CONNECTED SUM DECOMPOSITION OF COMPLEX PROJECTIVE HYPERSURFACES WITH QUADRATIC SINGULARITIES

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ABSTRACT

Nikita Yu. Netsvetaev, *Connected sum decomposition of complex projective hypersurfaces with quadratic singularities.*

We study the global topological structure of hypersurfaces in $\mathbb{C}P^{n+1}$, $n \geq 3$, with quadratic singularities and prescribed set of singular points. Under certain restrictions on the degree, we give a precise topological description of such a hypersurface by means of decomposing it into a connected sum. In this case the topological type of the hypersurface is determined by its dimension, degree, and the number of singular points.

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INTRODUCTION

We study the global topological structure of complex projective hypersurfaces (HSQS) with quadratic singularities. If not indicated otherwise, throughout the paper we assume that the dimension of the hypersurface is not equal to 2.

Let $A \subset \mathbb{C}P^{n+1}$ be a finite set, $s := \operatorname{card}(A)$ its cardinality. Let $\mathfrak{S}_n(A; d)$ denote the set of all HSQS's $X \subset \mathbb{C}P^{n+1}$ of degree d with $\operatorname{Sing} X = A$. It is not difficult to see that $\mathfrak{S}_n(A; d)$ is a Zariski open (maybe empty) set in a certain projective space, and hence it is irreducible and connected (see [2]). This implies that the topological type of a HSQS depends only on its degree and the location of its singular points, and not on the specific choice of the hypersurface itself.

Thus, we have a well-posed problem: for given n, A, and d, to describe the global topology of the hypersurfaces from $\mathfrak{S}_n(A; d)$.

This problem is most easy to handle if the number s of singular points is not too large, or the degree is not too samll, or the hypersurface is "typical" or "generic" in a certain sense. It turns out that in this situation the topology of the hypersurface is determined by the simplest possible invariants, namely by its dimension, degree, and the number of singular points. On the other hand, to describe the topological structure of an object means to produce its topological model, and this is not an easy task already for nonsingular hypersurfaces.

Let $X \subset \mathbb{C}P^{n+1}$ be a nonsingular hypersurface of degree d. It follows from what was said above that its differential type is determined by n and d: $X \cong X_n(d)$. As

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a model hypersurface we can take that given by the Fermat equation

$$X_n(d) := \left\{ (z_0 : z_1 : \cdots : z_{n+1}) \in \mathbb{C}P^{n+1} \,\middle|\, \sum_{i=0}^{n+1} z_i^d = 0 \right\}.$$

The structure of $X_n(d)$ was studied in sufficient detail by W. Browder, R. S. Kulkarni, A. S. Libgober, J. W. Wood, and others (see the references in [2]).

Let $n \neq 2$. Then $X_n(d)$ admits a (differential) connected sum decomposition of the form

$$X_n(d) \cong M_n(d) \# a(S^n \times S^n),$$

where $b_n(M_n(d)) = 0$ or 2, for n odd, and $b_n(M_n(d)) - |\operatorname{sign} X_n(d)| \leq 5$, for n even. (Here and below, $b_n(\cdot) := \operatorname{rk} H_n(\cdot)$ is the Betti number, and $\operatorname{sign}(\cdot)$ is the signature.)

It can be shown that the manifold $M_n(d)$ is determined uniquely up to diffeomorphism (see [7, 8]).

It is well known that in the vicinity of any quadratic singular point there exist suitable holomorphic coordinates $x_1, x_2, \ldots, x_{n+1}$ such that the hypersurface is given by $\sum_{i=1}^{n+1} x_i^2 = 0$. It is easy to see that the affine quadratic cone in $\mathbb{C}^{n+1}$ can be described as the result of contracting to a point the graph of the zero section in the tangent bundle of the n-sphere S^n. On the other hand, this tangent bundle is isomorphic to the normal bundle of the diagonal in $S^n \times S^n$. Therefore, the vicinity of the quadratic singular point is canonically homeomorphic (in a reasonable sense) to that of the singular point (i.e., the point having no Euclidean neighborhood) of the space ($S^n \times S^n$/ Diagonal), and so the latter space presents what can be called a "compact form" of the quadratic singularity. (It should be noted that here the orientation must be taken care of. Everywhere below we assume that $S^n \times S^n$ in which the diagonal is contracted is oriented properly.)

Definition. We call a HSQS X *topologically standard* if it admits a (differential) connected sum decomposition of the form

$$X = X_n(d; s) = M(n; d) \# (a - s)(S^n \times S^n) \# s(S^n \times S^n / \text{Diagonal}).$$

We see that the topological type of a topologically standard singular hypersurface is described with the same precision as that of a nonsingular hypersurface, whence the name.

Notation. Let $A = \{P_1, \ldots, P_s\} \subset \mathbb{C}P^{n+1}$. We define the number $\phi_n(A) = \phi_n(\{P_1, \ldots, P_s\})$ as the minimal possible degree of a hypersurface Y such that P_i is an isolated singular point of Y for every $i = 1, \ldots, s$.

If the points $P_1, \ldots, P_s$ are in "generic position," then the number $\phi_n(\{P_1, \ldots, P_s\})$ depends only on n and s, and we denote it by $\phi(n; s)$.

Examples. (1) Obviously, $\phi_n(A) \leq 2 \cdot \operatorname{card}(A) = 2s$. (For any singular point P_i of A we can find a quadratic cone Q_i with vertex at P_i, which does not pass through the remaining singular points of A; then $\bigcup_{i=1}^{s} Q_i$ is the required hypersurface of degree $2s$.)

(2) Let $n \geq 2$, $d \geq 3$. If the points of A are in general position (i.e., no $n+2$ of them lie in a hyperplane), then $\phi_n(A) \leq 3[\frac{\operatorname{card}(A)-1}{n+2}] + 3$. (This is proved similarly, we only should use singular cubics instead of singular quadrics.)

Our main result is as follows.

Theorem 0.1. *Let $A \subset \mathbb{C}P^{n+1}$, $n > 2$, be a finite set. If $d > \phi_n(A)$, then any hypersurface $X \in \mathfrak{S}_n(A; d)$ is topologically standard.*

In particular, let $\psi_n(A) := \min\{d : \mathfrak{S}_n(A; d) \neq \emptyset\}$. If $d > \psi_n(A)$, then any $X \in \mathfrak{S}_n(A; d)$ is topologically standard.

Corollary. *Let $X \subset \mathbb{C}P^{n+1}$, $n > 2$, be a hypersurface of degree $\neq 2$, with s quadratic singularities. If X satisfies one of the following conditions:*

(1) $\deg X > 2s$,
(2) $\deg X > \phi_n(\operatorname{Sing} X)$,
(3) *the singular points of X are in generic position and* $\deg X > \phi(n; s)$,
(4) *the singular points of X are in general position (in the usual sense) and* $\deg X \geq 3[\frac{s-1}{n+2}] + 4$ *(or, equivalently, $s \leq [\frac{\deg X - 1}{3}](n+2)$),*

then it is topologically standard.

