FRENET FRAMES AND TODA SYSTEMS

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Abstract. It is shown that the integrability conditions of the equations satisfied by the local Frenet frame associated with a holomorphic curve in a complex Grassmann manifold coincide with a special class of nonabelian Toda equations. A local moving frame of a holomorphic immersion of a Riemann surface into a complex Grassmann manifold is constructed and the corresponding connection coefficients are calculated.

1. Introduction

In the present paper we consider local differential geometry of a holomorphic curve ψ in the complex Grassmann manifold $G^k(\mathbb{C}^n)$. It is well known that the simplest way to study such a curve is to consider a holomorphic local lift ξ of ψ to the space $\mathrm{Mat}(n, k; \mathbb{C})$ of complex $n \times k$ matrices, defined on some open subset U of M. The natural object arising here is the so called Frenet frame being a mapping which associates a basis of $\mathbb{C}^n$ with a point of U. We give a procedure of the construction of the Frenet frame and derive the equations which are satisfied by the corresponding $\mathbb{C}^n$-valued functions on M. Then we show that the obtained equations are closely connected to nonabelian Toda systems based on the Lie group $\mathrm{GL}(n, \mathbb{C})$. The equations describing these systems are exactly integrable, so we get a possibility to construct holomorphic embeddings with prescribed properties. Note that our definition of the Frenet frame is different from the definition usually used for the problem under consideration, see, for example, [Li97]. Namely, we do not perform the complete orthogonalisation of the basis, and it is our definition that leads to an exactly integrable system. The paper is concluded with the explanation of how the Frenet frame can be used to construct a local moving frame associated with the mapping ψ and with the calculation of the corresponding connection coefficients.

There is a lot of papers discussing geometry of harmonic mappings of a Riemann surface into a projective space, see, for example, [EWo83, BWo92] and references therein. Since any holomorphic mapping from one Kähler manifold to another one is harmonic, most results of those paper are valid for holomorphic mappings. There are also some papers devoted to consideration of harmonic mappings from Riemann surfaces to Grassmann manifolds [EWo83, CWo85, BWo86, BSa87, CWo87, Wol88, Woo88, Val88, Uhl89]. In these papers the problems of classification and construction of harmonic mappings are mainly investigated. On the other hand, local differential geometry properties of holomorphic mappings from Riemann surfaces to Grassmann manifolds have not yet been discussed in a systematic way. In this respect we would like to mention the paper [Li97] which deals with differential geometry of holomorphic mappings from the two-sphere to a Grassmann manifold.

Note that the connection of abelian Toda systems based on the Lie group $\mathrm{SL}(n, \mathbb{C})$ with the equations satisfied by the Frenet frame associated with an immersion of a Riemann surface into a projective space appeared to be useful for the study of W-geometries [GMa92, GMa93, Ger93]. Moreover, this connection allows to give a simple proof of the infinitesimal Plücker relations. The abelian Toda systems based on other semisimple Lie groups can be associated with equations satisfied by the Frenet frames of holomorphic curves in the corresponding flag manifolds. This leads to the so-called generalised infinitesimal Plücker relations [RSa94, GSa96]. The results of the present paper show that there are some relations between the geometric characteristics of the holomorphic curves associated with a holomorphic curve in a Grassmann manifold, but here we have no direct analogues of the Plücker relations. This problem requires an additional consideration. It is also very interesting to investigate the corresponding W-geometries.

2. Frenet frames

2.1. **Holomorphic curves in Grassmann manifolds.** We start with the description of the realisation of complex Grassmann manifolds which is convenient for our purposes. Let $\mathrm{Mat}(n, k; \mathbb{C})$ denote the complex manifold of $n \times k$ complex matrices. Assume that $n > k$ and denote by $\mathrm{Mat}(n, k; \mathbb{C})^\times$ the open submanifold of $\mathrm{Mat}(n, k; \mathbb{C})$ formed by $n \times k$ matrices of rank k. The manifold $\mathrm{Mat}(n, k; \mathbb{C})^\times$ could be naturally identified with the set of all bases of k-dimensional subspaces of $\mathbb{C}^n$. Two points $p, p' \in \mathrm{Mat}(n, k; \mathbb{C})^\times$ correspond to bases of the same subspace of $\mathbb{C}^n$ if and only if there is an element g of the complex general linear group $\mathrm{GL}(k, \mathbb{C})$ such that $p' = pg$. Thus we have the free right action of the complex Lie group $\mathrm{GL}(k, \mathbb{C})$ on $\mathrm{Mat}(n, k; \mathbb{C})^\times$ connecting different bases of the k-dimensional subspaces of $\mathbb{C}^n$. The orbit space here is the Grassmann manifold $G^k(\mathbb{C}^n)$. The canonical projection $\pi : \mathrm{Mat}(n, k; \mathbb{C})^\times \to G^k(\mathbb{C}^n)$ is holomorphic and $\mathrm{Mat}(n, k; \mathbb{C})^\times \xrightarrow{\pi} G^k(\mathbb{C}^n)$ is a holomorphic principal $\mathrm{GL}(k, \mathbb{C})$-bundle.

Consider a holomorphic curve $\psi : M \to G^k(\mathbb{C}^n)$, where M is some connected Riemann surface. Any such a curve defines a holomorphic subbundle $\underline{\psi}$ of rank k of the trivial holomorphic fibre bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$. Here the fiber over the point $p \in M$ is $\psi(p)$. The mapping ψ is said to be *linearly full* if the bundle $\underline{\psi}$ is not contained in any proper trivial subbundle of $\underline{\mathbb{C}}^n$.

To describe geometry of the corresponding holomorphic curve in $G^k(\mathbb{C}^n)$ it is convenient to use local holomorphic lifts of the mapping ψ to $\mathrm{Mat}(n, k; \mathbb{C})^\times$. We say that a holomorphic mapping $\xi : U \to \mathrm{Mat}(n, k; \mathbb{C})^\times$, where U is an open subset of M, is a local holomorphic lift of ψ if it satisfies the relation

$$\pi \circ \xi = \psi|_U.$$

Since our consideration is local, we will use local holomorphic lifts defined in coordinate neighbourhoods of M. The corresponding local coordinate will be denoted by z. Moreover, to make our notations consistent with those used in description of Toda systems, we denote $z^+ = z$, $z^- = \bar{z}$ and

$$\partial_- = \partial/\partial z, \qquad \partial_+ = \partial/\partial \bar{z}.$$

Let $\xi : U \to \mathrm{Mat}(n, k; \mathbb{C})^\times$ be a lift of a holomorphic mapping $\psi : M \to G^k(\mathbb{C}^n)$. Denote by $f_1, \ldots, f_k$ the holomorphic mappings from U to $\mathbb{C}^n$ determined by the columns of the matrix-valued mapping ξ. For any $p \in U$ the linear subspace W_p of $\mathbb{C}^n$ spanned by the vectors $f_1(p), \ldots, f_k(p)$ and by the vectors given by the first and higher order derivatives of the functions $f_1, \ldots, f_k$ over z at the point p, is independent of the choice of the lift ξ and local coordinate z.

Proposition 2.1. *For any points p and p' of M one has*

$$W_p = W_{p'}.$$

Proof. Let p be an arbitrary point of U and $c = z(p)$. Expand the functions

$$f_1 \circ z^{-1}, \ \ldots, \ f_k \circ z^{-1} \tag{2.1}$$

in the power series at c, which converges absolutely in some disc $D(c, r) \subset z(U)$. Since the first and higher order derivatives of the functions $f_1 \circ z^{-1}, \ldots, f_k \circ z^{-1}$ over z can be also expanded at c in the power series which absolutely converge in $D(c, r_c)$, it is clear that for any $p' \in z^{-1}(D(c, r))$ one has $W_{p'} \subset W_p$. From the other hand, if $p' \in z^{-1}(D(c, r/2))$ then the functions (2.1) and their first and higher derivatives over z can be expanded at the point $c' = z(p')$ in a power series which converge at least in the disc $D(c', r/2)$. Since $p \in z^{-1}(D(c', r/2))$, one concludes that $W_p \subset W_{p'}$. Therefore, for all points p' such that $|z(p) - z(p')| < r/2$ one has $W_{p'} = W_p$.

Considering various local lifts of ψ we conclude that for any point $p \in M$ there is an open neighbourhood U_p such that $W_{p'} = W_p$ for all $p' \in U_p$. Now the assertion of the Proposition follows from the connectedness of M. □

Thus we have some constant subspace W of $\mathbb{C}^n$ which characterises the curve ψ. The mapping ψ can be considered as a mapping from M to $G^k(W)$, in this case it can be easily shown that ψ is linearly full. Therefore, without any loss of generality we can restrict ourselves to the consideration of linearly full mappings.

2.2. **Some properties of vector-valued holomorphic functions.** In this Section V is an n-dimensional complex vector space and U is an open connected subset of the complex plane $\mathbb{C}$. The results similar to ones discussed below in this Section for the case of polynomial mappings are proved in [Li95, Li97].

