COORDINATE-FREE CLASSIC GEOMETRIES
III. PLÜCKER MAP

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Abstract. We use the m-plücker map between grassmannians in order to study basic aspects of classic geometries.

1. Introduction

This paper links the (pseudo-)riemannian geometry of the nondegenerate piece $\mathrm{Gr}^0_{\mathbb{K}}(k,V)$ of a grassmannian to the structures discussed in [AGr] and [AGoG]. It is merely intended to illustrate how do the methods from the previous papers work in the differential geometry of grassmannians. Many of the presented results are known in particular cases.[1] We believe that our treatment provides additional clarity even in those cases.

It follows a brief description of the results. The m-plücker map is a minimal isometric embedding. The gauss equation provides the curvature tensor in the form of the $(2,1)$-symmetrization of the triple product exactly as in the projective case. $\mathrm{Gr}^0_{\mathbb{K}}(k,V)$ is proven to be einstein. Generic geodesics in $\mathrm{Gr}^0_{\mathbb{K}}(k,V)$ are described. Finally, we illustrate how a grassmannian classic geometry unexpectedly shows up in relation to convexity in real hyperbolic space.

It turns out that the hermitian metric actually plays no role in most of the proofs. The tangent vectors can usually be taken as footless or as observed from different points. Therefore, many definitions, for instance, those of isometric or minimal embeddings and of the gauss equation, may be restated in the terms of the product. This must be fruitful since the product embodies different (pseudo-)riemannian concepts in a single simple structure. In the spirit of [AGoG], it would be nice to understand what remains from these concepts after arriving at the absolute.

To prevent a possible scepticism of the reader, we have to say that the pseudo-riemannian metrics play a fundamental role in the study of the riemannian classical geometries: basic geometrical objects almost never form riemannian spaces. To illustrate this remark, the beautiful article [GuK] is to be mentioned, where the authors work in an ambient that in fact falls into our settings.

The differential geometry of grassmannians is a rather vast field (see, for instance, the survey [BoN]). We believe that it is reasonable to redemonstrate known facts in the area by using the language of our papers. Of course, we recognize that such a project involves a huge amount of work, but is probably worth the candle: besides giving each fact an appropriate generality, it would provide a better understanding of particular problems in classic geometries.

2000 *Mathematics Subject Classification.* 53A20 (53A35, 51M10).

First author partially supported by the Institut des Hautes Études Scientifiques (IHÉS).

Second author supported by the Max-Planck-Gesellschaft.

[1] If the hermitian form on V is definite, the classic geometry is sort of elliptic. Most of the known facts deal with this case. The hermitian algebra of the indefinite form requires additional effort thus making it nontrivial the case of 'hyperbolic' classic geometries.

2. Plücker-and-play

We assume that the reader is familiar with the notation from the beginning of [AGoG, Section 2]. Our purpose is to study the *m-plücker embedding*

$$E^m:\mathrm{Gr}_{\mathbb K}(k,V)\to\mathrm{Gr}_{\mathbb K}\big(\tbinom{k}{m},{\textstyle\bigwedge^m}V\big),\qquad p\mapsto{\textstyle\bigwedge^m}p,$$

where the vector space $\bigwedge^m V$ is equipped with the hermitian form given by the rule

$$\langle v_1\wedge\dots\wedge v_m,w_1\wedge\dots\wedge w_m\rangle:=\det\langle v_i,w_j\rangle.$$

Let $p\in M$. It is not difficult to see that the differential of the map $M\to\mathrm{Lin}_{\mathbb K}\big(\bigwedge^m P,\bigwedge^m V\big)$ at p sends the tangent vector $\overline t\in\mathrm T_pM=\mathrm{Lin}_{\mathbb K}(p,V)$ to $E^m\overline t\in\mathrm{Lin}_{\mathbb K}\big(\bigwedge^m p,\bigwedge^m V\big)$ defined by the rule $E^m\overline t:p_1\wedge\dots\wedge p_m\mapsto\sum_{i=1}^m p_1\wedge\dots\wedge\overline tp_i\wedge\dots\wedge p_m$ for all $p_1,\dots,p_m\in p$. Therefore, we can describe the differential of E^m at p as

$$E^m:\mathrm{Lin}_{\mathbb K}(p,V/p)\to\mathrm{Lin}_{\mathbb K}\big({\textstyle\bigwedge^m}p,{\textstyle\bigwedge^m}V/{\textstyle\bigwedge^m}p\big),$$

$$E^mt:p_1\wedge\dots\wedge p_m\mapsto\sum_{i=1}^m p_1\wedge\dots\wedge\overline tp_i\wedge\dots\wedge p_m+{\textstyle\bigwedge^m}p$$

for all $t:p\to V/p$ and $p_1,\dots,p_m\in p$, where $\overline t:p\to V$ is an arbitrary lift of t.