Let us give the sketch of the proof of Theorem 0.1. It proceeds in five steps. Let X be an HSQS such that $\deg(X) > \phi_n(\operatorname{Sing}(X))$. Then:

(1) the number of singular points of X is bounded from above by homological characteristics of $X_n(d)$ (see §3);
(2) X is homologically standard in a certain precisely defined sense (see §4);
(3) the arrangement of the vanishing cycles in the homology of a nonsingular hypersurface is standard (see §5);
(4) X admits a connected sum decomposition of described form (see §6);
(5) this decomposition is unique up to diffeomorphism by the cancellation theorems.

In §1 we discuss the notion of diffeomorphic singular hypersurface, which allows us to make more precise our statements on the topology of such hypersurfaces, and in §2 we mention some facts on nonsingular hypersurfaces that are necessary for our presentation.

§1. Diffeomorphism and rigid isotopy. Generic hypersurfaces

Though below we restrict our consideration mainly by hypersurfaces with quadratic singularities, all what follows can be transferred with appropriate changes to the case of arbitrary isolated singularities.

The following definition seems to be most suitable for the topological study of singular hypersurfaces.

Definition. Two hypersurfaces with quadratic singularities are *diffeomorphic* if there is a homeomorphism f between them such that

(1) f is a diffeomorphism outside the singular points and
(2) in the vicinities of the corresponding singular points, there are suitable holomorphic coordinates $x_1, x_2, \dots, x_{n+1}$; $y_1, y_2, \dots, y_{n+1}$ such that the hypersurfaces are given by $\sum_{i=1}^{n+1} x_i^2 = 0$ and $\sum_{i=1}^{n+1} y_i^2 = 0$, and f is given by $x_i = y_i$, $i = 1, \dots, n+1$.

In other words, f must be (locally) extendable to a diffeomorphism of some neighborhoods of the hypersurfaces which is holomorphic in the vicinities of singular points. We easily see that a diffeomorphism is also a PL-homeomorphism with respect to suitable triangulations of the hypersurfaces.

A sufficient condition for diffeomorphism is yielded by rigid isotopy.

Definition. By a *rigid isotopy* we mean a 1-parametric family of hypersurfaces with singularities of fixed type. (It is sufficient that the Milnor numbers of singular points do not vary during the isotopy.)

It would be interesting to find two hypersurfaces which are diffeomorphic but not rigidly isotopic.

In its turn, the simplest way to prove the rigid isotopy is to prove that the corresponding moduli space is irreducible. Thus, we see that all nonsingular hypersurfaces of the same dimension and degree are diffeomorphic. More generally, there are several cases where the rigid isotopy type (and so the topological and the diffeomorphism type) of a hypersurface is uniquely determined by its dimension, degree, and the number of singular points. For the simplicity we restrict our consideration by quadratic singularities only.

Notation. Let $P_1, \dots, P_s \in \mathbb{C}P^{n+1}$. We define $\psi_n(\{P_1, \dots, P_s\})$ as the minimal possible degree of a hypersurface Y with quadratic singularities such that $\operatorname{Sing} Y = \{P_1, \dots, P_s\}$. If the points $P_1, \dots, P_s$ are in a "generic position", then the number $\psi_n(\{P_1, \dots, P_s\})$ depends only on n and s, and we denote it by $\psi(n; s)$.

Proposition. *The rigid isotopy type of a hypersurface of degree d with s quadratic singular points $P_1, \dots, P_s \in \mathbb{C}P^{n+1}$ is uniquely determined in each of the following cases:*

(1) *the singular points P_i are fixed;*
(2) *the singular points P_i are in a generic position and the degree d is sufficiently high: $d \geq \psi(n, s)$;*
(3) *the singular points P_i are arbitrary and $d > 2s$.*

The proposition says that in each of the cases (1–3), the considered hypersurfaces constitute an irreducible family and so any two of them are rigidly isotopic.

Case (1) was noted by Dimca (cf. [2]). Cases (2) and (3) can be proved by means of extending his arguments.

In case (2) we shall speak about *generic* hypersurfaces.

§2. Nonsingular hypersurfaces

Let $X \subset \mathbb{C}P^{n+1}$ be a nonsingular hypersurface of degree d. Its differential type is determined by n and d: $X \cong X_n(d)$. As a model hypersurface we can take that given by the Fermat equation

$$X_n(d) = \{z_0^d + z_1^d + \cdots + z_{n+1}^d = 0\}.$$

The structure of $X_n(d)$ was studied in sufficient detail by Browder, Kulkarni, Libgober, Wood, and others (see the references in [2]). Here we mention the basic facts and also some results that are used below. It easily follows from the Lefschetz Hyperplane Section Theorem that if $i \neq n$, then $H_i X = H_i \mathbb{C}P^n$. Recall that

$$H_i \mathbb{C}P^n = \begin{cases} \mathbb{Z}, & \text{if } i \text{ is even, } i \leq 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Let y be the generator of $H^2 X = \mathbb{Z}$. For n even let $h \in H_n X$ be the homology class dual to $y^{n/2}$. (Sometimes we call h the *algebraic class.*) The class h is primitive (indivisible). Let $n \neq 2$. Then $X_n(d)$ admits a connected sum decomposition of the form

$$X_n(d; s) \cong M_n(d) \# a(S^n \times S^n),$$

where $b_n(M_n(d)) = 0$ or 2, for n odd, and $b_n(M_n(d)) - |\operatorname{sign} X_n(d)| \leq 5$, for n even. It can be shown that the manifold $M_n(d)$ is determined uniquely up to a diffeomorphism (see [7, 8]).

If not explicitly indicated otherwise, we always use the homology with integral coefficients; $\mathrm{rk}_n(d)$ and $\mathrm{Sign}_n(d)$ denote the middle Betti number and the signature of the manifold $X_n(d)$, and $b_n^+(d)$ and $b_n^-(d)$ are the positive and negative inertia indices of the (intersection) quadratic form of $X_n(d)$.

$$b_n^{\pm}(d) := b_n^{\pm}(X_n(d)) = \frac{1}{2}(\mathrm{rk}_n(d) \pm \mathrm{Sign}_n(d)).$$

It is well known (cf. [6]) that the rank of the middle homology group of the hypersurface $X_n(d) \subset \mathbb{C}P^{n+1}$ is given by

$$\mathrm{rk}_n(d) = \begin{cases} \dfrac{(d-1)^{n+2} - 1}{d} + 2, & \text{if } n \in 2\mathbb{Z}, \\[2ex] \dfrac{(d-1)^{n+2} + 1}{d} - 1, & \text{if } n \notin 2\mathbb{Z}. \end{cases}$$

(This is most easily proved by induction, using the fact that $X_n(d)$ covers $\mathbb{C}P^n$ with the branch locus $X_{n-1}(d)$.)

To calculate the signature, it is convenient to use the generating function of the sequence $\{\mathrm{Sign}_n(d)\}_{n=0}^{\infty}$ (cf. [4]):

$$\sum_{n=0}^{\infty} \mathrm{Sign}_n(d) z^n = \frac{1}{z} \cdot \frac{1}{1-z^2} \cdot \frac{(1+z)^d - (1-z)^d}{(1+z)^d + (1-z)^d}.$$

Besides $\mathrm{Sign}_n(d)$, we will use the numbers $C_n(d)$ recursively defined by the relations

$$C_0(d) = d, \quad C_{2k}(d) = \mathrm{Sign}_{2k}(d) - \mathrm{Sign}_{2k-2}(d).$$

It is easy to see that

$$\sum_{n=0}^{\infty} C_n(d) z^n = \frac{1}{z} \cdot \frac{(1+z)^d - (1-z)^d}{(1+z)^d + (1-z)^d}$$

(this is the generating function for the sequence $\{C_n(d)\}_{n=0}^{\infty}$).

