Characteristic classes and 2–modular representations for some sporadic simple groups – II

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Dedicated to the memory of J. Frank Adams, teacher, colleague and friend.

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0. Introduction

As in an earlier paper [Th] we are concerned with calculating the cohomology ring $H^*(G,\mathbb{Z})$ of a sporadic simple group G away form the prime 2. This is easiest when the prime ℓ concerned divides $|G|$ to the first power, for $H^*(G,\mathbb{Z})_{(\ell)}$ is then periodic and all one has to do is identify a maximal generator. We complete this part of our programme is section two below. However our main purpose is at least to begin the determination of $H^*(G,\mathbb{Z})_{(\ell)}$ when an ℓ–Sylow subgroup G_ℓ is elementary abelian, and the ℓ–torsion is detected by the subgroup of $H^*(G_\ell,\mathbb{Z})$ left invariant by the action of the normaliser $N(G_\ell)$ of G_ℓ in G. We do this for several of the Mathieu groups M_k and for Janko's group J_1, postponing possible consideration of the general case to a future paper. As elsewhere in the theory of simple groups M_{24} provides an excellent test for the general method, since $M_{24,3}$ is an elementary non–abelian group of order 27, and the complete description of the stable elements in its cohomology is not easy.

A further motive for writing this paper is the wish to understand the relation between $H^*(G,K_*\mathbb{F}_{2^t})$ and the modular representations of G over the finite field $\mathbb{F}_{2^t}$. In most of the cases we consider the Chern subring in ordinary cohomology localised away from 2 is generated by the classes of one or two representations of low degree. This suggests a simple structure for $R\mathbb{F}_{2^t}(G)$ as a λ–ring with conjugation, particularly when $t = 1$ and one exploits the prime factorisation of $K_{2k-1}(\mathbb{F}_2) \simeq \mathbb{Z}/2^k-1$. However with the exception of J_1, which behaves much like a group with periodic cohomology, our results only suggest ways of studying modular representations, since we are faced with the familiar convergence problems of the Atiyah–Hirzebruch spectral sequence. Indeed the generic situation for groups of composite order seems to be that there are universal cycles, which cannot be detected by Chern classes in either the characteristic zero or the modular case. However cohomology does at least make plain which representations are important for the λ–ring structure: as an elementary example consider M_{11}, which has irreducible 2–modular representations of degrees 1, 10, 44 and 16. Using eigenvalues it is easy to see that $\rho_{44} = \lambda^2(\rho_{10})-(1)$, but because of their characters when restricted to $M_{11,11}$ ρ_{16} and its conjugate cannot be obtained in this way. However $\rho_{16} + \overline{\rho_{16}} \cong \lambda^2(\rho_{10}) - \rho_{10} - (3)$, showing that this situation is simpler over the prime field $\mathbb{F}_2$. This is reflected in cohomology by the fact that

$$H^9(M_{11},\mathbb{Z}/_{4^5-1})_{(11)} \cong \mathbb{Z}/_{11}, \text{ but } H^9(M_{11},\mathbb{Z}/_{2^5-1})_{(11)} \text{ is trivial.}$$

As a harder example the reader may like to consider M_{23} in the same way.

The final section of this paper is devoted to 2–torsion in the cohomology of J_1. We include it as a supplement to the partial calculations already in the literature, see [Ch], and also because it represents one of the last contributions to mathematics by J. Frank Adams.

1. Mathieu groups

We recall that the five simple Mathieu groups were originally constructed as examples of multiply transitive groups; the two quintuply transitive groups $M_{12} \hookrightarrow S_{12}$ and $M_{24} \hookrightarrow S_{24}$ contain the other three examples as stabilising subgroups. For a description of the various ways in which the groups M_k have been described we refer the reader to the "Atlas" – we shall be mainly concerned with the second series:

$$PSL(3,\mathbb{F}_4) = M_{21} \hookrightarrow M_{22} \hookrightarrow M_{23} \hookrightarrow M_{24} .$$

The importance of the projective special linear group M_{21} is that it carries much of the structure of $H^*(M_k,\mathbb{Z})_{(3)}$, indeed for the first three groups $M_{k,3}$ is an elementary abelian group of rank 2. Furthermore, with N and Z as usual denoting normaliser and centraliser, we have

(i)	$Z(M_{k,3}) = M_{k,3}$	$(k = 21, 22 \text{ and } 23)$, and
(ii)	$N(M_{k,3})/Z(M_{k,3}) \cong Q_8$	(quaternion group, $k = 21, 22$) and
	$\cong SD_{16}$	(semi–dihedral group, $k = 23$) .

