Holomorphic operator-valued symbols for edge-degenerate pseudo-differential operators

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Abstract

The Mellin quantization assigns to an edge–degenerate pseudo–differential operator an operator based on the Mellin transform in the cone axis direction. This procedure is here presented in a new and more precise way, which leads to isomorphisms between edge–degenerate and holomorphic Mellin symbols. Furthermore, we introduce a suitable class of operator–valued edge symbols without asymptotics. They are, in particular, parameter dependent pseudo–differential operators on a cone with smooth closed base.

1 Introduction

In this paper we describe some essential elements of the pseudo–differential analysis on a manifold with edges. Such a manifold is a topological space with the structure of a smooth manifold outside the edge Y; near each point $y \in Y$ it looks like the product of an open set $\Omega \subset \mathbb{R}^q$ and a cone C with a smooth closed base. One basic idea in the approach of SCHULZE [Sza], [ES] to handle such a singular object is to formulate a pseudo–differential calculus via operator–valued symbols, that is, symbols taking values in an established algebra on the cone. In other words, the calculus is generated by iteration of calculi on manifolds of less singular type.

Following this idea, we introduce a class of edge symbols, which are operator families parametrized by $y \in Y$ and the corresponding covariable in $\mathbb{R}^q$ and which take their values in the cone algebra on C. Near the conical singularity these families are defined in terms of holomorphic Mellin symbols, whereas in the interior by standard pseudo–differential symbols; both types are degenerate in a certain sense. These two components of the operator–valued edge symbols are related by the so–called Mellin quantization, which maps a degenerate pseudo–differential symbol to a holomorphic one in a way that the corresponding operators differ by a smoothing remainder. The first variant of this quantization for classical symbols was introduced in [Szb] (cf. also [DS], [SSb]), using techniques of asymptotic summation. Here we present a new proof of this result and obtain exact formulas both for the holomorphic symbol and for the remainder, formulas that are valid also for non–classical symbols. Moreover, we achieve topological isomorphisms between the symbol classes involved.

The Mellin quantization immediatly leads to another very important tool of the theory in the spirit of [Szc], [ES], the so–called kernel cut–off. Roughly speaking, this construction ensures that (Mellin) pseudo–differential operators can be written, modulo smoothing remainders, with symbols that extend to entire functions in the covariable.

The material of this paper has applications in future works of the authors, e.g. [GSS], where we develop a general algebra of pseudo–differential operators on a wedge–shaped manifold, including a natural Fréchet topology.

Let us finally note that, for example, Plamenevskij [Pl], Melrose, Mendoza [MM] and Mazzeo [M] also deal with operators on singular manifolds. While the nature of the degenerate symbols considered by these authors has some intersection with the type discussed here, the structure and intention of our calculus is fundamentally different.

Basic notation

$$\mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}, \quad \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

A cut–off function is a non-negative function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega \equiv 1$ near $t = 0$. For $u \in C_0^\infty(\mathbb{R}^n)$, $v \in C_0^\infty(\mathbb{R}_+)$ the Fourier and Mellin transform respectively, are given by

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx, \quad \mathcal{M}v(z) = \int_0^\infty t^{z-1} v(t)\, dt.$$

These transforms can be extended to more general (distribution) spaces.

Let $U \subset \mathbb{R}^m$ be open, and set $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$. Then $S^\mu(U \times \mathbb{R}^n)$ consists of all $p \in C^\infty(U \times \mathbb{R}^n)$ with

$$\sup_{x \in K, \xi \in \mathbb{R}^n} \left\{ |D_\xi^\alpha D_x^\beta p(x,\xi)|\, \langle\xi\rangle^{|\alpha|-\mu} \right\} < \infty \tag{1}$$

for all $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^m$, and all compact sets $K \subset U$. This is a Fréchet space. Further, such a p is called *classical* if there are symbols $p_{(\mu-j)} \in S^{\mu-j}(U \times \mathbb{R}^n)$, which are homogeneous of degree $\mu - j$ in the covariable ξ for large $|\xi|$, and

$$r_N(p) := p - \sum_{j=0}^{N-1} \in S^{\mu-N}(U \times \mathbb{R}^n) \quad \text{for all } N \in \mathbb{N}_0.$$

The space of these symbols is denoted by $S^\mu_{c\ell}(U \times \mathbb{R}^n)$. In view of the homogeneity, the functions $p_{(\mu-j)}$ can be identified with elements of $C^\infty(U \times S^{n-1})$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. The projective limit with respect to the maps $p \mapsto p_{(\mu-j)}$ and $p \mapsto r_N(p)$, gives rise to a Fréchet topology on $S^\mu_{c\ell}(U \times \mathbb{R}^n)$; this topology is stronger than that induced by $S^\mu(U \times \mathbb{R}^n)$.

To a symbol $p \in S^\mu(U \times U \times \mathbb{R}^n)$ with $U \subset \mathbb{R}^n$ we associate in the standard way its pseudo–differential operator $\mathrm{op}(p) : C_0^\infty(U) \to C^\infty(U)$ by

$$[\mathrm{op}(p)u](x) = \iint e^{i(x-x')\xi} p(x, x', \xi) u(x')\, dx' đ\xi.$$

Here $đ\xi = (2\pi)^{-n}\, d\xi$. The space of these operators is denoted by $L^\mu(U)$. We also consider the space $L^\mu(U; \mathbb{R}^q)$ of parameter dependent operators, where the parameter is treated as an additional covariable.

Moreover, set $S^\mu(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^n) = S^\mu(\mathbb{R} \times U \times \mathbb{R}^n)|_{\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^n}$. This is a Fréchet space if we take as semi–norms the analogous expressions as in (1), where now K is a compact set in $\overline{\mathbb{R}}_+ \times U$.

We also consider operators on half spaces, where the action along the inner normal is formulated in terms of the Mellin transform. For real β set $\Gamma_\beta = \{z \in \mathbb{C} \mid \mathrm{Re}\, z = \beta\}$. Under the

identification $\beta + i\tau \mapsto \tau : \Gamma_\beta \to \mathbb{R}$ we obtain the symbol classes $S^\mu(U \times \Gamma_\beta \times \mathbb{R}^n)$, and $S^\mu(\overline{\mathbb{R}}_+ \times U \times \Gamma_\beta \times \mathbb{R}^n)$, where $\Gamma_\beta \times \mathbb{R}^n \cong \mathbb{R}^{1+n}$ serves as the space of covariables.
If $U \in \mathbb{R}^n$ and $h(t, t', x, x', z, \xi) \in S^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times U \times U \times \Gamma_{1/2-\gamma} \times \mathbb{R}^n)$ we define the operator $\mathrm{op}_M^\gamma(\mathrm{op}(h)) : C_0^\infty(\mathbb{R}_+ \times U) \to C^\infty(\mathbb{R}_+ \times U)$ by

$$[\mathrm{op}_M^\gamma(\mathrm{op}(h))u](t,x) = \int_{\Gamma_{1/2-\gamma}} \int_0^\infty \left(\frac{t}{t'}\right)^{-z} \mathrm{op}(h)(t,t',z)u(t')\frac{dt'}{t'} đz.$$

Here $đz = (2\pi i)^{-1}dz$, and for t' fixed, $u(t')$ is viewed as a function in $C_0^\infty(U)$.

2 Mellin quantization

For a closed compact smooth manifold X let us set

$$X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X);$$

this is interpreted as the infinite cone with base X and vertex represented by $\{0\}\times X$, contracted to one point. Moreover, define the open stretched cone

$$X^\wedge = \mathbb{R}_+ \times X$$

with a fixed splitting of coordinates (t, x). If $(\tilde{t}, \tilde{x}) \in X^\wedge$ is another splitting of coordinates then we require that the transformation $(t, x) \to (\tilde{t}(t, x), \tilde{x}(t, x))$ extends to a diffeomorphism $\overline{\mathbb{R}}_+ \times X \to \overline{\mathbb{R}}_+ \times X$, where $\tilde{t}(0, x) = 0$.
The local model of a manifold with edges is $X^\Delta \times \Omega$ with a smooth, closed, compact manifold X and an open set $\Omega \subset \mathbb{R}^q$ that corresponds to a coordinate neighborhood on the edge. We employ the coordinates (t, x, y) on the associated open stretched wedge $X^\wedge \times \Omega$. To describe the interior symbols we fix a chart on X with local coordinates $x \in V$, V being an open set in $\mathbb{R}^n$.
The typical differential operators on a wedge are of the form

$$t^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(t,y)(-t\partial_t)^j(tD_y)^\alpha, \tag{2}$$

with coefficients $a_{j\alpha}(t, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$, i.e., smooth functions with values in the differential operators on X. Among these operators (for $\mu = 2$) are the Laplace–Beltrami operators for (warped) wedge metrics (i.e., metrics of the form $dt^2 + t^2 g_X(t) + dy^2$ with metric $g_X(t)$ on X depending smoothly on $t \in \overline{\mathbb{R}}_+$). We can interpret (2) as a Mellin operator with the operator–valued symbol

$$H(t,y,z,\eta) = t^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(t,y)z^j(t\eta)^\alpha,$$

which is a holomorphic family of operators with

$$H(t,y,\beta+i\tau,\eta) \in L^\mu(X;\mathbb{R}_\tau)$$

for every $(t, y, \eta) \in \overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^q$ uniformly for β in compact intervals.
This motivates the following definitions:

Definition 2.1 Let $\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$ denote the space of all symbols p, which are of the form

$$p(t,x,y,\tau,\xi,\eta) = \tilde{p}(t,x,y,t\tau,\xi,t\eta)$$

with a symbol $\tilde{p} \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$. Such a symbol p is called *edge-degenerate.* Analogously we define the corresponding spaces of classical symbols, indicated by subsript $c\ell$, where $\tilde{p}$ is asked to be classical.