For fixed dimension of the hypersurface, the functions $\operatorname{Sign}_n(d)$ and $C_n(d)$ are polynomials in its degree. For n small they have the following form:

$$\begin{aligned}
\operatorname{Sign}_0(d) &= d,\\
\operatorname{Sign}_2(d) &= \frac{1}{3}d(4-d^2),\\
\operatorname{Sign}_4(d) &= \frac{1}{15}d(2d^4-10d^2+23),\\
\operatorname{Sign}_6(d) &= \frac{1}{315}d(528-308d^2+112d^4-17d^6),\\
\operatorname{Sign}_8(d) &= \frac{1}{2835}d(62d^8-510d^6+1806d^4-3590d^2+5067),\\
\operatorname{Sign}_{10}(d) &= \frac{1}{155925}d(292860-239327d^2+149600d^4-13640d^8-1382d^{10}),
\end{aligned}$$

$$\begin{aligned}
C_0(d) &= d,\\
C_2(d) &= \frac{1}{3}(d-d^3),\\
C_4(d) &= \frac{1}{15}(d^3-d)(3-2d^2),\\
C_6(d) &= \frac{1}{315}(d-d^3)(17d^4-53d^2+45),\\
C_8(d) &= \frac{1}{2835}(d^3-d)(62d^6-295d^4+503d^2-315),\\
C_{10}(d) &= \frac{1}{155925}(d-d^3)(14175-27702d^2+22568d^4-8848d^6+1382d^8).
\end{aligned}$$

In what follows we will need the following inequalities, which are proved in [6]:

$$|\operatorname{Sign}_n(d)| \leq |C_n(d)|, \tag{1}$$

$$|C_{n+2}(d)| \leq (d-1)^2|C_n(d)|. \tag{2}$$

We also need a certain connection between the homology of the hypersurfaces $X_n(d)$ and $X_n(d+1)$.

Let $X_d \subset \mathbb{C}P^{n+1}$ be a nonsingular hypersurface, let $H \subset \mathbb{C}P^{n+1}$ be a hyperplane transversal to it, and let X_{d+1} be a nonsingular hypersurface obtained by a small perturbation of the hypersurface $X_d \cup H$. We can assume that the perturbation takes place in a small tubular neighborhood of the intersection $X_d \cap H$, and that the complement is subject only to a small isotopy. Then the "compactified affine part" $X_d \setminus U(H)$ can be regarded as lying in X_{d+1}. The inclusion is not uniquely defined, but the ambiguity is of no importance for us here.

Teorema 2.1. *Let $X_d \subset \mathbb{C}P^{n+1}$ be a nonsingular hypersurface, let $H \subset \mathbb{C}P^{n+1}$ be a hyperplane, and let X_{d+1} be a perturbation of the hypersurface $X_d \cup H$.*

The "inclusion" $X_d \setminus U(H) \to X_{d+1}$ induces a monomorphism onto a direct summand on the level of the homology groups. Furthermore, if n is even, this summand together with the algebraic class h_{d+1} also generates a direct summand.

In other words:

(1) the algebraic class $h_{d+1} \in H_n X_{d+1}$ is indivisible.

(2) the inclusion homomorphism $H_n X'_d \to H_n X_{d+1}$ is a monomorphism onto a primitive sublattice; and

(3) the sublattice $\langle h_{d+1}\rangle \oplus H_n X'_d \subset H_n X_{d+1}$ is also primitive.

It is easy to see that the truth of the statement does not depend on the specific choice of X_d, H, and X_{d+1}, so that we can take, for example, $X_d = X_n(d)$, $H = \{z_0 = 0\}$, and consider some specific perturbation, which can be successfully modeled on the topological level. Nevertheless, the proof turns out to be rather bulky, and we present it elsewhere.

§3. Estimates of the number of singular points of a complex hypersurface

Throughout the section we assume that the natural numbers m and n are such that $m = n + 1$ (so that we could keep the standard notation for Arnold's numbers and denote the dimension of the hypersurface by n).

Let $X \subset \mathbb{C}P^{n+1}$ be an algebraic hypersurface of degree d, having only quadratic singular points. The problem on the maximum possible number $\mathfrak{N}_m(d)$ of singular points of such a hypersurface is classical. The exact value of $\mathfrak{N}_m(d)$ is known only in few cases: for points on the line we have $\mathfrak{N}_1(d) = [d/2]$; for plane curves we have

$$\mathfrak{N}_2(d) = \frac{1}{2}d(d-1);$$

for cubic hypersurfaces in all dimensions we have

$$\mathfrak{N}_m(3) = \binom{m+1}{[m/2]}.$$

Miyaoka obtained a very strong upper bound for singular surfaces in $\mathbb{C}P^3$. For hypersurfaces of dimension more than 2, the best upper bound for the number of singular points (originally conjectured by Arnold) is due to Varchenko [10].

Definition. The *Arnold number* $A_m(d)$ is defined by

$$A_m(d) \stackrel{\text{def}}{=} \operatorname{card}\left\{(x_1, \dots, x_m) \in \mathbb{Z}^m \cap (0,d)^m \;\middle|\; \frac{(m-2)d}{2} < \sum_{i=1}^{m} x_i \le \frac{md}{2}\right\}.$$

An equivalent way of defining $A_m(d)$ is

$$A_m(d) \stackrel{\text{def}}{=} \operatorname{card}\left\{(x_1, \dots, x_{m+1}) \in \mathbb{Z}^{m+1} \cap (0,d)^{m+1} \;\middle|\; \sum_{i=1}^{m+1} x_i = \left[\frac{md}{2}\right] + 1\right\}.$$

Proposition (Arnold's Conjecture). *Let $X \subset \mathbb{C}^{n+1}$ be a hypersurface of degree d and dimension n with s isolated singularities. Then $s < A_{n+1}(d)$. In other words,*

$$\mathfrak{N}_m(d) \leq A_m(d).$$

This estimate was shown to be sharp for $d = 3$, i.e., for cubic hypersurfaces, by Kalker [5] (it is easy to see that $A_m(3) = \binom{m+1}{[m/2]}$). S. V. Chmutov's hypersurfaces [1] show that it is asymptotically precise up to a factor of $\sqrt{3}$ (for large d and fixed m). For quartics a series of "good" examples was found by Goryunov; see [3], which contains an interesting discussion as well.

As before, $X_n(d)$ denotes the Fermat hypersurface of degree d in $\mathbb{C}P^{n+1}$. By $\mathrm{rk}_n(d)$ and $\mathrm{Sign}_n(d)$ we denote the middle Betti number and the signature of $X_n(d)$, and $b_n^+(d)$ and $b_n^-(d)$ are positive and negative inertia indices of the (intersection) quadratic form of $X_n(d)$.

