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On actions of a maximal torus on orbits of highest weight vectors

Dmitri I. Panyushev[1]

Introduction

The ground field k is algebraically closed and of characteristic zero. Let G be a simply connected semisimple algebraic group. Each irreducible finite-dimensional G-module contains a unique smallest nonzero orbit – the orbit of highest weight vectors. Following [11], the closure of this orbit is said to be an HV-variety of G. Fix a Borel subgroup B of G and a maximal torus T in B. Denote by $\mathcal{X}(T)$ for the character group of T and by $\mathcal{X}_+$ the monoid of dominant weights relative to B. For $\lambda\in\mathcal{X}_+$, V_λ is the irreducible G-module with highest weight λ and $C(\lambda)\subset V_\lambda$ is respective HV-variety.

In this paper we study the T-actions on HV-varieties. Our main results concern the algebra $k[C(\lambda)]^T$ of T-invariant regular functions on $C(\lambda)$:

• we describe the canonical module $\omega_{\lambda/\!\!/T}$ of $k[C(\lambda)]^T$;

• we find all the λ's such that the quotient map $\pi: C(\lambda)\to C(\lambda)/\!\!/T$ is equidimensional, see sect. 3;

• we find all the λ's such that $k[C(\lambda)]^T$ is a polynomial algebra, i.e., $C(\lambda)/\!\!/T$ is an affine space, see sect. 5.

To compute $\omega_{\lambda/\!\!/T}$, we use information about canonical module ω_λ of $k[C(\lambda)]$ obtained in [7], and the theory developed in [5]. Each dominant weight λ determines a closed face $\Gamma(\lambda)$ of the Weyl chamber and, conversely, to each face Γ, one associates a special weight λ_Γ on it. Thereby starting from λ, we get a dominant weight $\lambda_{\Gamma(\lambda)}$. Both ω_λ and $\omega_{\lambda/\!\!/T}$ are being described in terms of mutual position of λ and $\lambda_{\Gamma(\lambda)}$, see (1.3) and (1.7). The knowledge gained about $\omega_{\lambda/\!\!/T}$ is then used for classifying the dominant weights such that $k[C(\lambda)]^T$ is polynomial.

The algebras $k[C(\lambda)]$ and $k[C(\lambda)]^T$, as well as their canonical modules, are graded. Let $F_\lambda(t)$ (resp. $F_{\lambda/\!\!/T}(t)$) be the Poincaré series of the first (resp. second) algebra. Because $k[C(\lambda)]$ is Cohen-Macaulay, minus degree of $F_\lambda(t)$ as a rational function is equal to lowest degree of nonzero elements in ω_λ (and similarly for $F_{\lambda/\!\!/T}(t)$). Denote this number by $q(\lambda)$. As usual, the proof of a classification result consists of two steps: to rule out

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”infinity” and to look over the remaining finite list. To deal with the first step, we show that $\dim C(\lambda) \le q(\lambda) + \dim T$ whenever $k[C(\lambda)]^T$ is polynomial, see (5.1). The list of admissible weights obtained after ruling out “infinity” though is not finite, yet is rather small. Its nontrivial part consists of two infinite families of dominant weights of SL_{p+1}. We handle these cases by applying some ‘stability’ results for HV-varieties of SL_{p+1}, which may have a wider range of applications. Namely, we prove that some families of HV-varieties have isomorphic algebras of T-invariants and, moreover, this common algebra can be interpreted as the coordinate algebra of an HV-variety of SL_p, see (4.3),(4.4). This allows us to finish the proof of classification theorem (5.3). For all cases when $C(\lambda)/\!\!/T$ is an affine space we indicate the the representation of the Weyl group on $C(\lambda)/\!\!/T$. A classification of equidimensional quotient maps is based on some geometric considerations in sect. 2.

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1 The canonical module of $C(\lambda)/\!\!/T$

In this section we recall basic facts on HV-varieties and then obtain a description of the canonical module for their quotients by T.

1.1 Notation. G, B, T are as above. The simple roots corresponding to B are $\{\alpha_1, \dots, \alpha_p\}$, Δ (resp. Δ_+) is the set of roots (resp. positive roots), and $\mathcal{X}_+$ is the monoid of dominant weights. The character group $\mathcal{X}(T)$ is being considered as a lattice in the $\mathbb{Q}$-vector space $E = \mathcal{X}(T) \otimes \mathbb{Q}$. The sublattice of $\mathcal{X}(T)$ generated by Δ is Q, and the monoid generated by Δ_+ is Q_+. For any set of roots $\mathcal{P} \subset \Delta$ we use the notation $|\mathcal{P}|$ to denote the sum of these roots, $|\mathcal{P}| = \sum_{\alpha\in\mathcal{P}} \alpha$. We fix a W-invariant inner product $(\ ,\ )$ on E. For $\lambda \in \mathcal{X}_+$, the positive integers $l_i = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) = (\lambda, \alpha_i^\vee)$ are called the numerical labels of λ. Then $\lambda = \sum_{i=1}^p l_i \varphi_i$, where $\varphi_1, \dots, \varphi_p$ are the fundamental weights. Recall that $\varrho = \frac{1}{2}|\Delta_+| = \varphi_1 + \ldots + \varphi_p$.

The μ-weight space of V_λ is V_λ^μ. Star ‘*’ is used for denoting either the dual G-module or the highest weight of the dual module. For instance, the symbols V_λ^* and V_{λ^*} represent the same object. For $\lambda \in \mathcal{X}_+$, let v_λ be a B-eigenvector in V_λ. Then $C(\lambda) := \overline{Gv_\lambda}$ is an HV-variety of G.

1.2 Theorem [11],[4]**.**

(i) $k[C(\lambda)] = k[Gv_\lambda] = \bigoplus_{n=0}^{\infty} V_{n\lambda}^*$. *In particular* $C(\lambda)$ *is normal;*

(ii) $C(\lambda)$ *is Cohen–Macaulay with only rational singularities.* □

The algebra $k[C(\lambda)]$ is graded, the grading being defined via the action of the 1-dimensional torus of scalar transformations in $GL(V_\lambda)$. The Poincaré (or Hilbert) series of $k[C(\lambda)]$ is defined by

$$F_\lambda(t) := \sum_{n\geq 0} \dim V^*_{n\lambda} t^n \ .$$

This formal power series is the Taylor expansion of a rational function of the form $H_\lambda(t)/(1-t)^{\dim C(\lambda)}$. The integer $q(\lambda) := -\deg F_\lambda = \dim C(\lambda) - \deg H_\lambda$ plays an important role in the theory of the canonical module of $C(\lambda)$. One can compute it as follows. To any dominant weight λ one can attach a closed face of the Weyl chamber $\mathbb{Q}_+\mathcal{X}_+$. Given $\lambda = \sum_{i\in I} l_i\varphi_i$ $(l_i > 0)$, we set $\Gamma(\lambda) := \sum_{i\in I} \mathbb{Q}_i\varphi_i$. That is, $\Gamma(\lambda)$ is the minimal face of $\mathbb{Q}_+\mathcal{X}_+$ containing λ. On the other hand, if Γ is any face of $\mathbb{Q}_+\mathcal{X}_+$, one defines a special weight λ_Γ as follows:

$$\lambda_\Gamma = |\{\alpha \in \Delta_+, (\alpha, \Gamma) \neq 0\}| \ .$$

The following is easy to prove, see e.g. [7].

- λ_Γ is dominant. Moreover, it is the orthogonal projection of 2ϱ on Γ.
- Γ is the minimal face of $\mathbb{Q}_+\mathcal{X}_+$ containing λ_Γ.

It follows that $n\lambda - \lambda_{\Gamma(\lambda)} \in \mathcal{X}_+$ for any $\lambda \in \mathcal{X}_+$ and sufficiently large $n \in \mathbb{N}$. Denote by ω_λ the canonical module of $k[C(\lambda)]$. (All information on canonical modules needed for us can be found in [7] and [5].) Recall that it is a reflexive $k[C(\lambda)]$-module of rank 1. Since $k[C(\lambda)]$ is graded, ω_λ is graded as well and the following conditions are equivalent: (i) ω_λ is projective, (ii) ω_λ is free. In this case $C(\lambda)$ is said to be Gorenstein. Next, G acts naturally on ω_λ preserving the grading.