Definition 2.2 Let $S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ be the space of all functions $h \in C^\infty(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$, which are holomorphic in $z \in \mathbb{C}$ with values in $S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{n+q})$, and

$$h(t,x,y,z,\xi,\eta)|_{\Gamma_\beta} \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_\beta \times \mathbb{R}^{n+q}_{\xi,\eta})$$

for every $\beta \in \mathbb{R}$ and uniformly for β in compact intervals. This is a Fréchet space with the system of semi-norms

$$\sup_{c \leq \beta \leq c'} |h(t,x,y,\beta + i\tau,\xi,\eta)|,$$

where $|\cdot|$ runs over a system of semi-norms of $S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}_{\tau,\xi,\eta})$. Analogously to Definition 2.1 we also introduce $\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ as the space of all symbols h satisfying

$$h(t,x,y,z,\xi,\eta) = \tilde{h}(t,x,y,z,\xi,t\eta),$$

where $\tilde{h}(t,x,y,z,\xi,\eta) \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$.

Similarly to the cone theory it is essential to establish a suitable Mellin quantization in order to obtain (continuous) actions in the natural weighted edge Sobolev spaces. The local Mellin quantization provides a relation between (y,η)-dependent operator families $\mathrm{op}_{t,x}(p)(y,\eta)$, defined in terms of the Fourier transform in (t,x), and $\mathrm{op}_M^{1/2}(\mathrm{op}_x(g))(y,\eta)$ defined by means of the Mellin transform in t. Moreover, we can pass from Mellin symbols $g(t,t',x,y,\tau,z,\eta)$, $z \in \Gamma_0$, to symbols that extend in z holomorphically to the whole complex plane.

Theorem 2.3 (Mellin quantization) *Let $p(t,x,y,\varrho,\xi,\eta) \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$. Then there exists a symbol $h(t,x,y,z,\xi,\eta) \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ such that*

$$\mathrm{op}_{t,x}(p)(y,\eta) = \mathrm{op}_M^{\frac{1}{2}}(\mathrm{op}_x(h))(y,\eta) \mod C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+ \times V; \mathbb{R}^q)). \tag{3}$$

Moreover, the symbol $\tilde{h}$ allows the following asymptotic expansion

$$\tilde{h}(t,x,y,i\varrho,\xi,\eta) \sim \tilde{p}(t,x,y,-\varrho,\xi,\eta) + \sum_{k=1}^{\infty} \Big(\sum_{j=0}^{k} c_{kj} \partial_\varrho^{k+j} \tilde{p}(t,x,y,-\varrho,\xi,\eta) \varrho^j \Big) \tag{4}$$

with certain $c_{kj} \in \mathbb{R}$ that are independent of $\tilde{p}$; the asymptotic summation being carried out in $S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$. An analogous result is valid for classical symbols. In this case we have the homogeneous components

$$\tilde{h}_{(\mu-l)}(t,x,y,i\varrho,\xi,\eta) = \sum_{k=0}^{l} \sum_{j=0}^{k} c_{kj} (\partial_\varrho^{k+j} \tilde{p})_{(\mu-l-j)}(t,x,y,-\varrho,\xi,\eta) \varrho^j.$$

PROOF. Let $u \in C_0^\infty(\mathbb{R}_+, C^\infty(V))$, then

$$\begin{aligned}
&\mathrm{op}_{t,x}(p)(y,\eta)u(t) \\
&= \int_{-\infty}^{\infty}\int_0^\infty e^{i(t-t')\tau}\mathrm{op}_x(p)(t,y,\tau,\eta)u(t')\,dt'd\!\!{}^-\tau \\
&= \int_{-\infty}^{\infty}\int_0^\infty \left(\frac{t}{t'}\right)^{-i\tau} M(t,t')t'\mathrm{op}_x(p)(t,y,-M(t,t')\tau,\eta)u(t')\frac{dt'}{t'}d\!\!{}^-\tau
\end{aligned}$$

with the transformation $\tau \to -M(t,t')\tau$, where $M(t,t') := \frac{\log t - \log t'}{t-t'}$ for $t, t' \in \mathbb{R}_+$. We have $M \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and $M > 0$ (precise calculations and further properties of $M(t,t')$ can be found, e.g., in [SSa, Section 2.4]). If we set

$$g(t,t',x,y,i\tau,\xi,\eta) = M(t,t')t'p(t,x,y,-M(t,t')\tau,\xi,\eta)$$

then $g \in S^\mu(\mathbb{R}_+^2 \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q})$ and

$$\mathrm{op}_{t,x}(p)(y,\eta) = \mathrm{op}_M^{\frac{1}{2}}(\mathrm{op}_x(g))(y,\eta)$$

for every $(y,\eta) \in \Omega \times \mathbb{R}^q$. The operator $\mathrm{op}_x(g)(t,t',y,i\tau,\eta)$ is already an operator–valued Mellin symbol. The next step is to pass to a holomorphic symbol h being additionally independent of t'. The procedure is similar to that from the usual pseudo–differential calculus; we reduce g to a sum of properly supported and smoothing parts, and calculate the corresponding complete symbol h.

Let $\phi \in C_0^\infty(\mathbb{R}_+)$ with $\phi \equiv 1$ near to 1. For $z \in \mathbb{C}$ put now

$$h(t,x,y,z,\xi,\eta) = t^z\mathrm{op}_M^{\frac{1}{2}}(\phi(t'/t)g)(x,y,\xi,\eta)t^{-z}.$$

Then, applying the change of variables $t' \to tr$

$$\begin{aligned}
h(t,x,y,z,\xi,\eta) &= \int_{-\infty}^{\infty}\int_0^\infty \left(\frac{t}{t'}\right)^{-i\tau+z}\phi(t'/t)g(t,t',x,y,i\tau,\xi,\eta)\frac{dt'}{t'}d\!\!{}^-\tau \\
&= \int_{-\infty}^{\infty}\int_0^\infty r^{i\tau-z}\phi(r)g(t,tr,x,y,i\tau,\xi,\eta)\frac{dr}{r}d\!\!{}^-\tau,
\end{aligned}$$

the integrals being understood as oscillatory integrals. It is easy to see that

$$g(t,tr,x,y,i\tau,\xi,\eta) = rM(r,1)\tilde{p}(t,x,y,-M(r,1)\tau,\xi,t\eta),$$

and so

$$h(t,x,y,z,\xi,\eta) = \iint r^{i\tau-z}\phi(r)M(r,1)\tilde{p}(t,x,y,-M(r,1)\tau,\xi,t\eta)\,dr d\!\!{}^-\tau. \tag{5}$$

We next prove that $h(t,x,y,\delta+i\varrho,\xi,t^{-1}\eta) = \tilde{h}(t,x,y,\delta+i\varrho,\xi,\eta) \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}_{\varrho,\xi,\eta})$ uniformly in $\delta \in [c_1,c_2]$ for every $c_1 < c_2$. The symbol estimates of $\tilde{h}$ will be recovered from those of $\tilde{p}$, hence we actually have that $\tilde{h}$ depends continuously on $\tilde{p}$. With the change of variables $\tau \to \tau + \varrho$ we get

$$\begin{aligned}
&\tilde{h}(t,x,y,\delta+i\varrho,\xi,\eta) \\
&= \iint r^{1+i\tau}\underbrace{r^{-\delta}\phi(r)M(r,1)}_{=:\psi_\delta(r)}\tilde{p}(t,x,y,-M(r,1)(\tau+\varrho),\xi,\eta)\frac{dr}{r}d\!\!{}^-\tau \\
&= \iint (1+i\tau)^{-N}r^{1+i\tau}(-r\partial_r)^N(\psi_\delta(r)\tilde{p}(t,x,y,-M(r,1)(\tau+\varrho),\xi,\eta))\frac{dr}{r}d\!\!{}^-\tau.
\end{aligned}$$

The last integral converges for $N \in \mathbb{N}$ big enough. Now

$$(-r\partial_r)^N(\psi_\delta(r)\tilde{p}(t,x,y,-M(r,1)(\tau+\varrho),\xi,\eta) = \sum_{k=0}^{N} c_k(r)(\partial_\tau^k\tilde{p})(\ldots)(\tau+\varrho)^k$$

with $c_k \in C_0^\infty(\mathbb{R}_+)$. For the symbol estimates of $\tilde{h}$ we only have to investigate the derivatives of $\tilde{p}$. For $l \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^{n+q}$ and $\beta \in \mathbb{N}_0^{1+n+q}$

$$\begin{aligned} I_{l,\alpha,\beta} &:= \left|\partial_\varrho^l D_{\xi,\eta}^\alpha D_{t,x,y}^\beta[(\partial_\tau^k\tilde{p})(t,x,y,-M(r,1)(\tau+\varrho),\xi,\eta)(\tau+\varrho)^k]\right| \\ &\leq \sum_{j=0}^{l}\left|c_j\partial_\varrho^j[(\partial_\tau^k D_{\xi,\eta}^\alpha D_{t,x,y}^\beta\tilde{p})(\ldots)]\partial_\varrho^{l-j}(\tau+\varrho)^k]\right|. \end{aligned}$$

Since

$$\begin{aligned} &|\partial_\varrho^j[(\partial_\tau^k D_{\xi,\eta}^\alpha D_{t,x,y}^\beta\tilde{p})(t,x,y,-M(r,1)(\tau+\varrho),\xi,\eta)]| \\ &\qquad = |(\partial_\tau^{j+k} D_{\xi,\eta}^\alpha D_{t,x,y}^\beta\tilde{p})(\ldots)(-M(r,1))^j| \leq c(r)\,\langle\tau+\varrho,\xi,\eta\rangle^{\mu-j-k-|\alpha|} \end{aligned}$$

with $c \in C^\infty(\mathbb{R}_+)$, and because of $|\partial_\varrho^{l-j}(\tau+\varrho)^k| \leq c\,\langle\tau+\varrho,\xi,\eta\rangle^{j+k-l}$, we obtain together with Peetre's inequality

$$I_{l,\alpha,\beta} \leq \tilde{c}_{l,\alpha,\beta}(r)\,\langle\tau+\varrho,\xi,\eta\rangle^{\mu-l-|\alpha|} \leq c_{l,\alpha,\beta}(r)\,\langle\tau\rangle^{|\mu-l-|\alpha||}\,\langle\varrho,\xi,\eta\rangle^{\mu-l-|\alpha|}.$$