From the point of view of topology of singular hypersurfaces, for n even the relation between the number $\mathfrak{N}_m(d)$ and the inertia indices $b_n^+(d)$ and $b_n^-(d)$ is of interest.

The main results of this section are the two following statements.

Theorem 3.0. *The inequality $A_m(d) < \min\{b_n^+(d), b_n^-(d)\}$ holds if and only if $(n-4)(d-2) \geq 18$ and $(n, d) \neq (6, 12)$. In particular, it holds for $n \geq 6$, $d \geq 13$ and for $n \geq 22$, $d \geq 3$.*

Corollary. *The inequality $A_m(d-1) < \min\{b_n^+(d), b_n^-(d)\}$ holds for $n \geq 4$.*

As a corollary of Varchenko's estimate, we see that if the dimension of the hypersurface and its degree satisfy the restrictions given in the theorem and in the corollary, then the following inequality holds:

$$\mathfrak{N}_m(d) < \min\{b_n^+(d), b_n^-(d)\}. \qquad (*_{n,d})$$

Note that as Kalker's and Goryunov's examples show, this inequality does not hold for $d = 3$, $n \leq 20$ and for $d = 4$, $n \leq 11$, and also for $n \leq 2$.

The proof of Theorem 3.0 is based on concrete calculations and on the possibility to prove by induction some stronger inequalities. The latter involve the numbers $B_m(d)$, which, similarly to Arnold's numbers, are equal to the number of integral points lying on certain sections of the $(m+1)$-dimensional cube. Their definition, properties, and the relation to Arnold's numbers are given below. We also give a table of Arnold's numbers for not very large values of m and d.

Calculation, estimates, and values of Arnold's numbers. By definition, Arnold's number is the number of all integral points $(x_1, \ldots, x_{m+1})$ lying inside the $(m+1)$-cube $(0, d)^{m+1}$ on its section by the hyperplane $\left\{\sum_{i=1}^{m+1} x_i = [md/2] + 1\right\}$. It is also useful to consider the sections of the cube by arbitrary hyperplanes orthogonal to its main diagonal. We set

$$N_{m,d}(p) := \mathrm{card}\left\{(x_1, \ldots, x_m) \in \mathbb{Z}^m \cap (0, d)^m \,\middle|\, \sum_{i=1}^{m} x_i = p\right\},$$

i.e., $N_{m,d}(p)$ is the number of the integral points in the open m-cube $(0,d)^m$ that lie on the hyperplane $\sum_{i=1}^m x_i = p$, or, in other words, the number of decompositions of p into m natural summands $x_1, \ldots, x_m$ each of wich does not exceed $d-1$. Thus,

$$A_m(d) = N_{m+1,d}\left(\left[\tfrac{md}{2}\right]+1\right).$$

Arnold's numbers and the numbers $N_{m,d}(p)$ introduced above possess many interesting properties. Here we only prove those properties which are directly used in the proof of Theorem 3.0.

Lemma 3.1. *We have the following relations:*

$$N_{m,d}(p) = \sum_{k=p-d+1}^{p-1} N_{m-1,d}(k), \tag{3}$$

$$N_{m,d}(p) = \sum_{i=0}^{m} \binom{m}{i}\binom{p-i(d-1)-1}{m-1} \tag{4}$$

(it is clear that $N_{m,d}(p) = 0$ for $p < m$ and for $p > m(d-1)$).

Proof. Relation (3) is obvious.

(4) The number of decompositions of p into m natural summands is equal to $\binom{p-1}{m-1}$. From these decompositions, we should subtract those decompositions where one of the numbers x_i is no less than d. The total number of such summands is equal to $\binom{m}{1}\cdot\binom{p-(d-1)-1}{m-1}$. After that, we should add those decompositions where certain two numbers $x_i, x_j \geq d$, etc. □

Between the remaining numbers $N_{m,d}(p)$ we are mostly interested in the number

$$B_m(d) := N_{m+1,d}\left(\left[\tfrac{(m+1)d}{2}\right]\right),$$

which is equal to the number of integral points in the "middle" section of the cube.

Lemma 3.2. $B_m(d) = \max_p N_{m+1,d}(p)$.

This is obviously proved by induction if we use relation (3). □

Corollary. *The following inequalities hold:*

$$A_m(d) < B_m(d), \tag{5}$$

$$B_{m+1}(d) < (d-1)B_m(d). \tag{6}$$

In what follows we will need concrete values of the numbers $A_m(d)$, $B_m(d)$, and also of the ranks $\mathrm{rk}_n(d)$, signatures $\mathrm{Sign}_n(d)$ and the numbers $C_n(d)$. For example, $B_{25}(3) = 10\,400\,600$, $C_{24}(3) = 1\,417\,176$. The other values of the mentioned numbers, that are used in the proof of the main theorem, are given in the following tables.

d	5	6	7	8	9
$A_9(d)$	100 110	780 175	3 801 535	15 776 530	50 415 760
$\|\mathrm{Sign}_8(d)\|$	30 501	174 726	744 647	2 578 248	7 648 969
$\mathrm{rk}_8(d)$	209 717	1 627 606	8 638 027	35 309 408	119 304 649
$A_{11}(d)$	1 501 566	18 121 038	128 184 068	719 540 108	3,021 294 288
$\|\mathrm{Sign}_{10}(d)\|$	288 899	2 433 600	14 293 929	65 162 656	246 015 527
$\mathrm{rk}_{10}(d)$	3 355 445	40 690 106	310 968 907	1 730 160 902	7 635 497 417

d	10	11
$A_9(d)$	149 803 270	374 894 389
$\|\mathrm{Sign}_8(d)\|$	20 133 002	48 162 411
$\mathrm{rk}_8(d)$	348 678 442	909 090 911
$A_{11}(d)$	11 306 743 020	35 099 288 940
$\|\mathrm{Sign}_{10}(d)\|$	802 564 928	2 329 795 533
$\mathrm{rk}_{10}(d)$	28 242 953 650	90 909 090 911

n	12	14	16	18	20	22
$A_m(3)$	3 003	11 440	43 758	167 960	646 646	2 496 144
$\|\mathrm{Sign}_n(3)\|$	1 459	4 373	13 123	39 365	118 099	354 293
$rk_n(3)$	5 463	21 847	87 383	349 527	1 398 103	5 592 407
$A_m(4)$	585 690	4 969 152	42 422 022	363 985 680	3 136 046 298	27 114 249 960
$\|\mathrm{Sign}_n(4)\|$	114 244	665 856	3 880 900	22 619 536	131 836 324	768 398 400
$\mathrm{rk}_n(4)$	1 195 744	10 761 682	96 855 124	871 696 102	7 845 264 904	70 607 384 122

d	5	6	7	8	9
$B_{11}(d)$	1 703 636	19 611 175	144 840 476	786 588 243	3 409 213 016
$C_{10}(d)$	319 400	2 608 326	15 038 576	67 740 904	253 664 496
$B_{13}(d)$	25 288 120	454 805 755	4 836 766 584	35 751 527 189	202 384 723 528
$C_{12}(d)$	3 025 400	36 329 286	288 675 408	1 712 092 200	8 158 715 856

d	10	11
$B_{11}(d)$	12 434 998 005	39 581 170 420
$C_{10}(d)$	822 697 930	2 377 957 944
$B_{13}(d)$	934 263 293 679	3 671 331 273 480
$C_{12}(d)$	32 795 554 890	115 031 578 024

n	14	16	18	20	22
$B_m(4)$	5 196 627	44 152 809	377 379 369	3 241 135 527	27 948 336 381
$C_n(4)$	780 100	4 546 756	26 500 436	154 455 860	900 234 724
$B_m(5)$	379 061 020	5 724 954 544	86 981 744 944	1 327 977 811 076	20 356 299 454 276
$C_n(5)$	28 657 000	271 443 000	2 571 145 000	24 354 235 000	230 686 625 000