Given $p\in\mathrm{Gr}^0_{\mathbb K}(k,V)$, we have the orthogonal decomposition

$$\textstyle\bigwedge^m V=\bigoplus_{i=0}^m\bigwedge^i p^\perp\wedge\bigwedge^{m-i}p. \tag{2.1}$$

In particular, taking $p\in\mathrm{Gr}^0_{\mathbb K}(k,V)$ and $t\in\mathrm T_p\,\mathrm{Gr}^0_{\mathbb K}(k,V)=\mathrm{Lin}_{\mathbb K}(p,p^\perp)$, we obtain

$$E^mt:p_1\wedge\dots\wedge p_m\mapsto\sum_{i=1}^m p_1\wedge\dots\wedge tp_i\wedge\dots\wedge p_m \tag{2.2}$$

for all $p_1,\dots,p_m\in p$. Note that (2.2) makes sense for an arbitrary $t:V\to V$.

Define the linear map $B(t_1,t_2):\bigwedge^m V\to\bigwedge^m V$ by the rule

$$B(t_1,t_2)(v_1\wedge\dots\wedge v_m):=\sum_{i\ne j}v_1\wedge\dots\wedge t_1v_i\wedge\dots\wedge t_2v_j\wedge\dots\wedge v_m$$

for all $v_1,\dots,v_m\in V$, where $t_1,t_2:V\to V$. (In the above sum, t_2v_j appears before t_1v_i if $i>j$.)

2.3. Lemma. *Let $p\in\mathrm{Gr}_{\mathbb K}(k,V)$ and let $t,t_1,t_2:V\to V$. Then*

$$\big\langle E^mt(p\wedge\dots\wedge p_m),q\wedge v_2\wedge\dots\wedge v_m\big\rangle=\langle p_1\wedge\dots\wedge p_m,t^*q\wedge v_2\wedge\dots\wedge v_m\rangle,$$

$$\big\langle B(t_1,t_2)(p_1\wedge\dots\wedge p_m),q_1\wedge q_2\wedge v_3\wedge\dots\wedge v_m\big\rangle=$$
$$=\langle p_1\wedge\dots\wedge p_m,t_1^*q_1\wedge t_2^*q_2\wedge v_3\wedge\dots\wedge v_m\rangle+\langle p_1\wedge\dots\wedge p_m,t_2^*q_1\wedge t_1^*q_2\wedge v_3\wedge\dots\wedge v_m\rangle$$

for all $q,q_1,q_2\in p^\perp$, $p_1,\dots,p_m\in p$, and $v_2,\dots,v_m\in V$.

Proof is based on simple known identities involving determinants (marked with † and left without proof). We have

$$\left\langle E^m t(p_1\wedge\dots\wedge p_m), q\wedge v_2\wedge\dots\wedge v_m\right\rangle=\sum_{i=1}^m \det\begin{pmatrix} 0 & \langle p_1,v_2\rangle & \cdots & \langle p_1,v_m\rangle\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \langle p_{i-1},v_2\rangle & \cdots & \langle p_{i-1},v_m\rangle\\ \langle tp_i,q\rangle & \langle tp_i,v_2\rangle & \cdots & \langle tp_i,v_m\rangle\\ 0 & \langle p_{i+1},v_2\rangle & \cdots & \langle p_{i+1},v_m\rangle\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \langle p_m,v_2\rangle & \cdots & \langle p_m,v_m\rangle\end{pmatrix}\overset{\dagger}{=}$$

$$\overset{\dagger}{=}\det\begin{pmatrix}\langle tp_1,q\rangle & \langle p_1,v_2\rangle & \cdots & \langle p_1,v_m\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle tp_m,q\rangle & \langle p_m,v_2\rangle & \cdots & \langle p_m,v_m\rangle\end{pmatrix}=\det\begin{pmatrix}\langle p_1,t^*q\rangle & \langle p_1,v_2\rangle & \cdots & \langle p_1,v_m\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle p_m,t^*q\rangle & \langle p_m,v_2\rangle & \cdots & \langle p_m,v_m\rangle\end{pmatrix}=\langle p_1\wedge\dots\wedge p_m, t^*q\wedge v_2\wedge\dots\wedge v_m\rangle$$

and

$$\left\langle B(t_1,t_2)(p_1\wedge\dots\wedge p_m), q_1\wedge q_2\wedge v_3\wedge\dots\wedge v_m\right\rangle=\sum_{i\neq j}\det\begin{pmatrix} 0 & 0 & \langle p_1,v_3\rangle & \cdots & \langle p_1,v_m\rangle\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \langle p_{i-1},v_3\rangle & \cdots & \langle p_{i-1},v_m\rangle\\ \langle t_1p_i,q_1\rangle & \langle t_1p_i,q_2\rangle & \langle t_1p_i,v_3\rangle & \cdots & \langle t_1p_i,v_m\rangle\\ 0 & 0 & \langle p_{i+1},v_3\rangle & \cdots & \langle p_{i+1},v_m\rangle\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \langle p_{j-1},v_3\rangle & \cdots & \langle p_{j-1},v_m\rangle\\ \langle t_2p_j,q_1\rangle & \langle t_2p_j,q_2\rangle & \langle t_2p_j,v_3\rangle & \cdots & \langle t_2p_j,v_m\rangle\\ 0 & 0 & \langle p_{j+1},v_3\rangle & \cdots & \langle p_{j+1},v_m\rangle\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \langle p_m,v_3\rangle & \cdots & \langle p_m,v_m\rangle\end{pmatrix}\overset{\dagger}{=}$$