Furthermore, since $\partial_\tau^k\psi_\delta$ depends continuously on δ for every $k \in \mathbb{N}_0$, the symbol estimates of $\tilde{h}$ are uniform in $c_1 < \delta < c_2$. If we analogously look now at the semi-norms of $\tilde{h}$ in $S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{n+q})$, we can easily see that $\tilde{h}$ depends holomorphically on z.
From a standard method of the pseudo-differential calculus (cf. [Sh, Theorem 3.1]), we have for each $N \in \mathbb{N}$ the expansion

$$\begin{aligned} \tilde{h}(t,x,y,i\varrho,\xi,\eta) &= \sum_{k=0}^{N-1}\frac{1}{k!}(-t'\partial_{t'})^k\partial_\varrho^k\left\{\phi(t'/t)g(t,t',x,y,i\varrho,\xi,t^{-1}\eta)\right\}|_{t'=t} \\ &\quad + \int_0^1\frac{(1-\theta)^{N-1}}{(N-1)!}\,\tilde{h}_N(t,x,y,\varrho,\xi,\eta,\theta)d\theta, \end{aligned} \qquad (6)$$

where $\tilde{h}_N(t,x,y,\varrho,\xi,\eta,\theta)$ is given by

$$\int_{-\infty}^{\infty}\int_0^\infty\left(\frac{t}{t'}\right)^{-i\tau}(i\tau)^N\phi(t'/t)(\partial_\varrho^N g)(t,t',x,y,i(\varrho+\theta\tau),\xi,t^{-1}\eta)\frac{dt'}{t'}đ\tau.$$

Moreover,

$$\partial_\varrho^k g(t,t',x,y,i\varrho,\xi,t^{-1}\eta) = M(t,t')t'(\partial_\varrho^k\tilde{p})(t,x,y,-M(t,t')t\varrho,\xi,\eta)(-M(t,t')t)^k$$

so that the sum on the right-hand side of (6) equals

$$\sum_{k=0}^{N-1}\frac{1}{k!}(t'\partial_{t'})^k\left\{t^k\phi(t'/t)M(t,t')^{k+1}t'(\partial_\varrho^k\tilde{p})(t,x,y,-M(t,t')t\varrho,\xi,\eta)\right\}\Big|_{t'=t}, \qquad (7)$$

and furthermore $\tilde{h}_N(t,x,y,\varrho,\xi,\eta,\theta)$ equals

$$\int_{-\infty}^{\infty}\int_0^{\infty} r^{i\tau}(-i\tau)^N\phi(r)M(r,1)^{N+1}(\partial_\varrho^N\tilde{p})(t,x,y,-M(r,1)(\varrho+\theta\tau),\xi,\eta)\,dr\,d\!\!\,\bar{}\tau.$$

The last identity clearly yields, as above, that the remainder in the expansion (6) belongs to $S^{\mu-N}(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{R}^{1+n+q}_{\varrho,\xi,\eta})$, showing that the asymptotic summation of $\tilde{h}$ can be carried out in the space of symbols defined up to $t=0$. Finally, (4) follows from (7) by applying the Leibniz formula, the chain rule

$$\partial_{t'}^L(u\circ v)=\sum_{j=1}^{L}\sum_{\gamma_1+\ldots+\gamma_j=L}c_j(\partial_{t'}^j u)(v(t'))\,\partial_{t'}^{\gamma_1}v\ldots\partial_{t'}^{\gamma_j}v$$

with $L,\gamma_j,c_j\in\mathbb{N}$, and the identity $\partial_{t'}^k M(t,t')|_{t'=t}=(-1)^k\dfrac{k!}{k+1}\dfrac{1}{t^{k+1}}$. □

Remark 2.4 a) Let $\tilde{p}\in S^\mu(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{R}^{1+n+q})$ and $\tilde{h}\in S^\mu(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{C}\times\mathbb{R}^{n+q})$ be as in the preceding Theorem 2.3. A Taylor expansion shows that

$$\tilde{h}(t,x,y,\delta+i\varrho,\xi,\eta)=\sum_{k=0}^{N-1}\frac{\delta^k}{k!}(\partial_z^k\tilde{h})(t,x,y,i\varrho,\xi,\eta)\;+\;r_{N,\delta}(t,x,y,\varrho,\xi,\eta),$$

where the remainder $r_{N,\delta}$ equals

$$\frac{\delta^N}{(N-1)!}\int_0^1(1-\theta)^{N-1}(\partial_z^N\tilde{h})(t,x,y,\theta\delta+i\varrho,\xi,\eta)\,d\theta.$$

Since $(\partial_z^N\tilde{h})(t,x,y,\theta\delta+i\varrho,\xi,\eta)\in S^{\mu-N}(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{R}^{1+n+q}_{\varrho,\xi,\eta})$ uniformly in $0\le\theta\le 1$, $c\le\delta\le c'$, we have $r_{N,\delta}\in S^{\mu-N}(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{R}^{1+n+q})$ uniformly for δ in compact intervalls. From this we can draw the following conclusions:

- If $\tilde{h}\in S^{\mu-\delta}(\overline{\mathbb{R}}_+\times V\times\Omega\times\Gamma_0\times\mathbb{R}^{n+q})$ for some $\delta>0$, then $\tilde{h}\in S^{\mu-\delta}(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{C}\times\mathbb{R}^{n+q})$.
- $\tilde{h}(t,x,y,\delta+i\varrho,\xi,\eta)-\tilde{p}(t,x,y,-\varrho,\xi,\eta)\in S^{\mu-1}(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{R}^{1+n+q}_{\varrho,\xi,\eta})$ for all δ.
- If $\tilde{p}$, and thus $\tilde{h}$, are classical symbols, the homogeneous principal symbol (with respect to (ϱ,ξ,η)) of $\tilde{h}$ is independent of $\delta\in\mathbb{R}$.

b) Let h be as in (3). In view of the holomorphy of h in z we obtain as a consequence of the Cauchy Theorem that

$$\operatorname{op}_M^{\frac{1}{2}}(\operatorname{op}_x(h))(y,\eta)=\operatorname{op}_M^{\gamma}(\operatorname{op}_x(h))(y,\eta)\quad\text{on } C_0^\infty(\mathbb{R}_+\times V)$$

for arbitrary $\gamma\in\mathbb{R}$.

Corollary 2.5 *The proof of Theorem 2.3 supplies a map*

$$\tilde{S}^\mu(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{R}^{1+n+q})\longrightarrow\tilde{S}^\mu(\overline{\mathbb{R}}_+\times V\times\Omega\times\mathbb{C}\times\mathbb{R}^{n+q})\tag{8}$$

such that $p(t,x,y,\tau,\xi,\eta) \mapsto h(t,x,y,z,\xi,\eta)$ implies (3). Moreover, if we fix there a function ϕ, the corresponding mapping

$$\tilde{p} \mapsto \tilde{h} : S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}) \longrightarrow S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$$

is continuous.

Due to the Closed Graph Theorem the above continuity also holds when the symbols are classical. Remember that the Fréchet topology of $S^\mu_{c\ell}$ is stronger than that induced by S^μ.

Theorem 2.6 *The map (8) induces isomorphisms*

$$\begin{aligned} &\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})/\tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}) \\ \cong\ &\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})/\tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q}). \end{aligned}$$

Corresponding isomorphisms hold with subscript $c\ell$.

Proof. Let us denote by $m(p)(t,x,y,z,\xi,\eta)$ the Mellin quantization of a symbol p according to Theorem 2.3, where we fix the function ϕ of the proof. Because of (4) we have trivially injectivity of the induced mapping of m. For the surjectivity let

$$h(t,x,y,z,\xi,\eta) \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$$

be given. Choose $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ with $\tilde{\psi} \equiv 1$ near to 0. Set $\psi(t,t') := \tilde{\psi}(\log t' - \log t)$ and $\psi(r) := \tilde{\psi}(\log r)$ for $t, t', r \in \mathbb{R}_+$. Imitating the proof of Theorem 2.3 we set

$$q(t,t',x,y,\tau,\xi,\eta) = M(t,t')^{-1}t'^{-1}h(t,x,y,-iM(t,t')^{-1}\tau,\xi,\eta),$$

$$p(t,x,y,\tau,\xi,\eta) = e^{-it\tau}\mathrm{op}_t(\psi(t,t')q)e^{it\tau};$$

then similar calculations show

$$\begin{aligned} p(t,x,y,\tau,\xi,\eta) &= \iint e^{i(t-t')(\varrho-\tau)}\psi(t,t')q(t,t',x,y,\varrho,\xi,\eta)\,dt'đ\varrho \\ &= \iint r^{-i\varrho}e^{-i(1-r)t\tau}\psi(r)h(t,x,y,-i\varrho,\xi,\eta)\frac{dr}{r}đ\varrho \end{aligned}$$

due to the changes of variables $t' \to tr$ and $\varrho \to t^{-1}M(r,1)\varrho$. In this manner we obtain the corresponding symbol

$$\tilde{p}(t,x,y,\tau,\xi,\eta) = \iint r^{-i\varrho}e^{-i(1-r)\tau}\psi(r)\tilde{h}(t,x,y,-i\varrho,\xi,\eta)\frac{dr}{r}đ\varrho \tag{9}$$

in $S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$. According to (5) the Mellin symbol to p looks like

$$\begin{aligned} m(p)(t,x,y,i\sigma,\xi,\eta) &= \iint s^{i(\tau-\sigma)}\phi(s)M(s,1)\tilde{p}(t,x,y,-M(s,1)\tau,\xi,t\eta)\,dsđ\tau \\ &= \iint s^{-i\sigma}e^{i(s-1)\tau}\phi(s)\tilde{p}(t,x,y,-\tau,\xi,t\eta)\,dsđ\tau \end{aligned}$$