A mapping from U to V is called a *V-valued function* on U. Let f be a V-valued function on U. For any $w \in V^*$ one defines the complex function $w(f)$ by

$$w(f)(c) = w(f(c)).$$

If g is a V^*-valued function on U one defines the complex function $g(f)$ by

$$g(f)(c) = g(c)(f(c)).$$

The function f is said to be *holomorphic*, if for any $w \in V^*$ the function $w(f)$ is holomorphic.

Let $\{e_i\}$ be a basis of V, $\{e^i\}$ be the dual basis of V^*, and f be a V-valued function on U. It is easy to verify that for any $c \in U$ one has

$$f(c) = \sum_{i=1}^{n} e_i \, b^i(c),$$

where $b^i = e^i(f)$. Here it is customary to write

$$f = \sum_{i=1}^{n} e_i \, b^i. \tag{2.2}$$

It is clear that the function f is holomorphic if and only if the functions b^i are holomorphic.

If f is a holomorphic V-valued function on U and g is a holomorphic V^*-valued function on U, then $g(f)$ is a holomorphic function on U.

A point $c \in U$ is called a *zero* of a holomorphic V-valued function $f \not\equiv 0$ on U, if $f(c) = 0$. A zero of a holomorphic V-valued function $f \not\equiv 0$ is a common zero of the functions b^i entering a representation of type (2.2). Here, in general, the order of the zero is different for different values of the index i, moreover, it depends on the choice of the basis $\{e_i\}$. Nevertheless, the minimal value of order does not depend on the choice of the basis $\{e_i\}$ and it characterises the V-valued function f itself. The corresponding positive integer is called the *order* of zero of f.

A holomorphic V-valued function $f \not\equiv 0$ on U may have only a countable number of isolated zeros, and the set of zeros has no limit points in U.

If $c \in U$ is a zero of order μ of a holomorphic V-valued function $f \not\equiv 0$ on U, one can represent f as

$$f = g(z - c)^{\mu}.$$

Here g is a holomorphic V-valued function on U having the same zeros as f, except the zero at a.

Proposition 2.2. *Let $f \not\equiv 0$ be a holomorphic V-valued function on U. There exists a holomorphic V-valued function g on U having no zeros in U and such that there is valid the representation*

$$f = g \, d,$$

where d is a holomorphic function on U.

Proof. In accordance with the generalisation of the Weierstrass theorem [Mar77], there exists a holomorphic function d having the same zeros and with the same orders as f. It is easy to show that the function $g = f/d$ satisfies the requirement of the Proposition. □

The *rank* of a set $\{f_1, \dots, f_k\}$ of holomorphic V-valued functions on U is defined as

$$\operatorname{rank}\{f_1, \dots, f_k\} = \max_{c \in U} \operatorname{rank}\{f_1(c), \dots, f_k(c)\}.$$

The rank of a set $\{f_1, \dots, f_k\}$ of holomorphic V-valued functions on U equals k if and only if $f_1 \wedge \dots \wedge f_k \not\equiv 0$. We say that a set $\{f_1, \dots, f_k\}$ of holomorphic V-valued functions on U is of *constant rank* if

$$\operatorname{rank}\{f_1(c), \dots, f_k(c)\} = \operatorname{rank}\{f_1, \dots, f_k\}$$

for any $c \in U$.

Proposition 2.3. *Let the set $\{f_1, \dots, f_k\}$ of holomorphic V-valued functions on U be of rank k. Then there exists a set $\{g_1, \dots, g_k\}$ of V-valued holomorphic functions on U of constant rank k such that*

$$f_\alpha = \sum_{\beta=1}^{k} g_\beta \, d_\alpha^\beta, \qquad \alpha = 1, \dots, k,$$

where d_α^β are holomorphic functions on U.

Proof. We prove the Proposition by induction over k. The case $k = 1$ coincides with Proposition 2.2. Suppose that the Proposition is valid for some fixed k. Consider a set $\{f_1, \dots, f_k, f_{k+1}\}$ of holomorphic V-valued functions on U having rank $k + 1$. It is clear that the rank of the set $\{f_1, \dots, f_k\}$ equals k. Hence, one can find a set $\{g_1, \dots, g_k\}$ of holomorphic V-valued functions on U, satisfying the requirement of the Proposition.

Taking into account Proposition 2.2, represent f_{k+1} in the form

$$f_{k+1} = g \, d,$$

where the holomorphic V-valued function g has no zeros, and consider the holomorphic $\bigwedge^{k+1}(V)$-valued function $g_1 \wedge \dots \wedge g_k \wedge g$. If this function has no zeros, then we take g as g_{k+1} and get the set of functions $\{g_1, \dots, g_k, g_{k+1}\}$ which satisfies the requirement of the Proposition.

Suppose now that the function $g_1 \wedge \dots \wedge g_k \wedge g$ has zeros at the points of the set $\{c_\iota\}_{\iota \in \mathcal{I}}$ and the zero at c_ι is of order μ_ι. Since $g_1(c_\iota) \wedge \dots \wedge g_k(c_\iota) \wedge g(c_\iota) = 0$ and the vectors $g_1(c_\iota)$, ..., $g_k(c_\iota)$ are linearly independent one has

$$g(c_\iota) = \sum_{\beta=1}^{k} g_\beta(c_\iota) \, b_\iota^\beta,$$

where b_ι^β are some complex numbers. Using the generalisation of the Weierstrass theorem and the generalisation of the Mittag–Leffler theorem [Mar77], one can show that there exist functions b^β, $\beta = 1, \dots, k$, which are holomorphic on U and have the property

$$b^\beta(c_\iota) = b_\iota^\beta.$$

Define the holomorphic V-valued function

$$h = g - \sum_{\beta=1}^{k} g_\beta \, b^\beta.$$

It is clear that this function has zeros only at the points of the set $\{c_\iota\}$. Moreover, since

$$g_1 \wedge \cdots \wedge g_k \wedge \eta = g_1 \wedge \cdots \wedge g_k \wedge g,$$

the order of zero of h at the point c_ι is less or equal to the order of zero of $g_1 \wedge \cdots \wedge g_k \wedge g$. Represent h in the form

$$h = g' \, d',$$

where the holomorphic V-valued function g' has no zeros. Then the equality

$$g_1 \wedge \cdots \wedge g_k \wedge g = g_1 \wedge \cdots \wedge g_k \wedge g' \, d'$$

implies that for any $\iota \in \mathcal{I}$ either the point c_ι is not a zero of the function $g_1 \wedge \cdots \wedge g_k \wedge g'$, or it is a zero of it but of order less than μ_ι. If the function $g_1 \wedge \cdots \wedge g_k \wedge g'$ has no zeros, we take g' as g_{k+1} and from the equality

$$f_{k+1} = \left(\sum_{\beta=1}^{k} g_\beta \, b^\beta + g' \, d' \right) d$$

conclude that the set $\{g_1, \ldots, g_k, g_{k+1}\}$ satisfies the requirement of the Proposition. If it is again not the case, we repeat along the lines of this paragraph using the function g' instead of g.

Since the order of a zero is always finite, after a finite number of steps we come to the set $\{g_1, \ldots, g_k, g_{k+1}\}$ satisfying the requirement of the Proposition. □

From the proof of Proposition 2.3 it follows that the functions $g_1, \ldots, g_k$ can be chosen in such a way that the matrix $D = (d_\alpha^\beta)$ is upper triangular. Moreover, if the set $\{f_1, \ldots, f_l\}$, $l \leq k$, is of constant rank l, then we can choose $g_1 = f_1, \ldots, g_l = f_l$.

Proposition 2.4. *Let $\{f_1, \ldots, f_k\}$, $k < n$, be a set of holomorphic V-valued functions on U of constant rank k. Then there exists a set $\{f_{k+1}, \ldots, f_n\}$ of holomorphic V-valued functions on U such that the set $\{f_1, \ldots, f_k, f_{k+1}, \ldots, f_n\}$ is of constant rank n on U.*

Proof. Let c be some point of U. Denote $e_1 = f_1(c), \ldots, e_k = f_k(c)$. The vectors $e_1, \ldots, e_k$ are linearly independent, and one can find vectors $e_{k+1}, \ldots, e_n$ such that $\{e_1, \ldots, e_k\}$ is a basis of V. Consider the set $\{f_1, \ldots, f_k, e_{k+1}\}$ of holomorphic V-valued functions on U. It is clear that the rank of this set equals $k+1$. Applying arguments used in the proof of Proposition 2.3, one can show that there is a holomorphic V-valued function f_{k+1} on U such that the set $\{f_1, \ldots, f_k, f_{k+1}\}$ is of constant rank $k+1$ on U. Using such a construction repeatedly we find a set $\{f_1, \ldots, f_n\}$ satisfying the requirement of the Proposition. □

Proposition 2.5. *Let $\{f_1, \ldots, f_n\}$ be a set of holomorphic V-valued functions on U of constant rank n. Then there exists a unique set $\{f^1, \ldots, f^n\}$ of holomorphic V^*-valued functions on U such that*

$$f^i(f_j) = \delta^i_j, \qquad i, j = 1, \ldots, n.$$

Proof. Let $\{e_i\}$ be a basis of V, and $\{e^i\}$ be the dual basis of V^*. One has

$$f_i = \sum_{j=1}^{n} e_j\, b^j_i,$$

where b^j_i, $i, j = 1, \dots, n$, are holomorphic functions on U. Since for any $c \in U$ the vectors $f_1(c), \dots, f_n(c)$ are linearly independent, the matrix $(b^j_i(c))$ is nondegenerate. Hence, there are holomorphic functions d^i_j, $i, j = 1, \dots, n$, on U such that for any $c \in U$ one has

$$\sum_{m=1}^{n} d^i_m(c)\, b^m_j(c) = \delta^i_j, \qquad i, j = 1, \dots, n.$$

It is easy to get convinced that the holomorphic V^*-valued functions

$$f^i = \sum_{j=1}^{n} d^i_j\, e^j, \qquad i = 1, \dots, n,$$

satisfy the requirement of the Proposition. □

Proposition 2.6. *Let $\{f_1, \dots, f_k\}$ be a set of holomorphic V-valued functions on U of constant rank k, and g be a holomorphic V-valued function on U such that $f_1 \wedge \dots \wedge f_k \wedge g \equiv 0$. Then there is a unique representation*

$$g = \sum_{\alpha=1}^{k} f_\alpha\, b^\alpha,$$

where b^α, $\alpha = 1, \dots, k$, are holomorphic functions on U.