$$\overset{\dagger}{=}\det\begin{pmatrix}\langle t_1p_1,q_1\rangle & \langle t_2p_1,q_2\rangle & \langle p_1,v_3\rangle & \cdots & \langle p_1,v_m\rangle\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \langle t_1p_m,q_1\rangle & \langle t_2p_m,q_2\rangle & \langle p_m,v_3\rangle & \cdots & \langle p_m,v_m\rangle\end{pmatrix}+\det\begin{pmatrix}\langle t_2p_1,q_1\rangle & \langle t_1p_1,q_2\rangle & \langle p_1,v_3\rangle & \cdots & \langle p_1,v_m\rangle\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \langle t_2p_m,q_1\rangle & \langle t_1p_m,q_2\rangle & \langle p_m,v_3\rangle & \cdots & \langle p_m,v_m\rangle\end{pmatrix}=$$

$$=\det\begin{pmatrix}\langle p_1,t_1^*q_1\rangle & \langle p_1,t_2^*q_2\rangle & \langle p_1,v_3\rangle & \cdots & \langle p_1,v_m\rangle\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \langle p_m,t_1^*q_1\rangle & \langle p_m,t_2^*q_2\rangle & \langle p_m,v_3\rangle & \cdots & \langle p_m,v_m\rangle\end{pmatrix}+\det\begin{pmatrix}\langle p_1,t_2^*q_1\rangle & \langle p_1,t_1^*q_2\rangle & \langle p_1,v_3\rangle & \cdots & \langle p_1,v_m\rangle\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \langle p_m,t_2^*q_1\rangle & \langle p_m,t_1^*q_2\rangle & \langle p_m,v_3\rangle & \cdots & \langle p_m,v_m\rangle\end{pmatrix}=$$

$$=\langle p_1\wedge\dots\wedge p_m, t_1^*q_1\wedge t_2^*q_2\wedge v_3\wedge\dots\wedge v_m\rangle+\langle p_1\wedge\dots\wedge p_m, t_2^*q_1\wedge t_1^*q_2\wedge v_3\wedge\dots\wedge v_m\rangle \ \blacksquare$$

Let $t\in\operatorname{Lin}_{\mathbb K}(p,p^\perp)\subset\operatorname{Lin}_{\mathbb K}(V,V)$. It follows from (2.2) and Lemma 2.3 that the only nonvanishing component of $(E^mt)^*$ related to the decomposition (2.1) has the form $(E^mt)^*:p^\perp\wedge\bigwedge^{m-1}p\to\bigwedge^m p$,

$$\textbf{(2.4)}\qquad (E^mt)^*:q\wedge p_2\wedge\dots\wedge p_m\mapsto t^*q\wedge p_2\wedge\dots\wedge p_m,$$

where $q\in p^\perp$ and $p_2,\dots,p_m\in p$. In other words, $(E^mt)^*=E^mt^*$. Similar arguments are applicable to $B(t_1,t_2)$ with $t_1,t_2\in\operatorname{Lin}_{\mathbb K}(p,p^\perp)\subset\operatorname{Lin}_{\mathbb K}(V,V)$.

2.5. Proposition (compare to [BoN, Assertions 1–2])**.** *The m-plücker embedding provides an hermitian (hence, pseudo-riemannian) embedding $E^m:\operatorname{Gr}^0_{\mathbb K}(k,V)\to\operatorname{Gr}^0_{\mathbb K}\big(\binom{k}{m},\bigwedge^m V\big)$, assuming the metric on $\operatorname{Gr}^0_{\mathbb K}(k,V)$ reescaled by the factor $\binom{k-1}{m-1}$.*

Proof. Let $p \in \mathrm{Gr}^0_{\mathbb{K}}(k,V)$ and let $t_1,t_2 : p \to p^\perp$ be tangent vectors at p. By (2.2) and (2.4),

$$(E^m t_1)^* E^m t_2 : p_1 \wedge \dots \wedge p_m \mapsto \sum_{i=1}^m p_1 \wedge \dots \wedge t_1^* t_2 p_i \wedge \dots \wedge p_m$$

for all $p_1,\dots,p_m \in p$. As is easy to see, $\mathrm{tr}(E^m\varphi) = \binom{k-1}{m-1}\,\mathrm{tr}\,\varphi$ for every linear map $\varphi : p \to p$ and the map $E^m\varphi : \bigwedge^m p \to \bigwedge^m p$ defined as in (2.2). Hence,

$$\langle E^m t_1, E^m t_2\rangle = \mathrm{tr}\big((E^m t_1)^* E^m t_2\big) = \mathrm{tr}\,\big(E^m(t_1^* t_2)\big) = \tbinom{k-1}{m-1}\,\mathrm{tr}(t_1^* t_2) = \tbinom{k-1}{m-1}\langle t_1, t_2\rangle \ \blacksquare$$

Given $p \in \mathrm{Gr}^0_{\mathbb{K}}(k,V)$, denote by $\pi'[p]$ and $\pi[p]$ the orthogonal projectors corresponding to the decomposition $V = p \oplus p^\perp$. For $t \in \mathrm{Lin}_{\mathbb{K}}(V,V)$, define the tangent vector $t_p := \pi[p]t\pi'[p]$ at p.