with the transformation $\tau \to M(s,1)^{-1}\tau$. Inserting (9) yields

$$= \iint s^{-i\sigma} e^{i(s-1)\tau} \phi(s) \left\{ \iint r^{-i\varrho} e^{i(1-r)\tau} \psi(r) h(t,x,y,-i\varrho,\xi,\eta) \frac{dr}{r} đ\varrho \right\} ds đ\tau$$

$$= \lim_{\varepsilon\to 0} \iint s^{-i\sigma} e^{is\tau} \phi(s) \left\{ \iint r^{-i\varrho} e^{-ir\tau} \psi(r) \tilde{\psi}(\varepsilon\varrho) h(t,x,y,-i\varrho,\xi,\eta) \frac{dr}{r} đ\varrho \right\} ds đ\tau$$

$$= \lim_{\varepsilon\to 0} \int s^{-i\sigma} \phi(s) \left\{ \int e^{is\tau} \left\{ \int e^{-ir\tau} \psi(r) \left\{ \int r^{-i\varrho} \tilde{\psi}(\varepsilon\varrho) h(\dots)\, đ\varrho \right\} \frac{dr}{r} \right\} đ\tau \right\} ds$$

$$= \lim_{\varepsilon\to 0} \int s^{-i\sigma} \phi(s)\psi(s) \left\{ \int s^{-i\varrho} \tilde{\psi}(\varepsilon\varrho) h(t,x,y,-i\varrho,\xi,\eta)\, đ\varrho \right\} \frac{ds}{s}$$

$$= \iint s^{i(\varrho-\sigma)} \underbrace{\phi(s)\psi(s)}_{=:\chi(s)} h(t,x,y,i\varrho,\xi,\eta) \frac{ds}{s} đ\varrho.$$

Obviously, $\chi(s) \in C_0^\infty(\mathbb{R}_+)$, and $\chi \equiv 1$ near to 1. Now

$$\begin{aligned} & h(t,x,y,i\sigma,\xi,\eta) - m(p)(t,x,y,i\sigma,\xi,\eta) \\ &= h(t,x,y,i\sigma,\xi,\eta) - \int_{-\infty}^{\infty}\int_0^\infty s^{i(\varrho-\sigma)} \chi(s) h(t,x,y,i\varrho,\xi,\eta) \frac{ds}{s} đ\varrho \\ &= h(t,x,y,i\sigma,\xi,\eta) - \int_{-\infty}^{\infty} h(t,x,y,i(\varrho+\sigma),\xi,\eta) u(i\varrho) đ\varrho, \end{aligned}$$

where $u(i\varrho)$ is the Mellin transform of χ. The difference $h - m(p)$ belongs to $\tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q})$, and using Remark 2.4 a) we even obtain

$$(h - m(p))(t,x,y,z,\xi,\eta) \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q}).$$

In order to demonstrate that the symbol $(h-m(p))(t,x,y,i\sigma,\xi,t^{-1}\eta)$ is smoothing, we expand $\tilde{h}(t,x,y,i(\varrho+\sigma),\xi,\eta)$ in ϱ near to $\varrho = 0$, resulting in

$$\tilde{h}(t,x,y,i(\varrho+\sigma),\xi,\eta) = \sum_{k=0}^{N-1} \frac{(i\varrho)^k}{k!} (\partial_z^k \tilde{h})(t,x,y,i\sigma,\xi,\eta) \;+\; r_{N,\varrho}(t,x,y,i\sigma,\xi,\eta)$$

with the remainder

$$r_{N,\varrho} = \frac{(i\varrho)^N}{(N-1)!} \int_0^1 (1-\theta)^{N-1} (\partial_z^N \tilde{h})(t,x,y,i(\theta\varrho+\sigma),\xi,\eta)\, d\theta.$$

The property of the Mellin transform $\int (i\varrho)^k u(i\varrho) đ\varrho = ((-r\partial_r)^k \chi)(1)$ implies

$$\sum_{k=0}^{N-1} \int (\partial_z^k \tilde{h})(t,x,y,i\sigma,\xi,\eta)(i\varrho)^k u(i\varrho) đ\varrho = \tilde{h}(t,x,y,i\sigma,\xi,\eta).$$

Finally, as in the proof of Theorem 2.3, it is easy to verify that the symbol

$$(h - m(p))(t,x,y,i\sigma,\xi,t^{-1}\eta) = \int r_{N,\varrho}(t,x,y,i\sigma,\xi,\eta) u(i\varrho) đ\varrho$$

belongs to $S^{\mu-N}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q})$. This completes the proof. □

Definition 2.7 The space of holomorphic Fourier symbols $S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times i\mathbb{C} \times \mathbb{R}^{n+q})$ consists of all functions $p(t,x,y,z,\xi,\eta)$ such that

$$(t,x,y,z,\xi,\eta) \mapsto p(t,x,y,iz,\xi,\eta) \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q}).$$

Furthermore we introduce the corresponding space $\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times i\mathbb{C} \times \mathbb{R}^{n+q})$, consisting of all symbols p satisfying

$$p(t,x,y,z,\xi,\eta) = \tilde{p}(t,x,y,tz,\xi,t\eta)$$

for a certain $\tilde{p} \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times i\mathbb{C} \times \mathbb{R}^{n+q})$. The topology is that induced by $S^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$.

Theorem 2.8 (Kernel cut–off) *To each $q \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$ there exists a symbol $p \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times i\mathbb{C} \times \mathbb{R}^{n+q})$ such that*

$$q - p \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}),$$

i.e., q can be "approximated" by a holomorphic symbol.

PROOF. Let $h = m(q)$ be the Mellin quantization of q. If we define $\tilde{p}$ as in (9), where $\tau \in \mathbb{R}$ is replaced by $z \in \mathbb{C}$, the associated symbol p is an element of $\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times i\mathbb{C} \times \mathbb{R}^{n+q})$, and from the proof of Theorem 2.6 we know that

$$m(p) - m(q) = m(p) - h \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q}).$$

The injectivity then implies that $p - q \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$. □
To $h \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q})$ it also makes sense to define $p \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$ via formula (9). This gives rise to the "inverse Mellin quantization"

$$\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q}) \longrightarrow \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times i\mathbb{C} \times \mathbb{R}^{n+q}). \tag{10}$$

For this map we then obtain results corresponding to Theorems 2.6 and 2.8:

Theorem 2.9 *The map (10) induces isomorphisms*

$$\begin{aligned} &\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q})/\tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q}) \\ \cong\ &\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times i\mathbb{C} \times \mathbb{R}^{n+q})/\tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times i\mathbb{C} \times \mathbb{R}^{n+q}). \end{aligned}$$

Theorem 2.10 (Kernel cut–off) *To each $g \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q})$ there exists an $h \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ such that*

$$g - h \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \Gamma_0 \times \mathbb{R}^{1+n+q}),$$

i.e., each Mellin symbol given on a vertical line in $\mathbb{C}$ can be "approximated" by a holomorphic one.

3 Edge symbols

For $s \in \mathbb{R}$ is $H^s(\mathbb{R}^n)$ the standard Sobolev space, and for $s, \delta \in \mathbb{R}$ we define

$$H^{s,\delta}(\mathbb{R}^n) := \left\{ \langle \cdot \rangle^{-\delta} u \mid u \in H^s(\mathbb{R}^n) \right\}$$

with the norm $\|v\|_{H^{s,\delta}(\mathbb{R}^n)} = \| \langle \cdot \rangle^{\delta} v\|_{H^s(\mathbb{R}^n)}$.

Definition 3.1 Let $s, \gamma \in \mathbb{R}$. Then $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ is defined as the closure of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|^2 = \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \int (1 + |z|^2 + |\xi|^2)^s |(\mathcal{M}_{t\to z}\mathcal{F}_{x\to\xi}u)(z,\xi)|^2 \, dz \, d\xi. \tag{11}$$

Here $\mathcal{F}$ is the Fourier transform and $\mathcal{M}$ the Mellin transform.

Remark 3.2 The analysis of edge pseudo–differential operators will be formulated in terms of operator–valued symbols acting between suitable Sobolev spaces on the infinite cone. To define these spaces, we fix an open covering $\{U_1, \ldots, U_N\}$ of X with corresponding diffeomorphisms $\chi_j : U_j \to V_j \subset \mathbb{R}^n$ and $\tilde{\kappa}_j : U_j \to \tilde{V}_j \subset S^n$, where S^n is the unit sphere in $\mathbb{R}^{1+n}$. To the latter ones associate

$$\kappa_j : \mathbb{R}_+ \times U_j \to \mathbb{R}^{1+n}, \quad (t,x) \mapsto t\tilde{\kappa}_j(x).$$

Further let $\{\phi_1, \ldots, \phi_N\}$ be a subordinate partition of unity, and $\{\psi_j \in C_0^\infty(U_j);\, j = 1, \ldots, N\}$ be a system of functions satisfying $\phi_j\psi_j = \phi_j$ for all $j = 1, \ldots, N$.

Definition 3.3 For $s, \gamma \in \mathbb{R}$ let $\mathcal{K}^{s,\gamma}(X^\wedge)$ denote the closure of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\|u\|^2 = \sum_{j=1}^{N} \|(\omega\phi_j u) \circ (1 \times \chi_j)^{-1}\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} + \|((1-\omega)\phi_j u) \circ \kappa_j^{-1}\|^2_{H^s(\mathbb{R}^{1+n})}, \tag{12}$$

where the functions on the right–hand side are extended by zero outside their natural domains. This construction is (up to equivalent norms) independent of the specific choice of all involved data, and yields a scale of Hilbert spaces. Analogously we introduce the spaces $E^{s,\gamma}(X^\wedge)$ replacing the second term on the right–hand side of (12) by the norm $\|((1-\omega)\phi_j u) \circ \kappa_j^{-1}\|^2_{H^{s,s}(\mathbb{R}^{1+n})}$. That is, in $E^{s,\gamma}(X^\wedge)$ we deal away from $t = 0$ with weighted Sobolev spaces.

