A bound for the orders of the torsion groups of surfaces with $c_1^2 = 2\chi - 1$

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Abstract

We shall give a bound for the orders of the torsion groups of minimal algebraic surfaces of general type whose first Chern numbers are twice the Euler characteristics of the structure sheaves minus 1, where the torsion group of a surface is the torsion part of the Picard group. Namely, we shall show that the order is at most 3 if the Euler characteristic is 2, that the order is at most 2 if the Euler characteristic is greater than or equal to 3, and that the order is 1 if the Euler characteristic is greater than or equal to 7. Moreover for each integer $\lambda = 2$, 3 and 4, we shall construct a family of surfaces above whose torsion groups are isomorphic to the cyclic group of order 2, and whose Euler characteristics are λ.

0 Introduction

In the present paper, we shall give a bound for the orders of the torsion groups $\mathrm{Tors}(X) = \mathrm{TorPic}(X)$'s of minimal algebraic surfaces X's with $c_1^2 = 2\chi(\mathcal{O})-1$ and $\chi(\mathcal{O}) \geq 2$. Here as usual, c_1 and $\chi(\mathcal{O})$ are the first Chern class and the Euler characteristic of the structure sheaf, respectively, and the group $\mathrm{Tors}(X)$ is the torsion part of the Picard group of X. We shall also construct a family of X's as above with $\chi(\mathcal{O}) = \lambda$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ for each integer $2 \leq \lambda \leq 4$.

In classification theories of the numerical Godeaux surfaces (i.e., minimal surfaces of general type with $c_1^2 = 1$ and $\chi(\mathcal{O}) = 1$), one fixes the torsion group or the fundamental group as an additional invariant, and finds concrete descriptions for each case . For example, Miyaoka and Reid independently showed that if the torsion group $\mathrm{Tors}(X)$ is $\mathbb{Z}/5$, then the fundamental group

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is isomorphic to $\mathbb{Z}/5$, and the canonical model of the universal cover is a quintic surface in $\mathbb{P}^3$ ([11], [14]). It is well-known that the order $\sharp\mathrm{Tors}(X)$ is at most 5.

Similar theories have been developed for some other cases of invariants. For example, minimal surfaces with $c_1^2 = 2$ and $\chi(\mathcal{O}) = 2$ are classified in [4] and [3], while surfaces with $c_1^2 = 2\chi(\mathcal{O}) - 2$ having non-trivial torsion are studied in [5]. In [12], the author gave a complete description for minimal algebraic surfaces X's with $c_1^2 = 3$, $\chi(\mathcal{O}) = 2$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/3$.

In the present paper, we consider the case $c_1^2 = 2\chi(\mathcal{O}) - 1$, and give a bound for the orders of the torsion groups $\mathrm{Tors}(X)$'s. Namely, we shall show that $\sharp\mathrm{Tors}(X) \leq 3$ for $\chi(\mathcal{O}) = 2$, that $\sharp\mathrm{Tors}(X) \leq 2$ for $\chi(\mathcal{O}) \geq 3$, and that $\sharp\mathrm{Tors}(X) = 1$ for $\chi(\mathcal{O}) \geq 7$. We shall also construct a family of examples with $\chi(\mathcal{O}) = \lambda$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ for each integer $2 \leq \lambda \leq 4$. Note that the line $c_1^2 = 2\chi(\mathcal{O}) - 1$ is parallel to the Noether line, and that the case $\chi(\mathcal{O}) = 1$ on this line is that of the numerical Godeaux surfaces. The case $\chi(\mathcal{O}) = 2$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/3$ is the one for which the author gave a concrete description in [12]. Thus our bound is sharp for the cases $\chi(\mathcal{O}) = 2$, 3 and 4.

In order to obtain the bound for the orders of the torsion groups, we use a method due to Miyaoka and Reid ([11], [14]): we take an unramified cover corresponding to torsion divisors, and study its canonical map. We employ Horikawa's method ([6]) to study the canonical map. In order to construct the examples X's with $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$, we use a combination — though not exactly, but in a sense — of the Campedelli construction ([1, p.234]) and the Godeaux construction ([1, p.234]): we take double covers of $\mathbb{P}^1 \times \mathbb{P}^1$, and take their quotients by certain free actions of $\mathbb{Z}/2$.

In Section 1, we give some lemmas, and state our main results. In Section 2, we study the case $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$. In Section 3, we study the case $\sharp\mathrm{Tors}(X) = 4$ or 5, and give a proof of the bound. Finally in Section 4, we construct the families of X's with $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$.

The bound given in this paper is not best possible for the case $5 \leq \chi(\mathcal{O}) \leq 6$. In the subsequent paper, the author shall give a complete description for the surfaces of the case $\chi(\mathcal{O}) = 4$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$, and exclude the case $5 \leq \chi(\mathcal{O}) \leq 6$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$. Throughout this paper, we work over the complex number field $\mathbb{C}$.

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Notation

Let S be a compact complex manifold of dimension 2. We denote by $c_1(S)$, $p_g(S)$ and $q(S)$, the first Chern class, the geometric genus and the irregularity of S, respectively. The torsion group $\mathrm{Tors}(S) = \mathrm{TorPic}(S)$ is the torsion part of the Picard group of S. If V is a complex manifold, K_V is a canonical divisor of V. For a coherent sheaf $\mathcal{F}$ on V, we denote by $h^i(\mathcal{F})$ and $\chi(\mathcal{F})$, the dimension of the i-th cohomology group and the Euler characteristic of $\mathcal{F}$, respectively. Let $f : V \to W$ be a morphism to a complex manifold W, and D a divisor on W. We denote by $f^*(D)$ and $f_*^{-1}(D)$ the total transform and the strict transform of D, respectively. The symbol $\sim$ means a linear equivalence of divisors. We denote by $\Sigma_d \to \mathbb{P}^1$ the Hirzebruch surface of degree d. The divisors Δ_0 and Γ are its minimal section and its fiber, respectively. Throughout this paper, X is a minimal algebraic surface of general type with $c_1^2 = 2\chi(\mathcal{O}_X) - 1$.

1 Main results and some lemmas

The following is Deligne's lemma for our case. For a general form of this lemma, see for example [2, Theorem14].

Lemma 1.1. *Let X be a minimal algebraic surface of general type with $c_1^2 = 2\chi(\mathcal{O}_X) - 1$, and $\pi : Y \to X$ an unramified cover of finite degree m. Then $m \leq 6$ and $q(Y) = 0$.*

Proof. Apply Noether's inequality to Y, and use the unbranched covering trick. □

¿From the lemma above, we infer the following:

Lemma 1.2. *Let X be a surface as in Lemma 1.1 with $\chi(\mathcal{O}_X) = \lambda$. Then $p_g(X) = \lambda - 1$, $q(X) = 0$ and $\sharp\mathrm{Tors}(X) \leq 6$.*

The bound given in Lemma 1.2 is not sharp. In this paper, we sharpen the bound to some extent. Namely, we shall show the following theorem.

Theorem 1. *Let X be a minimal algebraic surface of general type with $c_1^2 = 2\chi(\mathcal{O}_X) - 1$. Then the following hold :*
i) if $\chi(\mathcal{O}_X) = 2$, then $\sharp\mathrm{Tors}(X) \leq 3$,
ii) if $\chi(\mathcal{O}_X) \geq 3$, then $\sharp\mathrm{Tors}(X) \leq 2$,
iii) if $\chi(\mathcal{O}_X) \geq 7$, then $\sharp\mathrm{Tors}(X) = 1$.

Moreover we construct a family of examples with $\chi(\mathcal{O}) = \lambda$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ for each integer $\lambda = 2$, 3 and 4. Namely, we shall show the following :

Theorem 2. *There exists a family of minimal algebraic surfaces X's of general type with $c_1^2 = 2\chi(\mathcal{O}_X) - 1$, $\chi(\mathcal{O}_X) = 4 - k$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ for each integer $0 \leq k \leq 2$.*

Meanwhile for the case $\chi(\mathcal{O}_X) = 2$ and $\sharp\mathrm{Tors}(X) = 3$, we have the following theorem. See [12] for a proof of Theorem 3.

Theorem 3 ([12]). *Let X be a minimal algebraic surface of general type with $c_1^2 = 3$, $\chi(\mathcal{O}_X) = 2$ and $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$. Then both the torsion group $\mathrm{Tors}(X)$ and the fundamental group $\pi_1(X)$ are $\mathbb{Z}/3$, and the canonical model of the universal cover of X is a complete intersection in $\mathbb{P}^4$ of type $(3,3)$. Moreover, if a canonical divisor of X is ample, then the number of moduli of X is 14.*

Corollary 1.1. *The bound given in Theorem 1 is sharp for the case $2 \leq \chi(\mathcal{O}_X) \leq 4$.*

By Lemma 1.2, we have only to consider the case $\sharp\mathrm{Tors}(X) \leq 6$. Following Miyaoka and Reid ([11], [14]), we take an unramified Galois cover Y of X corresponding to torsion divisors to show Theorem 1. We employ Horikawa's method ([6]) to study the canonical map Φ_{K_Y} of Y. In what follows, X is a minimal algebraic surface of general type with $c_1^2 = 2\chi(\mathcal{O}_X) - 1$ and $\chi(\mathcal{O}_X) = \lambda \geq 2$.

Let us give a lemma which we shall use in the construction of the families given in Theorem 2. Let W be a compact connected complex manifold, and G a group acting on W. Let B be an effective reduced divisor on W such that $B \sim nF$ for a non-trivial divisor F and an integer $n \geq 2$. Then we have a Galois cover $V \to W$ of mapping degree n with branch locus B. The variety V is a subvariety of the total space of the line bundle F. We assume that the divisors B and F are stable under the action of G. We say that an action of G on V is a lifting of the one on W, if the action of G on V and that on W are compatible with the projection $V \to W$. Let us give a criterion for the existence of a lifting. Let h be a meromorphic function on W corresponding to the principal divisor $B - nF$. Then $c_g = (g^*h)/h$ is a non-zero constant function for any $g \in G$, and $g \mapsto c_g$ gives a character c of G. Let $\mathrm{Char}(G)$ be the character group of G, and Ψ the endomorphism of $\mathrm{Char}(G)$ given by $\chi \mapsto n\chi$. We denote by $\mathrm{Im}(\Psi)$ the image of the morphism $\Psi : \mathrm{Char}(G) \to \mathrm{Char}(G)$.