Let $U \subset M$ be a *saturated* and *nondegenerate* open set. This means that $U\,\mathrm{GL}_{\mathbb{K}} P = U$ and $\pi U \subset \mathrm{Gr}^0_{\mathbb{K}}(k,V)$, where $\pi : M \to \mathrm{Gr}_{\mathbb{K}}(k,V)$ stands for the quotient map. A smooth map $X : U \to \mathrm{Lin}_{\mathbb{K}}(V,V)$ is said to be a *lifted field* over U if $X(p)_p = X(p)$ and $X(pg) = X(p)$ for all $p \in U$ and $g \in \mathrm{GL}_{\mathbb{K}} P$. In other words, π maps X onto a correctly defined smooth tangent field over the open $\pi U \subset \mathrm{Gr}^0_{\mathbb{K}}(k,V)$.

For $t \in \mathrm{Lin}_{\mathbb{K}}(V,V)$, define

$$\nabla_t X(p) := \Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X\big((1+\varepsilon t)p\big)\Big)_p.$$

Since $\pi'[pg] = \pi'[p]$ and $\pi[pg] = \pi[p]$ for all $p \in U$ and $g \in \mathrm{GL}_{\mathbb{K}} P$, the field $p \mapsto \nabla_{Y(p)} X$ is lifted for arbitrary lifted fields X and Y over U. Obviously, ∇ enjoys the properties of an affine connection; we assume $\mathrm{Gr}^0_{\mathbb{K}}(k,V)$ equipped with this *intrinsic* connection.

2.6. Proposition. *The connection induced by the m-plücker embedding coincides with the intrinsic one and the map*

$$B(t_1,t_2) : \mathrm{T}_p\mathrm{Gr}^0_{\mathbb{K}}(k,V) \times \mathrm{T}_p\mathrm{Gr}^0_{\mathbb{K}}(k,V) \to \big(E^m \mathrm{T}_p\mathrm{Gr}^0_{\mathbb{K}}(k,V)\big)^\perp$$

is the second fundamental form of the embedding.

Proof. Let $p \in \mathrm{Gr}^0_{\mathbb{K}}(k,V)$ and let $t \in \mathrm{Lin}_{\mathbb{K}}(p,p^\perp) \subset \mathrm{Lin}_{\mathbb{K}}(V,V)$. First, we need to establish some auxiliary formulae.

Denote $g(\varepsilon) := 1 + \varepsilon t$. Since $g^{-1}(\varepsilon)g(\varepsilon) = 1$ for small ε, $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g(\varepsilon) = t$ and $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\big(g^{-1}(\varepsilon)\big)^* = -t^*$. The projectors

$$\pi'(\varepsilon) := \pi'\big[\textstyle\bigwedge^m g(\varepsilon)p\big],\quad \pi(\varepsilon) := \pi\big[\textstyle\bigwedge^m g(\varepsilon)p\big]$$

satisfy

$$\pi'(\varepsilon)\big(g(\varepsilon)p_1 \wedge \dots \wedge g(\varepsilon)p_m\big) = g(\varepsilon)p_1 \wedge \dots \wedge g(\varepsilon)p_m,$$

$$\pi(\varepsilon)\Big(\big(g^{-1}(\varepsilon)\big)^* q \wedge g(\varepsilon)p_2 \wedge \dots \wedge g(\varepsilon)p_m\Big) = \big(g^{-1}(\varepsilon)\big)^* q \wedge g(\varepsilon)p_2 \wedge \dots \wedge g(\varepsilon)p_m$$

for all $q \in p^\perp$ and $p_1,\dots,p_m \in p$ because $\big(g^{-1}(\varepsilon)\big)^* q \in \big(g(\varepsilon)p\big)^\perp$. Taking derivatives, we obtain

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi'(\varepsilon)(p_1 \wedge \dots \wedge p_m) + \pi'\big[\textstyle\bigwedge^m p\big]\displaystyle\sum_{i=1}^m p_1 \wedge \dots \wedge tp_i \wedge \dots \wedge p_m = \sum_{i=1}^m p_1 \wedge \dots \wedge tp_i \wedge \dots \wedge p_m$$

and

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi(\varepsilon)(q \wedge p_2 \wedge \dots \wedge p_m) + \pi\big[\textstyle\bigwedge^m p\big]\displaystyle\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\big(g^{-1}(\varepsilon)\big)^* q \wedge g(\varepsilon)p_2 \wedge \dots \wedge g(\varepsilon)p_m\Big) =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\big(g^{-1}(\varepsilon)\big)^* q \wedge g(\varepsilon)p_2 \wedge \cdots \wedge g(\varepsilon)p_m\Big).$$