The proof of Theorem 3.0 for $d \leq 11$. We remind the reader that our aim is to find out for which n and d the inequality

$$A_m(d) < \min\{b_n^+(d), b_n^-(d)\} \qquad (*_{n,d})$$

holds; it is convenient to write it in the form

$$2A_m(d) + |\operatorname{Sign}_n(d)| < \operatorname{rk}_n(d). \qquad (**_{n,d})$$

We also consider the inequality

$$2B_m(d) + |C_n(d)| < \frac{(d-1)^{n+2}}{d}, \qquad (***_{n,d})$$

which is stronger than the previous one in view of estimates (1) and (5).

The idea of introducing the stronger inequality is that it possesses the following inductive property:

Lemma 3.3. *If inequality $(***_{n_0,d})$ holds, then inequality $(***_{n,d})$ holds for all even $n \geq n_0$.*

To prove this it is sufficient to use inequalities (2) and (6). □

Corollary. *Inequality $(***_{n,3})$ (and hence inequality $(**_{n,3})$ is satisfied for all even $n \geq 24$; in addition, inequality $(**_{22,3})$ holds.*

Let us check that inequality $(***_{24,3})$ is true. Indeed,

$$2B_{25}(3) + |C_{24}(3)| = 22218376 < \frac{2^{26}}{3}.$$

Inequality $(**_{22,3})$ is also checked by direct calculation. □

For the other $d \leq 11$ the argument follows the same pattern: for fixed d we check inequality $(***_{n,d})$ by direct calculation. If it turns out that it is true for some $n(d)$, then it is true for all $n \geq n(d)$. After that we check inequality $(**_{n,d})$ for $n < n(d)$.

In the table below we give the results of direct calculations; the plus sign means that the stronger inequality $(***_{n,d})$ holds, and the simple plus sign means that only inequality $(**_{n,d})$ is true. Let us emphasize once more that as follows from Lemma 3.3, all empty places in the table should be marked by the sign ⊕.

$n \setminus d$	3	4	5	6	7	8	9	10	11	12	13
6	−	−	−	−	−	−	−	−	+	−	⊕
8	−	−	−	−	+	+	+	+	+	⊕	
10	−	−	+	+	⊕	⊕	⊕	⊕	⊕		
12	−	−	⊕	⊕							
14	−	+	⊕								
16	−	⊕									
18	−	⊕									
20	−	⊕									
22	+	⊕									
24	⊕										

Thus we see that the inequality $A_m(d) < \min\{b_n^+(d), b_n^-(d)\}$ is fulfilled for $d = 3$ and $n \geq 22$; $d = 4$ and $n \geq 14$; $d = 5, 6$ and $n \geq 10$; $7 \leq d \leq 11$ and $n \geq 8$.

Arnold polynomials and the end of the proof. Let $n = 8$. In view of (4) we have the relation

$$B_9(d) = N_{10,d}(5d) = \binom{5d}{9} - 10\binom{4d+1}{9} + 45\binom{3d+2}{9} - 120\binom{2d+3}{9} + 210\binom{d+4}{9}.$$

In particular, $B_9(d)$ is a polynomial in d,

$$B_9(d) = \frac{1}{36288} d(d-1)(d-2)(15619d^6 - 93714d^5 + \\ + 248434d^4 - 368976d^3 + 328939d^2 - 169830d + 44568).$$

Consider the difference

$$\mathrm{Dif}_8(d) := \frac{(d-1)^{10} - 1}{d} - C_8(d) - 2B_9(d).$$

It can be checked that the substitution $d \mapsto d + 12$ in $\mathrm{Dif}_8(d)$ yields a polynomial with positive coefficients, and therefore for all $d \geq 12$ we have

$$2B_9(d) + C_8(d) < \frac{(d-1)^{10} - 1}{d} < \frac{(d-1)^{10}}{d}.$$

By Lemma 3.3, inequality $(***_{n,d})$, and hence inequality $(**_{n,d})$, is true for all even $n \geq 8$ and $d \geq 12$.

For what follows we need to express $A_m(d)$ as a function of d. As seen from the relation $A_m(d) = N_{m+1,d}([\frac{md}{2}] + 1)$, these functions are not polynomials for m odd, and this is precisely the case in which we are interested. For example, $A_1(d) = [d/2]$ (this is obvious). This is the reason why for m odd the case of d even is to be considered separately.

Using relation (4) for the numbers $N_{m,d}(p)$, we get

$$\begin{aligned}
A_2(d) &= \frac{1}{2} d(d-1), \\
A_3(2k) &= \frac{1}{6} k(23k^2 - 27k + 10), \\
A_3(2k+1) &= \frac{1}{6} k(23k^2 + 1), \\
A_4(d) &= \frac{1}{24} d(d-1)(11d^2 - 27d + 22), \\
A_5(2k) &= \frac{1}{60} k(841k^4 - 1910k^3 + 1745k^2 - 760k + 144), \\
A_5(2k+1) &= \frac{1}{60} k(841k^4 + 55k^2 + 4), \\
A_6(d) &= \frac{1}{360} d(d-1)(151d^4 - 695d^3 + 1295d^2 - 695d + 474), \\
A_7(2k) &= \frac{1}{5040}(259723k^6 - 863513k^5 + 1239637k^4 - \\
&\qquad - 985355k^3 + 462952k^2 - 124292k + 15888), \\
A_7(2k+1) &= \frac{1}{5040}(259723k^6 + 20230k^4 + 2107k^2 + 180).
\end{aligned}$$

For the sake of brevity, below we let $D_n(d)$ denote the difference between the right- and left-hand sides of inequality $(**_{n,d})$. It turns out that if we perform the substitutions $k \mapsto k+7$ and $k \mapsto k+5$ in the polynomials $D_6(2k)$ and $D_6(2k+1)$, respectively, then, as before, we get polynomials with positive coefficients. Hence the inequality considered is true for $d \geq 13$. Let us emphasize that it does not hold for $n = 6$, $d = 12$.

To complete the proof of the theorem it remains to consider the cases $n = 2, 4$ and $d \geq 12$; $n \geq 24$ and $d = 2$.

We have

$$\begin{aligned} D_2(2k) &= \frac{-1}{3}(k-1)(7k^2+28k-6), \\ D_2(2k+1) &= \frac{-1}{3}(7k^3+24k^2-7k-6), \\ D_4(2k) &= \frac{-1}{30}(9k^5+970k^4-2015k^3+1640k^2-664k+120), \\ D_4(2k+1) &= \frac{-1}{30}k(9k^4+800k^3-25k^2+40k-44), \end{aligned}$$

and so inequalities $(**_{2,d})$ and $(**_{4,d})$ do not hold.