1.3 Theorem [7, 3.6,5.1]**.**

(i) $q(\lambda) = \min\{n \mid n\lambda - \lambda_{\Gamma(\lambda)} \in \mathcal{X}_+\}$;

(ii) $(\omega_\lambda)_n \simeq V^*_{n\lambda-\lambda_{\Gamma(\lambda)}}$. *Hence,* $(\omega_\lambda)_n \neq \{0\}$ *iff* $n \geq q(\lambda)$;

(iii) *the* $k[C(\lambda)]$*-module structure on* ω_λ *is given by the tensor product of representations with the subsequent projection onto the Cartan component:*

$$V^*_{m\lambda} \times V^*_{n\lambda-\lambda_{\Gamma(\lambda)}} \to V^*_{(n+m)\lambda-\lambda_{\Gamma(\lambda)}} \ . \qquad \square$$

1.4 Corollary. (i) $C(\lambda)$ *is Gorenstein if and only if* $n\lambda = \lambda_{\Gamma(\lambda)}$ *for some* $n \in \mathbb{N}$;
(ii) $(q(n\lambda) - 1)n < q(\lambda) \leq q(n\lambda)n$ *for any* $n \in \mathbb{N}$.

Proof. (i) Obvious.
(ii) From (1.3)(i) for $n\lambda$ it follows that $q(n\lambda)n\lambda - \lambda_{\Gamma(\lambda)} \in \mathcal{X}_+$ and $(q(n\lambda)-1)n\lambda - \lambda_{\Gamma(\lambda)} \notin \mathcal{X}_+$. Then apply (1.3)(i) to λ. $\square$

1.5 Remarks. 1. An algorithm for computing weights λ_Γ is found in [7, 3.7]. It works as follows. If Γ is an edge of the Weyl chamber, say $\Gamma = \mathbb{Q}_+\varphi_i$, then $\lambda_\Gamma = q(\varphi_i)\varphi_i$ and since λ_Γ is the orthogonal projection of 2ϱ, we have $q(\varphi_i) = (2\varrho, \varphi_i)/(\varphi_i, \varphi_i)$. The integers $q(\varphi_i)$'s are said to be the *edge labels* of G. Suppose $\Gamma = \sum_{i\in I} \mathbb{Q}_i\varphi_i$ and $\#I \geq 2$. Let us say the nodes of the Dynkin diagram corresponding to I are white and the other ones are black. For each white node, consider the maximal connected subgraph that contains no other white nodes. It determines a simple subgroup G' of G together with an edge of the little Weyl chamber. Then the label of λ_Γ on this white node is equal to the respective edge label of G'. The black node labels of λ_Γ are equal to zero.
The table of edge labels for all simple groups is found in [7, 3.7]. However, it contains two errors for $\mathbf{E}_7$: one has to replace '14' by '13', and '28' by '14'. After these corrections one may see that $\operatorname{rk} G + 1 \leq q(\varphi_i) \leq h$ (h is the Coxeter number), and $q(\varphi_i) = h$ if and only if φ_i is minuscule. An a priori proof of these facts will be given in the Appendix.

2. The weight $\lambda_{\Gamma(\lambda)}$ corresponds to the anticanonical sheaf of the flag variety $\operatorname{Proj} C(\lambda)$. This is important for computing cohomology of invertible sheaves on $\operatorname{Proj} C(\lambda)$. For this reason the edge labels were also computed in Akhiezer's book [1, p. 125]. (However, the list contains one error for $\mathbf{E}_7$.)

3. Another algorithm for computing λ_Γ was given by D.Snow [9, §2].

The algebra of T-invariants $k[C(\lambda)]^T$ has the zero weight space $V^0_{n\lambda^*}$ as its n-th graded piece. The corresponding Poincaré series

$$F_{\lambda/\!\!/T}(t) := \sum_{n\geq 0} (\dim V^0_{n\lambda^*}) t^n$$

defines a rational function and we set $q(\lambda/\!\!/T) := -\deg F_{\lambda/\!\!/T}$. Let $\omega_{\lambda/\!\!/T}$ be the canonical module of $k[C(\lambda)]^T$. As usual, $C(\lambda)/\!\!/T := \operatorname{Spec} k[C(\lambda)]^T$. Because $C(\lambda)/\!\!/T$ has rational singularities by Boutot's theorem and in particular is Cohen-Macaulay, $q(\lambda/\!\!/T)$ is equal to the smallest degree of nonzero elements in $\omega_{\lambda/\!\!/T}$. In order to compute it, we apply a simplified version of a general result of F.Knop [5, Satz 6(d)]:

1.6 Theorem. *Suppose X is a T-variety having rational singularities and the T-action on X is stable. Then $\omega_{X/\!\!/T} = \pi_*(\omega_X)^T$, where $\pi : X \to X/\!\!/T$ is the quotient map.* (We consider the action of a connected group, hence $\Theta \simeq k$. The divisor D_s considered in [5] is equal to 0 whenever the action is stable.) □

The T-action on $C(\lambda)$ is always stable [8, 2.1] and making use of (1.3) we obtain the following description.

1.7 Theorem. *Let $\lambda \in \mathcal{X}_+$ and $\Gamma = \Gamma(\lambda)$ be the corresponding face of $\mathbb{Q}_+\mathcal{X}_+$. Then*
(i) $(\omega_{\lambda/\!\!/T})_n \simeq V^0_{n\lambda^*-\lambda^*_\Gamma}$*;*
(ii) *the $k[C(\lambda)]^T$-module structure on $\omega_{\lambda/\!\!/T}$ is given by the natural map*
$$V^0_{m\lambda^*} \times V^0_{n\lambda^*-\lambda^*_\Gamma} \to V^0_{(n+m)\lambda^*-\lambda^*_\Gamma} \quad .$$ □

Recall that $V_\mu^0 \neq 0$ if and only if $\mu \in Q$ and that $\lambda_\Gamma \in Q$ for any face Γ. Together with (1.4) it immediately implies

1.8 Corollary.
1. $q(\lambda /\!\!/ T) = \min\{n \in \mathbb{N} \mid n\lambda \in Q \text{ and } n\lambda - \lambda_\Gamma \in \mathcal{X}_+\}$*;*
2. Let $a \in \mathbb{N}$ *be the minimal number such that* $a\lambda \in Q$*. Then* $q(\lambda /\!\!/ T) = q(a\lambda)a$ *and* $q(\lambda) \leq q(\lambda /\!\!/ T) \leq q(\lambda) + a - 1$*;*
3. $q(\lambda) = q(\lambda /\!\!/ T)$ *if and only if* $q(\lambda)\lambda \in Q$*.* □

Let us give a review of the Gorenstein property for $C(\lambda) /\!\!/ T$.

1.9 Proposition.
(i) *If* $C(\lambda)$ *is Gorenstein, then* $q(\lambda) = q(\lambda /\!\!/ T)$ *and* $C(\lambda) /\!\!/ T$ *is also Gorenstein;*
(ii) *Let* $C(\lambda) /\!\!/ T$ *be Gorenstein and* $a \in \mathbb{N}$ *the smallest number such that* $a\lambda \in Q$*, then* $\dim V_\mu^0 = 1$*, where* $\mu = q(a\lambda)a\lambda - \lambda_\Gamma$*.*

Proof. (i) This follows from (1.4) and (1.7).
(ii) In the Gorenstein case, $\omega_{\lambda /\!\!/ T}$ is a free graded $k[C(\lambda)]^T$-module. Hence the homogeneous piece of the minimal degree has to be of dimension 1. □

The second assertion shows that if $C(\lambda) /\!\!/ T$ is Gorenstein and $C(\lambda)$ is not, then G has a non-trivial representation V_μ such that $\dim V_\mu^0 = 1$. A classification of such dominant weights for simple groups can be extracted from [2, thm. 3.1]. The list of admissible pairs (G, μ) is:

$$(\mathbf{A}_\mathrm{p}, n(p+1)\varphi_1)\ (n \in \mathbb{N}), \quad (\mathbf{B}_\mathrm{p}, \varphi_1), \quad (\mathbf{G}_2, \varphi_1)\ .$$

Whence one immediately obtains

1.10 Corollary. *Suppose* G *is any of the groups* $\mathbf{C}_\mathrm{p}$ $(p \geq 3), \mathbf{D}_\mathrm{p}, \mathbf{E}_\mathrm{p}$*, or* $\mathbf{F}_4$*. Then* $C(\lambda) /\!\!/ T$ *is Gorenstein if and only if* $C(\lambda)$ *is.* □

A direct verification shows that the same assertion holds for $\mathbf{G}_2$. For $\mathbf{A}_\mathrm{p}$ and $\mathbf{B}_\mathrm{p}$, there are non Gorenstein weights λ such that $C(\lambda) /\!\!/ T$ is Gorenstein. Their description is unknown at this writing. It is only clear that there are finitely many of them for $\mathbf{B}_\mathrm{p}$ and the theory developed in the next section proves that their number for $\mathbf{A}_\mathrm{p}$ is infinite.