Proof. For any $c \in U$ the vectors $f_1(c), \dots, f_k(c)$ are linearly independent, while the vectors $f_1(c), \dots, f_k(c), g(c)$ are linearly dependent, therefore one has

$$g = \sum_{\alpha=1}^{k} f_\alpha\, b^\alpha,$$

where b^α, $\alpha = 1, \dots, k$, are complex functions on U. Due to Proposition 2.4 one can construct holomorphic V-valued functions $f_{k+1}, \dots, f_n$ such that the set $\{f_1, \dots, f_n\}$ is of constant rank n. Let f^i, $i = 1, \dots, n$, be the holomorphic V^*-valued functions satisfying the requirement of Proposition 2.5. Using these functions, one gets

$$b^\alpha = f^\alpha(g), \qquad \alpha = 1, \dots, k.$$

Thus, the functions b^α, $\alpha = 1, \dots, k$, are holomorphic. □

Proposition 2.7. *Let the set $\{f_1, \dots, f_k\}$ of holomorphic V-valued functions on U be of rank $l \le k$. Then there is a set $\{g_1, \dots, g_l\}$ of holomorphic V-valued functions on U of constant rank l such that there is valid the representation*

$$f_\alpha = \sum_{\beta=1}^{l} g_\beta\, d^\beta_\alpha, \qquad \alpha = 1, \dots, k, \tag{2.3}$$

where d^β_α, $\alpha = 1, \dots, k$, $\beta = 1, \dots, l$, are holomorphic functions on U.

If $\{g'_1, \dots , g'_l\}$ is another set of holomorphic V-valued functions on U of constant rank l and for some holomorphic functions d'^{β}_{α} one has

$$f_\alpha = \sum_{\beta=1}^{l} g'_\beta \, d'^{\beta}_{\alpha}, \qquad \alpha = 1, \dots , k,$$

then for some holomorphic functions c^β_α, $\alpha, \beta = 1, \dots , l$, one has

$$g'_\alpha = \sum_{\beta=1}^{l} g_\beta \, c^\beta_\alpha, \qquad \alpha = 1, \dots , l. \tag{2.4}$$

Proof. Without loss of generality we can assume that the rank of the set $\{f_1, \dots , f_l\}$ equals l. Thus, in accordance with Proposition 2.3 one can find a set $\{g_1, \dots , g_l\}$ of holomorphic V-valued functions on U such that the set $\{g_1, \dots , g_l\}$ is of constant rank l and (2.3) is valid for $\alpha = 1, \dots , l$. It is not difficult to understand that for any α such that $l < \alpha \le k$ one has $g_1 \wedge \dots \wedge g_l \wedge f_\alpha \equiv 0$. Thus, due to Proposition 2.6 representation (2.3) is valid for all values of α.

To prove the second part of the Proposition, we suppose that the set $\{f_1, \dots , f_l\}$ is again of rank l. Then one has

$$f_1 \wedge \dots \wedge f_l = g_1 \wedge \dots \wedge g_l \det \widetilde{D}, \tag{2.5}$$

$$f_1 \wedge \dots \wedge f_l = g'_1 \wedge \dots \wedge g'_l \det \widetilde{D}', \tag{2.6}$$

where $\widetilde{D}$ and $\widetilde{D}'$ are the holomorphic $\mathrm{Mat}(l, \mathbb{C})$-valued functions formed by the functions d^β_α and d'^{β}_{α}, $\alpha, \beta = 1, \dots , l$, respectively. From (2.6) it follows that

$$f_1 \wedge \dots \wedge f_l \wedge g'_\alpha \equiv 0, \qquad \alpha = 1, \dots , l.$$

Using this relation and taking into account (2.5), one concludes that

$$g_1 \wedge \dots \wedge g_l \wedge g'_\alpha \equiv 0, \qquad \alpha = 1, \dots , l.$$

Now Proposition 2.6 implies that representation (2.4) is valid. □

Proposition 2.8. *Let $\{g_1, \dots , g_k\}$ be a set of V-valued holomorphic functions on U of constant rank k, and $\{f_1, \dots , f_l\}$ be a set of V-valued holomorphic functions on U such that the rank of the set $\{g_1, \dots , g_k, f_1, \dots , f_l\}$ equals $k + m$, $m \le l$. Then there exist a set $\{g_{k+1}, \dots , g_{k+m}\}$ of V-valued holomorphic functions on U such that the set $\{g_1, \dots , g_k, g_{k+1}, \dots , g_{k+m}\}$ is of constant rank $k + m$ and there is valid the representation*

$$f_\alpha = \sum_{\beta=1}^{k+m} g_\beta d^\beta_\alpha, \qquad \alpha = 1, \dots , l,$$

where d^β_α, $\alpha = 1, \dots , l$, $\beta = 1, \dots , k + m$, are holomorphic functions on U.

Proof. The validity of the Proposition follows from the proof of Proposition 2.3, from the discussion given after that proof, and from the proof of Proposition 2.7. □

2.3. **Construction of Frenet frame.** Let M be a Riemann surface, and U be an open subset of M. The *rank* of a holomorphic $\mathrm{Mat}(n,k;\mathbb{C})$-valued function ξ on U is defined as

$$\operatorname{rank}\xi = \max_{p\in U}\operatorname{rank}\xi(p).$$

We say that a holomorphic $\mathrm{Mat}(n,k;\mathbb{C})$-valued function ξ on U is of *constant rank* if

$$\operatorname{rank}\xi(p) = \operatorname{rank}\xi$$

for each $p \in U$. A $\mathrm{Mat}(n,k;\mathbb{C})$-valued holomorphic function ξ on U generates the set $\{f_1,\dots,f_k\}$ of $\mathbb{C}^n$-valued holomorphic functions on U determined by the columns of ξ. A $\mathrm{Mat}(n,k;\mathbb{C})$-valued holomorphic function ξ on U has rank l if and only if the corresponding set $\{f_1,\dots,f_k\}$ is of rank l. A $\mathrm{Mat}(n,k;\mathbb{C})$-valued holomorphic function ξ on U is of constant rank if and only if the corresponding set $\{f_1,\dots,f_k\}$ is of constant rank. Having these remarks in mind, we use the results of Section 2.2 for $\mathrm{Mat}(n,k;\mathbb{C})$-valued holomorphic functions.

Consider again a holomorphic curve $\psi : M \to G^k(\mathbb{C}^n)$ and some holomorphic lift $\xi : U \subset M \to \mathrm{Mat}(n,k;\mathbb{C})^\times$ of ψ. The function ξ considered as a $\mathrm{Mat}(n,k;\mathbb{C})$-valued function is of constant rank k.

Since, in general, the function ξ will be the first member of some finite sequence of holomorphic matrix-valued functions, it is convenient to denote $\xi_0 = \xi$ and $k_0 = k$. Consider the holomorphic $\mathrm{Mat}(n,2k_0;\mathbb{C})$-valued function $(\xi_0 \;\; \partial\xi_0/\partial z)$ on U. This function is of rank $k_0 + k_1$, where $0 \le k_1 \le k_0$. In the case $k_1 = 0$ due to Proposition 2.8 one has

$$\partial_-\xi_0 = \xi_0\, B_{00}, \tag{2.7}$$

where B_{00} is a holomorphic $\mathrm{Mat}(k_0;\mathbb{C})$-valued function on U. Therefore, all the derivatives of ξ_0 over z can be expressed via ξ_0 by relation similar to (2.7). If $k_1 > 0$, then again due to Proposition 2.8 there exists a holomorphic $\mathrm{Mat}(n,k_1;\mathbb{C})$-valued function ξ_1 of constant rank k_1 such that

$$\partial_-\xi_0 = \xi_0\, B_{00} + \xi_1\, B_{10}, \tag{2.8}$$

where B_{00} is a holomorphic $\mathrm{Mat}(k_0;\mathbb{C})$-valued function and B_{10} is a holomorphic $\mathrm{Mat}(k_1,k_0;\mathbb{C})$-valued function. The linear subspaces of $\mathbb{C}^n$ spanned by the columns of the matrix $(\xi_0(p)\,\xi_1(p))$, $p \in U$, do not depend on the choice of functions ξ_0 and ξ_1. Actually, they are determined only by the mapping ψ. Using these subspaces, we define a holomorphic subbundle $\underline{\psi}_1$ of $\underline{\mathbb{C}}^n$ called the *first osculating space* of ψ, which generates a holomorphic curve $\psi_1 : M \to G^{k_0+k_1}(\mathbb{C}^n)$ called the *first associated curve* of ψ.