From $t^*q \in p$ and from

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\big(g^{-1}(\varepsilon)\big)^* q \wedge g(\varepsilon)p_2 \wedge \cdots \wedge g(\varepsilon)p_m\Big) = -t^*q \wedge p_2 \wedge \cdots \wedge p_m + \sum_{i=2}^{m} q \wedge p_2 \wedge \cdots \wedge tp_i \wedge \cdots \wedge p_m,$$

we conclude that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi'(\varepsilon)(p_1 \wedge \cdots \wedge p_m) = \sum_{i=1}^{m} p_1 \wedge \cdots \wedge tp_i \wedge \cdots \wedge p_m, \tag{2.7}$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi(\varepsilon)(q \wedge p_2 \wedge \cdots \wedge p_m) = -t^*q \wedge p_2 \wedge \cdots \wedge p_m. \tag{2.8}$$

Let X be a lifted field over a neighbourhood of p. Denote $X(\varepsilon) := X\big(g(\varepsilon)p\big)$ and $s := X(0) = X(p)$. Define

$$E(\varepsilon) : \textstyle\bigwedge^m V \to \bigwedge^m V, \qquad v_1 \wedge \cdots \wedge v_m \mapsto \displaystyle\sum_{i=1}^{m} v_1 \wedge \cdots \wedge X(\varepsilon)v_i \wedge \cdots \wedge v_m.$$

Clearly, $E^m X(\varepsilon) = \pi(\varepsilon)E(\varepsilon)\pi'(\varepsilon)$. We conclude from (2.7), (2.8), and $st = 0$ that

$$\nabla_{E^m t} E^m X(p_1 \wedge \cdots \wedge p_m) = \Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi(\varepsilon)E(\varepsilon)\pi'(\varepsilon)\Big)_{\bigwedge^m p} p_1 \wedge \cdots \wedge p_m =$$

$$= \pi(0)\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi(\varepsilon)E(0) + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}E(\varepsilon) + E(0)\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi'(\varepsilon)\Big)p_1 \wedge \cdots \wedge p_m =$$

$$= \pi(0)\Big(-\sum_{i=1}^{m} p_1 \wedge \cdots \wedge t^*sp_i \wedge \cdots \wedge p_m + \sum_{i=1}^{m} p_1 \wedge \cdots \wedge \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}X(\varepsilon)p_i \wedge \cdots \wedge p_m +$$

$$+\sum_{i\neq j} p_1 \wedge \cdots \wedge sp_i \wedge \cdots \wedge tp_j \wedge \cdots \wedge p_m\Big) = \sum_{i=1}^{m} p_1 \wedge \cdots \wedge \pi[p]\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}X(\varepsilon)p_i \wedge \cdots \wedge p_m + B(s,t)p_1 \wedge \cdots \wedge p_m$$

(in the terms of the connection in $\mathrm{Gr}^0_{\mathbb{K}}\left(\binom{k}{m}, \bigwedge^m V\right)$). In other words,

$$\nabla_{E^m t} E^m X = E^m \nabla_t X + B\big(X(p), t\big).$$

The first term is tangent to the image of the m-plücker embedding and the second one is orthogonal to it ∎

2.9. Corollary. *The intrinsic connection is hermitian* (*pseudo-riemannian*)*.*

Proof. Taking $m = k$, the fact follows from Propositions 2.5, 2.6, and [AGr, Proposition 4.3] ∎

2.10. Corollary. *Let $p \in \mathrm{Gr}^0_{\mathbb{K}}(k,V)$ and let $t, t_1, t_2 : p \to p^\perp$ be tangent vectors to $\mathrm{Gr}^0_{\mathbb{K}}(k,V)$ at p. The curvature tensor is given by*

$$R(t_1,t_2)t = tt_1^*t_2 + t_2t_1^*t - tt_2^*t_1 - t_1t_2^*t.$$

Proof. Since the above formula provides the curvature tensor in the projective case [AGr, Subsection 4.4], it suffices to show that the curvature tensors in $\mathrm{Gr}^0_{\mathbb{K}}(k,V)$ and in $\mathrm{Gr}^0_{\mathbb{K}}\big(\binom{k}{m},\bigwedge^m V\big)$ given by this formula satisfy the gauss equation (see [KoN, Proposition VII.4.1]) related to the embedding E^m.

Let $t,t_1,t_2:p\to p^\perp$ be tangent vectors. Then, by Lemma 2.3,

$$E^m t(E^m t_1)^*E^m t_2(p_1\wedge\dots\wedge p_m)=E^m t\sum_{i=1}^m p_1\wedge\dots\wedge t_1^*t_2p_i\wedge\dots\wedge p_m=$$

$$=\sum_{i\neq j}p_1\wedge\dots\wedge tp_i\wedge\dots\wedge t_1^*t_2p_j\wedge\dots\wedge p_m+\sum_{i=1}^m p_1\wedge\dots\wedge tt_1^*t_2p_i\wedge\dots\wedge p_m.$$

for all $p_1,\dots,p_m\in p$. The last sum is exactly $E^m(tt_1^*t_2)(p_1\wedge\dots\wedge p_m)$. Hence,

$$\big(E^m t(E^m t_1)^*E^m t_2-E^m(tt_1^*t_2)\big)(p_1\wedge\dots\wedge p_m)=\sum_{i\neq j}p_1\wedge\dots\wedge tp_i\wedge\dots\wedge t_1^*t_2p_j\wedge\dots\wedge p_m=B(t,t_1^*t_2).$$