2 Geometric constructions

Because the coordinate algebra $k[C(\lambda^*)]$ is a multiplicity free G-module, $C(\lambda^*)$ has finitely many B-orbits. Their description is immediate due to Bruhat decomposition. Write v_μ for a μ-weight vector. Then $v_{-\lambda}$ is a lowest weight vector in V_{λ^*}. As usual, $w \mapsto l(w)$ is the length function on W and $W^\lambda \subset W$ is the set of representatives of minimal length for the coset space W/W_λ.

2.1 Lemma.
1. $C(\lambda^*) = (\bigsqcup_{w\in W^\lambda} B\cdot v_{-w\lambda}) \cup \{0\}$*;*
2. $\mathrm{codim}_{C(\lambda^*)} B\cdot v_{-w\lambda} = l(w)$*.* $\square$

Let $P = \tilde{L}N$ be a standard parabolic subgroup in G, where $\tilde{L}$ is the Levi subgroup containing T and N is the unipotent radical. Denote by L the derived group of $\tilde{L}$. Write I for the subset of $[1,p]$ corresponding to the simple roots of L. Recall that this establishes a one-to-one correspondence between the standard parabolic subgroups and the subsets of $[1,p]$. We may (and will) consider the φ_i $(i \in I)$ as the fundamental weights of L. Set $\varphi_P = \sum_{i\notin I}\varphi_i$. It is the minimal P-regular dominant weight (a dominant weight λ is called P-regular if the stabilizer of the line $\langle v_\lambda\rangle \in \mathbb{P}V_\lambda$ is exactly P). The weight φ_P defines a function $\mathcal{X}(T) \to \mathbb{Z}$, $\mu \mapsto (\varphi_P, \mu^\vee)$, which we call the level function.

Given $\lambda \in \mathcal{X}_+$, define a dominant weight λ' of L in the following way. If $\lambda = \sum_i l_i\varphi_i$, then $\lambda' = \sum_{i\in I} l_i\varphi_i$. The following visual procedure is related to this definition: λ being represented by the Dynkin diagram equipped with the numerical labels, one just erases all the nodes that are not in I. The remaining (may be not connected) diagram with remaining labels represents (the highest weight of) a representation of L.

2.2 Lemma. *Let λ be arbitrary dominant weight. Then*
1. $(V_\lambda)^N$ *is an irreducible L-module whose highest weight is* λ'*;*
2. $(V_\lambda)^N = \bigoplus_\mu V_\lambda^\mu$*, where μ runs over all weights such that* $(\mu^\vee, \varphi_P) = (\lambda^\vee, \varphi_P)$*.*

Proof. 1. Each highest weight vector of L in $(V_\lambda)^N$ is a highest weight vector of G as well, hence irreducibility. Clearly, $v_\lambda \in (V_\lambda)^N$ is a highest weight vector for L. The description of the embedding $L \subset G$ shows that the L-weight of v_λ is just λ'.
2. We think of V_λ as the direct sum of level spaces $\bigoplus_{(\mu^\vee,\varphi_P)=const} V_\lambda^\mu$. Since each level space is L-stable and the space of maximal level $(\lambda^\vee, \varphi_P)$ lies in the irreducible module $(V_\lambda)^N$, we are done. $\square$

Let us apply it to HV-varieties. We have

$$k[C(\lambda^*)]^N \simeq \oplus_{n=0}^\infty V_{n\lambda}^N = \oplus_{n=0}^\infty V_{n\lambda'} \simeq k[C(\lambda'^*]\ .$$

That is, $k[C(\lambda^*)]^N$ has the same graded L-module structure as $k[C(\lambda'^*)]$. Yet this does not imply that multiplication is the same.

2.3 Theorem.
1. $C(\lambda'^*)$ *is naturally being realized as a L-stable subvariety of* $C(\lambda^*)$*;*
2. $k[C(\lambda^*)]^N \simeq k[C(\lambda'^*)]$*.*

Proof. 1. $C(\lambda^*)$ contains $v_{-\lambda}$, a lowest weight vector. Then $\overline{Lv_{-\lambda}} \simeq C(\lambda'^*)$.
2. We think of $V_{n\lambda}$ as the set of functions of degree n on $C(\lambda^*)$. For each n, take in

$V_{n\lambda}$ the sum of all level spaces whose level is not maximal and call it J_n. Obviously, $J := \oplus_{n\ge 1} J_n$ is an ideal in $k[C(\lambda^*)]$. It then follows from (2.2)(2) that $k[C(\lambda^*)]/J \simeq k[C(\lambda^*)]^N$. Therefore J is prime and it defines an irreducible L-stable subvariety of $C(\lambda^*)$. Denote by $\tilde J$ the ideal generated by J_1. Then $\mathcal{V}(J) \subset \mathcal{V}(\tilde J) = C(\lambda^*) \cap J_1^\perp$, where $\mathcal{V}(\tilde J)$ is the set of zeros of $\tilde J$ and $J_1^\perp$ is the annihilator of J_1 in the dual space. Clearly, $J_1^\perp$ is the minimal level subspace in V_{λ^*}. It then contains a lowest weight vector $v_{-\lambda}$. Since $J_1^\perp$ is L-stable, $C(\lambda^*) \cap J_1^\perp \supset \overline{Lv_{-\lambda}} \simeq C(\lambda'^*)$. Let P_- be the opposite parabolic subgroup to P. From the decomposition $G = P_- W P_-$, it immediately follows that if $gv_{-\lambda} \in J_1^\perp$ then there is $\tilde g \in P_-$ such that $gv_{-\lambda} = \tilde g v_{-\lambda}$. Therefore $C(\lambda^*) \cap J_1^\perp = \overline{Lv_{-\lambda}}$. Because $\mathcal{V}(J) \ne \{pt\}$, we have $\mathcal{V}(J) = \mathcal{V}(\tilde J)$. Whence $J = \operatorname{rad} \tilde J$ and $k[C(\lambda^*)]/J \simeq k[C(\lambda'^*)]$. □

2.4 Corollary. *As a subvariety of $C(\lambda^*)$, $C(\lambda'^*)$ lies in the null-fibre of the quotient map $C(\lambda^*) \to C(\lambda^*)/\!\!/T$.*

Proof. By construction, $J_1^\perp = \sum_{\mu\in D} V_{\lambda^*}^\mu$, where $D = \{\mu \mid (\mu^\vee, \varphi_P) = -(\lambda^\vee, \varphi_P)\}$. Hence D belongs to an open half-space in E and, by the Hilbert–Mumford criterion, $J_1^\perp \cap C(\lambda^*) = C(\lambda'^*)$ lies in the null-fibre. □

Remarks. 1. If the reader do not like to see so many '*' in the statement of (2.3), then he may proceed in the dual way. Take $v_\lambda \in C(\lambda)$. Then $Lv_\lambda \simeq C(\lambda')$ and $k[\overline{Lv_\lambda}] \simeq k[C(\lambda)]^{N_-}$, where N_- is the unipotent radical of P_-. However, the same amount of '*' is required for the proof of the dual version of (2.2).
2. It follows from a recent result of A. Broer that $J = \tilde J$ in the proof of (2.3).