Consider the holomorphic $\mathrm{Mat}(n,k_0+2k_1;\mathbb{C})$-valued function $(\xi_0\,\xi_1\,\partial\xi_1/\partial z)$. It is of rank $k_0+k_1+k_2$, where $0 \le k_2 \le k_1$. Again, if $k_2 = 0$, one concludes that the derivatives of ξ_0 and ξ_1 over z can be expressed via ξ_0 and ξ_1. If $k_2 \neq 0$ one finds a holomorphic $\mathrm{Mat}(n,k_2;\mathbb{C})$-valued function ξ_2 of constant rank k_2 such that

$$\partial_-\xi_1 = \xi_0\, B_{01} + \xi_1\, B_{11} + \xi_2\, B_{21}.$$

After a finite number of steps we end up with the functions $\xi_0, \ldots, \xi_t$ satisfying the relations

$$\partial_- \xi_a = \sum_{b=1}^{a+1} \xi_b \, B_{ba}, \tag{2.9}$$

where $B_{t+1,t} \equiv 0$. Thus, all the derivatives of $\xi_0, \ldots, \xi_t$ over z can be expressed via $\xi_0, \ldots, \xi_t$. Each step of the construction gives us the corresponding subbundle $\underline{\psi}_a$, $a = 1, \ldots, t$, of $\underline{\mathbb{C}}^n$ and we get the flag of the subbundles

$$\underline{\psi}_0 \subset \underline{\psi}_1 \subset \cdots \subset \underline{\psi}_t.$$

Proposition 2.9. *For any $p \in M$ the fiber $\underline{\psi}_{tp}$ coincide with the linear subspace V defined in Section 2.1.*

Proof. The independence of $\underline{\psi}_{tp}$ of p can be proved along the lines of the proof of Proposition 2.1. So we have just one subspace of $\mathbb{C}^n$ which we denote by W'. It is quite clear that $W \subset W'$. Suppose that W is a proper subspace of W'. In this case there exists an element $w \in \mathbb{C}^{n*}$ such that $w(v) = 0$ for each $v \in W$, but $w(u) \neq 0$ for some element of W'. Identifying $\mathbb{C}^n$ with the space of complex $n \times 1$ matrices and $\mathbb{C}^{n*}$ with the space of complex $1 \times n$ matrices, we can write

$$w(v) = wv.$$

The condition $w(v) = 0$ for each $v \in W$ is equivalent to the requirement

$$w \, \partial_-^l \xi = 0$$

for $l = 0, 1, \ldots$. In particular, one has

$$w \, \xi = w \, \xi_0 = 0.$$

Further, using (2.8) one obtains

$$w \, \partial_- \xi_0 = w \, \xi_1 \, B_{10} = 0.$$

Since the $\mathrm{Mat}(k_1, k_0; \mathbb{C})$-valued function B_{10} is of rank k_1, one has

$$w \, \xi_1 = 0.$$

Similarly one shows that $w \, \xi_a = 0$ for all $a = 1, \ldots, t$, but this contradicts to our supposition. Thus, $W' = W$. □

For a linearly full mapping ψ one has $W = \mathbb{C}^n$, therefore, $\underline{\psi}_t = \underline{\mathbb{C}}^n$. In any case the tth associated curve is trivial.

Starting from this point, we assume that the linear space $\mathbb{C}^n$ is endowed with a positive definite hermitian scalar product

$$(v, u) = \sum_{i,j=1}^{n} \overline{v^i} \, h_{\bar{\imath} j} \, u^j,$$

that in matrix notation can be written as

$$(v, u) = v^\dagger h \, u.$$

Here and below we denote by $A^\dagger$ the hermitian conjugate of the matrix A.

Now we proceed to the construction of what we call the Frenet frame associated with the lift ξ of the mapping ψ. Consider a $\mathrm{Mat}(n, \mathbb{C})$-valued function

$$\mathit{\Pi}_0 = I_n - \xi_0 (\xi_0^\dagger \, h \, \xi_0)^{-1} \xi_0^\dagger \, h, \tag{2.10}$$

where I_n is the $n \times n$ unit matrix. For any $p \in U$ the matrix $\mathit{\Pi}_0(p)$ is a matrix of the operator of the orthogonal projection to the orthogonal complement of $\underline{\psi}_{0p}$ in $\mathbb{C}^n$, with respect to the canonical basis of $\mathbb{C}^n$. From (2.8) one immediately gets

$$\mathit{\Pi}_0 \, \partial_- \xi_0 = \mathit{\Pi}_0 \, \xi_1 \, B_{10}. \tag{2.11}$$

Denoting

$$\varphi_0 = \xi_0, \qquad \varphi_1 = \mathit{\Pi}_0 \, \xi_1, \tag{2.12}$$

and

$$\beta_0 = \varphi_0^\dagger \, h \, \varphi_0,$$

we can write (2.11) in the form

$$\partial_- \varphi_0 = \varphi_0 \, \beta_0^{-1} \partial_- \beta_0 + \varphi_1 \, B_{10}, \tag{2.13}$$

where we used the relation

$$\partial_- \beta_0 = \varphi_0^\dagger \, h \, \partial_- \varphi_0$$

which follows from the equality

$$\partial_+ \varphi_0 = 0.$$

The subspaces spanned by the linear combinations of the columns of the matrix $\varphi_1(p)$ do not depend on the choice of the lift $\xi = \xi_0$ and the function ξ_1. These subspaces generate the subbundle $\underline{\varphi}_1$ of $\underline{\mathbb{C}}^n$. Actually it is an orthogonal complement of the subbundle $\underline{\varphi}_0 = \underline{\psi}_0$ in $\underline{\psi}_1$. Thus, we have the following representation

$$\underline{\psi}_1 = \underline{\varphi}_0 \oplus \underline{\varphi}_1,$$

where the sum in the right hand side is orthogonal. Note that the orthogonality of the fibers of $\underline{\varphi}_0$ and $\underline{\varphi}_1$ is equivalent to the validity of the equality

$$\varphi_0^\dagger \, h \, \varphi_1 = 0.$$

Find now the equations satisfied by the function φ_1. Define the $\mathrm{Mat}(n, \mathbb{C})$-valued function $\mathit{\Pi}_1$ by

$$\mathit{\Pi}_1 = I_n - \varphi_1 \beta_1^{-1} \varphi_1^\dagger h,$$

where

$$\beta_1 = \varphi_1^\dagger \, h \, \varphi_1.$$

For any $p \in U$ the matrix $\mathit{\Pi}_1(p)$ is the matrix of the operator of the orthogonal projection to the orthogonal complement of $\underline{\varphi}_{1p}$ in $\mathbb{C}^n$. Since the subspaces $\underline{\varphi}_{0p}$ and $\underline{\varphi}_{1p}$ are orthogonal, we have

$$\mathit{\Pi}_0 \, \mathit{\Pi}_1 = \mathit{\Pi}_1 \, \mathit{\Pi}_0. \tag{2.14}$$

Using relations (2.12), (2.14) and the equality

$$\partial_- \xi_1 = \xi_0 \, B_{01} + \xi_1 \, B_{11} + \xi_2 \, B_{21},$$

one gets

$$\mathit{\Pi}_2 \, \partial_- \varphi_1 = \mathit{\Pi}_1 \, \partial_- \mathit{\Pi}_0 \, \xi_1 + \mathit{\Pi}_1 \, \mathit{\Pi}_0 \, \xi_2 \, B_{21}. \tag{2.15}$$

It follows from (2.10) and (2.13) that

$$\partial_- \mathit{\Pi}_0 = -\varphi_1 \, B_{10} \, \beta_0^{-1} \varphi_0^\dagger \, h.$$

Therefore, denoting

$$\varphi_2 = \mathit{\Pi}_1 \, \mathit{\Pi}_0 \, \xi_2,$$

we rewrite (2.15) as

$$\partial_- \varphi_1 = \varphi_1 \, \beta_1^{-1} \varphi_1^\dagger \, h \, \partial_- \varphi_1 + \varphi_2 \, B_{21}. \tag{2.16}$$