The group SD_{16} has presentation $\{s,t : s^8 = t^2 = 1 ,\ t^{-1}st = s^3\}$. The isomorphisms are not immediately apparent from the tables in the Atlas, but an alternative source is the paper of Z. Janko, [J]. Since the centraliser is as small as possible the action of the quotient group on $M_{k,3}$ is faithful. When $k = 21$ we write G for the normaliser, it is a split extension of the form

$$C_3^a \times C_3^b \rightarrowtail G \twoheadrightarrow Q_8^{s,t} \; .$$

We shall pick a convenient basis for the normal subgroup as a vector space over an extension field of $\mathbb{F}_3$ below. From now on we use the following notation:
Let K be a finite abelian group generated as a direct product by $a,b,\ldots$ The one–dimensional representation $\hat{\alpha}$ of K is faithful on the subgroup $\langle a \rangle$, maps the remaining generators $b,\ldots$ to 1, and $\alpha = c_1(\hat{\alpha}) \in H^2(K,\mathbb{Z})$. The group M_{24} has a representation τ, the Todd representation, in $GL(11,\mathbb{F}_2)$ described in [Td], which when lifted to characteristic zero has the partial character:

class	1	3^6	5^4	7_1^3	7_2^3	11	23_1	23_2	3^8
χ_τ	11	2	1	$\frac{1-\sqrt{-1}}{2}$	$\frac{1-\sqrt{-7}}{2}$	0	$\frac{-1+\sqrt{-23}}{2}$	$\frac{-1-\sqrt{-23}}{2}$	-1

Here 7^3 denotes a conjugacy class consisting of three disjoint 7–cycles with three 1–cycles omitted from the notation, etc. We shall also denote by τ its restriction to any of the smaller Mathieu groups contained in M_{24}.

Away from the primes 2 and 3 we have

THEOREM 1 (i) $H^*(M_{24},\mathbb{Z}[\frac{1}{6}])$ <u>is generated by the classes</u> c_3, c_4, c_{10} <u>and</u> c_{11} <u>of the</u> 11–<u>dimensional representation</u> τ (<u>suitably restricted to a representative family of Sylow subgroups</u>).

(ii) <u>If</u> $k = 11, 12, 22$ or 23 $H^*(M_k,\mathbb{Z}[\frac{1}{6}])$ <u>has the same generators away from the prime</u> 11. <u>In all four cases</u>

$$c_{10}(\tau \mid M_{k,11}) = c_5^2(\rho_k \mid M_{k,11}) \ ,$$

<u>where ρ_k can be identified from the table</u>

k	11	12	22	23
$\deg(\rho_k)$	16	16	280	896

Remark. The anomalous behaviour at 11 is explained by the splitting of a single conjugacy class of permutations on passing from a symmetric to a Mathieu group.

For a proof see [Th].

Let $\mathrm{Ch}(\ )_{(\ell)}$ denote the Chern subring of the even–dimensional cohomology localised at the prime ℓ.

THEOREM 2 <u>The subring</u> $\mathrm{Ch}(M_k)_{(3)}$ <u>of</u> $H^*(M_k,\mathbb{Z})_{(3)}$ <u>is generated by</u> $c_i(\tau \mid M_{k,3})$, $i = 3,4$. <u>At least when</u> $k = 22$ <u>or</u> 23 $\mathrm{Ch}(M_k)_{(3)}$ <u>is properly contained in</u> $H^*(M_k,\mathbb{Z})_{(3)}$.