We have $\mathcal{K}^{s,\gamma}(X^\wedge) \subset H^s_{loc}(X^\wedge)$, and

$$\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge).$$

Further let $k^\varrho(t)$ be a non–vanishing smooth weight function which equals t^ϱ near $t = 0$, and is identically 1 at infinity. Then we obtain

$$\mathcal{K}^{s,\gamma+\varrho}(X^\wedge) = k^\varrho \mathcal{K}^{s,\gamma}(X^\wedge)$$

for all $\gamma, \varrho, s \in \mathbb{R}$.

Definition 3.4 For an operator $A \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\varrho}(X^\wedge))$ we can define the *formal adjoint* with respect to the scalar product $(\cdot,\cdot)_{\mathcal{K}^{0,0}(X^\wedge)}$ as the unique operator $A^* \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{K}^{s,-\varrho}(X^\wedge), \mathcal{K}^{s-\mu,-\gamma}(X^\wedge))$ satisfying

$$(Au, v)_{\mathcal{K}^{0,0}(X^\wedge)} = (u, A^* v)_{\mathcal{K}^{0,0}(X^\wedge)}$$

for all $u, v \in C_0^\infty(X^\wedge)$.

Remark 3.5 For each $\lambda > 0$ define the linear mapping $\kappa_\lambda : C_0^\infty(X^\wedge) \longrightarrow C_0^\infty(X^\wedge)$ by

$$(\kappa_\lambda u)(t, x) := \lambda^{(n+1)/2} u(\lambda t, x),$$

where $n = \dim X$. These mappings extend by continuity to linear operators on $\mathcal{K}^{s,\gamma}(X^\wedge)$ for all $s, \gamma \in \mathbb{R}$, and the set $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a *(strongly continuous) group* of isomorphisms, that means

(i) $\kappa_\lambda \kappa_\varrho = \kappa_{\lambda\varrho}$ for all $\lambda, \varrho > 0$ and $\kappa_1 = \mathrm{id}_{\mathcal{K}^{s,\gamma}(X^\wedge)}$,

(ii) for each $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$ the function $\lambda \mapsto \kappa_\lambda u : \mathbb{R}_+ \to \mathcal{K}^{s,\gamma}(X^\wedge)$ is continuous.

The same is true on $E^{s,\gamma}(X^\wedge)$.

Definition 3.6 Let us fix a strictly positive function $\eta \mapsto [\eta]$ in $C^\infty(\mathbb{R}^q)$ with $[\eta] = |\eta|$ for $|\eta| > c$, for a constant $c > 0$. Let $\gamma, \delta, \nu \in \mathbb{R}$. The space $\mathrm{R}_G^\nu(\Omega \times \mathbb{R}^q, (\gamma, \delta))$ of *Green symbols* for the local edge calculus consists of all operator-valued functions

$$g(y, \eta) \in C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q))$$

such that

(i) for every cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$

$$g_\omega(y, \eta) := \omega g(y, \eta)\omega \in \bigcap_{s,r\in\mathbb{R}} C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), E^{r,\delta}(X^\wedge)));$$

(ii) the following symbol estimates hold

$$\|\kappa_{[\eta]}^{-1}\{D_\eta^\alpha D_y^\beta g_\omega(y, \eta)\}\kappa_{[\eta]}\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}, E^{r,\delta})} \le c[\eta]^{\nu-|\alpha|}$$

for all $s, r \in \mathbb{R}$, all multi-indices α, $\beta \in \mathbb{N}_0^q$ and all $y \in K$ for arbitrary $K \subset\subset \Omega$ and $\eta \in \mathbb{R}^q$, with constants $c = c(s, r, \alpha, \beta, K) > 0$;

(iii) the point-wise formal adjoint $g_\omega^*(y, \eta)$ satisfies analogous estimates with the norm in $\mathcal{L}(\mathcal{K}^{s,-\delta}, E^{r,-\gamma})$.

Henceforth we restrict ourselves to degenerate classical symbols. This is motivated by the following:

Lemma 3.7 *Let $p \in \tilde{S}_{cl}^\mu(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}) \cap S^{-\infty}(\mathbb{R}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$. Then it is even true that $p \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times U \times \mathbb{R}^{1+n+q})$.*

PROOF. Let $\tilde{p} \in S^{\mu}_{c\ell}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$ be the corresponding non-degenerate symbol to p, with the asymptotic expansion in homogeneous components

$$\tilde{p} \sim \sum_{j=0}^{\infty} \tilde{p}_{(\mu-j)} \quad \text{in} \quad S^{\mu}_{c\ell}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}).$$

Now $\tilde{p} \in S^{-\infty}(\mathbb{R}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$ implies that $\tilde{p}_{(\mu-j)} \equiv 0$ in $\mathbb{R}_+$, and by continuity also in $\overline{\mathbb{R}}_+$, for every j. Thus $\tilde{p} \in S^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$. □

For non-classical symbols the above result is in general not true. Set for instance $\tilde{p}(t, x, \tau, \xi, \eta) = q(\tau, \xi, \eta)\omega(t\tau, \xi, \eta)$, where $q(\tau, \xi, \eta) \in S^{\mu}(\mathbb{R}^{1+n+q})$ and $\omega(\tau, \xi, \eta) \in C_0^{\infty}(\mathbb{R}^{1+n+q})$ with $\omega \equiv 1$ near to 0. Note that $\omega(t\tau, \xi, \eta)$ is an element of $S^{-\infty}(\mathbb{R}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$, but if we demand symbol estimates up to $t = 0$, then it only belongs to $S^0(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$. Therefore, $p(t, x, \tau, \xi, \eta) = \tilde{p}(t, x, t\tau, \xi, t\eta)$ provides a counter-example.

In order to establish a notion of complete (interior) symbols for the edge symbols, we first have a look on the behaviour of a local symbol p under coordinate changes.

Remark 3.8 Let $\chi : V \to \tilde{V}$ be a diffeomorphism. It is well known that there exists an open neighbourhood U of the diagonal in $\tilde{V} \times \tilde{V}$ and a non-degenerate matrix-function $\Psi \in C^{\infty}(U, GL(n, \mathbb{R}))$ with

$$(\chi^{-1}(\tilde{x}) - \chi^{-1}(\tilde{y}))\Psi(\tilde{x}, \tilde{y})\tilde{\xi} = (\tilde{x} - \tilde{y})\tilde{\xi} \qquad \forall (\tilde{x}, \tilde{y}) \in U \; \forall \tilde{\xi} \in \mathbb{R}^n.$$

Let $\phi \in C^{\infty}(U)$ be properly supported, $\phi \equiv 1$ in a neighbourhood of the diagonal, and $\phi_1 \in C^{\infty}(V \times V)$ be given by $\phi_1(x, x') = \phi(\chi(x), \chi(x'))$. For $p \in \tilde{S}^{\mu}_{c\ell}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q})$ define $p_\chi \in \tilde{S}^{\mu}_{c\ell}(\overline{\mathbb{R}}_+ \times \tilde{V} \times \Omega \times \mathbb{R}^{1+n+q})$ by

$$\begin{aligned} p_\chi(t, \tilde{x}, y, \tau, \xi, \eta) \;\; = \;\; & \iint e^{iw\tilde{\xi}} \phi(\tilde{x}, \tilde{x} + w) \, |\det[(D\chi^{-1}(\tilde{x} + w))\Psi(\tilde{x}, \tilde{x} + w)]| \\ & p(t, \chi^{-1}(\tilde{x}), y, \tau, \Psi(\tilde{x}, \tilde{x} + w)(\xi + \tilde{\xi}), \eta) \, dw d\tilde{\xi}. \end{aligned}$$

Then the (y, η)-wise operator push-forward satisfies

$$(1 \times \chi)_* \mathrm{op}_{t,x}(p)(y, \eta) = \mathrm{op}_{t,\tilde{x}}(p_\chi)(y, \eta) + (1 \times \chi)_* \mathrm{op}_{t,x}((1 - \phi_1)p).$$

Clearly, the second term is an element of $C^{\infty}(\Omega, L^{-\infty}(\mathbb{R}_+ \times \tilde{V}; \mathbb{R}^q))$. Now assume that the operator of a symbol $p_1 \in \tilde{S}^{\mu}_{c\ell}(\overline{\mathbb{R}}_+ \times \tilde{V} \times \Omega \times \mathbb{R}^{1+n+q})$ also approximates $(1 \times \chi)_* \mathrm{op}_{t,x}(p)$ modulo $C^{\infty}(\Omega, L^{-\infty}(\mathbb{R}_+ \times \tilde{V}; \mathbb{R}^q))$. From the theory of parameter dependent pseudo-differential operators we conclude that $p_\chi - p_1 \in S^{-\infty}(\mathbb{R}_+ \times \tilde{V} \times \Omega \times \mathbb{R}^{1+n+q})$, hence $p_\chi - p_1 \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times \tilde{V} \times \Omega \times \mathbb{R}^{1+n+q})$ in view of Lemma 3.7. In other words, we have proved the following:

Lemma 3.9 *The operator push-forward $(1 \times \chi)_*$ induces an isomorphism*

$$\begin{aligned} & \tilde{S}^{\mu}_{c\ell}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}) / \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{R}^{1+n+q}) \\ \cong \;\; & \tilde{S}^{\mu}_{c\ell}(\overline{\mathbb{R}}_+ \times \tilde{V} \times \Omega \times \mathbb{R}^{1+n+q}) / \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times \tilde{V} \times \Omega \times \mathbb{R}^{1+n+q}), \end{aligned}$$

the so-called symbol push-forward. Furthermore, $((1 \times \chi)_*)^{-1} = (1 \times \chi^{-1})_*$.