Finally, for $d = 2$ we have $\mathrm{rk}_n(2) = 2$ and $A_m(2) = 1$, and so the inequality does not hold in this case, either. The theorem is proved. □

THE TABLE OF ARNOLD'S NUMBERS $A_m(d)$

$m\backslash d$	3	4	5	6	7	8	9	10	11	12
3	4	16	31	68	104	180	246	375	480	676
4	10	45	135	320	651	1190	2010	3195	4840	7051
5	15	126	456	1506	3431	7872	14412	27237	43917	73578
6	35	357	1918	7140	20993	52374	115788	233331	436975	771155
7	56	1016	6728	34000	113688	349840	848443	2006216	4038560	8110272
8	126	2907	27876	162585	689715	2345553	6780735	17309772	40051495	85578174

$m\backslash d$	3	4	5	6	7	8	9
9	210	8350	100110	780175	3801535	15776530	50415760
10	462	24068	411334	3755070	22967626	106413923	402102580
11	792	69576	1501566	18121038	128184068	719540108	3021294288
12	1716	201643	6137274	87648795	772661695	4875903487	24077093222
13	3003	585690	22675744	424803757	4352660949	33104865990	182346607184
14	6435	1704510	92348750	2062582590	26199964377	225151377325	1452698053500

$m\backslash d$	13	14	15	16	17	18	19
3	829	1106	1316	1688	1964	2445	2796
4	9945	13650	18305	24060	31076	39525	49590
5	109192	168518	235893	343008	459768	639009	828339
6	1295658	2088814	3250989	4908380	7217112	10367637	14589435
7	14457060	25976820	42506704	70461800	108202894	168730719	246714912
8	171206685	324080718	585208638	1014637230	1697957178	2754351747	4346415756

$m\backslash d$	10	11	12	13
9	149803270	374894389	905642810	1932328870
10	1299917322	3715101654	9608991865	22885637138
11	11306743020	35099288940	102187498128	260496554176
12	98553025974	347847754670	1088945997854	3087267546454
13	860623146306	3309636861545	11625405426558	35369058879168
14	7528026744360	32811494188975	124314245763015	419511854571210

$m\backslash d$	14	15	16	17
9	4054626785	7732603550	14651139020	25708340360
10	50856761241	106533818314	212085986002	403898132968
11	639330337978	1418801671728	3076919557296	6160867356084
12	8053317212824	19568682516215	44728193445130	96930278265548
13	101626040905767	262194141697515	651355236406812	1487024439732664
14	1284506121970565	3620350477963365	9500530948438950	23427637519453240

§4. Homologically standard hypersurfaces

Our main results in this section are the theorem stated in the end of the section and its corollary. To prove them we need some auxiliary definitions and results.

Let $X \subset \mathbb{C}P^{n+1}$ be a singular hypersurface of degree d (maybe having non-isolated singularities), and let $\widetilde{X} \subset \mathbb{C}P^{n+1}$ be a nonsingular hypersurface of the same degree, which is sufficiently close to X (i.e., is obtained from X by a "small perturbation").

For each *isolated* singular point P of X, its *Milnor lattice* L is defined as well as a homomorphism in: $L \to H_n(\widetilde{X})$ preserving the intersection form defined on L.

Definition. Let $\Sigma = \{P_1, \dots, P_k\} \subset \mathbb{C}P^{n+1}$ be a certain collection of isolated singular points of X, and let $L_1, \dots, L_k$ be their Milnor lattices. We say that Σ is *homologically standard* if the homomorphism

$$\mathrm{In}_\Sigma = \sum_{i=0}^{k} \mathrm{in}_i \colon \bigoplus_{i=0}^{k} L_i \to H_n(\widetilde{X})$$

is mono and its image $A \subset H_n(\widetilde{X})$ is a primitive sublattice, and moreover, for n even, the sublattice A together with the algebraic class $h \in H_n(\widetilde{X})$ generates a primitive sublattice, i.e., $\mathrm{Tors}(H_n(\widetilde{X})/(A \oplus \langle h \rangle)) = 0$.

Obviously, any subset of a homologically standard collection of singular points is homologically standard, too.

Proposition 4.1. *Let $\{X_t\}_{|t|<r}$, $X_t \subset \mathbb{C}P^{n+1}$, be a one-parametric family of (singular) hypersurfaces of given degree, and let $\Sigma = \{P_1, \dots, P_k\} \subset \mathbb{C}P^{n+1}$ be a homologically standard collection of isolated singular points of X_0. Then for t sufficiently small the collection of those singular points of X_t which are situated close to P_1, ..., P_k is homologically standard. More precisely, there are $\epsilon > 0$ and $t_0 > 0$ such that for every $t < t_0$ the set $\mathrm{Sing}(X_t) \cap U_\epsilon(\Sigma)$ is a homologically standard collection of singular points of X_t. (Here $U_\epsilon(\Sigma)$ denotes the ϵ-neighborhood of Σ.)*

Proof. It is sufficient to use the well-known fact (cf. [2]) that if a multi-singularity f is adjacent to a singularity g, then the Milnor lattice of f is a primitive sublattice of that of g. □

To prove our main technical result we need the following easy assertion, which also follows from the above-mentioned fact on the adjacent singularities.

Proposition 4.2. *In the case of an affine hypersurface with isolated singularities, the inclusion homomorphism of the Milnor lattices of the singularities to the homology of a close nonsingular hypersurface is a monomorphism onto a direct summand.* □

Our main technical result is the following.

Theorem 4.3. *Let $W \subset \mathbb{C}P^{n+1}$ be a singular hypersurface. If W contains a hyperplane, then any collection of isolated singular points of W is homologically standard.*

Proof. Let us present the perturbation in the form

$$W = X \cup H \rightsquigarrow X_d \cup H \rightsquigarrow X_{d+1} = \widetilde{X},$$

where X_d and X_{d+1} are nonsingular hypersurfaces. Let $X'_d = X_d \setminus H$ be the affine part of X_d. (It is obviously homotopy equivalent to $X_d \setminus U(H)$.) Then for the Milnor lattices we have homomorphisms

$$\bigoplus_{i=0}^{k} L_i \to H_n(X'_d) \to H_n(X_{d+1}).$$

By Proposition 4.2 and Theorem 2.1 each of the homomorphisms is an inclusion onto a direct summand. Hence their composition is also an inclusion onto a direct summand. This proves Theorem 4.3 for n odd.

For n even, Theorem 2.1 implies that the image of $H_n(X'_d)$ in $H_n(X_{d+1})$ generates together with the algebraic class a primitive sublattice, and by Proposition 2.2 the same is true for the image of the lattice L. □

Let $X \subset \mathbb{C}P^{n+1}$ be a hypersurface with isolated singularities. It easily follows from the Lefschetz Hyperplane Section Theorem that if $i \neq n$, $n+1$, then $H_i X = H_i \mathbb{C}P^n$. Let y be the generator of $H_2 X = \mathbb{Z}$. For n even let $h \in H_n X$ be the homology class dual to $y^{n/2}$. (We call it the *algebraic class*, also.)