Proof. We calculate the 3–primary part of $H^*(G,\mathbb{Z})$, where G is the normaliser of a representative 3–Sylow subgroup in $PSL(3,\mathbb{F}_4)$. The spectral sequence for the defining short exact sequence is trivial, so $H^*(G,\mathbb{Z})_{(3)} = H^*(C_3 \times C_3,\mathbb{Z})^{Q_8} = E_2^{*,0}$. The odd dimensional contribution is an exterior algebra on a 3–dimensional generator, compare [Le]. In even dimensions proceed as follows: Let V be a 2–dimensional vector space over $\mathbb{F}_3$ and consider the induced action of Q_8 on the symmetric algebra $S(V^*)$. Take coefficients in $\mathbb{F}_9$ rather than $\mathbb{F}_3$, so as to diagonalise the action of an element s of order 4 in Q_8. Here we use the usual presentation of Q_8 as
$\{s,t : s^4 = 1, s^2 = t^2, t^{-1}st = s^{-1}\}$, and

represent Q_8 in $SL(2,\mathbb{F}_9)$ by

$$s \longmapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad t \longmapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Having extended the scalars choose a basis of eigenvectors $\{A,B\}$ for $\mathbb{F}_9 \underset{\mathbb{F}_3}{\otimes} S(V^*) = \mathbb{F}_q[\alpha,\beta]$ with $sA = iA$ and $sB = -iB$. Formally one first chooses A and then takes B to be the image of A under the Frobenius map ψ. As an automorphism ψ fixes α and β, and on the coefficients $\psi(\lambda) = \lambda^3$. We may further suppose that over the extension field $\mathbb{F}_q$ the bases $\{A,B\}$ and $\{\alpha,\beta\}$ are related by the equations

$$A = i\alpha + \beta , \quad B = \alpha + i\beta .$$

The remark in the preamble about the choice of basis is now clear – G_3 is to be generated by a and b dual to the classes α and β. Now $\mathbb{F}_q[A,B]^{<s>}$ has an $\mathbb{F}_q$–basis consisting of all monomials A^jB^k with $j + 3k \equiv 0 \bmod 4$, which is equivalent to $(k-j) \equiv 0 \bmod 4$. Since t induces the automorphism $A \longmapsto -B$, $B \longmapsto A$, one type of invariant polynomial is "evenly symmetric" in A and B, i.e. one considers symmetric polynomials of the form

$$A^jB^k + A^kB^j = \sigma^t_{jk} \quad \text{, where}$$

j and k are both even, and $(k-j) \equiv 0 \bmod 4$. The second type must satisfy $A^jB^k - A^kB^j = \sigma^-_{jk}$, where j and k are both odd and $(k-j) \equiv 0(4)$. The first few invariant polynomials are $A^2B^2 = -(\alpha^2 + \beta^2)^2$, $A^4 + B^4 = -(\alpha^4 + \beta^4)$, $A^5B - AB^5 = (\alpha^2 + \beta^2)(\alpha^3\beta + \alpha\beta^3)$, ... One sees immediately that $S(V^*)^{Q_8}$ has two generators of degree 4, one of degree 6, ... On the other hand by counting dimensions we see that all but one of the irreducible representations of G factor through the quotient group Q_8, and the exception, obtained by induction form the trivial representation, restricts to the regular representation minus a trivial summand on $C_3 \times C_3$. An easy calculation now shows that $Ch(G)_{(3)}$ is generated by c_6 and c_8 of this restriction, and hence is properly contained in $H^{even}(G,\mathbb{Z})_{(3)}$.

This argument applies immediately to the Mathieu groups M_{21} and M_{22} since the stable elements in the cohomology of $M_{k,3}$ coincide with those invariant under the normaliser, see [Sw]. Inspection of the character table again shows that $Ch(M_{22})_{(3)}$ is generated by the Chern classes of the regular representation of $C_3 \times C_3$. For M_{23} the argument follows the same pattern, except that one replaces Q_8 by SD_{16}, represented over $\mathbb{F}_q$ by

$$s \longmapsto \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^3 \end{bmatrix}, \quad t \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{where } \zeta = 1-i$$

is a primitive 8th–root of unity. A basis of eigenvectors is given by $\{A,B\}$, where $sA = \zeta A$, $sB = \zeta^3 B$, and because t has order 2 rather than 4 the invariant polynomials are $A^jB^k + A^kB^j$ with $j + 3k \equiv 0 \bmod 8$. As one would except this subalgebra is smaller than for M_{22}, but A^2B^2 still provides a generator in degree 4, which is not describable as a Chern class.