Lemma 1.3. *The action of G on W lifts to one on V, if and only if $c \in \mathrm{Im}(\Psi)$. If $c \in \mathrm{Im}(\Psi)$, then there exist exactly $\sharp\ker(\Psi)$ liftings of the action of G, where $\ker(\Psi)$ is the kernel of the morphism Ψ.*

Proof. See Appendix. $\square$

For a proof of the following theorem, see [6].

Theorem 4 (Horikawa). *Let S be a minimal algebraic surface of general type with $p_g \geq 3$ whose canonical system $|K|$ is not composite with a pencil. We denote by $\pi : \tilde{S} \to S$ a composite of quadric transformations such that the variable part $|L|$ of $|\pi^* K|$ is free from base points. Then $K^2 \geq L^2 \geq 2p_g - 4$. Moreover, if $K^2 = L^2$, then the canonical system $|K|$ has no base points. If $L^2 = 2p_g - 4$, then any general member of $|L|$ is a non-singular hyperelliptic curve.*

2 The case $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$

In this section, we study the case $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$.

Proposition 2.1. *Let X be a surface as in Lemma 1.1 with $\mathbb{Z}/3 \subset \mathrm{Tors}(X)$. Then $\chi(\mathcal{O}_X) = \lambda \leq 2$*

Let X be a surface as in Lemma 1.1 with $\mathbb{Z}/3 \simeq T \subset \mathrm{Tors}(X)$. We assume $\chi(\mathcal{O}_X) = \lambda \geq 3$, and derive a contradiction. Assume that $\chi(\mathcal{O}_X) = \lambda \geq 3$. We have an unramified Galois triple cover $\pi : Y \to X$ corresponding to the subgroup $T \simeq \mathbb{Z}/3$. We have

$$K_Y^2 = 3(2\lambda - 1), \quad p_g(Y) = 3\lambda - 1, \quad q(Y) = 0$$

by Lemma 1.1. We denote by $G = \mathrm{Gal}(Y/X)$ the the Galois group of $\pi : Y \to X$. We study the canonical map $\Phi_{K_Y} : Y - - \to \mathbb{P}^{3\lambda-2}$ and the canonical image $Z = \Phi_{K_Y}(Y)$ of Y using the action on Y of the Galois group G. Let $|M|$ and F be the variable part and the fixed part of $|K_Y|$, respectively. We have a natural isomorphism

$$H^0(Y, \mathcal{O}_Y(mK_Y)) \simeq \bigoplus_{\chi \in \mathrm{Char}(G)} H^0(X, \mathcal{O}_X(mK_X - D_\chi))$$

for each $m \geq 1$, where D_χ's are the torsion divisors corresponding to the characters χ's of G. In particular, $|K_Y|$ is spanned by the pull-back of divisors on X, and so are $|M|$ and F. Let $p : \tilde{Y} \to Y$ be the shortest one among all composites of quadric transformations such that the variable part $|L|$ of $|p^* M|$ is free from base points. We have

$$|p^* K_Y| = |p^* M| + p^* F = |L| + E + p^* F,$$

where E is an exceptional divisor. We take $p : \tilde{Y} \to Y$ in such a way that the action G on Y lifts to one on $\tilde{Y}$. Since $|M|$ and F are spanned by the pull-back of divisors on X, we have $M^2 \equiv MF \equiv F^2 \equiv 0 \bmod 3$. Moreover we have $E^2 \equiv 0 \bmod 3$, since E is stable under the action of G on $\tilde{Y}$. It follows

$$L^2 = M^2 + E^2 \equiv LE = -E^2 \equiv MF \equiv K_Y F \equiv 0 \mod 3. \tag{1}$$

Lemma 2.1. *The canonical map $\Phi_{K_Y} : Y - - \to \mathbb{P}^{3\lambda-2}$ is a rational map of degree 2 onto a nondegenerate surface of minimal degree $3\lambda - 3$ in $\mathbb{P}^{3\lambda-2}$.*

Proof. The canonical map Φ_{K_Y} is not birational, since we have $K_Y^2 - (3p_g(Y) - 7) = 7 - 3\lambda < 0$. Moreover, by this together with $q(Y) = 0$, the canonical system $|K_Y|$ is not composite with a pencil (see [9, Theorem 1.1]). It follows

$$\deg \Phi_{K_Y} = \frac{L^2}{\deg Z} \le \frac{K_Y^2}{p_g(Y) - 2} = 2 + \frac{1}{\lambda - 1},$$

hence $\deg \Phi_{K_Y} = 2$. So by (1), we have $L^2 \equiv 0 \bmod 6$. Meanwhile by Theorem 4, we have

$$K_Y^2 = 3(2\lambda - 1) \ge L^2 \ge 2p_g(Y) - 4 = K_Y^2 - 3.$$

Thus we have $L^2 = 6\lambda - 6$, hence the assertion follows. □

By the lemma above, we have $LE + MF + K_Y F = 3$, where each term of the right side of this equality is a non-negative integer divisible by 3. Moreover by the Riemann-Roch theorem, we have $M^2 + MK_Y = 2M^2 + MF \equiv 0 \bmod 2$, hence $MF \equiv 0 \bmod 6$. It follows $MF = 0$. Thus by Hodge's index theorem, we have $LE = 3$ and $F = 0$, which implies that the morphism $p : \tilde{Y} \to Y$ is the blowing-up of Y at three simple base points of the canonical system $|K_Y|$. We denote by $\Phi_L = \Phi_{K_Y} \circ p : \tilde{Y} \to \mathbb{P}^{3\lambda-2}$ the morphism associated with the linear system $|L|$.

We put $n = p_g(Y) - 1 = 3\lambda - 2$. Note that we have $n \equiv 1 \bmod 3$. Thus from a classification of surfaces of minimal degree (see [7, Lemma 1.2] or [13]), we infer the following :

Lemma 2.2. *Let $Z = \Phi_{K_Y}(Y) \subset \mathbb{P}^n$ be the canonical image of our surface Y. Then Z is either*
Case 1) the Hirzebruch surface Σ_d embedded into $\mathbb{P}^n$ by $|\Delta_0 + \frac{n-1+d}{2}\Gamma|$, where $n - d - 3$ is a non-negative integer, or
Case 2) a cone over a rational curve of degree $n - 1$ in $\mathbb{P}^{n-1}$, that is, the image of the Hirzebruch surface Σ_{n-1} by $|\Delta_0 + (n-1)\Gamma|$.

Note that the action of $G = \mathrm{Gal}(Y/X)$ on $\tilde{Y}$ induces one on Z. We exclude both Case 1) and Case 2) in Lemma 2.2.

Lemma 2.3. *The case* 1) *in Lemma 2.2 is impossible.*

Proof. Assume that the canonical image $Z = \Phi_{K_Y}(Y)$ is the Hirzebruch surface Σ_d embedded into $\mathbb{P}^n$ by $|\Delta_0 + \frac{n-1+d}{2}\Gamma|$. There exist an unramified Galois triple cover $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ and a composite $r : \tilde{X} \to X$ of quadric transformations such that $r \circ \tilde{\pi} = \pi \circ p$ and $\mathrm{Gal}(\tilde{Y}/\tilde{X}) \simeq \mathrm{Gal}(Y/X)$ hold. If $d \geq 1$, then the minimal section Δ_0 is the unique $(-d)$-curves on Z. If $d = 0$, then we can take Δ_0 in such a way that Δ_0 is stable under the action of G on Z, since we have $G \simeq \mathbb{Z}/3$. In both cases, there exists a member $\Delta_1 \in |\Delta_0|$ stable under the action of G on Z. Let $\Phi_\Gamma : Z \to C_0 = \mathbb{P}^1$ be the morphism associated with the linear system $|\Gamma|$. Then the action of G on Z induces one on C_0. There exists a point on C_0 stable under the action of G on C_0, since C_0 is rational. Thus there exists a member $\Gamma_1 \in |\Gamma|$ stable under the action of G on Z. The total transforms $\Phi_L^*(\Delta_1)$ and $\Phi_L^*(\Gamma_1)$ are both stable under the action of G on $\tilde{Y}$, hence the pull-back by $\tilde{\pi}$ of divisors on $\tilde{X}$. Thus we have $\Phi_L^*(\Delta_1) \cdot \Phi_L^*(\Gamma_1) \equiv 0 \bmod 3$, which contradicts the equality $\Phi_L^*(\Delta_1) \cdot \Phi_L^*(\Gamma_1) = 2$. □

Lemma 2.4. *The case* 2) *in Lemma 2.2 is impossible.*

Proof. Assume that the canonical image $Z = \Phi_{K_Y}(Y)$ is the image of the Hirzebruch surface Σ_{n-1} by $|\Delta_0 + (n-1)\Gamma|$. In this case, Z is a cone over a rational curve $C_0 \simeq \mathbb{P}^1$ of minimal degree $n-1$ in $\mathbb{P}^{n-1}$. We denote by p_0 the vertex of Z. Let Λ_0 be the linear system which consists of the pull-back by Φ_L of all hyperplanes containing p_0 in $\mathbb{P}^n$. We denote by Λ and F', the variable part and the fixed part of Λ_0, respectively. Then $\Lambda = |(n-1)D|$ holds for a linear pencil $|D|$ without fixed components (see [7, Lemma 1.5]). We have $LF' = 0$, since $\Phi_L(F') \subset \{p_0\}$. Thus we obtain $2(n-1) = L^2 = (n-1)((n-1)D^2 + DF')$, hence $D^2 = 0$ and $DF' = 2$. Meanwhile we have

$$\Lambda_0 = \mathbb{P}(V) \subset |L| = \mathbb{P}(H^0(\mathcal{O}_{\tilde{Y}}(L)))$$

for a linear subspace $V \subset H^0(\mathcal{O}_{\tilde{Y}}(L))$. Since the vertex p_0 is stable under the action of G on Z, the subspace V is stable under the action of G on $H^0(\mathcal{O}_{\tilde{Y}}(L))$. We therefor infer, since $G \simeq \mathbb{Z}/3$, that V is spanned by eigenvectors of τ_0^*, where τ_0 is a generator of G. Thus Λ_0 is spanned by divisors stable under the action of G on $\tilde{Y}$, hence so are Λ and F'. Since $D^2 = 0$, we have a morphism $\Phi_\Lambda : \tilde{Y} \to \mathbb{P}^{n-1}$ associated with the linear system $\Lambda = |(n-1)D|$. Here the image $\Phi_\Lambda(\tilde{Y}) = C_0 \subset \mathbb{P}^{n-1}$ is a nonsingular rational curve of minimal degree $n-1$, and the surface Z is a cone over C_0. The action of G on $\tilde{Y}$ induces one on C_0, since F' is stable under the action of G on $\tilde{Y}$. This action on C_0 has a fixed points, since $C_0 \simeq \mathbb{P}^1$. It follows that there exists a member $C \in |D|$ stable under the action of G on $\tilde{Y}$. Now we derive a contradiction as follows. Both F' and C are stable under the action of G on $\tilde{Y}$. Then by the same method as in the proof of Lemma 2.3, we obtain $DF' = CF' \equiv 0 \bmod 3$, which contradicts the equality $DF' = 2$. $\square$

This completes the proof of Proposition 2.1.