Definition. We say that the hypersurface X is *homologically standard*, if it satisfies the following conditions (1–3):

(1) $\operatorname{Tors} H_n X = 0$,
(2) $H_{n+1} X = H_{n+1} \mathbb{C}P^n$,
(3) for n even, the class $h \in H_n X$ is indivisible.

Remarks. Let X be a HSQS. 1. If X is homologically standard, then all its homology groups are uniquely determined by its dimension, degree, and the number of singular points: $H_i X = H_i \mathbb{C}P^n$ for $i \neq n$, and $H_n X$ is a free Abelian group of rank $b_n X = b_n X(n,d) - s$.

2. Condition (2) is automatically fulfilled, if n is even.
3. Condition (3) is fulfilled, if $\deg X \notin 2\mathbb{Z}$.

Proposition 4.4. *A hypersurface X with isolated singularities is homologically standard if and only if the collection of all singular points of X is homologically standard.* □

Our main aim in this section is the following theorem.

Theorem 4.5. *Let $X \subset \mathbb{C}P^{n+1}$ be a hypersurface with quadratic singularities. If there exists a hypersurface Y of degree less than* $\deg X$ *and such that each singular point of X is an isolated singular point of Y, then X is homologically standard.*

Proof. Let T be any hypersurface of degree $\deg X - \deg Y$, which does not pass through the singular points of X. Consider the reducible hypersurface $W = Y \cup T$. It follows from Theorem 4.3 that $\operatorname{Sing} X$ is a homologically standard collection of singularities of W.

Assume that X is given by $f = 0$, and W by $g = 0$. Consider the pencil of hypersurfaces $\{X_t\}$, with X_t given by $t \cdot f + (1-t)g = 0$. Obviously, $X_0 = W$ and $X_1 = X$. It is clear that for all t except a finite number of values, say, $t_1, \dots, t_K$, the hypersurface X_t has only quadratic singular points, $\operatorname{Sing} X_t = \operatorname{Sing} X$, and so X_t is diffeomorphic to X.

By the above proposition, X_t is homologically standard for every small $t \notin \{t_1, \dots, t_K\}$, and hence for all such t. □

For the definition of numbers $\phi_n(\{P_1, \dots, P_s\})$ and $\phi(n; s)$ used in the statement of the following corollary see §1.

Corollary. *Let $X \subset \mathbb{C}P^{n+1}$, $n \geq 1$, be a hypersurface with s quadratic singularities. If X satisfies one of the following conditions:*

(1) $\deg X > 2s$,
(2) $\deg X > \phi_n(\operatorname{Sing} X)$,
(3) *The singular points of X are in a generic position and* $\deg X > \phi(n; s)$,
(4) *the singular points of X are in the general position (in the usual sense) and* $\deg X \geq 3[\frac{s-1}{n+2}] + 4$ *(or, equivalently,* $s \leq [\frac{\deg X}{3}](n+2)$*),*

then it homologically standard.

§5. An auxiliary algebraic assertion

We need the following algebraic assertion, which is easily deduced from the known results on integral quadratic forms. Because of its importance for this paper, we call it a theorem.

Theorem 5.0. *Let L be a unimodular lattice of signature (t_+L, t_-L), let $h \in L$ be a characteristic element with $h^2 > 0$, and let the number $\epsilon = \pm 1$ be fixed. Let the elements*

$$a_1, \dots, a_r \in h^\perp; \quad a_i \cdot a_j = 2\epsilon\delta_{ij}; \quad i, j = 1, \dots, r,$$

be such that the sublattice $A := \langle h; a_1, \dots, a_r \rangle$ generated by $a_1, \dots, a_r$ and h is primitive, i.e., $\operatorname{Tors}(L/A) = 0$. *Assume that the following inequalities hold:*

$$(t_+L, t_-L) \geq (r, r+1), \tag{$*$}$$

$$\operatorname{rk} L = t_+L + t_-L \geq 2r + 4, \tag{$**$}$$

$$(t_+L, t_-L) \geq \begin{cases} (r+2, 1), & \epsilon = +1, \\ (2, r+1), & \epsilon = -1. \end{cases} \tag{$***$}$$

Then there exist elements $b_1, \dots, b_r \in h^\perp$ such that

$$a_i \cdot b_j = \delta_{ij}; \quad b_i \cdot b_j = 0; \quad i, j = 1, \dots, r.$$

Remark. It is sufficient to demand that the inequality $(*)$ hold simultaneously with the inequality

$$\epsilon(t_+L - t_-L) \geq 3 + \epsilon, \tag{$****$}$$

or that the inequality $(t_+L, t_-L) \underset{\neq}{\geq} (r+2, r+1)$ be fulfilled.

To prove the theorem we need several lemmas. (For the sake of brevity we state them in the form which is convenient for us. Also note that Lemmas 5.2 and 5.3 are given certainly not in the most general form.)

Lemma 5.1 (Nikulin [9], Corollary 1.13.5 (splitting-off)). *Let S be an even indefinite lattice. If $\operatorname{rk} S \geq \ell(D(S)) + 3$, then $S \cong U \oplus T$ for a certain lattice T. (Here U is the hyperbolic plane with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\ell(\cdot)$ denotes the smallest possible number of generators.)*

Corollary 1. *Let S be an even indefinite lattice of signature $(t_{(+)}, t_{(-)})$. If*

$$\min\{t_{(+)}, t_{(-)}\} \geq r \quad \textit{and} \quad \operatorname{rk} S \geq 2r + \ell(D(S)) + 1,$$

then

$$S \cong \underbrace{U \oplus \cdots \oplus U}_{r \text{ summands}} \oplus T$$

for a certain lattice T. (Here U is the hyperbolic plane with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\ell(\cdot)$ denotes the smallest possible number of generators.)

Proof. By Lemma 5.1, the lattice S contains a hyperbolic plane U. We can split it off and proceed by induction. Finally, we will find the required sublattice $\underbrace{U \oplus \cdots \oplus U}_{r \text{ summands}} \subset S$. As any unimodular sublattice, it splits off as a direct summand. □

Corollary 2. *Let L be a unimodular lattice of signature (t_+L, t_-L), and let $h \in L$ be a characteristic element with $h^2 > 0$. If*

$$(t_+L, t_-L) \geq (r+1, r) \quad \textit{and} \quad \operatorname{rk} L \geq 2r + 3,$$

then

$$h^\perp \cong \underbrace{U \oplus \cdots \oplus U}_{r \text{ summands}} \oplus T$$

for a certain lattice T.