The situation for the largest Mathieu group M_{24} is more complicated, since $M_{24,3}$ is a non–abelian group of order 27 and exponent 3. This cohomology of this group has been worked out by G. Lewis, see [Le], and using this multiplicative relations one can give a surprisingly simple description of the Chern subring. However the determination of the 3–primary part of $H^*(M_{24},\mathbb{Z})$ is harder, since we can no longer apply Swan's normaliser theorem.

2. Other sporadic simple groups

In this section we consider the twelve sporadic simple groups omitted from our previous paper [Th]. Loosely speaking these fall into three classes – the Fischer groups, those closely related to the Monster, and the oddments J_3, Ru, O'N, Ly and J_4. Because the last five groups are best described by means of faithful modular representations, our method works particularly well for them. However we start by sumarising the information for primes $\ell \geq 5$ dividing the order to the first power only, i.e. for which $H^*(G,\mathbb{Z})_{(\ell)}$ is periodic. A blank space means that the prime concerned does not divide the order; a space containing a dash (–) means that the Sylow subgroup G_ℓ is not cyclic. An asterisk (*) against an entry means that a maximal generator in cohomology may be taken to be the appropriate Chern class of a non–trivial irreducible representation of smallest degree.

G \ ℓ	5	7	11	13	17	19	23	29	31	37	43	47	67
J_3	4				16	18*							
Ru	–	12*		26*				28					
O'N	8*	–	22*			12			30				
Ly	–	12*	10*						12	36			44
J_4	8*	6*	–				44*	56*	20	24	28		
HN	–	12*	20*			18							
Th	–	–		24*		36*			30				
B	–	–	20*	26*	32*	36*	22		30			46	
He	–	–			16								
Fi_{22}	–	12*	10	12									
Fi_{23}	–	12*	20*	12	32*		22						
F'_{24}.	–	–	20*	26*	32*		22	28					

For the first groups we can summarise the information from our table in the following result:

THEOREM 3 Let the pair (G,q) be as shown

J_3	Ru	O'N	Ly	J_4
2	5	7	5	11

,

and let R be the coefficient ring $\mathbb{Z}[\frac{1}{6}]$ if $G = J_3$ and $\mathbb{Z}[\frac{1}{6q}]$ otherwise. Then $\hat{H}^*(G,R)$ is a sum of polynomial algebras, each of which is generated by a Chern class of a restricted irreducible representation.

Proof. This follows the lines of the argument in [Th], and depends on an examination of (a) the character tables and (b) the listed maximal subgroups of G in the Atlas.

Remarks on the individual groups.

Ru: Perhaps the most revealing representation is that of the related group 2.G in the orthogonal group $O_{28}(\mathbb{Z}[i])$ reduced modulo 5. So far as odd torsion in cohomology is concerned 2.G behaves like G, and the Chern classes of this $\mathbb{F}_5$–representation pick up maximal generators for 7 and 13, and the square of a maximal generator for 29.

Ly: This is perhaps the most interesting group among the oddments, since the period for the prime 31 (equal to $\frac{2}{5}(31-1)$) is so low. This is explained by Ly containing $G_2(\mathbb{F}_5)$ as a maximal subgroup (this group of Lie type has periodic cohomology for the primes 5, 7 and 31, the period for both the latter being 12). The remaining maximal subgroups of interest to us are the cyclic by cyclic extensions $67 : 22$ and $37 : 18$, and the semi–direct product $3^5 : (2 \times M_{11})$, which detect 67–, 37– and 11–torsion respectively. However in order to realise a maximal generator for 31 as a Chern class one must go in the Atlas to χ_{39} taking the value 43 110 144 at the identity.

J_4: This is usually thought of as a subgroup of $GL_{112}(\mathbb{F}_2)$. However comparison with other groups in this class suggests that one look for a more geometrically motivated representation over the Galois field $\mathbb{F}_{11}$.