3 Equidimensional quotient maps

In this section we find all λ's such that the quotient map $\pi_\lambda : C(\lambda) \to C(\lambda)/\!\!/T$ is equidimensional. The classification argument turns out to be very close to that used in [8] for describing the weights satisfying a stronger condition that $k[C(\lambda)]$ is a free $k[C(\lambda)]^T$-module. Denote by $\mathfrak{N}(\lambda)$ the fibre of π_λ containing zero ("null-fibre"). As is well known, π_λ is equidimensional if and only if $\mathfrak{N}(\lambda)$ is of the expected dimension $\dim C(\lambda) - \dim C(\lambda)/\!\!/T = \dim T$. In order to prove that π_λ is not equidimensional for most λ's, one has to be able to find out sufficiently large pieces of $\mathfrak{N}(\lambda)$. These will be supplied by (2.4). A simple but helpful observation is that π_λ is equidimensional if and only if $\pi_{n\lambda}$ $(n \ge 1)$ is. (For $C(\lambda) \to C(n\lambda)$ is a finite T-equivariant map.) Thus, it suffices to find only primitive dominant weights with this property.

3.1 Theorem. *Let G be a simple algebraic group and λ a primitive element in $\mathcal{X}_+$. Then π_λ is equidimensional if and only if λ occurs in the following list:*
$\mathbf{A}_\mathrm{p}$ — $\varphi_1, \varphi_p, \varphi_1+\varphi_p$; $\mathbf{B}_\mathrm{p}(p\ge 2), \mathbf{C}_\mathrm{p}(p\ge 2), \mathbf{D}_\mathrm{p}(p\ge 3)$ — φ_1.

Proof. "if". This is proven in [8, 2.6].
"only if". Suppose $\dim \mathfrak{N}(\lambda) = \dim T$. From (2.4) and the next remark, it follows that any erasing procedure applied to λ gives us a pair (L, λ') such that the corresponding subvariety $C(\lambda')$ lies in $\mathfrak{N}(\lambda)$. The strongest constraint corresponds to erasing a single node such that each connected component of the remaining diagram possesses nonzero labels (such a node is called λ-admissible in [8]). Then $\mathrm{rk}\, L = \mathrm{rk}\, G - 1$ and λ' determines a locally faithful irreducible representation of L. Hence each λ-admissible node yields a constraint of the form $\dim C(\lambda') \le \mathrm{rk}\, L + 1$. It is easy to see that this holds if and only if each simple factor L_i of L is isomorphic to some SL_{n_i} and the respective representation of L_i is just a symmetric power of either the simplest representation or its dual (compare with [8, 2.5]). Thus, making use of a different approach, we are arriving at essentially the same conclusion as in the first part of the proof of Theorem 2.6 in [8]: if λ is primitive and π_λ is equidimensional, then λ' has a rather specific form and therefore λ must be contained in the following list:
$\mathbf{A}_{\mathrm{p}}$—φ_1, φ_p, $a\varphi_1 + b\varphi_p$; $\mathbf{B}_{\mathrm{p}}$ $(p \ge 2)$, $\mathbf{C}_{\mathrm{p}}$ $(p \ge 2)$, $\mathbf{D}_{\mathrm{p}}$ $(p \ge 3)$—φ_1; $\mathbf{C}_2, \mathbf{G}_2$—$a\varphi_1 + b\varphi_2$, where $\gcd(a, b) = 1$.
This already is almost the required list. To eliminate extra 'ab'-cases, we use the description of B-orbits in HV-varieties (2.1). It suffices to detect one of them lying in $\mathfrak{N}(\lambda)$ and having dimension $\ge \mathrm{rk}\, G + 1$. From the Hilbert–Mumford criterion, it follows that $Bv_{-w\lambda} \subset \mathfrak{N}(\lambda)$ *iff* the shifted cone $-w\lambda + \mathbb{Q}_+ Q_+$ does not contain zero *iff* $w\lambda \not\in \mathbb{Q}_+ Q_+$. Since dimensions of B-orbits are computable, the task for the rank two cases reduces more or less to drawing pictures, and the general case of $\mathbf{A}_{\mathrm{p}}, a\varphi_1 + b\varphi_p$ with $a \ne b$ is similar to that with $p = 2$. This approach can be used for computing true dimension of $\mathfrak{N}(\lambda)$, for each irreducible component of $\mathfrak{N}(\lambda)$ is W-conjugate to a B-stable one. □

Remark. Comparing with [8, 2.6], we obtain the following *a posteriori* assertion:

If λ is primitive and π_λ is equidimensional, then $k[C(\lambda)]^T$ is a polynomial algebra. However, we shall see in the next section that there are primitive λ's such that $k[C(\lambda)]^T$ is polynomial, but π_λ is not equidimensional.

4 Stability results for SL_{p+1}

Throughout this section $G = SL_{p+1}$ with the obvious choice of B and T. We shall use the standard numeration of roots and weights, in particular $\varphi_i^* = \varphi_{p+1-i}$. Let $\lambda = \sum_{i=1}^p l_i\varphi_i$ be a dominant weight of SL_{p+1}. The associated partition $\underline{\lambda} = (\lambda_1, \cdots, \lambda_p)$ is defined by $\lambda_i = l_i + \cdots + l_p$, and $|\underline{\lambda}| := \sum_i \lambda_i$. Then the dominant weights of V_λ are in one-to-one correspondence with partitions $\underline{\mu} = (\mu_1, \cdots, \mu_{p+1})$ with at most $p+1$ nonzero parts such that $|\underline{\lambda}| = |\underline{\mu}|$ and $\underline{\lambda} \ge \underline{\mu}$. Arbitrary weights of V_λ correspond to strings of integers which are permutations of such $\underline{\mu}$'s. Denote by $\underline{\lambda}'$ the truncated partition $(\lambda_2, \cdots, \lambda_p)$. It determines a dominant weight λ' of SL_p, which is said to be truncated, too. As we shall see below, this notation is compatible with that used in sect. 2. Denoting $\varphi_1', \ldots, \varphi_{p-1}'$

the dominant weights of SL_p, we have $\lambda' = l_2\varphi'_1 + \ldots + l_p\varphi'_{p-1}$. We write $V_{\lambda'}$ for the respective SL_p-module. Clearly, $(n\lambda)' = n\lambda'$. The following is a representation-theoretic translation—with the same proof—of [6, I.6, Ex.2(c)]:

4.1 Lemma. *$\underline{\lambda}$ and $\underline{\mu}$ being as above, assume that each part of $\underline{\mu}$ $\geq \lambda_2$. Then* $\dim V^{\mu}_{\lambda} = \dim V_{\lambda'}$.

Proof. As is well known, $\dim V^{\mu}_{\lambda}$ is the number of (column-strict) tableaux of shape $\underline{\lambda}$ and weight $\underline{\mu}$. In any such tableau T, the 1's must occupy the first μ_1 squares in the top row, and hence T is determined by its entries in the remaining rows, which constitute a tableau of shape $\underline{\lambda}'$ and arbitrary weight. The word "arbitrary" means that all the weights of $V_{\lambda'}$ are taken into account, not only dominant ones. $\square$

4.2 Corollary. *Suppose $\lambda = \sum_{i=1}^{p} l_i\varphi_i \in Q \cap \mathcal{X}_+$ is such that $|\underline{\lambda}| \geq (p+1)\lambda_2$ or, what is the same,*

$$(\diamond) \qquad l_1 \geq (p-1)l_2 + \ldots + l_p \quad .$$

Then $\dim V^0_{m\lambda} = \dim V^0_{m\lambda + n(p+1)\varphi_1} = \dim V_{m\lambda'}$ *for any* $(m,n) \in \mathbb{N}^2$.

Proof. The condition of being in Q means $|\underline{\lambda}|/(p+1) \in \mathbb{N}$. Then the zero weight is given by partition whose parts are equal to $|\underline{\lambda}|/(p+1)$. Clearly, the truncated partitions associated to $m\lambda$ and $m\lambda + n(p+1)\varphi_1$ are the same. $\square$

This leads to the following observations. First, all the algebras $k[C(\lambda^* + n(p+1)\varphi_p)]^T$ $(n = 0, 1, \ldots)$ have the same Poincaré series whenever λ satisfies $(\diamond)$. Second, this series coincides with that of $k[C(\lambda'^*)]$, where $C(\lambda'^*) \subset V_{\lambda'^*}$ is HV-variety of SL_p. Because these coincidences by no means prove that the algebras in question are isomorphic, our aim is to recover some geometry which is hidden behind it.