3 The case $\sharp\mathrm{Tors}(X) = 4$ or 5 and a proof of Theorem 1

In this section, we exclude the case $\sharp\mathrm{Tors}(X) = 4$ and the case $\sharp\mathrm{Tors}(X) = 5$. Moreover we shall give a proof of Theorem 1.

Proposition 3.1. *Let X be a surface as in Lemma 1.1 with $\chi(\mathcal{O}_X) = \lambda \geq 2$. Then $\sharp\mathrm{Tors}(X) \neq 4$.*

To prove the proposition above, we assume $\sharp\mathrm{Tors}(X) = 4$, and derive a contradiction. Let X be a surface as in Lemma 1.1 with $\sharp\mathrm{Tors}(X) = 4$ and $\chi(\mathcal{O}_X) = \lambda \geq 2$. We have an unramified Galois quadruple cover $\pi : Y \to X$ corresponding to the torsion group $\mathrm{Tors}(X)$. We have

$$K_Y^2 = 4(2\lambda - 1), \quad p_g(Y) = 4\lambda - 1, \quad q(Y) = 0$$

by Lemma 1.1. Since $K_Y^2 = 2p_g(Y) - 2$, our surface Y is of one of the several types given in [10]. We shall exclude all the types for our Y using the action on Y of the Galois group $\mathrm{Gal}(Y/X)$ of π. First, we have the following lemma.

Lemma 3.1. *The canonical system $|K_Y|$ has no base points. Moreover, the canonical map $\Phi_{K_Y} : Y \to \mathbb{P}^{4\lambda-2}$ is a holomorphic map of degree 2 onto its image $Z = \Phi_{K_Y}(Y)$, hence* $\deg Z = 4\lambda - 2$.

Proof. We study the canonical map $\Phi_{K_Y} : Y - - \to \mathbb{P}^{4\lambda-2}$ using the Galois group $G = \mathrm{Gal}(Y/X)$. Let $|M|$ and F be the variable part and the fixed part of the linear system $|K_Y|$, respectively. We denote by $p : \tilde{Y} \to Y$ the shortest one among all composites of quadric transformations such that the variable part $|L|$ of $|p^*M|$ is free from base points. We have

$$|p^*K_Y| = |p^*M| + p^*F = |L| + E + p^*F,$$

where E is an exceptional divisor. By the same method as in Section 2, we obtain $L^2 \equiv 0 \bmod 4$. Meanwhile, by the results given in [10, Section 3] (or Theorem 4), we have

$$K_Y^2 = 4(2\lambda - 1) \geq L^2 \geq 2p_g(Y) - 4 = 4(2\lambda - 1) - 2.$$

Thus we obtain $L^2 = K_Y^2 = 4(2\lambda - 1)$, which implies that $|K_Y|$ is free from base points. We have $\deg \Phi_{K_Y} = 2$, since $p_g(Y) \geq 6$ (see [10, Section 3]). $\square$

Lemma 3.2. *The case $\lambda \geq 3$ is impossible. If $\lambda = 2$, then there exists a composite of three quadric transformations $W \to \mathbb{P}^2$ such that $Z \subset \mathbb{P}^6$ is the image of the morphism $\Phi_{-K_W} : W \to \mathbb{P}^6$ associated with the anti-canonical system $|-K_W|$.*

Proof. By the results given in [10, Section 3], Z is either i) the Veronese embedding into $\mathbb{P}^8$ of a quadric in $\mathbb{P}^3$ for $n = 8$, or ii) the image of $\mathbb{P}^2$ by the rational map associated with the linear system $|3l - \sum_{i=1}^{9-n} x_i|$, where l is a line on $\mathbb{P}^2$ and x_i's are points on $\mathbb{P}^2$ which are possibly infinitely near. The case i) does not occur for our Z, since $n = 4\lambda - 2 \equiv 2 \mod 4$. Thus our Z is as in the case ii). By $n = 4\lambda - 2 \leq 9$, we obtain $\lambda = 2$ and $n = 6$. $\square$

Thus our surface Y is of the type found in Theorem 3.2. of [10]. In what follows, we assume $\lambda = 2$.

Proposition 3.2. *Let $\Phi_{K_Y} : Y \to Z$ and $\Phi_{-K_W} : W \to Z$ be the morphisms as in the case $\lambda = 2$ of Lemma 3.2. Then there exists a unique morphism $f : Y \to W$ such that $\Phi_{K_Y} = \Phi_{-K_W} \circ f$. The branch locus B of f is a member of the linear system $|-4K_W|$. Moreover, the double cover Y' of W branched along B has at most rational double points as its singularities, and Y is the minimal desingularization of Y'.*

Proof. See Horikawa [10, Theorem 3.2]. $\square$

Lemma 3.3. *Let $f : Y \to W$ be the morphism as in Proposition 3.2. Then the action of the Galois group $G = \mathrm{Gal}(Y/X)$ on Y induces one on W. This action on W has the following two properties :*

i) W has no (-1)-curves which are stable under the action of G,

ii) W has no (closed) points which are stable under the action of G.

Proof. The first assertion is trivial; since $\Phi_{-K_W} : W \to Z$ is the minimal desingularization, the natural action of G on Z induces one on W.

Let us show that this action on W has the property i). Assume that W has a (-1)-curve C' stable under the action of G. Then $f^*(C')$ is stable under the action of G on Y, hence a pull-back of a divisor on X. In particular, we have $f^*(C')^2 \equiv 0 \mod 4$, which contradicts $f^*(C')^2 = -2$.

Next, we show that this action on W has the property ii). Assume that W has a point x stable under the action of G. We denote by $q_x : W_x \to W$ the quadric transformation at x. Take the fiber product $Y \times_W W_x$ of Y and W_x over W, the reduction of this fiber product, the normalization of this reduction, and the minimal desingularization of this normalization. We denote by Y_x, this minimal desingularization. Then there exists a morphism $f_x : Y_x \to W_x$ and a composite $p_x : Y_x \to Y$ of quadric transformations satisfying $f \circ p_x = q_x \circ f_x$. The action of G on Y induces one on Y_x and one on W_x. Now let E_x be the exceptional curve of the first kind appearing by $q_x : W_x \to W$. Then the total transform $f_x^*(E_x)$ is stable under the action of G on Y_x. Thus by the same method as in the proof of Lemma 2.3, we infer $f_x^*(E_x)^2 \equiv 0 \bmod 4$, which contradicts $f_x^*(E_x)^2 = -2$. □

Corollary 3.1. *The Galois group $G = \mathrm{Gal}(Y/X)$ is not isomorphic to $\mathbb{Z}/4$.*

Proof. Assume that $G \simeq \mathbb{Z}/4$. Take an automorphism τ_0 of W corresponding to a generator of G. Then τ_0 has fixed points, since we have $\chi(\mathcal{O}_W) = 1$. □

By the corollary above, we have only to exclude the case $G \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In what follows, we assume that G is isomorphic to Klein's four-group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Let $q : W = W_3 \to \mathbb{P}^2 = W_0$ be the composite of three quadric transformations as in Lemma 3.2. We have $q = q_1 \circ q_2 \circ q_3$, where $q_i : W_i \to W_{i-1}$ is a quadric transformation of W_{i-1} at x_i for each $1 \leq i \leq 3$. We denote by E_i the exceptional curve of the first kind appearing by q_i. By E_i', we denote the strict transform on W of E_i. We use the same symbol E_i for the total transform on W of the exceptional divisor E_i. We determine the configuration of x_i's using the following lemma.

Lemma 3.4. *Let C be a $(-d)$-curve on W which is not exceptional with respect to q. If $0 \leq d \leq 2$, then C is a strict transform on W of a line on $\mathbb{P}^2$ passing exactly $d+1$ of the three points x_i's.*

Proof. Let l be a line on $\mathbb{P}^2$. We have $C \sim m_0 q^*(l) - \sum_{i=1}^3 m_i E_i$ for certain integers m_i's, where $q^*(l)$ is the total transform of l by q. Note that $m_i \geq 0$ for any $1 \leq i \leq 3$, since C is not exceptional with respect to q. We have

$$m_0^2 - \sum_{i=1}^{3} m_i^2 = -d, \qquad 3m_0 - \sum_{i=1}^{3} m_i = 2 - d, \tag{2}$$

since $C^2 = -d$ and $CK_W = d - 2$. By the equalities above, we obtain

$$5\sum_{i=1}^{3} m_i^2 + \sum_{1 \leq i < j \leq 3} (m_i - m_j)^2 + \sum_{i=1}^{3} (m_i + d - 2)^2 = 9d + 4(2-d)^2 \leq 18,$$

hence $\sum_{i=1}^3 m_i^2 \leq 3$. Thus, we have $m_i^2 = m_i$ for any $1 \leq i \leq 3$. From this together with the equalities (2), we infer $m_0 = 1$ and $\sum_{i=1}^3 m_i = d+1$. Thus, we have the assertion. □

Now, we study the configuration of x_i's. First, we consider the case in which no two of the three points x_i's are infinitely near. This case is divided into the following two cases : the case 1-1) and the case 1-2).