Further calculations along the lines of those carried out for the Mathieu groups in the previous section seem possible, although not very rewarding. With the exceptions of HN, Ly and B, the orders of which are divisible by 5^6, all the groups on our list have the property that, if $\ell \geq 5$, then ℓ divides the order to at most the third power. Thus, if ℓ^2 is the highest power occuring, calculation of both $Ch(G)_{(\ell)}$ and $H^*(G,\mathbb{Z})_{(\ell)}$ as in Theorem 2 seems to be straightforward. A Sylow subgroup G_ℓ is necessarily abelian, and the image of the restriction map coincides with the subgroup invariant under the action of the normaliser $N(G_\ell)$. For ℓ^3 dividing the order one is again forced to use Lewis' calculations for the non–abelian group of order ℓ^3 and exponent ℓ. The situation is straightforward enough in principle, although certainly numerically complicated. The groups most accessible to this attack would seem to be Fi_{22} and Fi_{23}.

3. Janko's first group J_1 (revisited)

In our previous paper [Th] we exploited the fact that away from the prime two J_1 behaves like a group with periodic cohomology to calculate $H^*(J_1,\mathbb{Z}[\frac{1}{2}])$. With the exception of $\ell = 11$ the ℓ–periods all divide 12, which points to the importance of the $\mathbb{F}_{11}$–representation φ used originally by Janko to define the group. Indeed the dimension of φ equals 7 and is minimal for a positive non–trivial representation over any field. The values of φ on the different conjugacy classes are given by:

class	1	2	3	$5^{(1)}$	$5^{(2)}$	7	$10^{(1)}$	$10^{(2)}$	$15^{(1)}$	$15^{(2)}$	$19^{(i)}_{i=1,2,3}$
φ	7	-1	1	$\frac{-(1+\sqrt{5}}{2}$	$\frac{-(1-\sqrt{5}}{2}$	-1	$\frac{3-\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$	$1+\sqrt{5}$	$1-\sqrt{5}$	$1+\lambda(a_i)$

Here λ is one of 3 irreducible characters of degree 6 for the normaliser $N(J_{1,19})$, and a_1, a_2, a_3 represent three conjugacy classes of elements of order 19.

All elements of order two are conjugate, a 2–Sylow subgroup is elementary abelian of order 8, and any positive representation of J_1 must restrict to a direct sum of copies of the trivial and regular representations. For example $\varphi | J_{1,2}$ equals $\rho_{reg} - (1)$. The calculations are completed in even dimensions by

THEOREM 4 (J.F. Adams) $\hat{H}^{even}(J_{1,2},\mathbb{Z})_{(2)}$ *may be presented by* 5 *generators* x,y,z,u,v *of dimensions* 6,8,10,12,14 *respectively, and two relations* $r_{20} = 0$, $r_{24} = 0$, *where*

$$r_{20} = x^2y + xv + yu + z^2 ,$$
$$r_{24} = x^4 + x^2u + xyz + y^3 + zv + u^2 .$$

Proof. This is a more complicated version of that of Theorem 2, and we again use the symmetric algebra $S(V^*)$ associated with the 3–dimensional vector space V over $\mathbb{F}_2$. Write $S(V^*)$ as a polynomial algebra $\mathbb{F}_2[\alpha,\beta,\gamma]$, and let K be a subgroup of order 21 in $GL(V) \cong GL(3,\mathbb{F}_2)$ acting in the obvious way on $S(V^*)$. This is an accurate model for the cohomology of J_1, since the normaliser of $J_{1,2}$ is a cyclic–by–cyclic extension of the form $7:3$.

In order to find generators for $S(V^*)^K$ we embed $\mathbb{F}_2[\alpha,\beta,\gamma]$ in $\mathbb{F}_8[\alpha,\beta,\gamma]$, and let ψ be the Frobenius automorphism as in section 1. Over $\mathbb{F}_8$ we can find a new basis $\{A,B,C\}$ of $S(V^*)$ consisting of linearly independent eigenvectors corresponding to the eigenvalues η, η^2 and η^4 for an element $k \in K$ of order 7.

Step 1 $S(V^*)^K$ has an $\mathbb{F}_2$–base consisting of the symmetric sums

$$\sigma_{ijk} = A^iB^jC^k + B^iC^jA^k + C^iA^jB^k \ ,$$

where $i + 2j + 4k \equiv 0 \bmod 7$.