Therefore, the gauss equation takes the form[2]

$$\big\langle E^m w,B(t,t_1^*t_2)+B(t_2,t_1^*t)-B(t,t_2^*t_1)-B(t_1,t_2^*t)\big\rangle=\big\langle B(t_1,w),B(t_2,t)\big\rangle-\big\langle B(t_2,w),B(t_1,t)\big\rangle,$$

where $w:p\to p^\perp$. So, it suffices to show that

$$(E^m w)^*B(t,t_1^*t_2)+(E^m w)^*B(t_2,t_1^*t)=\big(B(t_1,w)\big)^*B(t_2,t),$$

$$(E^m w)^*B(t,t_2^*t_1)+(E^m w)^*B(t_1,t_2^*t)=\big(B(t_2,w)\big)^*B(t_1,t).$$

We prove only the first identity. By Lemma 2.3,

$$(E^m w)^*B(t,t_1^*t_2)(p_1\wedge\dots\wedge p_m)=\sum_{i\neq j}p_1\wedge\dots\wedge w^*tp_i\wedge\dots\wedge t_1^*t_2p_j\wedge\dots\wedge p_m,$$

$$(E^m w)^*B(t_2,t_1^*t)(p_1\wedge\dots\wedge p_m)=\sum_{i\neq j}p_1\wedge\dots\wedge w^*t_2p_i\wedge\dots\wedge t_1^*tp_j\wedge\dots\wedge p_m,$$

and

$$(B(t_1,w))^*B(t_2,t)(p_1\wedge\dots\wedge p_m)=\big(B(t_1,w)\big)^*\sum_{i\neq j}p_1\wedge\dots\wedge t_2p_i\wedge\dots\wedge tp_j\wedge\dots\wedge p_m=$$

$$=\sum_{i\neq j}p_1\wedge\dots\wedge t_1^*t_2p_i\wedge\dots\wedge w^*tp_j\wedge\dots\wedge p_m+\sum_{i\neq j}p_1\wedge\dots\wedge w^*t_2p_i\wedge\dots\wedge t_1^*tp_j\wedge\dots\wedge p_m\ \blacksquare$$

2.11. Corollary (compare to [BoN, Assertions 1–2])**.** *The m-plücker embedding is minimal.*

Proof. Let $e_1,\dots,e_k$ and $f_1,\dots,f_{n-k}$ be orthonormal bases in p and $p^\perp$. We define $t_{ij}e_j:=f_i$ and $t_{ij}e_m:=0$ if $m\neq j$, getting in this way an orthonormal basis in the tangent space at p. As is easy to see, $B(t_{ij},t_{ij})=0$. It remains to apply [dCa, Definition 2.10] $\blacksquare$

[2]Strictly speaking, we should take the (pseudo-)riemannian metric in the equality. However, the gauss equation turns out to be valid in a sense which is even stronger than the hermitian one.

2.12. Corollary (compare to [BoN, pp. 53 and 63]). *$\mathrm{Gr}^0_{\mathbb K}(k,V)$ is einstein. The corresponding constant is $n-2$ in the case of $\mathbb K=\mathbb R$ and $2n$ in the case of $\mathbb K=\mathbb C$, where $n=\dim_{\mathbb K}V$.*

Proof. We use the following elementary fact: Let $T:V\to V$ be an $\mathbb R$-linear map. Then $\mathrm{tr}_{\mathbb R}T=2\operatorname{Re}\mathrm{tr}_{\mathbb C}T$ if T is $\mathbb C$-linear and $\mathrm{tr}_{\mathbb R}T=0$ if T is $\mathbb C$-antilinear.

The ricci tensor is given by $\mathrm{ricci}(t_1,t):=\mathrm{tr}\big(t_2\mapsto R(t_1,t_2)t\big)$, where $t,t_1,t_2:p\to p^\perp$. Considering each term of the curvature tensor in Corollary 2.10, it is easy to see that

$$\mathrm{tr}(t_2\mapsto tt_1^*t_2)=k\,\mathrm{tr}(tt_1^*)=k\,\mathrm{tr}(t^*t_1),\quad \mathrm{tr}(t_2\mapsto t_2t_1^*t)=(n-k)\,\mathrm{tr}(t_1^*t)=(n-k)\,\mathrm{tr}(t^*t_1),$$
$$\mathrm{tr}(t_2\mapsto tt_2^*t_1)=\mathrm{tr}(t_2\mapsto t_1t_2^*t)=\mathrm{tr}(t^*t_1)$$