Similarly one gets

$$\partial_+ \mathit{\Pi}_0 = h^{-1} \, (\partial_- \mathit{\Pi}_0)^\dagger \, h = \varphi_0 \, \beta_0^{-1} D_{01} \varphi_1^\dagger \, h,$$

where

$$D_{01} = -B_{10}^\dagger$$

is an antiholomorphic $\mathrm{Mat}(k_1, k_2; \mathbb{C})$-valued function. Hence,

$$\partial_+ \varphi_1 = \varphi_0 \, \beta_0^{-1} D_{01} \, \beta_1.$$

This equation, in particular, implies that

$$\partial_- \beta_1 = \varphi_1^\dagger \, h \, \partial_- \varphi_1.$$

Therefore, equation (2.16) can be written as

$$\partial_- \varphi_1 = \varphi_1 \, \beta_1^{-1} \partial_- \beta_1 + \varphi_2 \, B_{21}.$$

As before, the columns of the matrices $\varphi_2(p)$ generate a subbundle $\underline{\varphi}_2$ and we have the orthogonal decomposition

$$\underline{\psi}_2 = \underline{\varphi}_0 \oplus \underline{\varphi}_1 \oplus \underline{\varphi}_2.$$

In general, defining inductively the functions

$$\beta_a = \varphi_a^\dagger \, h \, \varphi_a,$$

the projectors

$$\mathit{\Pi}_a = I_n - \varphi_a \beta_a^{-1} \varphi_a^\dagger \, h$$

and the functions

$$\varphi_{a+1} = \mathit{\Pi}_a \cdots \mathit{\Pi}_0 \, \xi_{a+1},$$

we get the orthogonal decompositions

$$\underline{\psi}_a = \bigoplus_{b=0}^{a} \underline{\varphi}_b.$$

Note that the sequence of subbundles $\underline{\varphi}_1, \ldots, \underline{\varphi}_t$ and the sequence of the mappings from M to the corresponding Grassmann manifolds are partial cases of the so called harmonic sequences of subbundles and mappings intensively used to classify and construct harmonic mappings from a Riemann surface to a Grassmann manifold.

Theorem 2.1. *The functions φ_a, $a = 0, \ldots, t$, satisfy the equations*

$$\partial_- \varphi_a = \varphi_a \, \beta_a^{-1} \partial_- \beta_a + \varphi_{a+1} \, B_{a+1,a}, \tag{2.17}$$

$$\partial_+ \varphi_a = \varphi_{a-1} \, \beta_{a-1}^{-1} D_{a-1,a} \, \beta_a, \tag{2.18}$$

where $B_{t+1,t} \equiv 0$, $D_{-1,0} \equiv 0$ and

$$D_{a-1,a} = -(B_{a,a-1})^\dagger. \tag{2.19}$$

Proof. Assume that that we proved the validity of equations (2.17), (2.18) for all $a \le b < t$. Using these equations for $a = b$, one has

$$\partial_- \Pi_b = -\varphi_{b+1} \, B_{b+1,b} \, \beta_b^{-1} \varphi_b^\dagger h + \varphi_b \, B_{b,b-1} \, \beta_{b-1}^{-1} \, \varphi_{b-1}^\dagger h,$$
$$\partial_+ \Pi_b = -\varphi_{b-1} \, \beta_{b-1}^{-1} \, D_{b-1,b} \, \varphi_b^\dagger h + \varphi_b \, \beta_b^{-1} D_{b,b+1} \, \varphi_{b+1}^\dagger h.$$

Now differentiating the definition of φ_{b+1} over z^- and z^+, we can easily show that equations (2.17), (2.18) are valid for $a = b + 1$. □

Consider the $\mathrm{Mat}(n, \mathbb{C})$-valued mapping $\varphi = (\varphi_0 \cdots \varphi_t)$. For any $p \in U$ the matrix $\varphi(p)$ is nondegenerate. Denote by $f_1, \ldots, f_n$ the $\mathbb{C}^n$-valued functions determined by the columns of φ. For any $p \in U$ the vectors $f_1(p), \ldots, f_n(p)$ are linearly independent and form a basis in $\mathbb{C}^n$. We call the set of mappings $f_1, \ldots, f_n$, or the mapping φ which generates this set, the *Frenet frame* of ψ associated with the lift ξ.

3. Connection with Toda systems

3.1. $\mathbb{Z}$-graded Lie algebras.

To get the equations describing a Toda system [LSa92, RSa94, RSa97a, RSa97b] one starts with a complex Lie group G whose Lie algebra $\mathfrak{g}$ is endowed with a $\mathbb{Z}$-gradation

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m.$$

Introduce the following subalgebras of $\mathfrak{g}$

$$\tilde{\mathfrak{h}} = \mathfrak{g}_0, \qquad \tilde{\mathfrak{n}}_- = \bigoplus_{m<0} \mathfrak{g}_m, \qquad \tilde{\mathfrak{n}}_+ = \bigoplus_{m>0} \mathfrak{g}_m.$$

and denote by $\tilde{H}$ and $\tilde{N}_\pm$ the connected Lie subgroups of G corresponding to the subalgebras $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{n}}_\pm$ respectively.

Suppose that $\widetilde{H}$ and $\widetilde{N}_\pm$ are closed subgroups of G and, moreover,

$$\tilde{H} \cap \tilde{N}_\pm = \{e\}, \qquad \tilde{N}_- \cap \tilde{N}_+ = \{e\},$$
$$\tilde{N}_- \cap \tilde{H}\tilde{N}_+ = \{e\}, \qquad \tilde{N}_-\tilde{H} \cap \tilde{N}_+ = \{e\},$$

where e is the unit element of G. This is, in particular, true for the finite-dimensional complex reductive Lie groups, see, for example, [Hum75]. Let the set $\tilde{N}_-\tilde{H}\tilde{N}_+$ be dense in G, then for any element g which belongs to $\tilde{N}_- \tilde{H} \tilde{N}_+$ one can write the following unique decomposition

$$g = n_- h n_+^{-1}, \tag{3.1}$$

where $n_- \in \tilde{N}_-$, $h \in \tilde{H}$ and $n_+ \in \tilde{N}_+$. This is again true for the finite-dimensional complex reductive Lie groups. Decomposition (3.1) is called the *Gauss decomposition.*

There is a simple classification of possible $\mathbb{Z}$-gradations for complex semisimple Lie algebras, see, for example, [GOV94, RSa97b]. In this case for any $\mathbb{Z}$-gradation of a such an algebra $\mathfrak{g}$, there exists a unique element $q \in \mathfrak{g}$ which has the following property. An element $x \in \mathfrak{g}$ belongs to the subspace $\mathfrak{g}_m$ if and only if $[q, x] = mx$. The element q is called the *grading operator.*

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r. Denote by h_i and $i = 1, \ldots, r$, some set of the Cartan generators of $\mathfrak{g}$. For any set of r nonnegative numbers l_i the element

$$q = \sum_{i,j=1}^{r} h_i (k^{-1})_{ij} l_j, \tag{3.2}$$

where $k = (k_{ij})$ is the Cartan matrix of $\mathfrak{g}$, is the grading operator of some $\mathbb{Z}$-gradation of $\mathfrak{g}$. The numbers l_i can be considered as the labels assigned to the vertices of the corresponding Dynkin diagram. The well-known canonical gradation of $\mathfrak{g}$ arises when one chooses all the numbers l_i equal to 2. Actually, if two sets of labels are connected by an automorphism of the Dynkin diagram, we get two $\mathbb{Z}$-gradations connected by an 'external' automorphism of $\mathfrak{g}$. If it is not the case, then different sets of labels give $\mathbb{Z}$-gradations which cannot be connected by an automorphism of $\mathfrak{g}$.

If all the labels of the Dynkin diagram of a semisimple Lie algebra $\mathfrak{g}$ are different from zero, then the subgroup $\mathfrak{g}_0$ coincides with the Cartan subalgebra $\mathfrak{h}$. In this case the subgroup $\tilde{H}$ is abelian and we obtain the so called *abelian Toda equations.* In all other cases the subgroup $\tilde{H}$ is nonabelian and the corresponding Toda equations are called *nonabelian.*

If we deal with a reductive Lie algebra, we choose as the grading operator any grading operator of its maximal semisimple subalgebra.