This is proved by showing that monomials of the form $A^iB^jC^k$ are k–invariant, and then taking the sum in order to allow for the group extension.

Step 2 Write

$$\begin{aligned} x &= \sigma_{111} = ABC \\ y &= \sigma_{130} = AB^3 + BC^3 + CA^3 \\ z &= \sigma_{320} = A^3B^2 + B^3C^2 + C^3A^2 \\ u &= \sigma_{510} = A^5B + B^5C + C^5A \\ v &= \sigma_{700} = A^7 + B^7 + C^7 \ . \end{aligned}$$

Step 3 Use induction on the degree of the symmetric sums σ_{ijk} to show that the five polynomials above actually do generate the invariant elements. Direct calculation shows that they also belong to $\mathbb{F}_2[\alpha,\beta,\gamma]$, rather than to the polynomial ring over $\mathbb{F}_8$. Furthermore the two relations r_{20} and r_{24} are satisfied. (This can be proved more slickly using Steenrod operations.)

Step 4 The relations are exhaustive. We have to show that the ring epimorphism

$$R = \mathbb{F}_2[x,y,z,u,v]/(r_{20},r_{24}) \xrightarrow{f} S(V^*)^K$$

is a monomorphism. We begin by localising so as to invert $x = ABC$.

LEMMA 5 The map $R \longrightarrow R(x^{-1})$ is mono.

Proof. One first shows by successive formation of quotients that the sequence x,y,v,r_{20},r_{24} in $\mathbb{F}_2[x,y,z,u,v]$ is regular. From this it follows that multiplication by x is (1–1) on the quotient ring R .

Now extend the scalars in the localised ring from $\mathbb{F}_2$ to $\mathbb{F}_8$, noting that we have one generator and one relation less. Replace r_{24} by

$$r'_{24} = x^4 + x^2u + y^3 + \frac{yzu + z^3}{x} + u^2 \text{ , and write}$$

$U = A^2/_B$, $V = B^2/_C$, $W = C^2/_X$. Then U,V,W are fixed by k and permuted by an element h of order 3 in K . Write $y/_z = U + V + W = g_1$, $z/_x = UV + VW + WU = g_2$ and $x = UVW = g_3$. Then $u/_a = U^2V + V^2W + W^2U = g_4$, say, and

$$R(x^{-1}) = \frac{\mathbb{F}_2\,[g_1,g_2,g_3,g_4]}{(r_{12})}\,(x^{-1}) \text{ , where}$$

$$r_{12} = g_4^2 + (g_1g_2 + g_3)g_4 + (g_1^3g_3 + g_2^3 + g_3^2) \ .$$

Given the algebraic independence of U, V and W it is now clear that f is a monomorphism after inversion of x and extension of scalars. Given Lemma 5 the same is true for the original map.

COROLLARY 5 The 2–primary part of the Chern subring $Ch(J_1)_{(2)}$ is properly contained in $H^{even}(J_1,\mathbb{Z})_{(2)}$.

Proof. This is a matter of evaluating the total Chern class of the regular representation of an elementary abelian group of rank 3. It turns out that the only non–vanishing classes are

$$c_4 = \alpha^4 + \dots + \alpha^2\beta^2 + \dots + \alpha\beta\gamma(\alpha + \beta + \gamma) \ ,$$
$$c_6 = \alpha^2\beta^4 + \dots + \alpha\beta\gamma(\alpha^3 + \beta^3 + \gamma^3 + \alpha\beta\gamma) \text{ and}$$
$$c_7 = \alpha^4(\beta^2\gamma + \gamma^2\beta) + \dots \ .$$

This calculation serves as a useful check on that in Theorem 4, and the existence of the classes x and z of degrees 6 and 10 shows that there are invariant elements other than Chern classes. Furthermore, and the same argument applies to the Mathieu groups, comparison of spectral sequences shows that the class x (for example) is a universal cycle in the Atiyah–Hirzebruch spectral sequence converging to the completed representation ring $R(J_1)^{\wedge}$. Here no localisation of coefficients is involved, and we have yet further examples for which the Grothendieck filtration of R(G) is definitely finer than the topological.

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