1-1). The case in which no line on $\mathbb{P}^2$ includes the set $\{x_1,\ x_2,\ x_3\}$. In this case, W has no (-2)-curves. There exist exactly six (-1)-curves on W, that is, three exceptional divisors of the first kind appearing by q and three (-1)-curves coming from lines on $\mathbb{P}^2$.

1-2). The case in which the three points x_1, x_2 and x_3 lie on a line on $\mathbb{P}^2$. In this case, $\{E_1, E_2, E_3\}$ is the set of all (-1)-curves on W. The Galois group G acts on the set $\{E_1, E_2, E_3\}$. Since G is isomorphic to Klein's four-group, at least one of the E_i's is stable under the action of G on W, which contradicts the property i) in Lemma 3.3. Thus the case 1-2) is excluded.

Second, we consider the case in which x_1 and x_2 are distinct points on $\mathbb{P}^2$, and the point x_3 is infinitely near to x_2. Let $L_{1,2}$ be the unique line on $\mathbb{P}^2$ passing the two points x_1 and x_2. This case is divided into the following two cases : the case 2-1) and the case 2-2).

2-1). The case in which x_3 does not lie on the strict transform $(q_1 \circ q_2)_*^{-1}(L_{1,2})$ by $q_1 \circ q_2$ of the line $L_{1,2}$. Let $L_{2,3}$ be the line on $\mathbb{P}^2$ whose strict transform $(q_1 \circ q_2)_*^{-1}(L_{2,3})$ passes x_3. In this case, the strict transform E_2' is the unique (-2)-curve on W, and there are exactly four (-1)-curves on W: E_1, E_3, $q_*^{-1}(L_{1,2})$, and $q_*^{-1}(L_{2,3})$. Let $q' : W \to W'$ be the blowing down of two (-1)-curves E_3 and $q_*^{-1}(L_{1,2})$. Then W' is isomorphic to the Hirzebruch surface Σ_0 of degree 0, where we may assume $\Delta_0 = q'_*(E_2')$ and $\Gamma = q'_*(E_1)$ are a minimal section and a fiber of $\Sigma_0 \to \mathbb{P}^1$, respectively. Note that E_3 and $q_*^{-1}(L_{1,2})$ are the only (-1)-curves on W intersecting E_2'. Thus $E_3 + q_*^{-1}(L_{1,2})$ is stable under the action of $G = \mathrm{Gal}(Y/X)$ on W, which induces an action on W'. Since $G \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$, there exists a non-trivial element $g_0 \in G$ such that both E_3 and $q_*^{-1}(L_{1,2})$ are stable by the corresponding involution $g_0|_W$ of W. Then $q'(E_3)$, $q'(q_*^{-1}(L_{1,2}))$, and $\Delta_0 = q'_*(E_2')$ are all stable by the involution $g_0|_{W'}$ of W' corresponding to g_0. From this, we see easily that at least one out of the three curves E_1, E_2', and $q_*^{-1}(L_{1,2})$ lies in the fixed locus of $g_0|_W$. Moreover, if E_2' is in the fixed locus of $g_0|_W$, then so is a curve $q'^*(\Delta)$, where $\Delta \in |\Delta_0|$ is a certain member distinct from Δ_0. This shows that the branch divisor $B \in |-4K_W|$ of f intersects the fixed locus of $g_0|_W$, since $q'^*(\Delta)$, E_1, and $q_*^{-1}(L_{1,2})$ are non-singular rational curves with selfintersection 0, -1, and -1, respectively. Take a point $x \in B$ fixed by $g_0|_W$. Then since the double cover Y' in Proposition 3.2 has at most rational double points as its singularities, the set $f^{-1}(x)$ includes a fixed point of the automorphism $g_0 \in \mathrm{Gal}(Y/X)$, which contradicts the definition of $\pi : Y \to X$. Thus the case 2-1) is excluded.

2-2). The case in which x_3 lies on the strict transform $(q_1 \circ q_2)_*^{-1}(L_{1,2})$ by

$q_1 \circ q_2$ of the line $L_{1,2}$. In this case, $\{E_2', q_*^{-1}(L_{1,2})\}$ is the set of all (-2)-curves on W, where $q_*^{-1}(L_{1,2})$ is the strict transform of $L_{1,2}$ by q. The curve E_3 is the unique (-1)-curve intersecting all (-2)-curves on W, hence stable under the action of G. This contradicts the property i) in Lemma 3.3. Thus the case 2-2) is excluded.

Finally, we consider the case in which all x_i's are infinitely near, namely, the case in which x_2 is infinitely near to x_1, and x_3 is infinitely near to x_2. Let $L_{1,2}$ be the unique line on $\mathbb{P}^2$ such that x_2 lies on the strict transform ${q_1}_*^{-1}(L_{1,2})$. Note that W has no (-3)-curves, since the anti-canonical system $|-K_W|$ has no fixed components. Thus, x_3 does not lie on the strict transform ${q_2}_*^{-1}(E_1)$. This case is divided into the following two cases: the case 3-1) and the case 3-2).

3-1). The case in which x_3 does not lie on the strict transform $(q_1 \circ q_2)_*^{-1}(L_{1,2})$ by $q_1 \circ q_2$. In this case, $\{E_1', E_2'\}$ is the set of all (-2)-curves on W, hence $E_1' \cap E_2'$ is a point stable under the action of G on W. This contradicts the condition ii) in Lemma 3.3. Thus the case 3-1) is excluded.

3-2). The case in which x_3 lies on the strict transform $(q_1 \circ q_2)_*^{-1}(L_{1,2})$ by $q_1 \circ q_2$. In this case, E_3 is the unique (-1)-curves on W, hence stable under the action of G on W. This contradicts the property i) of Lemma 3.3, and the case 3-2) is excluded.

Thus we have the following.

Lemma 3.5. *The three points x_i's are in a general position. Namely x_1, x_2 and x_3 are distinct three points of $\mathbb{P}^2$ which do not lie on a line. The surface W has exactly six (-1)-curves, that is, E_i's for $1 \leq i \leq 3$ and the strict transforms $q_*^{-1}(L_{i,j})$ for $1 \leq i < j \leq 3$, where $L_{i,j}$ is the unique line on $\mathbb{P}^2$ passing x_i and x_j.*

By Lemma 3.3, we obtain a group homomorphism $\mu : G \to \mathrm{Aut}(W)$ corresponding to the action of G on W, where $\mathrm{Aut}(W)$ is the group of analytic automorphisms of W. We study the conjugacy class of $\mu(G)$ in $\mathrm{Aut}(W)$. Let $(X_0 : X_1 : X_2)$ be a homogeneous coordinate of $\mathbb{P}^2$ such that $x_1 = (1 : 0 : 0)$, $x_2 = (0 : 1 : 0)$ and $x_3 = (0 : 0 : 1)$. For $(a, b) \in \mathbb{C}^\times \times \mathbb{C}^\times$, we denote by $\varphi_{(a,b)}$ the automorphism of W corresponding to the automorphism $(X_0 : X_1 : X_2) \mapsto (X_0 : aX_1 : bX_2)$ of $\mathbb{P}^2$. Then we have an exact sequence

$$0 \to \mathbb{C}^\times \times \mathbb{C}^\times \to \mathrm{Aut}(W) \to D_6 \to 0, \tag{3}$$

where D_6 is the dihedral group of degree 6. Here, the morphism $\mathbb{C}^\times \times \mathbb{C}^\times \to \mathrm{Aut}(W)$ is given by $(a, b) \mapsto \varphi_{(a,b)}$, and the morphism $\alpha : \mathrm{Aut}(W) \to D_6$ corresponds to transitions of six (-1)-curves on W. Let φ_σ and φ_τ be automorphisms of W which correspond to the Cremona transform $(X_0 : X_1 :$

$X_2) \mapsto (X_2X_0 : X_0X_1 : X_1X_2)$ and the automorphism $(X_0 : X_1 : X_2) \mapsto (X_0 : X_2 : X_1)$, respectively. Then we have the following equalities:

$$(\varphi_\sigma)^6 = \mathrm{id}_W, \qquad (\varphi_\tau)^2 = \mathrm{id}_W, \qquad \varphi_\sigma \circ \varphi_\tau \circ \varphi_\sigma \circ \varphi_\tau = \mathrm{id}_W,$$

where the morphism id_W is the unit of the automorphism group of W. Thus the short exact sequence (3) splits. Putting $\sigma = \alpha(\varphi_\sigma)$ and $\tau = \alpha(\varphi_\tau)$, we see that σ and τ form a set of generators of D_6.

1). First, we consider the case $\sharp(\alpha \circ \mu)(G) = 4$. In this case, the subgroup $(\alpha \circ \mu)(G)$ is a Sylow 2-subgroup of D_6, hence conjugate to a subgroup $\langle \sigma^3, \tau \rangle \subset D_6$ generated by σ^3 and τ. Thus, replacing $q : W \to \mathbb{P}^2$ if necessary, we can take the morphism q in such a way that the subgroup $\mu(G)$ is generated by $\varphi_\sigma^3 \circ \varphi_{(bc^2,b)}$ and $\varphi_\tau \circ \varphi_{(c,\frac{1}{c})}$ in $\mathrm{Aut}(W)$, where $b \in \mathbb{C}$ and $c \in \mathbb{C}$ are certain non-zero constants. Then two points of W corresponding to $(1 : 1/(c\sqrt{b}) : 1/\sqrt{b})$ and $(1 : -1/(c\sqrt{b}) : -1/\sqrt{b}) \in \mathbb{P}^2$ are stable under the action of G on W, which contradicts the property ii) in Lemma 3.3. Thus the case 1) is excluded.

Second, we consider the case $\sharp(\alpha \circ \mu)(G) = 2$. The dihedral group D_6 has exactly three conjugate classes which are represented by elements of order 2, namely, those represented by σ^3, τ and $\sigma^3\tau$ respectively.