in the case of $\mathbb K=\mathbb R$ and that

$$\mathrm{tr}_{\mathbb C}(t_2\mapsto tt_1^*t_2)=k\,\mathrm{tr}(tt_1^*),\quad \mathrm{tr}_{\mathbb C}(t_2\mapsto t_2t_1^*t)=(n-k)\,\mathrm{tr}(t_1^*t),$$
$$\mathrm{tr}_{\mathbb R}(t_2\mapsto tt_1^*t_2)=2k\operatorname{Re}\mathrm{tr}(t^*t_1),\quad \mathrm{tr}_{\mathbb R}(t_2\mapsto t_2t_1^*t)=2(n-k)\operatorname{Re}\mathrm{tr}(t^*t_1),$$
$$\mathrm{tr}_{\mathbb R}(t_2\mapsto tt_2^*t_1)=\mathrm{tr}_{\mathbb R}(t_2\mapsto t_1t_2^*t)=0$$

in the case of $\mathbb K=\mathbb C$ ■

2.13. Generic geodesics. Let $p\in\mathrm{Gr}^0_{\mathbb K}(k,V)$ and let $t\in\mathrm{Lin}_{\mathbb K}(p,p^\perp)\subset\mathrm{Lin}_{\mathbb K}(V,V)$ be a tangent vector at p. Assume that $t^*t:p\to p$ has no isotropic eigenvectors. We are going to describe the geodesic determined by t.

Since the map $t^*t:p\to p$ is self-adjoint and has no isotropic eigenvectors, there exists an orthonormal basis $p_1,\dots,p_k$ in p formed by eigenvectors of t^*t and the corresponding eigenvalues $\lambda_1,\dots,\lambda_k$ are real. Put $W_j:=\mathbb Rp_j+\mathbb Rtp_j$. The W_j's are pairwise orthogonal because the tp_j's are pairwise orthogonal. Being restricted to W_j, the form is real and does not vanish. So, W_j provides a geodesic $\mathrm G_j\subset\mathbb P_{\mathbb K}V$ if $tp_j\ne0$. By [AGr, Lemma 2.1], $\mathrm G_j$ is respectively spherical, hyperbolic, or euclidean exactly when $\lambda_j>0$, $\lambda_j<0$, or $\lambda_j=0$. (If $tp_j=0$, $\mathrm G_j$ is a single point in $\mathbb P_{\mathbb K}V$.)

Let t_j be the tangent vector to $\mathrm G_j$ at p_j given by $t_j:p_j\mapsto tp_j$. Every geodesic $\mathrm G_j$ admits a local *uniformly* parameterized lift $p_j(s)$ to V with respect to t_j. This means that the tangent vector $p_j(s)\mapsto\dot p_j(s)$ at $p_j(s)$ is the parallel displacement of t_j from $p_j(0)=p_j$ to $p_j(s)$ along $\mathrm G_j$ (in particular, $\dot p_j(s)\in p_j(s)^\perp\cap W_j$) and that $\langle p_j(s),p_j(s)\rangle$ is constant in s. If $\mathrm G_j$ is not euclidean, such a parameterization is readily obtainable from those in [AGr, Subsection 3.2]. In the euclidean case, $p_j(s):=p_j+stp_j$ is the desired parameterization [AGr, Corollary 5.9]. Note that $\ddot p_j(s)\in\mathbb Rp_j(s)$. This is obvious in the euclidean case and is otherwise implied by the fact that $\langle\dot p_j(s),\dot p_j(s)\rangle$ is constant and $\ddot p_j(s)\in W_j$.

As in [AGoG, Section 2], we fix a k-dimensional $\mathbb K$-vector space P. Let $b_1,\dots,b_k\in P$ be a basis and let $p(s):P\to V$ be the linear map given by the rule $p(s):b_j\mapsto p_j(s)$.

2.13.1. Lemma. *The curve $\mathrm G:s\mapsto p(s)$ is a geodesic in $\mathrm{Gr}^0_{\mathbb K}(k,V)$ and t is its tangent vector at p.*

Proof. The tangent vector to $\mathrm G$ at $p(s)$ is given by the linear map $t(s)\in\mathrm{Lin}_{\mathbb K}\big(p(s),p(s)^\perp\big)\subset\mathrm{Lin}_{\mathbb K}(V,V)$, $t(s):p_j(s)\mapsto\dot p_j(s)$, because $\dot p_j(s)\in p(s)^\perp$ and the W_j's are pairwise orthogonal.

In the definition of ∇, taking the derivative of $X\big(c(\varepsilon)\big)$ at $\varepsilon=0$, where $c(\varepsilon):=(1+\varepsilon t)p$, amounts to taking the derivative of $X\big(p(s)\big)$ at s because $\dot c(0)=\dot p(s)$. Therefore, $\nabla_{\dot{\mathrm G}(s)}\dot{\mathrm G}(s)=\pi\big[p(s)\big]\dot t(s)\pi'\big[p(s)\big]$. Taking the derivative of $t(s)p_j(s)=\dot p_j(s)$, we obtain $\dot t(s)p_j(s)+t(s)\dot p_j(s)=\ddot p_j(s)$. Since $t(s)\big(p(s)^\perp\big)=0$ and $\dot p_j(s)\in p(s)^\perp$, we have $\pi\big[p(s)\big]\dot t(s)p_j(s)=\pi\big[p(s)\big]\ddot p_j(s)=0$ due to $\ddot p_j(s)\in\mathbb Rp_j(s)$ ■

We call $\mathrm G_j$ a *spine* of $\mathrm G$. We may interpret a point $\mathrm G(s)$ as a linear subspace in $\mathbb P_{\mathbb K}V$ spanned by the $p_j(s)'s$. Moving along the geodesic $\mathrm G$ in $\mathrm{Gr}^0_{\mathbb K}(k,V)$ is the same as moving along the spines with velocities given by[3] $\sqrt{|\lambda_j|}$. The equality $tp_j=0$ says that $\mathrm G_j$ is a point fixed during the movement.