3.2. **Toda equations and their general solution.** Let M be a simply connected complex one dimensional manifold. Consider a reductive complex Lie group G whose Lie algebra $\mathfrak{g}$ is endowed with a $\mathbb{Z}$-gradation. Let l be a positive integer, such that the grading subspaces $\mathfrak{g}_m$ for $-l < m < 0$ and $0 < m < l$ are trivial, and c_- and c_+ be some fixed mappings from M to the subspaces $\mathfrak{g}_{-l}$ and $\mathfrak{g}_{+l}$, respectively, such that c_- is holomorphic and c_+ is antiholomorphic. Restrict ourselves to the case when G is a matrix Lie group. In this case the Toda equations are the matrix partial differential equations of the form

$$\partial_+(\gamma^{-1} \partial_- \gamma) = [c_-, \gamma^{-1} c_+ \gamma], \tag{3.3}$$

where γ is a mapping from M to $\tilde{H}$. These equations are the zero curvature condition for the connection $\omega = \omega_- dz^- + \omega_+ dz^+$ on the trivial principal G-bundle $M \times G$, where

$$\omega_- = \gamma^{-1} \partial_- \gamma + c_-, \qquad \omega_+ = \gamma^{-1} c_+ \gamma.$$

Since M is simply connected, then there exists a mapping $\varphi : M \to G$ such that

$$\partial_- \varphi = \varphi(\gamma^{-1}\partial_-\gamma + c_-), \qquad \partial_+\varphi = \varphi\gamma^{-1}c_+\gamma.$$

To obtain the general solution of Toda equations one can use the following procedure [LSa92, RSa94, RSa97a, RSa97b]. Choose some mappings $\gamma_\pm$ from M to $\tilde{H}$ such that γ_- is holomorphic and γ_+ is antiholomorphic. Then integrate the equations

$$\mu_\pm^{-1}\partial_\pm\mu_\pm = \gamma_\pm c_\pm \gamma_\pm^{-1}, \qquad \partial_\mp\mu_\pm = 0. \tag{3.4}$$

The Gauss decomposition (3.1) induces the corresponding decomposition of mappings from M to G. In particular, one obtains

$$\mu_+^{-1}\mu_- = \nu_-\eta\nu_+^{-1}, \tag{3.5}$$

where the mapping η takes values in $\tilde{H}$, and the mappings $\nu_\pm$ take values in $\tilde{N}_\pm$. It can be shown that the mapping

$$\gamma = \gamma_+^{-1}\eta\gamma_- \tag{3.6}$$

satisfies the Toda equations, and any solution to these equations can be obtained by the described procedure. Note that for the corresponding mapping φ one has the following expression [RSa94, RSa97a, RSa97b]

$$\varphi = g\mu_+\nu_-\eta\gamma_- = g\mu_-\nu_+\gamma_-, \tag{3.7}$$

where g is an arbitrary constant element of G.

We will be interested in the so called hermitian solutions of Toda equations [RSa94, RSa97b][1]. Let the Lie algebra $\mathfrak{g}$ be endowed with some antilinear antiautomorphism σ which can be extended to the antiholomorphic antiautomorphism Σ of the Lie group G. Suppose that

$$\sigma(\mathfrak{g}_m) = \mathfrak{g}_{-m}, \qquad m \in \mathbb{Z}.$$

In this case one has

$$\Sigma(\widetilde{H}) = \widetilde{H}, \qquad \Sigma(\widetilde{N}_\pm) = \widetilde{N}_\mp.$$

Assume that the mappings $c_\pm$ entering the Toda equations are subjected to the condition

$$\sigma(c_-) = -c_+.$$

In this case, if γ is a solution to Toda equations, then $\Sigma \circ \gamma$ is also a solution of the same equations, and we can consider solutions of Toda equations having the property

$$\Sigma \circ \gamma = \gamma. \tag{3.8}$$

To get such solutions using the general procedure of integration of the Toda equations described above, we should start with the mappings $\gamma_\pm$ which satisfy the relation

$$\gamma_+ = \Sigma \circ \gamma_-^{-1}$$

and choose the solutions of (3.4) for which

$$\mu_+ = \Sigma \circ \mu_-^{-1}.$$

[1] Actually in [RSa94, RSa97b] we use the term 'real solutions'.

It can be shown that in such a way we get all the solutions of the Toda equations satisfying (3.8). We call such solutions *hermitian*. The mapping φ corresponding to a hermitian solution of Toda equations satisfies the relation

$$(\Sigma \circ \varphi)\Sigma(g)g\varphi = \gamma,$$

where g is the element of G entering (3.7).

3.3. **Nonabelian Toda equations associated with Lie group** $\mathrm{GL}(n, \mathbb{C})$ **and Frenet frames.** In this Section, generalising the consideration of [RSa97c], we give the general form of nonabelian Toda equations based on the Lie group $\mathrm{GL}(n, \mathbb{C})$ and establish their connection to the equations of the Frenet frame.

The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of the Lie group $\mathrm{GL}(n, \mathbb{C})$ is reductive and can be represented as the direct product of the simple Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ of rank $n-1$ and a one dimensional Lie algebra isomorphic to $\mathfrak{gl}(1, \mathbb{C})$ and composed by the $n \times n$ complex matrices which are multiplies of the unit matrix. Consider the general $\mathbb{Z}$-gradation of $\mathfrak{sl}(n, \mathbb{C})$ arising when we choose the labels of the corresponding Dynkin diagram as follows

$$\overbrace{\overset{0}{\circ} - \cdots - \overset{0}{\circ}}^{k_0-1} - \overset{s_1}{\circ} - \overbrace{\overset{0}{\circ} - \cdots - \overset{0}{\circ}}^{k_1-1} - \overset{s_2}{\circ} - \overset{0}{\circ} - \cdots - \overset{0}{\circ} - \overset{s_t}{\circ} - \overbrace{\overset{0}{\circ} - \cdots - \overset{0}{\circ}}^{k_t-1}$$

It is convenient to take as a Cartan subalgebra for $\mathfrak{sl}(n, \mathbb{C})$ the subalgebra consisting of diagonal $n \times n$ matrices with zero trace. Here the standard choice of the Cartan generators is

$$(h_i)_{kl} = \delta_{ki}\delta_{li} - \delta_{k,i+1}\delta_{l,i+1}, \qquad i = 1, \dots, n-1.$$

With such a choice of Cartan generators, using (3.2), we obtain the following block matrix form of the grading operator

$$q = \begin{pmatrix} \rho_0 I_{k_0} & 0 & \cdots & 0 & 0 \\ 0 & \rho_1 I_{k_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho_{t-1} I_{k_{t-1}} & 0 \\ 0 & 0 & \cdots & 0 & \rho_t I_{k_t} \end{pmatrix},$$

where

$$\rho_a = \frac{1}{n}\left(- \sum_{b=1}^{a} s_b \sum_{c=0}^{b-1} k_c + \sum_{b=a+1}^{t} s_b \sum_{c=b}^{t} k_c \right).$$

We use this grading operator to define a $\mathbb{Z}$-gradation of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. It is not difficult to describe the arising grading subspaces of $\mathfrak{gl}(n, \mathbb{C})$ and the relevant subgroups of $\mathrm{GL}(n, \mathbb{C})$. To this end we consider an element x of $\mathfrak{gl}(n, \mathbb{C})$ as a $t \times t$ block matrix $(x_{ab})_{a,b=0}^{t}$, where x_{ab} is $k_a \times k_b$ matrix. For fixed $a \neq b$, the block matrices x having only the block x_{ab} different from zero belong to the grading subspace $\mathfrak{g}_m$

with

$$m = \sum_{c=a+1}^{b} s_c, \quad a < b, \qquad m = -\sum_{c=b+1}^{a} s_c, \quad a > b.$$

The block-diagonal matrices form the grading subspace $\mathfrak{g}_0$.

Using the same block matrix representation for the elements of $\mathrm{GL}(n, \mathbb{C})$, we see that the subgroup $\widetilde{H}$ consists of all block-diagonal nondegenerate matrices, and the subgroups $\widetilde{N}_-$ and $\widetilde{N}_+$ consist, respectively, of all block lower and upper triangular matrices with unit matrices on the diagonal.

The equations related to Frenet frame arise in the case when all integers s_a, $a = 1, \ldots, t$, are equal to 2. In this case the mappings $c_\pm$ have to take values in the subspaces $\mathfrak{g}_{\pm 2}$ and their general form is

$$c_- = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_{10} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & B_{t,t-1} & 0 \end{pmatrix}, \qquad c_+ = \begin{pmatrix} 0 & D_{01} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & D_{t-1,t} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Parametrise the mapping γ as

$$\gamma = \begin{pmatrix} \beta_0 & 0 & \cdots & 0 & 0 \\ 0 & \beta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{t-1} & 0 \\ 0 & 0 & \cdots & 0 & \beta_t \end{pmatrix}.$$

The corresponding Toda equations are

$$\partial_+(\beta_a^{-1}\partial_-\beta_a) = -\beta_a^{-1} D_{a,a+1}\beta_{a+1}B_{a+1,a} + B_{a,a-1}\beta_{a-1}D_{a-1,a}\beta_a, \tag{3.9}$$

where $B_{t+1,t} \equiv 0$ and $D_{-1,0} \equiv 0$.

Represent the mapping φ entering (3.8) in the block form $\varphi = (\varphi_0 \cdots \varphi_t)$, where φ_a is a $\mathrm{Mat}(n, k_a; \mathbb{C})$-valued function. Then it is clear that relations (3.8) literally coincide with equations (2.17), (2.18). Thus, the integrability conditions of the equations satisfied by the Frenet frame are the Toda equations (3.9).

Not any solution of equations (3.9) gives us a mapping φ describing the Frenet frame. First of all we should consider the Toda systems for which relations (2.19) are satisfied. These relations are equivalent to the equality $(c_-)^\dagger = -c_+$. Moreover, for the Frenet frame one has $\beta_a^\dagger = \beta_a$, which is equivalent to $\gamma^\dagger = \gamma$. Hence, we should consider only hermitian solutions to the Toda equations corresponding to the antiholomorphic antiautomorphism of $\mathrm{GL}(n, \mathbb{C})$ generated by the ordinary hermitian conjugation of matrices. Finally, for the Frenet frame one has $\varphi^\dagger h\, \varphi = \gamma$. Therefore, constructing the mapping φ with the help of (3.7) one should take the constant element g of G satisfying the relation $g^\dagger g = h$.