2-1). The case in which the subgroup $(\alpha \circ \mu)(G)$ is conjugate to $\langle \sigma^3 \rangle$ in D_6. In this case, we can take the morphism q in such a way that the subgroup $\mu(G)$ is generated by $\varphi_\sigma^3 \circ \varphi_{(a,b)}$ and $\varphi_{(c,d)}$ in $\mathrm{Aut}(W)$, where a, b, c, and d are certain non-zero complex numbers with $c^2 = d^2 = 1$. We have $(c,d) \neq (1,1)$, since the case $(c,d) = (1,1)$ violates the property ii) in Lemma 3.3. Thus by equalities

$$\begin{aligned}
&\varphi_\sigma \circ (\varphi_\sigma^3 \circ \varphi_{(a,b)}) \circ \varphi_\sigma^{-1} = \varphi_\sigma^3 \circ \varphi_{(\frac{a}{b},a)}, \quad \varphi_\sigma \circ \varphi_{(-1,1)} \circ \varphi_\sigma^{-1} = \varphi_{(-1,-1)}, \\
&\varphi_\sigma^{-1} \circ (\varphi_\sigma^3 \circ \varphi_{(a,b)}) \circ \varphi_\sigma = \varphi_\sigma^3 \circ \varphi_{(b,\frac{b}{a})}, \quad \varphi_\sigma^{-1} \circ \varphi_{(1,-1)} \circ \varphi_\sigma = \varphi_{(-1,-1)},
\end{aligned}$$

we have only to consider the case $(c,d) = (-1,-1)$. In this case, the (-1)-curve $E_1 = q^{-1}(x_1)$ is a component of the fixed locus of the automorphism $\varphi_{(c,d)} = \varphi_{(-1,-1)}$. We denote by $\psi_0 \in G = \mathrm{Gal}(Y/X)$ the automorphism of Y corresponding to $\varphi_{(-1,-1)}$. By equivalence $B \sim -4K_W$, where the curve B is the branch divisor of $f : Y \to W$ as in Proposition 3.2, we see that $B \cap E_1 \neq \emptyset$, which shows existence of a fixed point $x \in B$ of $\varphi_{(-1,-1)}$. Now since the double cover Y' in Proposition 3.2 has at most rational double points as its singularities, the set $f^{-1}(x)$ includes a fixed point of the automorphism $\psi_0 \in G$ of Y. This contradicts the definition of $\pi : Y \to X$, hence the case 2-1) is excluded.

2-2). The case in which the subgroup $(\alpha \circ \mu)(G)$ is conjugate to $\langle \tau \rangle$ in D_6. In this case, replacing q if necessary, we can take the morphism q in such a way that the subgroup $(\alpha \circ \mu)(G)$ is generated by τ in D_6. Then the (-1)-curves E_1 and $q_*^{-1}(L_{2,3})$ are stable under the action of G on W, which contradicts the property i) in Lemma 3.3. Thus the case 2-2) is excluded.

2-3). The case in which the subgroup $(\alpha \circ \mu)(G)$ is conjugate to $\langle \sigma^3 \circ \tau \rangle$ in D_6. In this case, replacing q if necessary, we can take the morphism q in such a way that the subgroup $(\alpha \circ \mu)(G)$ is generated by $\sigma^3 \circ \tau$ in D_6. Then by equalities $(\sigma^3 \circ \tau)(q_*^{-1}(L_{1,2})) = E_2$ and $(\sigma^3 \circ \tau)(q_*^{-1}(L_{1,3})) = E_3$, we see that two points $q_*^{-1}(L_{1,2}) \cap E_2$ and $q_*^{-1}(L_{1,3}) \cap E_3$ are stable under the action of G on W. This contradicts the property ii) in Lemma 3.3. Hence the case 2-3 is excluded.

3). Finally, we consider the case in which $\sharp(\alpha \circ \mu)(G) = 1$. In this case, any (-1)-curves on W are stable under the action of G on W, which contradicts the property i) in Lemma 3.3. Hence the case 3) is excluded.

Thus we have proved the following lemma, which, together with Lemma 3.2, completes the proof of Proposition 3.1:

Lemma 3.6. *The case $\lambda = 2$ in Lemma 3.2 is impossible.*

Let us exclude the case $\sharp\mathrm{Tors}(X) = 5$.

Proposition 3.3. *Let X be a surface as in Lemma 1.1 with $\chi(\mathcal{O}_X) = \lambda \geq 2$. Then $\sharp\mathrm{Tors}(X) \neq 5$.*

Proof. Let X be a surface as in Lemma 1.1 with $\chi(\mathcal{O}_X) = \lambda \geq 2$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/5$, and $\pi : Y \to X$ the unramified Galois cover of degree 5 corresponding to $\mathrm{Tors}(X)$. Then Y is a minimal algebraic surface of general type with $K_Y^2 = 2p_g(Y) - 3$ and $p_g(Y) = 5\lambda - 1 \geq 9$. Thus the canonical system $|K_Y|$ has a unique base point ([8, Section 1]), and this base point is a fixed point of any automorphism of Y. This contradicts the assumption that $\pi : Y \to X$ is an unramified Galois cover of degree 5. □

Proof of Theorem 1.

Now we are ready to prove Theorem 1. Let X be a minimal algebraic surface as in Theorem 1 with $\chi(\mathcal{O}_X) \geq 2$. Since $\mathbb{Z}/3 \subset \mathbb{Z}/6$, we have $\sharp\mathrm{Tors}(X) \neq 6$ by Proposition 2.1 and Theorem 3. Thus by Lemma 1.2, Propositions 2.1, 3.1 and 3.3, we have i) and ii) in Theorem 1. The bound iii) immediately follows from the following theorem due to Xiao. □

Theorem 5 (Xiao, Corollary 4 in [15]). *Minimal regular surfaces of general type with $c_1^2 < (8/3)(\chi(\mathcal{O}) - 2)$ are algebraically simply connected.*

4 A family of X's with $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$

In this section, we construct a family of X's as in Lemma 1.1 with $\chi(\mathcal{O}_X) = 4 - k$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ for each integer $0 \leq k \leq 2$. Let $W = \mathbb{P}^1 \times \mathbb{P}^1$ be the Hirzebruch surface of degree 0, and $(X_0 : X_1)$ and $(Y_0 : Y_1)$ homogeneous coordinates of $\mathbb{P}^1$. We define an involution ι_0 of W by

$$\iota_0 : ((X_0 : X_1), (Y_0 : Y_1)) \mapsto ((X_1 : X_0), (Y_1 : Y_0)).$$

We put $x = X_1/X_0$ and $y = Y_1/Y_0$. Let G be a group of automorphisms of W generated by ι_0. Then $G \simeq \mathbb{Z}/2$ acts naturally on W, and W has exactly 4 fixed points of ι_0, namely $p_1 : (x, y) = (1, 1)$, $p_2 : (x, y) = (1, -1)$, $p_3 : (x, y) = (-1, 1)$ and $p_4 : (x, y) = (-1, -1)$. Let $q : W_0 \to W$ be the blowing-up of W at $2k + 2$ points $w_1, \ldots, w_{2k+2}$, where $\{w_{2j+1}\}_{0 \le j \le k}$ is a set of distinct $k + 1$ points on $W \setminus \{p_1, \ldots, p_4\}$, and $w_{2j+2} = \iota_0(w_{2j+1})$ for each integer $0 \le j \le k$. The action of G on W lifts to one on W_0. We denote by $E^0_i = q^{-1}(w_i)$ the exceptional curve of the first kind lying over w_i for $1 \le i \le 2k + 2$. Let $q' : W_2 \to W_0$ be the blowing-up of W_0 at two points w'_1 and w'_2, where $w'_1 \in E^0_1$ and $w'_2 = \iota_0(w'_1) \in E^0_2$. We denote by $E^\vee_i = {q'}^{-1}(w'_i)$ the exceptional curve of the first kind lying over w'_i for $i = 1, 2$. We use the same symbol E^0_i for the total transform on W_2 of the divisor E^0_i. We put $\bar{q} = q \circ q' : W_2 \to W$. Note that the action of G on W lifts to one on W_2.

Lemma 4.1. *Assume that the configuration of the $k+1$ points w_{2j+1}'s $(0 \le j \le k)$ and that of w'_1 are sufficiently general. Then there exists a reduced curve B'_2 on W_2 satisfying the following five conditions* :

1) $B'_2 \in |\bar{q}^*(8\Delta_0 + 8\Gamma) - \sum_{i=1,2} 3(E^0_i + E^\vee_i) - \sum_{3 \le i \le 2k+2} 4E^0_i|$,
2) $B'_2 \cap {q'}^{-1}_*(E^0_i) = \emptyset$ *for* $i = 1, 2$,
3) $B'_2 \cap \bar{q}^{-1}(\{p_1, \ldots, p_4\}) = \emptyset$,
4) B'_2 *has at most negligible singularities,*
5) B'_2 *is stable under the action of* G *on* W_2.

Note that $\sum_{3 \le i \le 2k+2} 4E^0_i = 0$ if $k = 0$. We shall give a proof of the lemma above at the end of this section. We define a reduced curve B_2 on W_2 by

$$B_2 = B'_2 + \sum_{i=1,2} {q'}^{-1}_*(E^0_i).$$

Then B_2 is stable under the action of G, and singularities of B_2 are at most negligible ones. Moreover we have $B_2 \sim 2F_2$, where

$$F_2 \sim \bar{q}^*(4\Delta_0 + 4\Gamma) - \sum_{i=1,2} (E^0_i + 2E^\vee_i) - \sum_{3 \le i \le 2k+2} 2E^0_i.$$

Let $f_2 : Y_2 \to W_2$ be the double cover of W_2 with branch locus B_2, and $\tilde{Y} \to Y_2$ the minimal desingularization of Y_2. Then we obtain a surjective morphism $f : \tilde{Y} \to W_2$ of mapping degree 2 with branch locus B_2. We have

$f^*(q'^{-1}_*(E_i^0)) = 2E_i$ for a (-1)-curve E_i on $\tilde{Y}$ for each $i = 1, 2$. We denote by $p : \tilde{Y} \to Y$ the blowing-down of the two (-1)-curves E_1 and E_2. Then we see easily that

$$K_Y^2 = 2(2(4-k)-1), \qquad \chi(\mathcal{O}_Y) = 2(4-k). \tag{4}$$

Lemma 4.2. *Assume that the configuration of the $k+1$ points w_{2j+1}'s $(0 \le j \le k)$ and that of w_1' are sufficiently general. Then the fixed part of the canonical system $|K_{\tilde{Y}}|$ is $\sum_{i=1,2} 2E_i$, and the variable part of $|K_{\tilde{Y}}|$ is free from base points. In particular, Y is minimal.*

Proof. Since W is a rational surface, we have $|K_{\tilde{Y}}| = f^*|K_{W_2} + F_2|$, where

$$F_2 + K_{W_2} \sim \bar{q}^*(2\varDelta_0 + 2\varGamma) - \sum_{i=1,2} E_i^\vee - \sum_{3 \le i \le 2k+2} E_i^0.$$

We study the linear system $|K_{W_2} + F_2|$. We denote by L_{w_i} the unique member of $|\varGamma|$ passing w_i, and by M_{w_i} the unique member of $|\varDelta_0|$ passing w_i, where $1 \le i \le 2k+2$.