Proof. It is sufficient to note that that the number of generators of the discriminant group of the lattice $h^\perp$ is equal to 1: $\ell(D(h^\perp)) = \ell(D(\langle h \rangle)) = 1$, and apply the previous Corollary. □

Lemma 5.2 (Nikulin [9], Theorem 1.13.2). *Any even indefinite lattice S is uniquely determined up to isomorphism by its signature $(t_{(+)}, t_{(-)})$ and discriminant group $(D(S))$ (with fixed discriminant form on it) if $\operatorname{rk} S \geq \ell(D(S)) + 2$.*

Lemma 5.3 (Nikulin [9], Theorem 1.14.2). *Let S be an even indefinite lattice with discriminant group $(D(S))$. If $\operatorname{rk} S \geq \ell(D(S)) + 2$, then the natural homomorphism $O(S) \to O(D(S))$ is surjective. (Here $O(D(S))$ denotes the group of those automorphisms of $D(S)$ which preserve the discriminant form.)*

Lemma 5.4. *Let L be a unimodular lattice and let A and A_1 be two nondegenerate sublattices of L which are isomorphic to each other and have isomorphic orthogonal complements. If the natural homomorphism $O(A^\perp) \to O(D(A^\perp))$ is epi, then any* *isomorphism between the lattices A and A_1 can be extended to an automorphism of the lattice L.*

Proof. Let us fix an isomorphism $f\colon A \to A_1$. We will use the fact that there is a canonical isomorphism $D(A) \cong D(A^\perp)$ (it changes the sign of the discriminant forms). Consider the induced isomorphism $f_\#\colon D(A) \to D(A_1)$ and the corresponding isomorphism $f_\#^\perp\colon D(A^\perp) \to D(A_1^\perp)$. Since the lattices $A^\perp$ and $A_1^\perp$ are isomorphic and $O(A^\perp) \to O(D(A^\perp))$ is an epimorphism, $f_\#^\perp$ is induced by an isomorphism $f^\perp\colon A^\perp \to A_1^\perp$. Consider the isomorphism $\phi = f \oplus f^\perp\colon A \oplus A^\perp \to A_1 \oplus A_1^\perp$ and the corresponding isomorphism $\phi \otimes \mathbb{R}\colon L \otimes \mathbb{R} \to L \otimes \mathbb{R}$. Since $f_\#$ and $f_\#^\perp$ coincide, it easily follows that $(\phi \otimes \mathbb{R})(L) = L$, and hence the required extension is yielded by $(\phi \otimes \mathbb{R})|_L$. □

Lemma 5.5. *If under the assumptions of the theorem the inequalities $(*)$ and $(**)$ are fulfilled, then the arrangement of the vectors a_i in the lattice L is defined uniquely up to an automorphism of L leaving h fixed.*

Proof. Consider the lattice $W := A^\perp$. Since the element h is characteristic, W is an even lattice. Inequality $(***)$ is equivalent to the fact that $t_\pm \geq W$ (i.e., the lattice W is indefinite). Due to inequality $(*)$ we have

$$\operatorname{rk} W = \operatorname{rk} L - \operatorname{rk} A = \operatorname{rk} L - r - 1 \geq r + 3,$$

and

$$\ell(D(W)) = \ell(D(A)) \leq \operatorname{rk} A = r + 1.$$

Thus, $\operatorname{rk} W \geq \ell(D(W)) + 2$. Therefore, by Lemma 5.2, the lattice W is uniquely defined, and by Lemma 5.3 the natural homomorphism $O(W) \to O(D(W))$ is epi. Hence, by Lemma 5.4, every two embeddings of the lattice A in L coincide up to automorphism of L. □

Proof of Theorem 5.0. By Corollary 1 from Lemma 5.1, $h^\perp$ contains a sublattice isomorphic to $\underbrace{U \oplus \cdots \oplus U}_{r \text{ summands}}$. Let us take in the found hyperbolic planes some generators a_i', b_i' with matrix $\begin{pmatrix} 2\epsilon & 1 \\ 1 & 0 \end{pmatrix}$, $i = 1, \ldots, r$. It remains to use Lemma 5.5, which implies that there is an automorphism $\phi\colon L \to L$ such that $a_i' \mapsto a_i$, and we can set $b_i = \phi(b_i')$, $i = 1, \ldots, r$. □

§6. Connected sum decomposition of homologically standard hypersurfaces

Proposition 6.1. *Let X be a homologically standard hypersurface with quadratic singularities and let one of the following conditions (1-3) be fulfilled:*

(1) *n is odd,*
(2) *n is even, $(n-4)(d-2) > 17$, and $(n, d) \neq (6, 12)$.*
(3) *n is even, $n \geq 4$, and $s < \min\{b_+(X_n(d)), b_-(X_n(d))\}$,*

Then X is topologically standard, i.e., we have a connected sum decomposition

$$X \cong X_n(d; s) \cong M_n(d) \# (a - s)(S^n \times S^n) \# s(S^n \times S^n / \Delta), \qquad (*)$$

where Δ is the diagonal and $M_n(d)$ and a depend on n and d only.

Remark. Note that condition (2) implies condition (3). Also, it is obvious that the connected sum decomposition $(*)$ implies the inequality $\mathrm{card}(\mathrm{Sing}(X)) \leq \min\{b_+(X_n(d)), b_-(X_n(d))\}$.

Proof. We restrict our consideration by the case of n even, the case of n odd being in essence the same or even more simple.

In view of Theorem 5.0, the homological standardness of the hypersurface implies that the arrangement of the vanishing cycles in the homology group of a close nonsingular hypersurface is standard, which in turn allows us to describe the topological structure of the singular hypersurface.

Let s be the number of singular points of X and let $a_1, \dots, a_s \in H_n(X_n(d))$ be the vanishing cycles (we identify the Fermat hypersurface $X_n(d)$ with a nonsingular hypersurface close to X). Each vanishing cycle a_i by is realized by a "vanishing n-sphere" $S_i \subset X_n(d)$. (Note that the spheres are pairwise disjoint.)

Applying the theorem of the preceding section, we obtain some classes $b_1, \dots, b_s \in H_n(X_n(d))$. They can be realized by "dual" spheres $S'_1, \dots, S'_s$, which can be taken pairwise disjoint and such that $S'_i \cap S_j = \emptyset$ for $i \neq j$ and S'_i transversally intersects S_j at precisely one point.

Now the boundary of the union $S'_i \cup S_i$ is an $(2n+1)$-sphere, which splits off a connected summand $S^n \times S^n$ containing the sphere S_i as the diagonal. Due to cancellation theorems (see [7]), this decomposition is unique up to diffeomorphism, q.e.d. □

References

1. S. V. Chmutov, *Examples of projective surfaces with many singularities*, J. Alg. Geom. **1** (1992), 191–196.
2. A. Dimca, *Singularities and Topology of Hypersurfaces*, Springer, 1992.
3. V. V. Goryunov, *Singular quartics with many nodes*, Advances Soviet Math. **21** (1994), 147–161.
4. F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer, 1966.
5. A. Kalker, *Cubic Fourfolds with Fifteen Ordinary Double Points*, Ph. D. Thesis, Leiden, 1986.
6. R. S. Kulkarni and J. W. Wood, *Topology of nonsingular complex hypersurfaces*, Adv. Math. **35** (1980), 239–263.
7. N. Yu. Netsvetaev, *Diffeomorphism and stable diffeomorphism of simply connected manifolds*, Algebra i Analiz **2** (1990), no. 2, 112–120; English transl. in Leningrad Math. J. **2** (1991), 312–320.
8. ———, *Diffeomorphism criteria for smooth manifolds and algebraic varieties*, Contemporary Math. **431**, (Part 3) (1992), 453–459.
9. V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Izv. Akad. Nauk SSSR. Ser. Mat. **43** (1979), 111–117.
10. A. N. Varchenko, *On semicontinuity of the spectrum and an upper bound for the number of singular points of projective hypersurfaces*, Doklady Akad. Nauk SSSR **270** (1983).