions. A *symplectic realization* of a Poisson manifold P is a pair (W, ρ), where W is a symplectic manifold and ρ is a Poisson morphism from W to P. A symplectic realization $\rho : W \to P$ is called *complete* if ρ is complete as a Poisson map and ρ is said to be *full* if it is a submersion. There are many ways to construct symplectic realizations, but we are interested in just one particular way of doing it. Precisely, we want to realize Poisson manifolds as quotients of symplectic manifolds by group actions. In applications, as Weinstein [W1] pointed out, the symplectic manifold W may represent a collection of states such that the points belonging to the same orbit of G-action are considered to be physically indistinguishable. Thus the set of "true physical states" is none other than the Poisson manifold W/G, and the group G is called a *gauge group*. This motivates the definition below.

Definition 4 *A Poisson manifold P is symplectically realizable with a gauge group G if there exists a symplectic manifold W and a Lie group G freely acting on W by symplectomorphisms, such that $W/G \cong P$.*

An upshot of this definition is that a) if P is symplectically realizable with a gauge group G, $\rho : W \to P$ is a quotient map and necessarily a complete full surjective symplectic realization; b) P and $\mathfrak{g}^*$ are Morita equivalent.

Theorem 3 *The Kontsevich quantization algebra of a symplectically realizable Poisson manifold with a nilpotent gauge group possesses a trace functional.*

Proof. Applying Theorem 1 we obtain a trace functional on $\mathfrak{g}^*$ in view of the hypothesis, and Theorem 2 now ensures that $\mathsf{Tr}_{P,\mu}$ satisfies the trace property, where μ is the volume form obtained from the standard translation-invariant form on $\mathfrak{g}^*$. Q.E.D.

References

[BFFLS] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Deformation theory and quantization I and II. Ann. Phys. **237**, 61-110, 111-151 (1978)

[CaFe] A. Cattaneo, G. Felder, A Path Integral Approach to the Kontsevich Quantization Formula. `q-alg/9902090v3` (1999)

[CFS] A. Connes, M. Flato, D. Sternheimer, Closed Star Products and Cyclic Cohomology. Lett. Math. Phys. **24**, 1-12 (1992)

[F] B. Fedosov, A Simple Geometrical Construction of Deformation Quantization. J. Diff. Geom. **40**, 213-238 (1994)

[GG] V. Ginzburg, A. Golubev, Holonomy on Poisson Manifolds and the Modular Class. J. of AMS, to appear.

[Ka] V. Kathotia, Kontsevich's Universal Formula for Deformation Quantization and the Campbell-Baker-Hausdorff Formula I. `QA/9811174v2` (1998)

[Ko] M. Kontsevich, Deformation Quantization of Poisson Manifolds I. `q-alg/9709040` (1997)

[V] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations.* Englewood Cliffs, NJ: Prentice-Hall 1974

[W1] A. Weinstein, The Local Structure of Poisson Manifolds. J. Diff. Geom. **18**, 523-557 (1983)

[W2] A. Weinstein, The Modular Automorphism Group of a Poisson Manifold. J. Geom. Phys. **23**, 379-394 (1997)

[X] P. Xu, Morita Equivalence of Poisson Manifolds. Comm. Math. Phys., **142**, 493-509 (1991)