First, we give a proof for the case $k = 0$ or 1. Assume that $k = 0$ or 1. The linear system $|\bar{q}^*(\varDelta_0 + \varGamma) - \sum_{i=1,2} E_i^\vee| + |\bar{q}^*(\varDelta_0 + \varGamma) - \sum_{3 \le i \le 2k+2} E_i^0|$ is a subsystem of $|F_2 + K_{W_2}|$. Note that both $L_{w_1} + M_{w_2}$ and $L_{w_2} + M_{w_1}$ are members of $|\varDelta_0 + \varGamma|$ passing w_1 and w_2. Thus the fixed part of $|\bar{q}^*(\varDelta_0 + \varGamma) - \sum_{i=1,2} E_i^\vee|$ is $\sum_{i=1,2} q'^{-1}_*(E_i^0)$, and the variable part of $|\bar{q}^*(\varDelta_0 + \varGamma) - \sum_{i=1,2} E_i^\vee|$ is free from base points. Moreover $|\bar{q}^*(\varDelta_0 + \varGamma) - \sum_{3 \le i \le 2k+2} E_i^0|$ is free from base points. Thus the assertion follows for the case $k = 0$ or 1.

Next we give a proof for the case $k = 2$. Take a member C_1 of $|2\varDelta_0 + \varGamma|$ passing the 5 points w_1, w_3, w_4, w_5 and w_6. This is possible, since $\dim |2\varDelta_0 + \varGamma| = 5$. Let C_2 be a member of $|2\varDelta_0 + \varGamma|$ passing the 5 points w_1, w_2, w_3, w_5 and w_6. Then the 4 members $C_1 + L_{w_2}$, $C_2 + L_{w_4}$, $\iota_0^*(C_1) + L_{w_1} = \iota_0^*(C_1 + L_{w_2})$ and $\iota_0^*(C_2) + L_{w_3} = \iota_0^*(C_2 + L_{w_4})$ of $|2\varDelta_0 + 2\varGamma|$ pass the 6 points $w_1, \ldots, w_6$, hence they are corresponding to members of $|K_{W_2} + F_2|$. We use these 4 divisors to study the canonical system $|K_{\tilde{Y}}|$.

Let C_1', C_1'' and D be effective divisors on W satisfying $C_1 = C_1' + D$ and $\iota_0^*(C_1) = C_1'' + D$, where C_1' and C_1'' have no common irreducible components. Then we have $\iota_0^*(D) = D$ and $C_1'' = \iota_0^*(C_1')$. Let us show that $D = 0$, namely, that C_1 and $\iota_0^*(C_1)$ have no common irreducible components, on the assumption that the configurations of w_{2j+1}'s $(0 \le j \le 2)$ are sufficiently general. We see easily that if the configuration of the 3 points w_{2j+1}'s $(0 \le j \le 2)$ is sufficiently general, then the following five conditions are satisfied :

i) no members of $|2\varDelta_0 + \varGamma|$ stable under ι_0 pass the 3 points w_1, w_3, w_5,

ii) each member of $|\varDelta_0|$ contains at most one out of the 6 points $w_1, \ldots, w_6$,

iii) each member of $|\varGamma|$ contains at most one out of the 6 points $w_1, \dots, w_6$,
iv) no members of $|\varDelta_0 + \varGamma|$ stable under ι_0 pass the 2 points w_3 and w_5,
v) no members of $|2\varDelta_0 + \varGamma|$ passing the 4 points $w_3, \dots, w_6$ are tangent to L_{w_1} at w_1.

Assume that $D \in |2\varDelta_0 + \varGamma|$. Then $D \in |2\varDelta_0 + \varGamma|$ is stable under ι_0, and passes the 3 points w_1, w_3, w_5, which contradicts the condition i). Thus we have $D \notin |2\varDelta_0 + \varGamma|$.

Assume that $D \in |2\varDelta_0|$. Then we have $C_1 \in |\varDelta_0| + |\varDelta_0| + |\varGamma|$, which contradicts the conditions ii) and iii). Thus we have $D \notin |2\varDelta_0|$.

Assume that $D \in |\varDelta_0 + \varGamma|$. Then $C'_1 \in |\varDelta_0|$ contains at most one out of the 5 points w_1, $w_3, \dots, w_6$ by the condition ii). Thus, since $\iota_0^*(D) = D$, the divisor D passes the 4 points w_3, w_4, w_5 and w_6. This contradicts the condition iv). Thus we have $D \notin |\varDelta_0 + \varGamma|$.

Assume that $D \in |\varDelta_0|$. Then D is a member of $|\varDelta_0|$ stable under ι_0. Note that $w_1, \dots, w_6 \in W \setminus \{p_1, \dots, p_4\}$, where $\{p_1, \dots, p_4\}$ is the set of all fixed points of ι_0 on W. Thus by the condition ii), the divisor D contains none of the 6 points $w_1, \dots, w_6$. It follows that both C'_1 and $C''_1 = \iota_0^*(C'_1)$ contain the 4 points w_3, w_4, w_5 and w_6, which contradicts $C'_1 \cdot C''_1 = 2$. Thus we have $D \notin |\varDelta_0|$.

Assume that $D \in |\varGamma|$. Then we have $C_1 \in |\varDelta_0| + |\varDelta_0| + |\varGamma|$, which contradicts the conditions ii) and iii). Thus we have $D \notin |\varGamma|$.

Thus, by the argument above, the divisors C_1 and $\iota_0^*(C_1)$ have no common irreducible components. Moreover C_1 and L_{w_1} have no common irreducible components by the conditions ii) and iii). By the condition v), we have $C_1 \cap L_{w_1} = w_1 + w_7$ for a certain point $w_7 \neq w_1$ on W. It follows

$$(C_1 + L_{w_2}) \cap (\iota_0^*(C_1) + L_{w_1}) = w_7 + w_8 + \sum_{1 \le i \le 6} w_i,$$

where $w_8 = \iota_0(w_7)$. ¿From this we infer that the fixed part of $|K_{W_2} + F_2|$ is $\sum_{i=1,2} {q'}_*^{-1}(E_i^0)$, and that the base locus of the variable part of $|K_{W_2} + F_2|$ is at most $\bar{q}^{-1}(\{w_7, w_8\})$ on the assumption that the configuration of the 4 points w_1, w_3, w_5 and w'_1 are sufficiently general.

By the same method as in the case of C_1, we see that if the configuration of w_1, w_3, w_5 and w'_1 are sufficiently general, then

$$(C_2 + L_{w_4}) \cap (\iota_0^*(C_2) + L_{w_3}) = w'_7 + w'_8 + \sum_{1 \le i \le 6} w_i,$$

where $w'_7 \in L_{w_3}$ and $w'_8 \in L_{w_4}$ are certain points on W. It follows that the base locus of the variable part of $|K_{W_2} + F_2|$ is at most $\bar{q}^{-1}(\{w'_7, w'_8\})$. Thus the assertion follows for the case $k=2$, since we have $\{w_7, w_8\} \cap \{w'_7, w'_8\} = \emptyset$. $\square$

In what follows, we assume that the configuration of the $k+1$ points w_{2j+1}'s $(0 \le j \le k)$ and that of w'_1 are sufficiently general as in Lemma 4.2, hence that Y is minimal. We put

$$F_2 = \bar{q}^*(\sum_{i=1,2} 2(L_{w_i} + M_{w_i})) - \sum_{i=1,2}(E_i^0 + 2E_i^\vee) - \sum_{3 \le i \le 2k+2} 2E_i^0,$$

where L_{w_i} and M_{w_i} are the divisors as in the proof of Lemma 4.2. Then the divisors B_2 and F_2 are stable under the action of G. Let h be a meromorphic function on W_2 corresponding to the principal divisor $B_2 - 2F_2$. Then $c_{\iota_0} = (\iota_0^* h)/h$ is a non-zero constant. We use the same symbol p_i for the point on W_2 lying over the fixed point $p_i \in W$ of ι_0. Since $\{p_1, \ldots, p_4\} \cap \mathrm{supp}(B_2 - 2F_2) = \emptyset$, we infer $h(p_1) \ne 0$, hence $c_{\iota_0} = 1$. Thus by Lemma 1.3, there exist exactly two liftings to Y_2 of the action of G on W_2.

Lemma 4.3. *There exists a unique free action of G on $\tilde{Y}$ which is obtained by lifting the action on W_2. This action on $\tilde{Y}$ induces one on Y free from fixed points.*

Proof. The fiber $f_2^{-1}(p_i)$ is a set of 2 points for each $1 \le i \le 4$. We take the unique lifting to Y_2 of the action of G such that the induced action of G on $f_2^{-1}(p_1)$ is free from fixed points. We obtain an action of G on $f_2^{-1}(p_i)$ by restricting this lifting. Since $\{p_1, \ldots, p_4\}$ is the set of all fixed points of the action of G on W_2, it only needs to show that the action of G on $f_2^{-1}(p_i)$ is free for any $2 \le i \le 4$.