4.3 Theorem. *Suppose $\lambda = \sum_{i=1}^{p} l_i\varphi_i$ is a dominant weight lying in Q and satisfying condition $(\diamond)$. Then $C(\lambda^*)/\!\!/T \simeq C(\lambda^* + n(p+1)\varphi_p)/\!\!/T$ and $C(\lambda)/\!\!/T \simeq C(\lambda + n(p+1)\varphi_1)/\!\!/T$ for any $n \in \mathbb{N}$.*

Proof. By duality, it is enough to establish either of the conclusions and, by induction, it is enough to treat case $n = 1$. To prove that $k[C(\lambda^*)]^T \simeq k[C(\lambda^* + (p+1)\varphi_p)]^T$, we consider a bi-graded algebra which contains both of them. Define C to be the orbit closure

$$C = \overline{G(v_{(p+1)\varphi_p}, v_{\lambda^*})} \subset V_{(p+1)\varphi_p} \oplus V^*_{\lambda} \ .$$

As was shown in [11], C is normal and $k[C]$ is bi-graded, the graded piece of degree (n,m) being $k[C]_{n,m} = V_{n(p+1)\varphi_1 + m\lambda}$. Moreover, given $(n_0, m_0) \in \mathbb{N}^2$, we have

$$\bigoplus_{n=0}^{\infty} k[C]_{nn_0, nm_0} \simeq k[C(n_0(p+1)\varphi_p + m_0\lambda^*)] \ .$$

Associated to the projections of C onto $C((p+1)\varphi_p)$ and $C(\lambda^*)$, there are the graded monomorphisms $i_1 : k[C((p+1)\varphi_p)]^T \to k[C]^T$ and $i_2 : k[C(\lambda^*)]^T \to k[C]^T$ such that $i_1(k[C((p+1)\varphi_p)]^T_n) = k[C]^T_{n,0}$ and $i_2(k[C(\lambda^*)]^T_m) = k[C]^T_{0,m}$. Since $\dim k[C((p+1)\varphi_p)]^T_n = 1$ for any n and $k[C]$ is bi-graded,

$$i_1 \otimes i_2 : k[C((p+1)\varphi_p)]^T \otimes k[C(\lambda^*)]^T \to k[C]^T$$

is a monomorphism. Because $\dim k[C]^T_{n,m} = \dim k[C]^T_{0,m}$ for any n by (4.2), we see that $i_1 \otimes i_2$ is surjective, i.e., an isomorphism. Let y be a nonzero element in 1-dimensional space $k[C]^T_{1,0}$. Then

$$k[C(\lambda^* + (p+1)\varphi_p)]^T = \bigoplus_{n=0}^{\infty} k[C]^T_{n,n} = \bigoplus_{n=0}^{\infty} y^n k[C]^T_{0,n} \simeq \bigoplus_{n=0}^{\infty} k[C]^T_{0,n} = k[C(\lambda^*)]^T \ . \qquad \square$$

Example. For the first layer representations, i.e., those with $|\underline{\lambda}| = p+1$, condition $(\diamond)$ is satisfied iff $\underline{\lambda}$ is a hook iff $\lambda = (p+1-i)\varphi_1 + \varphi_i$.

4.4 Theorem.
1. For any $\lambda \in Q \cap \mathcal{X}_+$, there is a graded monomorphism $k[C(\lambda^*)]^T \to k[C(\lambda'^*)]$;
2. If λ satisfies condition $(\diamond)$, then

$$k[C(\lambda^*)]^T \simeq k[C(\lambda'^*)] \quad .$$

Hence $C(\lambda)/\!\!/T$, $C(\lambda^*)/\!\!/T$, $C(\lambda'^*)$, $C(\lambda')$ are isomorphic as abstract varieties.

Before proving this result, we deduce a consequence of it and make some preparations.

4.5 Corollary. *Suppose either $\lambda = (p-1)\varphi_1 + \varphi_2 + n(p+1)\varphi_1$ or $\lambda = \varphi_1 + \varphi_p + n(p+1)\varphi_1$. Then $k[C(\lambda)]^T$ is a polynomial algebra of dimension p generated by the functions of degree 1.*

Proof. Under this assumption, the truncated weight λ' corresponds to either the simplest representation of SL_p or its dual. In both cases, $C(\lambda')$ is an affine space of dimension p. $\square$

¿From now on, we consider SL_p as the subgroup L of G whose simple roots are $\alpha_2, \ldots, \alpha_p$. Let P be the standard parabolic subgroup whose semisimple part is L. Then the unipotent radical N of P is p-dimensional, it corresponds to the roots $\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \ldots + \alpha_p$. Let $e_1, \ldots, e_p$ be a basis of $\mathfrak{n} = \operatorname{Lie} N$ consisiting of respective root vectors. Because N is commutative, the e_i's pairwise commute in the enveloping algebra $\mathcal{U}(\mathfrak{g})$. Set $A = e_1 + \ldots + e_p \in \mathcal{U}(\mathfrak{g})$. We may consider A and all its powers as operators in any representation space of G.

4.6 Lemma.
1. $A^m(V^0_\lambda) = 0$ for all $m > (\lambda^\vee, \varphi_1)$;
2. $A^{(\lambda^\vee,\varphi_1)}$ injectively maps V^0_λ into $(V_\lambda)^N$.

Proof. Recall from sect. 2 that associated to a parabolic subgroup we have the level function, which is now just $\mu \mapsto (\mu^\vee, \varphi_1)$, and V_λ may be thought of as the union of the level spaces. Each application of A takes a weight vector one level up. Since the maximal level is $l_1 = (\lambda^\vee, \varphi_1)$, the first part follows. The only assertion that still requires proof is injectivity of A^{l_1}. Take any $v \in V_\lambda^0$. Then $A^{l_1}v$ is the sum of "monomials" of the form $e_1^{m_1} \cdots e_p^{m_p} v$, where $m_1 + \ldots + m_p = l_1$. The ordering of e_i's is immaterial, because they commute. Since the weights of e_i's are linearly independent, different nonzero monomials belong to *different* weight subspaces. It then suffices to prove that at least one monomial is not zero. If all such monomials were equal to zero, then one would find on a *smaller* level a vector which would being annihilated by all operators e_i $(i = 1, \ldots, p)$, i.e., would belong to $(V_\lambda)^N$. However, this contradicts to (2.2). □

After these preparations we come back to
Proof of (4.4). Recall that $l_1 = (\lambda^\vee, \varphi_1)$ and

$$\bigoplus_{n\ge 0} V^0_{n\lambda} \simeq k[C(\lambda^*)]^T \text{ and } \bigoplus_{n\ge 0} V_{n\lambda'} \simeq k[C(\lambda'^*)] \ .$$

We define the map $\mathcal{A} : k[C(\lambda^*)]^T \to k[C(\lambda^*)]^N \simeq k[C(\lambda'^*)]$ as follows. If $v \in V^0_{n\lambda}$ then $\mathcal{A}(v) = \frac{1}{(l_1 n)!} A^{l_1 n} v$. By (4.6) $\mathcal{A}$ is injective and one needs only to prove that it respects multiplication. This follows at once from the Leibniz rule and condition (4.6)(1), which shows that only the required summand survives ($v_i \in V^0_{n_i\lambda}$):

$$\mathcal{A}(v_1 \cdot v_2) = \frac{1}{(l_1(n_1+n_2))!} A^{l_1(n_1+n_2)}(v_1 \cdot v_2) =$$

$$= \frac{1}{(l_1(n_1+n_2))!} \sum_{i=0}^{l_1(n_1+n_2)} \binom{l_1(n_1+n_2)}{i} A^i v_1 \cdot A^{l_1(n_1+n_2-i)} v_2$$
$$= \frac{1}{(l_1(n_1+n_2))!} \binom{l_1(n_1+n_2)}{l_1 n_1} A^{l_1 n_1} v_1 \cdot A^{l_1 n_2} v_2 = \mathcal{A}v_1 \cdot \mathcal{A}v_2 \ .$$