4. Geometrical interpretation of Frenet frames

4.1. **Invariant metrics on principal fibre bundles.** Let $P \xrightarrow{\pi} N$ be a principal fibre G-bundle, and g^* be a metric on P which is invariant with respect to the standard right action of G on P. The metric g^* determines a connection on P for which the horizontal subspace $\mathcal{H}_p \subset T_p(P)$, $p \in P$, is defined as the orthogonal complement to the vertical subspace $\mathcal{V}_p \subset T_p(P)$ tangent to the fiber through p. For any vector field X on N there is a unique horizontal vector field X^* on P satisfying the relation

$$d\pi \circ X^* = X \circ \pi. \tag{4.1}$$

The vector field X^* is called the *horizontal lift* of X. The vector field X^* is invariant with respect to the standard right action of G on P. Below we call a vector field on P invariant with respect to the standard right action of G simply an invariant vector field. Any invariant horizontal vector field on P is the horizontal lift of a unique vector field on N. By definition, for any vector field X on N and a function f on N one has

$$\pi^*(X(f)) = X^*(\pi^* f). \tag{4.2}$$

Denote by $X^\perp$ the component of the vector field X on P orthogonal to the fibers, that is actually the horizontal component of X. For any vector fields X and Y on N the relation

$$[X, Y]^* = [X^*, Y^*]^\perp \tag{4.3}$$

is valid.

The metric g^* determines a metric g on N defined by the equality

$$\pi^*(g(X, Y)) = g^*(X^*, Y^*). \tag{4.4}$$

Proposition 4.1. *The Levi–Civita connections ∇^* and ∇ associated with g^* and g satisfy the equality*

$$(\nabla_Y X)^* = (\nabla^*_{Y^*} X^*)^\perp.$$

Proof. The validity of the assertion of the Proposition follows immediately from the Koszul formula

$$\begin{aligned} 2g(\nabla_Y X, Z) &= Y(g(X, Z)) + X(g(Z, Y)) - Z(g(Y, X)) \\ &- g(Y, [X, Z]) + g(X, [Z, Y]) + g(Z, [Y, X]) \end{aligned}$$

after taking into account definition (4.4) and relations (4.2), (4.3). □

Let ψ be an immersion of a manifold M to N. A mapping $X : N \to T(M)$ is said to be a vector field on ψ if

$$\mathrm{pr}_M \circ X = \psi,$$

where pr_M is the canonical projection of $T(M)$ to M. A vector field X' defined on some open subset of M is called an extension of a vector field X on ψ if

$$X' \circ \psi = X.$$

Here and below writing relations containing extensions of vector fields we imply the appropriate restrictions of the domains.

The covariant derivative $\nabla_Y X$ of the vector field X on ψ along the vector field Y on M is defined as

$$\nabla_Y X = \nabla_{(d\psi \circ Y)'} X' \circ \psi, \tag{4.5}$$

where X' and $(d\psi \circ Y)'$ are some arbitrary extensions of the vector fields X and $d\psi \circ Y$.

Let now $\xi : U \subset M \to P$ be a local lift of ψ. The mapping ξ is an immersion of U to P. We call a vector field X on ξ *vertical* if for any $p \in U$ the vector X_p is tangent to the fiber through p. Similarly, a vector field X on ξ is said to be *horizontal* if for any $p \in U$ the tangent vector X_p is horizontal, or, in other words, orthogonal to the fibre through $\xi(p)$.

Denote the induced connection generated by ψ also by ∇.

Proposition 4.2. *For any vector field X on ψ and a vector field Y on M the relation*

$$\nabla_Y X = d\pi \circ \nabla^*_{(d\psi \circ Y)'^*} X'^* \circ \xi$$

is valid.

Proof. From Proposition 4.1 it follows that

$$(\nabla_{(d\psi \circ Y)'} X')^* = (\nabla^*_{(d\psi \circ Y)'^*} X'^*)^{\perp}.$$

Relation (4.1) gives

$$\nabla_{(d\psi \circ Y)'} X' \circ \pi = d\pi \circ \nabla^*_{(d\psi \circ Y)'^*} X'^*.$$

The statement of the Proposition follows now from definition (4.5). □

Proposition 4.3. *Let X be a horizontal vector field on ξ. There is an extension X' of X such that X' is horizontal and invariant.*

Proof. The vector field $d\pi \circ X$ is a vector field on ψ. Let $(d\pi \circ X)'$ be an arbitrary extension of $d\pi \circ X$. Consider the vector field

$$X' = (d\pi \circ X)'^*. \tag{4.6}$$

This vector field is, by definition, horizontal and invariant. Let us show that it is an extension of X. Indeed, from (4.6) one gets

$$d\pi \circ X' \circ \xi = (d\pi \circ X)' \circ \psi = d\pi \circ X.$$

Since, $X' \circ \xi$ and X are horizontal vector fields, then $X' \circ \xi = X$. Thus, X' is an extension of X which satisfies the requirements of the Proposition. □

Below by an extension of a vector field X on ξ we mean an extension of X which satisfies the requirements of Proposition 4.3.

Proposition 4.4. *Let X be a horizontal vector field on ξ, and Y be a vector field on M. For the vector field $Z = d\pi \circ X$ on ψ one has*

$$\nabla_Y Z = d\pi \circ \nabla^*_{(d\xi \circ Y)^{\perp\prime}} X' \circ \xi.$$

Proof. Since X' is horizontal and invariant, there is a unique vector field Z' on N such that

$$Z'^* = X'.$$

It can be easily shown that Z' is an extension of Z. From Proposition 4.2 it follows that

$$\nabla_Y Z = d\pi \circ \nabla^*_{(d\psi \circ Y)'^*} X' \circ \xi.$$

Further, the vector field $(d\xi \circ Y)^{\perp'}$ is horizontal and invariant. Therefore, there is a unique vector field Y' such that

$$(d\xi \circ Y)^{\perp'} = Y'^*.$$

It is not difficult to demonstrate that Y' is an extension of the vector field $d\psi \circ Y$. Thus, we see that the assertion of the Proposition is true. □

4.2. **Construction of a local moving frame.** Let us return to the case of holomorphic curves in Grassmann manifolds. Denote by w^i_α, $i = 1, \dots, n$, $\alpha = 1, \dots, k$, the standard coordinate functions on $\mathrm{Mat}(n, k; \mathbb{C})$ and their restrictions to $\mathrm{Mat}(n, k; \mathbb{C})^\times$. A generic vector field on $\mathrm{Mat}(n, k; \mathbb{C})^\times$ has a unique representation

$$X = \sum_{i=1}^{n} \sum_{\alpha=1}^{k} \left(X^i_\alpha \frac{\partial}{\partial w^i_\alpha} + X^{\bar\imath}_{\bar\alpha} \frac{\partial}{\partial \bar w^{\bar\imath}_{\bar\alpha}} \right).$$

Recall that $\mathrm{Mat}(n, k; \mathbb{C})^\times$ is the principal $\mathrm{GL}(k, \mathbb{C})$-bundle over $G^k(\mathbb{C}^n)$. The vertical subspaces are generated by the Killing vector fields corresponding to the right action of the Lie group $\mathrm{GL}(k, \mathbb{C})$ on $\mathrm{Mat}(n, k; \mathbb{C})^\times$. The vector fields

$$K^\beta_\alpha = \sum_{i=1}^{n} w^i_\alpha \frac{\partial}{\partial w^i_\beta}, \qquad \alpha, \beta = 1, \dots, k,$$

and

$$\bar K^{\bar\beta}_{\bar\alpha} = \sum_{i=1}^{n} \bar w^{\bar\imath}_{\bar\alpha} \frac{\partial}{\partial \bar w^{\bar\imath}_{\bar\beta}}, \qquad \alpha, \beta = 1, \dots, k,$$

form bases in the spaces of the holomorphic and antiholomorphic Killing vector fields. A vector field X on $\mathrm{Mat}(n, k; \mathbb{C})^\times$ is invariant with respect to the right action of $\mathrm{GL}(k, \mathbb{C})$ if and only if

$$[K^\beta_\alpha, X] = 0, \qquad [\overline{K}^{\bar\beta}_{\bar\alpha}, X] = 0, \qquad \alpha, \beta = 1, \dots, k,$$

that is equivalent to

$$\sum_{i=1}^{n} w^i_\alpha \frac{\partial X^j_\gamma}{\partial w^i_\beta} = X^j_\alpha \delta^\beta_\gamma, \qquad \sum_{i=1}^{n} w^i_\alpha \frac{\partial X^{\bar\jmath}_{\bar\gamma}}{\partial w^i_\beta} = 0,$$
$$\sum_{i=1}^{n} \bar w^{\bar\imath}_{\bar\alpha} \frac{\partial X^j_\gamma}{\partial \bar w^{\bar\imath}_{\bar\beta}} = 0, \qquad \sum_{i=1}^{n} \bar w^{\bar\imath}_{\bar\alpha} \frac{\partial X^{\bar\jmath}_{\bar\gamma}}{\partial \bar w^{\bar\imath}_{\bar\beta}} = X^{\bar\jmath}_{\bar\alpha} \delta^{\bar\beta}_{\bar\gamma}.$$