Analogous statements are valid in the class of holomorphic Mellin symbols. The symbol $h_\chi \in \tilde{S}^\mu_{c\ell}(\overline{\mathbb{R}}_+ \times \tilde{V} \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ associated to $h \in \tilde{S}^\mu_{c\ell}(\overline{\mathbb{R}}_+ \times V \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ is defined in the same way as p_χ.

Remark 3.10 From the Mellin quantization (5), it is obvious that $p \in \tilde{S}^\mu_{c\ell}(\overline{\mathbb{R}}_+ \times \tilde{V} \times \Omega \times \mathbb{R}^{1+n+q})$ and $h = m(p) \in \tilde{S}^\mu_{c\ell}(\overline{\mathbb{R}}_+ \times \tilde{V} \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ implies $m(p_\chi) = h_\chi$. Thus, understood as operations on the corresponding equivalence classes, Mellin quantization and symbol push–forward are commuting, i.e.

$$m \circ (1 \times \chi)_* = (1 \times \chi)_* \circ m.$$

Let us fix again an open covering $\{U_1, \dots, U_N\}$ of X by coordinates neighbourhoods. Further let $\{\chi_1, \dots, \chi_N\}$ be the corresponding charts and $\{\phi_1, \dots, \phi_N\}$, $\{\psi_1, \dots, \psi_N\}$ as in Remark 3.2.

Definition 3.11 The set of complete symbols of order $\mu \in \mathbb{R}$ is defined as

$$\Sigma^\mu = \{(p_1, \dots, p_N) | \, p_j \in \tilde{S}^\mu_{c\ell}(\overline{\mathbb{R}}_+ \times V_j \times \Omega \times \mathbb{R}^{1+n+q}),\ p_j = (1 \times (\chi_j \circ \chi_i^{-1}))_* p_i\}.$$

Here the symbol push–forward corresponding to $\chi_j \circ \chi_i^{-1} : \chi_i(U_i \cap U_j) \to \chi_j(U_i \cap U_j)$ is as in Lemma 3.9.

Taking into account the previous remark, we can associate to a complete symbol $(p_1, \dots, p_N)$ the tuple $(h_1, \dots, h_N)$ via formula (5) in a convenient way.

At present, we turn to a class of operator–valued symbols that are parameter dependent families of pseudo–differential operators on the infinite cone $X^\wedge$.

Definition 3.12 For $\gamma, \mu \in \mathbb{R}$ let

$$\mathbf{R}^\mu(\Omega \times \mathbb{R}^q, (\gamma, \gamma - \mu))$$

be the space of all

$$a(y,\eta) = \omega_0(t[\eta])\, a_h(y,\eta)\, \omega_1(t[\eta]) + (1 - \omega_0(t[\eta]))\, a_p(y,\eta)\, (1 - \omega_2(t[\eta])) + g(y,\eta),$$

with

$$a_h(y,\eta) = \sum_{j=1}^N \Phi_j\, (1 \times \chi_j^{-1})_*\, t^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}} (\mathrm{op}_x(h_j))(y,\eta)\, \Psi_j \tag{13}$$

$$a_p(y,\eta) = \sum_{j=1}^N \Phi_j\, (1 \times \chi_j^{-1})_*\, t^{-\mu} \mathrm{op}_{t,x}(p_j)(y,\eta)\, \Psi_j, \tag{14}$$

where $(p_1, \dots, p_N) \in \Sigma^\mu$, $(h_1, \dots, h_N)$ the corresponding holomorphic tuple, Φ_j and Ψ_j are the multiplicators with ϕ_j and ψ_j, and $g \in \mathbf{R}^\mu_G(\Omega \times \mathbb{R}^q, (\gamma, \gamma - \mu))$. Further, $\omega_i \in C_0^\infty(\overline{\mathbb{R}}_+)$ are arbitrary cut–off functions satisfying $\omega_0(1 - \omega_1) = 0$, $\omega_2(1 - \omega_0) = 0$. Such an element $a(y,\eta)$ is called *edge symbol (without asymptotics)*.

Constructing to $(p_1, \dots, p_N)$ an edge symbol as above (without Green remainder) we obtain a map

$$\mathrm{Op} : \Sigma^\mu \to \mathbf{R}^\mu(\Omega \times \mathbb{R}^q, (\gamma, \gamma - \mu)).$$

Our next aim is to prove the following:

Theorem 3.13 *The mapping* Op *induces an isomorphism*

$$\Sigma^\mu / \Sigma^{-\infty} \to \mathbf{R}^\mu(\Omega \times \mathbb{R}^q, (\gamma, \gamma - \mu)) / \mathbf{R}^\mu_G(\Omega \times \mathbb{R}^q, (\gamma, \gamma - \mu)) \tag{15}$$

The inverse is denoted by σ^μ. It associates to an equivalence class of edge symbols the local pseudo–differential symbols of a representative.

To this end we need some further results.

Remark 3.14 For $\gamma \in \mathbb{R}$ the mapping $S_\gamma : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{1+n})$,

$$(S_\gamma u)(r, x) = e^{(\gamma - \frac{n+1}{2})r} u(e^{-r}, x),$$

induces isomorphisms $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^{1+n})$ for all $s \in \mathbb{R}$. If $h(t, t', x, x', z, \xi) \in C^\infty(\mathbb{R}^2_+ \times \mathbb{R}^{2n} \times \Gamma_{\frac{n+1}{2} - \gamma} \times \mathbb{R}^n)$, a (formal) computation shows that $\mathrm{op}_M^{\gamma - n/2}(\mathrm{op}_x(h)) = S_\gamma^{-1} \mathrm{op}_{r,x}(h_\gamma) S_\gamma$ with

$$h_\gamma(r, r', x, x', \tau, \xi) = h(e^{-r}, e^{-r'}, x, x', \tfrac{n+1}{2} - \gamma + i\tau, \xi).$$

We now define $S^\mu_F(\mathbb{R}^2_+ \times \mathbb{R}^{2n} \times \Gamma_{\frac{n+1}{2} - \gamma} \times \mathbb{R}^n)$ as the space of all such h with

$$\sup \left\{ \left| (t\partial_t)^k (t'\partial_{t'})^{k'} \partial_x^\beta \partial_{x'}^{\beta'} \partial_\tau^l \partial_\xi^\alpha h(t, t', x, x', \tfrac{n+1}{2} - \gamma + i\tau, \xi) \right| \langle \tau, \xi \rangle^{|\alpha| + l - \mu} \right\} < \infty$$

for all $k, k', l \in \mathbb{N}_0$, $\beta, \beta', \alpha \in \mathbb{N}_0^n$, and the supremum being taken over all $(t, t') \in \mathbb{R}^2_+$, $(x, x') \in \mathbb{R}^{2n}$, $(\tau, \xi) \in \mathbb{R}^{1+n}$. This is a Fréchet space. Then the Calderón–Vaillancourt Theorem for global pseudo–differential operators in $\mathbb{R}^{1+n}$ implies that $\mathrm{op}_M^{\gamma - n/2}(\mathrm{op}_x(\cdot))$ induces continuous mappings

$$S^\mu_F(\mathbb{R}^2_+ \times \mathbb{R}^{2n} \times \Gamma_{\frac{n+1}{2} - \gamma} \times \mathbb{R}^n) \to \mathcal{L}(\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n), \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n))$$

for each $s \in \mathbb{R}$. In particular, the operator–norm of $\mathrm{op}_M^{\gamma - n/2}(\mathrm{op}_x(h))$ can be estimated from above by a finite number of semi–norms introduced before.

Lemma 3.15 *For $h \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$, and $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$ set*

$$H(y, \eta) = \omega(t)\omega_0(t[\eta])\, \Phi\, t^{-\mu} \mathrm{op}_M^{\gamma - n/2}(\mathrm{op}_x(h))(y, \eta)\, \Psi\, \omega_1(t[\eta])\omega(t).$$

Then for each $s, r \in \mathbb{R}$

(i) $H \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n), \mathcal{H}^{r,\gamma-\mu}(\mathbb{R}_+ \times \mathbb{R}^n)))$,

and for each compact set $K \subset \Omega$ and all multi–indices α, β

(ii) $\|\kappa_{[\eta]}^{-1}\{D_\eta^\alpha D_y^\beta H(y,\eta)\}\kappa_{[\eta]}\|_{\mathcal{L}(\mathcal{H}^{s,\gamma},\mathcal{H}^{r,\gamma-\mu})} \le c[\eta]^{\mu-|\alpha|}$

for all $y \in K$, $\eta \in \mathbb{R}^q$, *with a constant* $c = c(s,r,\alpha,\beta,K)$.