Thus, $\mathcal{A}$ is a graded algebra monomorphism and the first part is proved. If λ satisfies $(\diamond)$, then $\mathcal{A}$ is one-to-one due to (4.2), hence $C(\lambda^*)/\!\!/T \simeq C(\lambda'^*)$. Finally, HV-varieties sitting in dual modules are isomorphic as abstract varieties, hence $C(\lambda'^*) \simeq C(\lambda')$. □

Remark. Theorem 4.3 follows from (4.4) as well. We included the separate proof of the former because it is more direct and can be transformed into a "tableau-free" proof. In fact, one may avoid appeal to tableaux and prove (4.3) together with the first equality in (4.2) by means of the following argument. We work with C as in (4.3). The first part of that proof established that $i_1 \otimes i_2$ is a monomorphism for any λ. Geometrically, this means that the natural map

$$(i_1 \otimes i_2)^\star : C/\!\!/T \to C((p+1)\varphi_p)/\!\!/T \times C(\lambda^*)/\!\!/T$$

is dominant. Then one can prove in a purely geometric way that $(i_1 \otimes i_2)^\star$ is *finite* whenever λ satisfies $(\diamond)$. It is striking to see how $(\diamond)$ arises from the application of the

Hilbert–Mumford criterion to the T-action on C. Then coming back to algebra, we have $\dim V_\lambda^0 = \dim V^0_{\lambda+n(p+1)\varphi_1}$ for $n \gg 0$. By monotonicity of weight multiplicities, one then concludes this holds for any $n \in \mathbb{N}$ and thus $i_1 \otimes i_2$ is an isomorphism. The end of proof is the same as in (4.3).

5 HV-varieties with free algebra of T-invariants

In this section we find all the dominant weights of simple groups such that $C(\lambda)/\!\!/T$ is an affine space and describe the representation of the Weyl group that affords $C(\lambda)/\!\!/T$.

5.1 Lemma. *Suppose $C(\lambda)/\!\!/T$ is an affine space and $a \in \mathbb{N}$ is the minimal number such that $a\lambda \in Q$. Then*

(i) $\dim C(\lambda) \le \dim T + q(\lambda)$;

(ii) $\dim C(\lambda) < \dim T + q(\lambda)/a + 1$.

Proof. It follows from stability of the action that $\dim C(\lambda)/\!\!/T = \dim C(\lambda) - \dim T$. If $C(\lambda)/\!\!/T$ is an affine space then $q(\lambda/\!\!/T)$ is the sum of degrees of free generators of $k[C(\lambda)]^T$. Whence $q(\lambda/\!\!/T) \ge \dim C(\lambda) - \dim T$. If $\lambda \in Q$, then $q(\lambda/\!\!/T) = q(\lambda)$, which proves (i) in this particular case. In general, we have $C(a\lambda)/\!\!/T \simeq C(\lambda)/\!\!/T$ and by the preceding part the inequality $\dim C(a\lambda) \le \dim T + q(a\lambda)$ holds. Obviously, $\dim C(a\lambda) = \dim C(\lambda)$. From (1.4)(ii) it follows that $q(a\lambda) < q(\lambda)/a + 1$, which proves (ii). Finally, (i) is a consequence of (ii). $\square$

The condition (ii) is stronger, but it is easier to deal first with (i) in classification arguments.

5.2 Lemma. *If condition (5.1)(i) holds for $\lambda = \sum_{i\in I} l_i\varphi_i$ $(l_i \ne 0)$ and $\#I \ge 2$, then there is $i_0 \in I$ such that $\dim C(\varphi_{i_0}) \le \dim T + q(\varphi_{i_0}) - 2$.*

Proof. Since $\dim C(\lambda) > \dim C(\varphi_j)$ for any $j \in I$, it suffices to prove that $q(\lambda) < q(\varphi_{i_0})$ for some $i_0 \in I$. From (1.3)(i) it follows that $q(\lambda) \le q(\sum_{i\in I}\varphi_i)$. Hence we may assume that $\lambda = \sum_{i\in I}\varphi_i$. Recall that $q(\varphi_i)$ is being computed as follows. Set $\mathcal{P}_i = \{\alpha \in \Delta_+ \mid (\alpha, \varphi_i) > 0\}$. Then $|\mathcal{P}_i| = q(\varphi_i)\varphi_i$, i.e., $q(\varphi_i) = (|\mathcal{P}_i|, \varphi_i)/(\varphi_i, \varphi_i)$. Similarly, set $\mathcal{P}_\lambda = \cup_{i\in I}\mathcal{P}_i$. Then $|\mathcal{P}_\lambda| = \sum_{i\in I} c_i\varphi_i$ and $q(\sum_{i\in I}\varphi_i) = \max\{c_i\}$. Since $(|\mathcal{P}_\lambda|, \varphi_i) = (|\mathcal{P}_i|, \varphi_i)$, we see that $c_i < q(\varphi_i)$ for every $i \in I$. Thus, the inequality $q(\lambda) < q(\varphi_{i_0})$ holds for i_0 such that $c_{i_0} = \max\{c_i\}$. $\square$

The main result of this section is the following

5.3 Theorem. *Let $C(\lambda)$ be an HV-variety of a simple group G. Then $C(\lambda)/\!\!/T$ is an affine space if and only if either λ or λ^* occur in the following table.*

(The column 'dim' contains dimension of $C(\lambda)/\!\!/T$, the next one contains degrees of basic invariants, and the Weyl group representation on $C(\lambda)/\!\!/T$ is indicated in the last column.

The number d is equal to $\gcd(n,p{+}1)$. The notation for representations will be explained below.)

Group	λ	dim	degrees	repr
$\mathbf{A}_{\mathrm{p}}$	$n\varphi_1$	1	$(p+1)/d$	$(sign)^{n/d}$
	$(p-1)\varphi_1+\varphi_2+n(p+1)\varphi_1$	p	$1,\ldots,1$	$R\otimes(sign)^n$
	$\varphi_1+\varphi_p+n(p+1)\varphi_1$	p	$1,\ldots,1$	$R\otimes(sign)^{n+1}$
$\mathbf{B}_{\mathrm{p}}$	φ_1	p	$1,2,\ldots,2$	$R(\mathfrak{S}_p)\oplus(sign)$
$(p\geq 2)$	$2\varphi_1$	p	$1,\ldots,1$	$R(\mathfrak{S}_{\mathfrak{p}})\oplus\mathbf{1}$
$\mathbf{C}_{\mathrm{p}}$	φ_1	p	$2,\ldots,2$	R
$(p\geq 2)$	$2\varphi_1$	p	$1,\ldots,1$	R
$\mathbf{D}_{\mathrm{p}}$	φ_1	$p-1$	$2,\ldots,2$	$R(\mathfrak{S}_{\mathfrak{p}})$
$(p\geq 3)$	$2\varphi_1$	$p-1$	$1,\ldots,1$	$R(\mathfrak{S}_{\mathfrak{p}})$

Proof. Let us say that λ is *good* whenever $C(\lambda)/\!\!/T$ is an affine space. First, we shall prove that all good weights occur in the table. The general strategy is as follows. Let λ be good. Then it satisfies (5.1) and if it contains several fundamental weights then one of those satisfies even the stronger condition (5.2). So, we find all fundamental weights satisfying (5.1) and then check whether is it possible to add something without disturbing this condition. We apply (5.1)(i) for an initial estimate and then (5.1)(ii) for "fine tuning".