Let L_s be a member of $|\mathit{\Gamma}|$ given by $x - s = 0$, and M_s a member of $|\mathit{\Delta}_0|$ given by $y - s = 0$ for each $s \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Then we have $\bar{q}^{-1}(L_1) \simeq \mathbb{P}^1$ and $\mathcal{O}_{\bar{q}^{-1}(L_1)}(F_2) \simeq \mathcal{O}_{\mathbb{P}^1}(4)$. Putting $U_i = \bar{q}^{-1}(L_1) \setminus \{p_i\}$ $(i = 1, 2)$, we have $\bar{q}^{-1}(L_1) = \bigcup_{i=1,2} U_i$. For each $i = 1, 2$, we take a coordinate z_i on U_i such that $\iota_0 : z_i \mapsto -z_i$ on U_i and $z_1 z_2 = 1$ on $U_1 \cap U_2$ hold. Note that the fixed point $p_1 \in U_2$ is given by $z_2 = 0$, and that the fixed point $p_2 \in U_1$ is given by $z_1 = 0$. Let $\bigcup_{i=1,2} U_i \times \mathbb{C}$ be the total space of the line bundle $\mathcal{O}_{\bar{q}^{-1}(L_1)}(F_2)$. We take a fiber coordinate ζ_i on $U_i \times \mathbb{C}$ such that

$$\zeta_1 = \frac{\zeta_2}{z_2^4}. \tag{5}$$

Let $g_i = 0$ be a defining equation of $B_2|_{\bar{q}^{-1}(L_1)}$ on U_i such that $g_1 = g_2/z_2^8$. Then $f_2^{-1}(\bar{q}^{-1}(L_1))$ is a subvariety of $\bigcup_{i=1,2} U_i \times \mathbb{C}$ locally defined by $\zeta_i^2 - g_i = 0$. Since B_2 is stable under the action of G, the function $\iota_0^* g_1/g_1 = \iota_0^* g_2/g_2$ is holomorphic on $\bar{q}^{-1}(L_1)$, hence a constant. ¿From this together with $g_2(p_1) \ne$

0, we infer $\iota_0^* g_i = g_i$ for $i = 1, 2$. Thus, since the action of G on $f_2^{-1}(p_1)$ is non-trivial, the automorphism of $f_2^{-1}(\bar{q}^{-1}(L_1))$ corresponding to $\iota_0 \in G$ is given by $(z_2, \zeta_2) \mapsto (-z_2, -\zeta_2)$ on $U_2 \times \mathbb{C}$. By this together with (5), we see that this automorphism is given by $(z_1, \zeta_1) \mapsto (-z_1, -\zeta_1)$ on $U_1 \times \mathbb{C}$. Thus the action on $f_2^{-1}(p_2)$ is free from fixed points.

Note that we have $p_1, p_3 \in \bar{q}^{-1}(M_1)$ and $p_3, p_4 \in \bar{q}^{-1}(L_{-1})$. Using M_1 and L_{-1} in place of L_1, we see that the action of G on $f_2^{-1}(p_i)$ is free for $i = 3, 4$ in the same way. Thus the assertion follows. □

Proposition 4.1. *Let X be a quotient of Y by the free action of G given in Lemma 4.3. Then X is a minimal algebraic surface of general type with $c_1^2 = 2\chi(\mathcal{O}) - 1$, $\chi(\mathcal{O}) = 4 - k$ and* $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$.

Proof. Since the projection $\pi : Y \to X$ is an unramified Galois cover of degree 2, we infer from (4) and Lemma 4.2 that X is a minimal surface with $c_1^2 = 2\chi(\mathcal{O}) - 1$, $\chi(\mathcal{O}) = 4 - k$ and $\mathbb{Z}/2 \subset \mathrm{Tors}(X)$. The isomorphy $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ follows from Theorem 1. □

Finally, we give a proof of Lemma 4.1. We take the homogeneous coordinates $(X_0 : X_1)$ and $(Y_0 : Y_1)$ as in the beginning of this section such that w_1 is given by $(x, y) = (0, 0)$. Let C_3 be the unique member of $|\Delta_0 + \Gamma|$ whose strict transform on W_0 passes w_1'. Then C_3 is defined by $\mu x + \nu y = 0$ for certain constants μ and $\nu \in \mathbb{C}$. The point w_{2j+1} is given by $(x, y) = (\alpha_j, \beta_j)$ for each integer $1 \leq j \leq k$, where α_j and $\beta_j \in \mathbb{C}$ are certain constants.

Put $\eta^{\iota_0}(X_0, X_1; Y_0, Y_1) = \eta(X_1, X_0; Y_1, Y_0)$ for each homogeneous polynomial $\eta(X_0, X_1; Y_0, Y_1) \in H^0(\mathcal{O}_W(l\Delta_0 + m\Gamma))$ of bidegree (l, m). Then $\eta \mapsto \eta^{\iota_0}$ gives an involution of $H^0(\mathcal{O}_W(l\Delta_0 + m\Gamma))$, and this involution induces an action of $G = \langle \iota_0 \rangle \simeq \mathbb{Z}/2$ on $H^0(\mathcal{O}_W(l\Delta_0 + m\Gamma))$. Let $V^+_{(l,m)}$ be the space consisting of all elements in $H^0(\mathcal{O}_W(l\Delta_0 + m\Gamma))$ stable under this action. We denote by $\Lambda^+_{(l,m)} = \mathbb{P}(V^+_{(l,m)})$ the subsystem of $|l\Delta_0 + m\Gamma|$ corresponding to the subspace $V^+_{(l,m)}$. If D is an effective divisor on W_2, we denote by $\Lambda^+_{(l,m)}(D)$ the space consisting of all members C's of $\Lambda^+_{(l,m)}$ such that $\bar{q}^* C - D$ is effective. We put $\tilde{\Lambda}^+_{(l,m)}(D) = \bar{q}^*(\Lambda^+_{(l,m)}(D)) - D$. Moreover we put $\Lambda^+ = \Lambda^+_{(8,8)}(\sum_{i=1,2} 3(E_i^0 + E_i^\vee) + \sum_{3 \leq i \leq 2k+2} 4E_i^0)$ and $\tilde{\Lambda}^+ = \tilde{\Lambda}^+_{(8,8)}(\sum_{i=1,2} 3(E_i^0 + E_i^\vee) + \sum_{3 \leq i \leq 2k+2} 4E_i^0)$.

Proof of Lemma 4.1.

First, we give a proof for the case $k = 1$. In what follows, we assume that α_1, β_1, μ and ν are sufficiently general. Then we have $\dim \Lambda^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee) + \sum_{i=3,4} E_i^0) = 1$. It is easily verified that the base locus of this linear pencil is $\{w_i\}_{1 \leq i \leq 4} \cup \{w_9, w_{10}\}$, where the point w_9 is given by

$$x = \frac{\beta_1(\mu\beta_1 + \nu\alpha_1)}{\mu\alpha_1 + \nu\beta_1} \quad \text{and} \quad y = \frac{\alpha_1(\mu\beta_1 + \nu\alpha_1)}{\mu\alpha_1 + \nu\beta_1},$$

and $w_{10} = \iota_0(w_9)$. We use the same symbol w_i for the point on W_2 lying over $w_i \in W$, where $i = 9, 10$. It is also easily verified that $\tilde{\Lambda}^+_{(2,2)}(\sum_{i=3,4} E_i^0)$ is free from base points. Thus from

$$3\tilde{\Lambda}^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee) + \sum_{i=3,4} E_i^0) + \tilde{\Lambda}^+_{(2,2)}(\sum_{i=3,4} E_i^0) \subset \tilde{\Lambda}^+,$$

we infer that the base locus of $\tilde{\Lambda}^+$ is at most $\{w_9, w_{10}\}$. Meanwhile, since $\iota_0^*(C_3)$ passes w_1, we have

$$2(C_3 + \iota_0^*(C_3) + L_{\alpha_1} + L_{1/\alpha_1} + M_{\beta_1} + M_{1/\beta_1}) \in \Lambda^+,$$

where L_s and M_s are the divisors as in the proof of Lemma 4.3 for each $s \in \mathbb{C} \cup \{\infty\}$. Thus, since $C_3 + \iota_0^*(C_3) + L_{\alpha_1} + L_{1/\alpha_1} + M_{\beta_1} + M_{1/\beta_1}$ passes neither w_9 nor w_{10}, we infer that the linear system $\tilde{\Lambda}^+$ is free from base points. By Bertini's theorem, any general member B_2' of $\tilde{\Lambda}^+$ satisfies all the conditions given in Lemma 4.1.

Next, we give a proof for the case $k = 0$. In what follows, we assume that μ and ν are sufficiently general. Then we have $\dim \Lambda^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee)) = 2$. It is easily verified that $\tilde{\Lambda}^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee))$ is free from base points. We therefor infer, since we have

$$3\tilde{\Lambda}^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee)) + \bar{q}^*\Lambda^+_{(2,2)} \subset \tilde{\Lambda}^+,$$

that $\tilde{\Lambda}^+$ is free from base points. Thus any general member B_2' of $\tilde{\Lambda}^+$ satisfies all the conditions given in Lemma 4.1.

Finally, we give a proof for the case $k = 2$. In what follows, we assume that α_1, α_2, β_1, β_2, μ and ν are sufficiently general. Then we see easily that $\dim \Lambda^+_{(2,2)}(\sum_{1 \le i \le 6} E_i^0) = 1$, and that the base locus of this linear system is $\{w_i\}_{1 \le i \le 6} \cup \{w_{11}, w_{12}\}$, where the point w_{11} is given by

$$x = \frac{(\beta_1\beta_2 - 1)(\alpha_1\beta_2 - \alpha_2\beta_1)}{(\beta_1 - \beta_2)(\alpha_1\alpha_2 - \beta_1\beta_2)} \quad \text{and} \quad y = \frac{(\alpha_1\alpha_2 - 1)(\alpha_1\beta_2 - \alpha_2\beta_1)}{(\alpha_1 - \alpha_2)(\alpha_1\alpha_2 - \beta_1\beta_2)},$$

and $w_{12} = \iota_0^*(w_{11})$. We use the same symbol w_i for the point on W_2 lying over $w_i \in W$, where $i = 11, 12$. The linear system $\Lambda^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee) + \sum_{3 \le i \le 6} E_i^0)$ has a unique member C_4. The divisor C_4 is smooth at $w_1, \ldots, w_6, w_{11}$ and w_{12}, since any distinct 2 members of $\Lambda^+_{(2,2)}(\sum_{1 \le i \le 6} E_i^0)$ intersect each other transversally at these 8 points. We denote by $\bar{C}_4$ the strict transform on W_2 of C_4. Then we have $\bar{C}_4 = \bar{q}^*(C_4) - \sum_{i=1,2}(E_i^0 +$

$E_i^{\vee}) - \sum_{3\le i\le 6} E_i^0$. It is also easily verified that $\dim \tilde{\Lambda}^+_{(2,2)}(\sum_{3\le i\le 6} E_i^0) = 2$, and that this linear system has no base points. Thus from

$$3\Lambda^+_{(2,2)}(\sum_{i=1,2}(E_i^0+E_i^{\vee}) + \sum_{3\le i\le 6} E_i^0) + \Lambda^+_{(2,2)}(\sum_{3\le i\le 6} E_i^0) \subset \Lambda^+,$$

we infer that the base locus of $\tilde{\Lambda}^+$ is at most $\bar{C}_4$.