PROOF. Since $t^{-\mu} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n) \to \mathcal{H}^{s,\gamma-\mu}(\mathbb{R}_+ \times \mathbb{R}^n)$ is continuous, and $\kappa_{[\eta]}^{-1} t^{-\mu}\kappa_{[\eta]} = [\eta]^\mu t^{-\mu}$, without loss of generality we can assume that $\mu = 0$.
Setting

$$h_1(y,\eta,t,t',x,x',z,\xi) = \omega(t)\omega_0(t[\eta])\omega_1(t'[\eta])\omega(t')\phi(x)\psi(x')h(t,x,y,z,\xi,\eta),$$

we obtain that $h_1 \in C^\infty(\Omega \times \mathbb{R}^q, S_F^\mu(\mathbb{R}_+^2 \times \mathbb{R}^{2n} \times \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^n))$. Hence (i) holds in view of $H(y,\eta) = \mathrm{op}_M^{\gamma-n/2}(\mathrm{op}_x(h_1))(y,\eta)$ and Remark 3.14. Now let $\tilde{h}$ be associated with h and define $h_2(y,\eta,t,t',x,x',z,\xi)$ by

$$\omega(t[\eta]^{-1})\omega_0(t)\omega_1(t')\omega(t'[\eta]^{-1})\phi(x)\psi(x')\tilde{h}(t[\eta]^{-1},x,y,z,\xi,t\eta[\eta]^{-1})$$

Then $\kappa_{[\eta]}^{-1}H(y,\eta)\kappa_{[\eta]} = \mathrm{op}_M^{\gamma-n/2}(\mathrm{op}_x(h_2))(y,\eta)$. The norm of the latter operator is estimated from above by a finite number of terms

$$c\sup\left\{|(t\partial_t)^k\partial_x^\sigma\partial_\tau^l\partial_\xi^\delta(\tilde{h}(t[\eta]^{-1},x,y,\tfrac{n+1}{2}-\gamma+i\tau,\xi,t\eta[\eta]^{-1}))|\,\langle\tau,\xi\rangle^{|\delta|+l-\mu}\right\},$$

where the supremum is taken over all $t \in \operatorname{supp}\omega_0$, $x \in \operatorname{supp}\phi$, $y \in K$, $(\tau,\xi,\eta) \in \mathbb{R}^{1+n+q}$, and c is a constant independent of $\tilde{h}$. Since

$$(t\partial_t)^L(f(t[\eta]^{-1},t\eta[\eta]^{-1})) = \sum_{k+|\alpha|\le L} t^{k+|\alpha|}a_{k\alpha}(\eta)(\partial_t^k\partial_\eta^\alpha f)(t[\eta]^{-1},t\eta[\eta]^{-1})$$

with certain $a_{k\alpha} \in S^0(\mathbb{R}^q)$, the above supremum can be estimated in terms of semi-norms of $\tilde{h}$ in $S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$. This shows (ii) for $\alpha = \beta = 0$. The general case is treated analogously with help of Leibniz formula. Note that ∂_η^α generates a factor $t^{|\alpha|}$, and $\kappa_{[\eta]}^{-1}t^{|\alpha|}\kappa_{[\eta]} = [\eta]^{-|\alpha|}t^{|\alpha|}$. □

Remark 3.16 If $S^{\mu,m}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ denotes the Fréchet space of all symbols $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ with

$$\sup_{x,x',\xi\in\mathbb{R}^n}\left\{|\partial_x^\beta\partial_{x'}^{\beta'}\partial_\xi^\alpha p(x,x',\xi)|\,\langle x\rangle^{|\beta|-m}\,\langle\xi\rangle^{|\alpha|-\mu}\right\} < \infty$$

for all $\alpha,\beta,\beta' \in \mathbb{N}_0^n$, then $\mathrm{op}(\cdot)$ induces for each $s,\delta \in \mathbb{R}$ continuous mappings

$$S^{\mu,m}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{L}(H^{s,\delta}(\mathbb{R}^n), H^{s-\mu,\delta-m}(\mathbb{R}^n)),$$

i.e., the operator norm can be estimated in terms of semi-norms of the symbol.

Lemma 3.17 *Let* $p \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \Omega \times \mathbb{R}^{1+n+q})$ *and* $\phi,\psi \in C_0^\infty(\mathbb{R}^n)$. *Set*

$$P(y,\eta) = \omega(t)(1-\omega_0(t[\eta]))\,\Phi\,t^{-\mu}\mathrm{op}_{t,x}(p)(y,\eta)\,\Psi\,(1-\omega_1(t[\eta]))\omega(t).$$

Then for each $s,\delta,\nu,m \in \mathbb{R}$

(i) $P \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H^{s,\delta}(\mathbb{R}^{1+n}), H^{s-\nu,\delta-m}(\mathbb{R}^{1+n})))$,

and for each compact set $K \subset \Omega$ and all multi-indices α, β

(ii) $\|\kappa_{[\eta]}^{-1}\{D_\eta^\alpha D_y^\beta P(y,\eta)\}\kappa_{[\eta]}\|_{\mathcal{L}(H^{s,\delta},H^{s-\nu,\delta-m})} \leq c[\eta]^{\mu-|\alpha|}$

for all $y \in K$, $\eta \in \mathbb{R}^q$, with a constant $c = c(s,\delta,\nu,m,\alpha,\beta,K)$.

PROOF. The calculations are similar to those of the previous lemma. We first define $p_1(y,\eta,t,x,t',x',\tau,\xi)$ by

$$t^{-\mu}\omega(t)(1-\omega_0(t[\eta]))(1-\omega_1(t'[\eta]))\omega(t')\phi(x)\psi(x')p(t,x,y,\tau,\xi,\eta),$$

then $p_1 \in C^\infty(\Omega\times\mathbb{R}^q, S^{-\infty,-\infty}(\mathbb{R}^{1+n}_{t,x}\times\mathbb{R}^{1+n}_{t',x'}\times\mathbb{R}^{1+n}))$ due to the compact support in (t,x,t',x'); besides $P(y,\eta) = \mathrm{op}_{t,x}(p_1)(y,\eta)$. The assertion (i) then follows from Remark 3.16. For the estimates in (ii) we have to consider the symbol $p_2(y,\eta,t,x,t',x',\tau,\xi)$ given by

$$[\eta]^\mu t^{-\mu}\omega(\frac{t}{[\eta]})(1-\omega_0(t))(1-\omega_1(t'))\omega(\frac{t'}{[\eta]})\phi(x)\psi(x')\tilde{p}(\frac{t}{[\eta]},x,y,t\tau,\xi,t\frac{\eta}{[\eta]}),$$

where $\tilde{p}$ is the non-degenerate symbol associated to p. It is again smoothing like p_1, so in particular, $p_2 \in C^\infty(\Omega\times\mathbb{R}^q, S^{\nu,m}(\mathbb{R}^{1+n}\times\mathbb{R}^{1+n}\times\mathbb{R}^{1+n}))$ for every $\nu, m \in \mathbb{R}$. Further, $\kappa_{[\eta]}^{-1}P(y,\eta)\kappa_{[\eta]} = \mathrm{op}_{t,x}(p_2)(y,\eta)$, and the operator-norm can be majorized by a finite number of expressions like

$$c\sup_{t,x,\tau,\xi}\Big\{|\partial_t^k\partial_x^\sigma\partial_\tau^l\partial_\xi^\delta\omega(t[\eta]^{-1})(1-\omega_0(t))t^{-\mu}\tilde{p}(t[\eta]^{-1},x,y,t\tau,\xi,t\eta[\eta]^{-1})|$$

$$\times\langle\tau,\xi\rangle^{l+|\delta|-\nu}\langle t,x\rangle^{k+|\sigma|-m}\Big\}[\eta]^\mu, \tag{16}$$

where $c > 0$ is independent of $\tilde{p}$, and t runs over $[c_1,\infty)$ for $c_1 > 0$. Looking first at the t-derivatives we observe

$$\begin{aligned}&\partial_t^k[\omega(t[\eta]^{-1})(1-\omega_0(t))t^{-\mu}\tilde{p}(t[\eta]^{-1},x,y,t\tau,\xi,t\eta[\eta]^{-1})]\\ =\ &\sum_{j=0}^k c_{jk}\partial_t^{k-j}[\omega(t[\eta]^{-1})(1-\omega_0(t))t^{-\mu}]\partial_t^j\tilde{p}(t[\eta]^{-1},x,y,t\tau,\xi,t\eta[\eta]^{-1}).\end{aligned}$$

Moreover,

$$\partial_t^j\tilde{p}(t[\eta]^{-1},x,y,t\tau,\xi,t\eta[\eta]^{-1}) = \sum_{|\gamma|\leq j}a_\gamma(\eta)\tau^{\gamma_2}(\partial_{t,\tau,\eta}^\gamma\tilde{p})(t[\eta]^{-1},x,y,t\tau,\xi,t\eta[\eta]^{-1}),$$

where $a_\gamma(\eta) \in S^0(\mathbb{R}^q)$ and $\gamma = (\gamma_1,\gamma_2,\gamma_3) \in \mathbb{N}_0^{1+1+q}$. Combining these two formulas with the symbol estimates of $\tilde{p}$ as an element in S^{-L}, $L > 0$, we obtain that the quantities (16) are dominated by a finite sum of terms of the form

$$\sup_{t,\tau,\xi}\Big\{\varphi(t)\langle t\tau,\xi,t\eta[\eta]^{-1}\rangle^{-L-\varrho}\langle\tau,\xi\rangle^{\varrho_1}\langle t\rangle^{\varrho_2}\Big\}[\eta]^\mu,$$

where $\varrho > 0$, $\varrho_1, \varrho_2 \in \mathbb{R}$, and $\varphi(t)$ is compactly supported or has at most polynomial growth at infinity. If we choose L big enough, the claim follows for $\alpha = \beta = 0$ once we use the elementary inequality

$$\langle t\tau, \xi, t\eta[\eta]^{-1}\rangle^{-L-\varrho} \langle t, \xi\rangle^{\varrho_1} \le c\,\langle \tau, \xi, t\rangle^{-L-\varrho} \langle t, \xi\rangle^{\varrho_1} \le \tilde{c}\,\langle t\rangle^{-L+\varrho_3}$$

with $\varrho_3 \in \mathbb{R}$. The case of arbitrary α and β follows similarly. □

Let $a = \omega_0 a_h \omega_1 + (1-\omega_0)a_p(1-\omega_1) + g$ be an edge symbol, cf. Definition 3.12. On the one hand, a_p is defined via pull–backs under $1 \times \chi_j$ of operators $\mathrm{op}(p_j)$, on the other hand, the cone Sobolev spaces are formulated (away from $t = 0$) via the push–forward under κ_j. Hence, to obtain continuity of a_p between cone Sobolev spaces, one can consider the push–forward of $\mathrm{op}(p_j)$ under $\kappa_j \circ (1 \times \chi_j^{-1})$ (inverse polar coordinates). For our purposes it is more convenient to analyze the behaviour of weighted Sobolev spaces in $\mathbb{R}^{1+n}$ under this change of coordinates. To this end we fix the following notations:

Let $V \subset \tilde{V}$ be open sets in $\mathbb{R}^n$ such that V is relatively compact in $\tilde{V}$. Let $\theta : \tilde{V} \to \tilde{U} \subset S^n$ be a diffeomorphism and set $\Theta(t,x) = t\theta(x) : \mathbb{R}_+ \times \tilde{V} \to \mathbb{R}^{1+n}$. Note that $\Theta^{-1}(y) = (|y|, \theta^{-1}(|y|^{-1}y))$. For $\varepsilon > 0$ set $V_\varepsilon =]\varepsilon, \infty[\times V$ and $V_\varepsilon^\triangle = \{t\theta(x) \mid t > \varepsilon,\, x \in V\}$.