I. $G=\mathbf{A}_{\mathrm{p}}$.
1. Here $q(\varphi_i)=p+1$ for any i and $\dim C(\varphi_i)=(p+1-i)i+1$. Hence (5.1)(i) gives $i^2-i(p+1)+2p\geq 0$. Analyzing it, we obtain that φ_1, φ_2, φ_{p-1}, φ_p satisfies (5.1)(i) for any p, φ_3 – for $p=5,6$, and φ_4 – for $p=6$. By duality, we may always assume that λ contains φ_i with $i\leq(p+1)/2$.
2. Consider the weights containing φ_1. Obviously, all multiples of φ_1 are good weights (in this case $\dim C(\lambda)/\!\!/T=1$). Look at other possibilities with φ_1. Assume that $\lambda=c_1\varphi_1+\varphi_i+(\ldots)$ satisfies (5.1)(i). Because $\dim C(\lambda)\geq\dim C(\varphi_1+\varphi_i)$ and $q(\lambda)\leq q(\varphi_1{+}\varphi_i)$, $\varphi_1{+}\varphi_i$ also satisfies it. Since $\dim C(\varphi_1{+}\varphi_i)=(p{+}2)i{-}i^2$ and $q(\varphi_1{+}\varphi_i)\leq p$, one obtains the inequality $(i-2)(i-p)\geq 0$ in this case. That is, either $i=2$ or $i=p$. One checks $\varphi_1+\varphi_2+\varphi_p$ $(p\geq 3)$ does not satisfy (5.1). Hence, there are two possibilities:

(a) $\lambda=c_1\varphi_1+c_2\varphi_2$. Here $\dim C(\lambda)=2p$ and $\lambda_\Gamma=2\varphi_1+p\varphi_2$. Therefore: (5.1)(i) holds for λ iff $q(\lambda)=p$ iff $c_2=1$. Applying more precise inequality (5.1)(ii), we have $2p<p+\frac{p}{a}+1$. Whence $a=1$, i.e., $\lambda\in Q$. This means that $c_1=(p-1)+n(p+1)$, $n=0,1,\ldots$.

(b) $\lambda=c_1\varphi_1+c_p\varphi_p$. Here $\dim C(\lambda)=2p$ and $\lambda_\Gamma=p\varphi_1+p\varphi_2$. Therefore: (5.1)(i) holds for λ iff $q(\lambda)=p$ iff either $c_1=1$ or $c_p=1$. Again, inequality (5.1)(ii) shows that $\lambda\in Q$ and this gives us weights $\varphi_1+\varphi_p+n(p+1)\varphi_1$, $n=0,1,\ldots$ or their duals.
3. Next, we are to consider dominant weights that contain φ_2 and do not contain both

φ_1 and φ_p; in particular, $p \geq 3$. Let $\lambda = \varphi_2$. Then $a \geq (p+1)/2$ and from (5.1)(ii) it follows that $p \leq 3$. Since $\mathbf{A}_3 = \mathbf{D}_3$, this case is relevant to items 8,9 in the table. Next, assume that $\lambda = \varphi_2 + \varphi_i + (\ldots)$ $(2 < i < p)$ and (5.1)(i) holds for it. Then

$$2p - 1 = \dim C(\varphi_2) < \dim C(\lambda) \leq p + q(\lambda) < p + q(\varphi_2) = 2p + 1 \ .$$

This means $\dim C(\lambda) = \dim C(\varphi_2) + 1$, which is possible only for $\lambda = \varphi_2 + c_1\varphi_1$. Thus, this case gives us no new admissible weights.
4. The last case in question is $\lambda = \varphi_3 + (\ldots)$ for $p = 5, 6$. Since we may assume that λ do not contain φ_1, φ_2, φ_{p-1}, φ_p, it remains only a few possibilities. One easily checks that (5.1)(ii) is never satisfied for them.

For other simple groups we play the same game and the work turns out to be shorter. We present only some details for the symplectic and the exceptional groups.

II. $G = \mathbf{C}_{\rm p}$, $(p \geq 2)$.
Here $q(\varphi_i) = 2p+1-i$ and $\dim C(\varphi_i) = (2p+1-2i)i + (i^2 - i)/2 + 1$, $(1 \leq i \leq p)$. The inequality (5.1)(i) holds in the following cases: φ_1 – for any p, φ_2 – for $p \leq 3$, and φ_3 – for $p = 3$. As a result, besides the items in the table, we obtain that (5.1)(ii) holds only for $\lambda = \varphi_2$ and $p = 3$. However, $k[C(\varphi_2)]^T$ is not a polynomial ring for Sp_6. Indeed, $\varphi_2 \in Q$ and $q(\varphi_2 /\!\!/ T) = \dim C(\varphi_2)/\!\!/T = 5$. Since $\dim V^0_{\varphi_2} = 2$, the algebra $k[C(\varphi_2)]^T$ can not be generated by 5 elements of degree 1.

III. For the exceptional groups, the only fundamental (and even dominant) weights satisfying (5.1)(i) are $(\mathbf{E}_6,\ \varphi_1)$ and $(\mathbf{G}_2,\ \varphi_1)$. However, $\varphi_1 \notin Q$ for $\mathbf{E}_6$ and (5.1)(ii) does not hold in this case. As for the $\mathbf{G}_2$-case, because $C(\varphi_1)$ is a hypersurface in 7-dimensional space, it is not hard to prove that $C(\varphi_1)/\!\!/T$ is a hypersurface, too.

We are left to show that all weights λ in the table are good. Items 1 and 4–9 are related to the simplest representations of classical groups and therefore they are easy. It has already been observed in [8, 2.6]. As for the two infinite strings, items 2 and 3, these cases are covered by (4.5). However, a direct "tableau-free" proof for $\lambda = \varphi_1 + \varphi_p$ is found in [8, 3.1]. □

Remark. One may note that $\dim C(\lambda)/\!\!/T \leq \dim T$ for all good λ's. However, the converse is not true; e.g., for $G = SL_3$ this inequality holds for any λ.

The Weyl group $W = N(T)/T$ acts naturally on $C(\lambda)/\!\!/T$. When λ is good, one obtains a representation of W. Let us call it the quotient representation. Next, we describe them for the λ's in the table.
Whenever all free generators of $k[C(\lambda)]^T$ are of the *same* degree, say m, the quotient map has the following interpretation. Denote by π the composition

$$C(\lambda) \hookrightarrow V_\lambda \xrightarrow{\phi} V_{m\lambda} \xrightarrow{\tau} V^0_{m\lambda} \ ,$$

where τ is the T-equivariant projection and ϕ corresponds to the natural homomorphism $V_{m\lambda}^* \to S^m V_\lambda^* = k[V_\lambda]_m$. Then π is the quotient map, i.e. $V_{m\lambda}^0 \simeq C(\lambda)/\!\!/T$. Therefore the quotient representation is just the zero weight representation. This applies to all but one items in the table, the remaining case is easily treated by hand. We use the following notation for representations in the table: R is the standard reflection representation of the corresponding Weyl group; 'sign' is the one-dimensional representation $w \mapsto (-1)^{l(w)}$; $R(\mathfrak{S}_\mathrm{p})$ is the reflection ($(p-1)$-dimensional) representation of the symmetric group $\mathfrak{S}_\mathrm{p}$; $\mathbf{I}$ is the trivial 1-dimensional representation.
As a by-product, one can see when the quotient representation is a reflection representation, i.e., when $C(\lambda)/\!\!/N(T)$ is an affine space:

5.4 Proposition. *Suppose that both $C(\lambda)/\!\!/T$ and $C(\lambda)/\!\!/N(T)$ are affine spaces. Then λ or λ^* is contained in the following list:* $\mathbf{A}_\mathrm{p}$ – $n\varphi_1$ *(n/d is even),* $\varphi_1+\varphi_p+n(p+1)\varphi_1$, *($n$ is even),* $(p-1)\varphi_1+\varphi_2+n(p+1)\varphi_1$, *($n$ is odd);* $\mathbf{B}_\mathrm{p}$ – $2\varphi_1$*;* $\mathbf{C}_\mathrm{p}, \mathbf{D}_\mathrm{p}$ – $\varphi_1, 2\varphi_1$. □

It is worth noting that there are weights such that $C(\lambda)/\!\!/N(T)$ is an affine space, while $C(\lambda)/\!\!/T$ is not. For example, this is the case for $G = SL_6$ and $\lambda = 2\varphi_3$, where $C(2\varphi_3)/\!\!/T$ is a hypersurface (see [3, 3.3]).