The linear system $\Lambda^+_{(4,4)}(\sum_{i=1,2} 3E_i^0 + \sum_{3\le i\le 6} 2E_i^0)$ has a unique member C_5. By the same method as in the proof of Lemma 4.2, we see that C_4 and C_5 have no common irreducible components. Thus we have

$$C_4 \cap C_5 = w_{13} + w_{14} + \sum_{i=1,2} 3w_i + \sum_{3\le i\le 6} 2w_i, \tag{6}$$

where w_{13} is a point on W and $w_{14} = \iota_0(w_{13})$. If $\{w_{13}, w_{14}\} = \{w_3, w_4\}$ holds for general $\alpha_1, \dots, \nu$, then we have $\{w_{13}, w_{14}\} = \{w_5, w_6\}$ for general $\alpha_1, \dots, \nu$, which is a contradiction. Thus we have $\{w_{13}, w_{14}\} \cap \{w_3, w_4\} = \emptyset$. In the same way, we see $\{w_{13}, w_{14}\} \cap \{w_5, w_6\} = \emptyset$. By the defining equation of C_5, we obtain $\mathrm{mult}_{w_1} C_5 = 3$. Thus, since the defining equation of C_5 is independent of μ and ν, we infer that $\{w_{13}, w_{14}\} \cap \{w_1, w_2\} = \emptyset$ for general μ and ν. Moreover by the defining equation of C_4 and that of C_5, we obtain $C_4 \cap \{p_1, \dots, p_4\} = \emptyset$ and $C_5 \cap \{w_{11}, w_{12}\} = \emptyset$. It follows

$$\{w_{13}, w_{14}\} \cap \{w_1, \dots, w_6, w_{11}, w_{12}, p_1, \dots, p_4\} = \emptyset. \tag{7}$$

Let us use the same symbol w_i for the point on W_2 lying over $w_i \in W$ for $i = 13, 14$. Then from (6), (7) and

$$2\Lambda^+_{(4,4)}(\sum_{i=1,2} 3E_i^0 + \sum_{3\le i\le 6} 2E_i^0) \subset \Lambda^+,$$

we infer that the base locus of $\tilde{\Lambda}^+$ is at most $\bar{C}_4 \cap \bar{C}_5 = \{w_{13}, w_{14}\}$, where $\bar{C}_5 = \bar{q}^*(C_5) - \sum_{i=1,2} 3E_i^0 - \sum_{3\le i\le 6} 2E_i^0$ is the strict transform on W_2 of C_5.

Now let us show that w_{13} and w_{14} are at most ordinary double points of general members of $\tilde{\Lambda}^+$ using the argument above. Let C_6 be a general member of $\Lambda^+_{(2,2)}(\sum_{1\le i\le 6} E_i^0)$. We denote by $\bar{C}_6 = \bar{q}^*(C_6) - \sum_{1\le i\le 6} E_i^0$ the strict transform on W_2 of C_6. Then since

$$\begin{aligned}\Lambda^+_{(2,2)}(\sum_{1\le i\le 6} E_i^0) + \Lambda^+_{(2,2)}(\sum_{i=1,2}(E_i^0+E_i^{\vee}) + \sum_{3\le i\le 6} E_i^0) \\ + \Lambda^+_{(4,4)}(\sum_{i=1,2} 3E_i^0 + \sum_{3\le i\le 6} 2E_i^0) \subset \Lambda^+,\end{aligned}$$

the divisor $\sum_{4\leq i\leq 6}\bar{C}_i+\sum_{i=1,2}2{q'}_*^{-1}(E_i^0)$ is a member of $\tilde{\Lambda}^+$. By $\bar{C}_4\cap\bar{C}_6=\{w_{11},w_{12}\}$ together with (6) and (7), we infer that both w_{13} and w_{14} are ordinary double points of $\sum_{4\leq i\leq 6}\bar{C}_i+\sum_{i=1,2}2{q'}_*^{-1}(E_i^0)$. Thus w_{13} and w_{14} are at most ordinary double points of general members of $\tilde{\Lambda}^+$. Hence any general member B_2' of $\tilde{\Lambda}^+$ satisfies all the conditions given in Lemma 4.1. □

Remark 1. Note that if $k=0$ or 1, then the isomorphism class of the quartet $(W_0,\iota_0|_{W_0},q'_*(B_2),\sum_{i=1,2}E_i^0)$ depends only on the isomorphism class of X. This is verified as follows. In the construction of X above, the morphism $\pi:Y\to X$ is the unramified double cover corresponding to $\mathrm{Tors}(X)$, and $p:\tilde{Y}\to Y$ is the shortest one among all composites of quadric transformations such that the variable part of $p^*|K_Y|$ is free from base points. The morphism $\Phi_{-K_{W_0}}\circ q'\circ f$ is the canonical map of $\tilde{Y}$, where $\Phi_{-K_{W_0}}:W_0\to\mathbb{P}^{6-2k}$ is the anti-canonical map of W_0. We have $\deg\Phi_{-K_{W_0}}=1$ for $k=0,1$ and $\deg\Phi_{-K_{W_0}}=2$ for $k=2$. Thus if $k=0$ or 1, then W_0 is the minimal desingularization of the normalization of the canonical image $Z=\Phi_{K_{\tilde{Y}}}(\tilde{Y})\subset\mathbb{P}^{6-2k}$, since $\Phi_{-K_{W_0}}$ contracts no (-1)-curves. Now since the divisor $\sum_{i=1,2}E_i^0$ on W_0 is the image by $q'\circ f$ of the fixed part of $p^*|K_Y|$, we infer from the argument above that the isomorphism class of the quartet $(W_0,\iota_0|_{W_0},q'_*(B_2),\sum_{i=1,2}E_i^0)$ depends only on the isomorphism class of X. Note also that $q':W_2\to W_0$ is the blowing-up of W_0 at all non-negligible singularities of $q'_*(B_2)$.

Appendix

Let us give a proof of Lemma 1.3. We use the same symbol g for the automorphism of W corresponding to $g\in G$. Let $\{U_i\}_{i\in I}$ be an open covering of W such that the divisor F is given by $f_i=0$ on U_i, where f_i is a meromorphic function on U_i. We take $\{U_i\}_{i\in I}$ in such a way that there exists a left action of G on I such that $g(U_i)=U_{g\cdot i}$ for any $g\in G$. Let $\cup_{i\in I}U_i\times\mathbb{C}$ be the total space of the line bundle F, such that $(p,\zeta_i)\in U_i\times\mathbb{C}$ and $(p,\zeta_j)\in U_j\times\mathbb{C}$ give the same point on $\cup_{i\in I}U_i\times\mathbb{C}$, if and only if $\zeta_i=(f_i/f_j)(p)\zeta_j$. We denote by $\pi:\cup_{i\in I}U_i\to W$ the natural projection.

We take a system $(h_i)_{i\in I}$ of defining equations of B such that $h_i=(f_i/f_j)^nh_j$ on $U_i\cap U_j$ hold. Here h_i is a holomorphic function on U_i for each i. Then the variety V is defined by $\zeta_i^n-h_i=0$ on $U_i\times\mathbb{C}$. Since $h_i/f_i^n=h_j/f_j^n$ gives a meromorphic function on W corresponding to the principal divisor $B-nF$, we have

$$g^*h_{g\cdot i}=c_g(g^*f_{g\cdot i}/f_i)^nh_i \tag{8}$$

on U_i for each $g\in G$, where $c:g\mapsto c_g$ is the Character of G given in Lemma 1.3. Take a constant $c'_g\in\mathbb{C}^\times$ satisfying ${c'_g}^n=c_g$. Then

$$(p,\zeta_i)\mapsto(g(p),\zeta_{g\cdot i})=(g(p),c'_g(g^*f_{g\cdot i}/f_i)(p)\zeta_i)$$

gives an automorphism of $\cup_{i\in I}U_i \times \mathbb{C}$. This automorphism induces that of V, say ψ_g, since (8) holds.

Now assume that the action of G on W lifts to that on V. We denote by φ_g the automorphism of $\cup_{i\in I}U_i \times \mathbb{C}$ corresponding to $g \in G$. Then from $\varphi_g = (\varphi_g \circ \psi_g^{-1}) \circ \psi_g$ and $\pi \circ (\varphi_g \circ \psi_g^{-1}) = \pi$, we infer that φ_g is given by

$$(p, \zeta_i) \mapsto (g(p), \zeta_{g\cdot i}) = (g(p), \chi_g(g^* f_{g\cdot i}/f_i)(p)\zeta_i), \tag{9}$$

where $\chi_g \in \mathbb{C}^\times$ is a constant such that $\chi_g^n = c_g$. Since $g \mapsto \varphi_g$ is an action of G, we see that $\chi : g \mapsto \chi_g$ is a character of G. Thus we have $c \in \mathrm{Im}(\Psi)$.

Assume conversely that $c \in \mathrm{Im}(\Psi)$. We define an automorphism $\varphi_{\chi,g}$ of V by $(p, \zeta_i) \mapsto (g(p), \zeta_{g\cdot i}) = (g(p), \chi_g(g^* f_{g\cdot i}/f_i)(p)\zeta_i)$ for each $\chi \in \Psi^{-1}(c)$ and $g \in G$. Then it is easily verified that $\varphi_\chi : g \mapsto \varphi_{\chi,g}$ is a lifting of the action of G on W. The set $\{\varphi_\chi\}_{\chi\in\Psi^{-1}(c)}$ is that of all liftings of the action of G on W. $\square$

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