To construct an invariant metric on $\mathrm{Mat}(n,k;\mathbb{C})^\times$ define first the $\mathrm{Mat}(k,\mathbb{C})$-valued function $\varDelta = (\varDelta_{\bar\alpha\beta})$ on $\mathrm{Mat}(n,k;\mathbb{C})^\times$ by

$$\varDelta_{\bar\alpha\beta} = \sum_{i,j=1}^{n} \overline{w}^{\bar\imath}_{\bar\alpha} h_{\bar\imath j} w^j_\beta.$$

It is quite clear that the metric g^* on $\mathrm{Mat}(n,k;\mathbb{C})^\times$ given by

$$g^* = \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{k} d\overline{w}^{\bar\imath}_{\bar\alpha} \otimes h_{\bar\imath j}\, dw^j_\beta\, (\varDelta^{-1})^{\beta\bar\alpha} + \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{k} h_{\bar\jmath i}\, dw^i_\alpha\, (\varDelta^{-1})^{\alpha\bar\beta} \otimes d\overline{w}^{\bar\jmath}_{\bar\beta}$$

is invariant with respect to the right action of $\mathrm{GL}(k,\mathbb{C})$ on $\mathrm{Mat}(n,k;\mathbb{C})^\times$. Hence, we can define the metric g on $G^k(\mathbb{C}^n)$ generated by g^*. This metric is proportional to the standard Kähler metric on $G^k(\mathbb{C}^n)$.

Proposition 4.5. *For any horizontal vector fields X and Y on $\mathrm{Mat}(n,k;\mathbb{C})^\times$ one has*

$$\begin{aligned}(\nabla_Y X)^\perp \;=\; & \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{k} Y^j_\beta \left(\frac{\partial X^i_\alpha}{\partial w^j_\beta}\frac{\partial}{\partial w^i_\alpha} + \frac{\partial X^{\bar\imath}_{\bar\alpha}}{\partial w^j_\beta}\frac{\partial}{\partial \overline{w}^{\bar\imath}_{\bar\alpha}} \right)^\perp \\ & + \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{k} Y^{\bar\jmath}_{\bar\beta} \left(\frac{\partial X^i_\alpha}{\partial \overline{w}^{\bar\jmath}_{\bar\beta}}\frac{\partial}{\partial w^i_\alpha} + \frac{\partial X^{\bar\imath}_{\bar\alpha}}{\partial \overline{w}^{\bar\jmath}_{\bar\beta}}\frac{\partial}{\partial \overline{w}^{\bar\imath}_{\bar\alpha}} \right)^\perp .\end{aligned}$$

Proof. A vector field X on $\mathrm{Mat}(n,k;\mathbb{C})^\times$ is orthogonal to the fibers if and only if

$$\sum_{i,j=1}^{n} \overline{w}^{\bar\jmath}_{\bar\beta} h_{\bar\jmath i} X^i_\alpha = 0, \qquad \sum_{i,j=1}^{n} X^{\bar\imath}_{\bar\alpha} h_{\bar\imath j} w^j_\beta = 0.$$

The Christoffel symbols of the metric g^* are

$$\varGamma^{i\beta\gamma}_{\alpha jk} = -\frac{1}{2}\sum_{m=1}^{n}\sum_{\delta=1}^{k} \left(\delta^i_j \delta^\gamma_\alpha (\varDelta^{-1})^{\beta\bar\delta} \overline{w}^{\bar m}_{\bar\delta} h_{\bar m k} + \delta^i_k \delta^\beta_\alpha (\varDelta^{-1})^{\gamma\bar\delta} \overline{w}^{\bar m}_{\bar\delta} h_{\bar m j} \right),$$

$$\varGamma^{i\bar\beta\gamma}_{\alpha\bar\jmath k} = \varGamma^{i\gamma\bar\beta}_{\alpha k\bar\jmath} = -\frac{1}{2}\left(\sum_{m=1}^{n} \delta^i_k (\varDelta^{-1})^{\gamma\bar\beta} h_{\bar\jmath m} w^m_\alpha - \sum_{\delta=1}^{k} \delta^\gamma_\alpha h_{\bar\jmath k} w^i_\delta (\varDelta^{-1})^{\delta\bar\beta} \right),$$

$$\varGamma^{\bar\imath\bar\beta\bar\gamma}_{\bar\alpha\bar\jmath\bar k} = \overline{\varGamma^{i\beta\gamma}_{\alpha jk}}, \qquad \varGamma^{\bar\imath\beta\bar\gamma}_{\bar\alpha j\bar k} = \varGamma^{\bar\imath\bar\gamma\beta}_{\bar\alpha\bar k j} = \overline{\varGamma^{i\bar\beta\gamma}_{\alpha\bar\jmath k}}, \qquad \varGamma^{i\bar\beta\bar\gamma}_{\alpha\bar\jmath\bar k} = \overline{\varGamma^{\bar\imath\beta\gamma}_{\bar\alpha jk}} = 0.$$

Using the above relations one sees that the assertion of the Proposition is true. □

For an arbitrary vector field X on ξ one has the representation

$$X = \sum_{i=1}^{n}\sum_{\alpha=1}^{k} \left[X^i_\alpha \left(\frac{\partial}{\partial w^i_\alpha} \circ \xi \right) + X^{\bar\imath}_{\bar\alpha} \left(\frac{\partial}{\partial \overline{w}^{\bar\imath}_{\bar\alpha}} \circ \xi \right) \right],$$

where X^i_α and $X^{\bar\imath}_{\bar\alpha}$ are some functions on M.

Let φ be the Frenet frame of ψ associated with the lift ξ. Define the set of vector fields $\widetilde{E}^\alpha_i$, $i = 1,\dots,n$, $\alpha = 1,\dots,k$, on ξ by

$$\widetilde{E}^\alpha_i = \sum_{j=1}^{n} \varphi^j_i \left(\frac{\partial}{\partial w^j_\alpha} \circ \xi \right).$$

The vector fields $\widetilde{E}^\alpha_\beta$, $\alpha, \beta = 1, \dots, k$, are vertical, while the vector fields $\widetilde{E}^\alpha_\mu$, $\alpha = 1, \dots, k$, $\mu = k+1, \dots n$, are horizontal.

The vector fields

$$E^\alpha_\mu = d\pi \circ \widetilde{E}^\alpha_\mu, \qquad \alpha = 1, \dots, k, \quad \mu = k+1, \dots, n,$$

form a local basis in the space of vector fields on ψ. Therefore, one has

$$\nabla_{\partial_\pm} E^\alpha_\mu = \sum_{\nu=k+1}^{n} \sum_{\beta=1}^{k} E^\beta_\nu (\Lambda_\pm)^{\nu\alpha}_{\beta\mu}.$$

Rewrite the equations (2.17) and (2.18) as

$$\partial_\pm f_i = \sum_{j=1}^{n} f_j (\lambda_\pm)^j_i$$

where the $\mathbb{C}^n$-valued functions $f_1, \dots, f_n$ are generated by the columns of the mapping φ.

Theorem 4.1. *The connection coefficients $(\Lambda_\pm)^{\nu\alpha}_{\beta\mu}$ are connected with the functions $(\lambda_\pm)^{\nu\mu}$ by the relations*

$$(\Lambda_-)^{\nu\alpha}_{\beta\mu} = \delta^\alpha_\beta (\lambda_-)^\nu_\mu - (\beta_0^{-1} \partial_- \beta_0)^\alpha_\beta \delta^\nu_\mu, \qquad (\Lambda_+)^{\nu\alpha}_{\beta\mu} = \delta^\alpha_\beta (\lambda_+)^\nu_\mu.$$

Proof. The statement of the Theorem follows from Propositions 4.4 and 4.5. $\square$

In conclusion of this Section we give the expressions for the metrics induced on M by the mappings ψ_a, $a = 0, \dots, t-1$. From the equations satisfied by the Frenet frame it follows immediately that for the metric g_a induced on ath associated curve one has

$$g_a(\partial_+, \partial_-) = -\operatorname{tr}(\beta_a^{-1} D_{a,a+1} \beta_{a+1} B_{a+1,a}) = \operatorname{tr}(\beta_a^{-1} B^\dagger_{a+1,a} \beta_{a+1} B_{a+1,a}).$$

Toda equations (3.9) give

$$\partial_+ \partial_- \ln \det \beta_a = g_a(\partial_+, \partial_-) - g_{a-1}(\partial_+, \partial_-).$$

This equality implies that the function $\ln(\det \beta_0 \cdots \det \beta_a)$ is a Kähler potential of the metric g_a.

Acknowledgments

It is a pleasure to thank Yu. I. Manin and M. V. Saveliev for their interest to the work and for fruitful discussions. The author is grateful to the Max–Planck–Institut für Mathematik in Bonn where this work was completed for the warm hospitality and financial support. This work was also partially supported by the Russian Foundation for Basic Research under grant no. 95–01–00125, and by the INTAS grant no. 96–690.

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