6 Examples and problemes

6.1 Example: $G = SL_3$. In this case all the weight multiplicities have an explicit simple description (see e.g. [8, 4.4]). It is then not hard to write down the Poincaré series $F_{\lambda/\!\!/T}$ for arbitrary λ. Suppose $\lambda = a\varphi_1 + b\varphi_2$, $a \ge b$. Then

$$F_{\lambda/\!\!/T}(t) = \begin{cases} \dfrac{1+(3b-1)t^3}{(1-t^3)^2}, & \text{if} \quad a-b \equiv \pm 1 \pmod 3 \\ \dfrac{1+(b-1)t}{(1-t)^2}, & \text{if} \quad a-b \equiv 0 \pmod 3 \ . \end{cases}$$

¿From Stanley's characterisation of Gorenstein domains it follows that $C(\lambda)/\!\!/T$ is Gorenstein if and only if $b = 0$ or $b = 1$ & $a \equiv b \pmod 3$ or $b = 2$ & $a \equiv b \pmod 3$. The first two series give us affine spaces. As for the third one, theorem 4.4 applies and one obtains that $C(2\varrho + 3n\varphi_1)/\!\!/T$ is a hypersurface (a quadric in $\mathbb{A}^3$). Moreover, $C(2\varrho + 3n\varphi_1)/\!\!/N(T)$ is an affine space. Together with the example at the end of the previous section, this gives some hope that the following might be true:

If $C(\lambda)/\!\!/N(T)$ is an affine space, then $C(\lambda)/\!\!/T$ is a complete intersection.

6.2 Duality for $\mathbf{A}_\mathrm{p}$. Looking through the table, one sees that the weights $\varphi_1+\varphi_p$ and $(p-1)\varphi_1+\varphi_2$, which generate two infinite series, correspond to conjugate partitions. Brion has suggested that it might be explained by means of an involution on the ring of symmetric functions which takes each elementary symmetric function e_r to the complete symmetric function h_r (see [6, ch. I]). Following this hypothesis, one may address the following question:

Let λ be a dominant weight such that $|\underline{\lambda}| = p+1$ and $\widehat{\lambda}$ the weight corresponding to the conjugate partition. Is it true that $C(\lambda)/\!\!/ \simeq C(\widehat{\lambda})/\!\!/T$?
At least, this is the case, if $\underline{\lambda}$ is a hook. For, then (4.4) applies and it turns out that λ' and $\widehat{\lambda}'$ correspond to dual fundamental representation of SL_p.

6.3 Isotypic components. The most interesting problem concerning the actions of T on $C(\lambda)$'s is the study of isotypic components. For any $\mu \in \mathcal{X}(T)$, one defines μ-isotypic component in $k[C(\lambda^*)]$ by $k[C(\lambda^*)]^{(\mu)} := \oplus_{n=0}^{\infty} V_{n\lambda}^{\mu}$. It is a graded $k[C(\lambda^*)]^T$-module whose structure has a lot to do with asymptotic behaviour of μ-weight multiplicities. The main question is:

When $k[C(\lambda^*)]^{(\mu)}$ is a Cohen-Macaulay (=CM) $k[C(\lambda^*)]^T$-module ?

Of course, if the quotient map $\pi_T : C(\lambda^*) \to C(\lambda^*)/\!\!/T$ is equidimensional, then all isotypic components are CM. But this is a common property of actions of reductive groups and unlike the case of representations [10], no other general results of qualitative nature are known for HV-varieties.
Next, we show how one can find many examples of non CM isotypic components for $G = SL_{p+1}$. We retain notation of sect. 4. Suppose $\lambda \in \mathcal{X}_+ \cap Q$ and $(\diamond)$ is satisfied. Then λ can be written as $\lambda = m(p+1)\varphi_1 + \tilde{\lambda}$, where $m \geq 0$ and one has equality in $(\diamond)$ for $\tilde{\lambda}$. Let μ be arbitrary nonzero weight of $V_{m(p+1)\varphi_1}$. Then μ is a weight of V_λ as well. Denote by $F_{\lambda^*|\mu}(t)$ the Poincaré series of $k[C(\lambda^*)]^{(\mu)}$.

Fact. $F_{\lambda^*|\mu}(t) = F_{\lambda^*/\!\!/T}(t) - 1$. Therefore in many cases $k[C(\lambda^*)]^{(\mu)}$ is not CM.

Indeed, if we treat μ as a weight of $V_{n\lambda}$, then each part of $\underline{\mu} \ \geq n\lambda_2$. From (4.1), it then follows that $\dim V_{n\lambda}^{\mu} = \dim V_{n\lambda'} = \dim V_{n\lambda}^{0}$ for $n \geq 1$. Since $k[C(\lambda^*)]^T$ is CM, we have

$$F_{\lambda^*/\!\!/T}(t) = \frac{H(t)}{\prod_{i=1}^{r}(1-t^{d_i})} ,$$

where $H(t) \in \mathbb{Z}[t]$ has nonnegative coefficients and $d_1, \ldots, d_r$ are degrees of a homogeneous system of parameters. It is easy to see that pretty often $H(t) - \prod_{i=1}^{r}(1-t^{d_i})$, the numerator of $F_{\lambda^*|\mu}(t)$, has negative coefficients (e.g. if $k[C(\lambda^*)]^T$ is polynomial or r is even) and therefore $k[C(\lambda^*)]^{(\mu)}$ can not be CM.

A Appendix

Here we prove the properties of edge labels mentioned in (1.5). Along with the root system Δ, we shall consider the dual system $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$. The notation $\mathrm{ht}\,\alpha^\vee$ refers to the height of a root in $\Delta^\vee$.

A.1 Theorem. *Let G be a simple algebraic group and h the Coxeter number of its root system. Then*

(i) $\mathrm{rk}\, G+1 \le q(\varphi_i) \le h$ *for any* i*;*

(ii) $q(\varphi_i)=h$ *if and only if* φ_i *is minuscule.*

Proof. Recall that $\mathcal{P}_i=\{\alpha\in\Delta_+ \mid (\alpha,\varphi_i)>0\}$ and $|\mathcal{P}_i|=q(\varphi_i)\varphi_i$. Let W_i be the subgroup of W generated by all simple reflections, except for s_i, and let $w_i\in W_i$ be the longest element. Then $w_i(\mathcal{P}_i)=\mathcal{P}_i$ and $w_i(\Delta_+\setminus\mathcal{P}_i)=-(\Delta_+\setminus\mathcal{P}_i)$. Therefore $\varrho+w_i\varrho=|\mathcal{P}_i|$. Take any $\alpha\in\mathcal{P}_i$. Then $(\varrho,\alpha^\vee)+(\varrho,w_i(\alpha^\vee))=(|\mathcal{P}_i|,\alpha^\vee)=q(\varphi_i)(\varphi_i,\alpha^\vee)$. That is, $\mathrm{ht}\,\alpha^\vee+\mathrm{ht}\,w_i(\alpha^\vee)=q(\varphi_i)(\varphi_i,\alpha^\vee)$. Taking $\alpha=\alpha_i$, one obtains $1+\mathrm{ht}\,w_i(\alpha_i^\vee)=q(\varphi_i)$. Since $\mathrm{ht}\,\alpha^\vee\le h-1$ for any $\alpha^\vee\in\Delta^\vee$, we have $q(\varphi_i)\le h$. Furthermore, $q(\varphi_i)=h$ if and only if $w_i(\alpha_i^\vee)$ is the highest root in $\Delta^\vee$. In this case, the equality $1=(\varphi_i,\alpha_i^\vee)=(\varphi_i,w_i(\alpha_i^\vee))$ implies that $|(\varphi_i,\alpha^\vee)|\le 1$ for any $\alpha\in\Delta$, i.e., φ_i is minuscule. On the other side, $w_i(\alpha_i^\vee)$ is the root of maximal length among the roots such that $(\varphi_i,\alpha^\vee)=1$. Since this set contains the root $(\sum_{i=1}^p\alpha_i)^\vee$, we have $\mathrm{ht}\,w_i(\alpha_i^\vee)\ge p=\mathrm{rk}\,G$. $\square$

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ul. Akad. Anokhina, d.30, kor.1, kv.7
Moscow 117602 Russia

Current address: *Max-Planck-Institut für Mathematik*
Gottfried-Claren-Str. 26
D-53225 Bonn, Deutschland

e-mail: panyush@dpa.msk.ru panyush@mpim-bonn.mpg.de