For an open set $U \subset \mathbb{R}^{1+n}$ let $\bar{H}^{s,\delta}(U)$ denote the closure of $C_0^\infty(U)$ in $H^{s,\delta}(\mathbb{R}^{1+n})$. Then, for $s \in \mathbb{N}_0$, we have

$$\|u\|^2_{\bar{H}^{s,\delta}(U)} \sim \sum_{|\alpha| \le s} \|D^\alpha(\langle\cdot\rangle^\delta u)\|^2_{L^2(U)}. \tag{17}$$

Lemma 3.18 *For each integer s, and $\delta \in \mathbb{R}$ the push-forward of functions by Θ and Θ^{-1}, respectively, induces continuous maps*

$$\Theta_* : \bar{H}^{s,\delta}(V_\varepsilon) \longrightarrow \bar{H}^{s,\delta-n/2+\min(0,s)}(V_\varepsilon^\triangle),$$
$$\Theta_*^{-1} : \bar{H}^{s,\delta}(V_\varepsilon^\triangle) \longrightarrow \bar{H}^{s,\delta+n/2-\max(0,s)}(V_\varepsilon).$$

PROOF. Since V is bounded, $\langle t, x\rangle \sim \langle t\rangle \sim \langle\Theta(t,x)\rangle$ on $\mathbb{R}_+ \times V$. Furthermore

$$|D_t^k D_x^\alpha \Theta(t,x)| \le c\,\langle t, x\rangle$$

uniformly on $\mathbb{R}_+ \times V$. If we denote with $D\Theta$ the Jacobian of Θ, we obtain

$$\det D\Theta(t,x) = t^n f(x),$$

with a function $f \in C_b^\infty(V)$. In particular,

$$|D_t^k D_x^\alpha(\det D\Theta(t,x))| \le c\,\langle t, x\rangle^n$$

uniformly on $\mathbb{R}_+ \times V$. On $V^\triangle$ the functions $y \mapsto \theta^{-1}(|y|^{-1}y)$ and $y \mapsto |y|$ are homogeneous of degree 0 and 1, respectively. Then $\langle\Theta^{-1}(y)\rangle \sim \langle y\rangle$ on $V_\varepsilon^\triangle$ and

$$|D^\alpha\Theta^{-1}(y)| \le c\,\langle y\rangle^{1-|\alpha|}, \quad |D^\alpha(\det D\Theta^{-1}(y))| \le c\,\langle y\rangle^{-n-|\alpha|}$$

uniformly on $V_\varepsilon^\triangle$. For $s \in \mathbb{N}_0$ the lemma now follows from elementary calculations, using the norm representation (17). For $s < 0$ note that the L^2 scalar product induces a dual pairing $H^{s,\delta}(\mathbb{R}^{1+n}) \times H^{-s,-\delta}(\mathbb{R}^{1+n}) \to \mathbb{C}$. Then the result follows from

$$(\Theta_* u, v)_{L^2} = (u, (\Theta_*^{-1}v)|\det D\Theta|)_{L^2}, \quad (\Theta_*^{-1}v, u)_{L^2} = (v, (\Theta_* u)|\det D\Theta^{-1}|)_{L^2}$$

for $u \in C_0^\infty(V_\varepsilon)$, $v \in C_0^\infty(V_\varepsilon^\triangle)$, and the fact that $|\det D\Theta| : \bar{H}^{s,\delta}(V_\varepsilon) \to \bar{H}^{s,\delta-n}(V_\varepsilon)$, $|\det D\Theta^{-1}| : \bar{H}^{s,\delta}(V_\varepsilon^\triangle) \to \bar{H}^{s,\delta+n}(V_\varepsilon^\triangle)$ are continuous. □

Remark 3.19 A consequence of Lemma 3.18 is that a linear operator $A : C_0^\infty(V_\varepsilon) \to C_0^\infty(V_\varepsilon)$ extends continuously to operators $\bar{H}^{s,\delta}(V_\varepsilon) \to \bar{H}^{r,\varrho}(V_\varepsilon)$ for all real s, r, δ, ϱ if and only if $\Theta_* A$ extends to continuous operators $\bar{H}^{s,\delta}(V_\varepsilon^\Delta) \to \bar{H}^{r,\varrho}(V_\varepsilon^\Delta)$ for all s, r, δ, ϱ. This behaviour carries over to families $A(y,\eta)$, where we now require smooth dependence on (y,η) as a function with values in continuous operators between weighted Sobolev spaces.

Theorem 3.20 *Let $(p_1, \ldots, p_N) \in \Sigma^{-\infty}$. Then the edge symbol $c(y,\eta) = \mathrm{Op}(p_1, \ldots, p_N)$ as in Definition 3.12 belongs to $\mathbf{R}_G^\mu(\Omega \times \mathbb{R}^q, (\gamma, \gamma - \mu))$.*

PROOF. To $(p_1, \ldots, p_N) \in \Sigma^{-\infty}$ we associate $(h_1, \ldots, h_N)$ and observe that $h_j \in \tilde{S}^{-\infty}(\overline{\mathbb{R}}_+ \times V_j \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ for all j. The edge symbol $c(y,\eta)$ is in $C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q))$, and in view of the local considerations in Lemma 3.15 and Lemma 3.17 it satisfies (i) and (ii) of Definition 3.6. At last the property (iii) holds since the local interior symbols of the formal adjoint $c_\omega^*(y,\eta)$ are of the form

$$(t')^{-\mu}\omega(t')\omega_0(t'[\eta])\omega_1(t[\eta])\omega(t)\phi(x')\psi(x)\overline{h(t',x',y,n+1-\bar{z},\xi,\eta)},$$

$$(t')^{-\mu+n/2}t^{-n/2}\omega(t')(1-\omega_0(t'[\eta]))(1-\omega_1(t[\eta]))\omega(t)\tilde{\phi}(x')\tilde{\psi}(x)\overline{p(t',x',y,\tau,\xi,\eta)},$$

and the calculations are similar. □

Proof of Theorem 3.13.

Clearly the map is surjective. Now let $(p_1, \ldots, p_N) \in \Sigma^\mu$ and the associated edge symbol $a(y,\eta)$ be an element of $\mathbf{R}_G^\mu(\Omega \times \mathbb{R}^q, (\gamma, \gamma - \mu))$. Then, in particular, $a \in C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q))$. In view of Theorem 2.3, formula (3), the standard theory implies that $p_j \in \tilde{S}^{-\infty}(\mathbb{R}_+ \times V_j \times \Omega \times \mathbb{R}^{1+n+q})$ for all $j = 1, \ldots, N$. Hence $(p_1, \ldots, p_N) \in \Sigma^{-\infty}$ because of Lemma 3.7. Finally Theorem 3.20 provides the injectivity, which completes the proof. □

References

[DS] Ch. Dorschfeldt and B.-W. Schulze, Pseudo-differential operators with operator-valued symbols in the Mellin-edge approach, Ann. Global Anal. and Geom. **12** (1994), 135–171.

[ES] Ju. V. Egorov and B.-W. Schulze, *Pseudo-differential operators, singularities, applications*, Birkhäuser Verlag, Basel. (to appear 1997)

[GSS] J. B. Gil, B.-W. Schulze, J. Seiler, A Fréchet algebra of edge pseudo– differential operators. (in preparation)

[Hw] I. L. Hwang, The L^2-boundness of pseudodifferential operators, Trans. Amer. Math. Soc. **302** (1987), 55–76.

[M] R. Mazzeo, Elliptic theory of differential edge operators I, Comm. Part. Diff. Equ. **16** (1991), 1615-1664.

[MM] R. Melrose and G. Mendoza, Elliptic operators of totally characteristic type, MSRI Preprint, Berkeley, 1983.

[Pl] B. A. Plamenevskij, *Algebras of pseudo-differential operators*, Mathematics and its Applications (Soviet Series) **43**, Kluwer Academic Publishers, Dordrecht, Boston, London, 1989; originally published in Russian, Nauka, Moscow, 1986.

[SSa] E. Schrohe and B.-W. Schulze, Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities II. In: M. Demuth et al. (eds.), *Boundary Value Problems, Schrödinger Opertors, Deformation Quantization*, Advances in Partial Differential Equations **2**, Akademie Verlag, Berlin, 1995, 70 – 205.

[SSb] E. Schrohe and B.-W. Schulze, Mellin operators in a pseudodifferential calculus for boundary value problems on manifolds with edges. In: R. Mennicken and C. Tretter (eds.), *IWOTA95 Proceedings*, Operator Theory: Advances and Applications, Birkhäuser, Basel. (to appear)

[Sza] B.-W. Schulze, Pseudo-differential operators on manifolds with edges. Symp. 'Partial Differential Equations', Holzhau 1988, Teubner-Texte Math. **112**, Leipzig, 1989, 259–288.

[Szb] B.-W. Schulze, Mellin representations of pseudo-differential operators on manifolds with corners, Ann. Global Anal. and Geom. **8** (1990), 261–297.

[Szc] B.-W. Schulze, The Mellin pseudo-differential calculus on manifolds with corners. Symp. 'Analysis in Domains and on Manifolds with Singularities', Breitenbrunn 1990, Teubner-Texte Math. **131**, Leipzig, 1992, 271–289.

[Sh] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer, Berlin, Heidelberg, New York, 1987.

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