[3]Well, involving an euclidean spine is more subtle.

A generic tangent vector t provides a choice of a basis formed by the eigenvectors of t^*t. In other words, if $2k \le n$, the intention of moving in some generic direction automatically chooses a certain reference frame.

2.14. Comments and questions. Many of the above facts admit a form not involving the hermitian metric.

- The first formula displayed in the proof of Proposition 2.5 says that $(E^m t_1)^* E^m t_2 = E^m(t_1^* t_2)$.
- The gauss equation in Corollary 2.10 follows from the much simpler one $(E^m w)^* B(t, t_1^* t_2) + (E^m w)^* B(t_2, t_1^* t) = \big(B(t_1, w)\big)^* B(t_2, t)$.
- The proof of minimality actually does not require the self-adjoint operator S_η from [dCa, Definition 2.10].
- What is the geometrical meaning of the other two symmetrizations of the trilinear product $tt_2^* t_1$?
- What about other functors in place of $\bigwedge^m$?

3. Convexity of some real hyperbolic polyhedra

This section illustrates how grassmannians appear in a typical situation that does not seem to involve them at the first glance. Here we deal with the real hyperbolic geometry $\mathbb{H}^4_{\mathbb{R}}$, that is, with $\mathbb{P}_{\mathbb{R}} V$, where V is an $\mathbb{R}$-vector space and the form has signature $++++-$. (The calculus in what follows may seem a little bit concise. On the other hand, it requires no specific knowledge in the area.)

A known problem on real hyperbolic disc bundles is to find the greatest value of $|e/\chi|$, where e stands for the Euler number of the bundle and χ, for the Euler characteristic of the base closed surface [GLT]. By now, the best value $|e/\chi| = 1/2$ [Kui], [Luo] is obtained via constructing a fundamental polyhedron without faces of codimension > 2 that is strongly convex in the sense that its disjoint faces lie in disjoint totally geodesic hypersurfaces. It is worthwhile trying polyhedra that are convex in the usual sense.

Such a polyhedron can be described in the terms of a finite number of positive points $p_1, \dots, p_n \in \mathbb{P}_{\mathbb{R}} V$. The face F_i is a *segment* in the hyperplane $H_i := p_i^\perp \cap \overline{\mathrm{B}}V$, i.e., the part of H_i between the disjoint planes E_{i-1} and E_i, where $E_i := F_i \cap F_{i+1} = \mathrm{Span}(p_i, p_{i+1})^\perp \cap \overline{\mathrm{B}}V$ for all i (the indices are modulo n). In the terms of the gramian matrix $U(p_1, \dots, p_n) := [u_{ij}]$, $u_{ij} := \langle p_i, p_j\rangle$, assuming that $u_{ii} = 1$, the strong convexity means $|u_{i(i+1)}| < 1 < |u_{ij}|$ for all $j \ne i-1, i, i+1$. In what follows, we obtain a criterion of the usual convexity.

It is convenient to use the following notation:

$$\textbf{(3.1)}\qquad \langle i_1 i_2, j_1 j_2\rangle := \det\begin{pmatrix} u_{i_1 j_1} & u_{i_1 j_2}\\ u_{i_2 j_1} & u_{i_2 j_2}\end{pmatrix}, \qquad \langle i_1 i_2 i_3, j_1 j_2 j_3\rangle := \det\begin{pmatrix} u_{i_1 j_1} & u_{i_1 j_2} & u_{i_1 j_3}\\ u_{i_2 j_1} & u_{i_2 j_2} & u_{i_2 j_3}\\ u_{i_3 j_1} & u_{i_3 j_2} & u_{i_3 j_3}\end{pmatrix}.$$

The fact that $H_i \cap H_{i+1} \ne \varnothing$ can be written as $\big\langle i(i+1), i(i+1)\big\rangle > 0$. The fact that E_{i-1} and E_i are disjoint is equivalent to $\mathrm{Span}(p_{i-1}, p_i, p_{i+1})^\perp \cap \overline{\mathrm{B}}V = \varnothing$, i.e., to $\big\langle (i-1)i(i+1), (i-1)i(i+1)\big\rangle < 0$ by the Sylvester criterion.

3.2. Lemma. *The segment F_i can be described as*

$$F_i = \Big\{x \in H_i \mid \big\langle (i-1)i, i(i+1)\big\rangle \langle x, p_{i-1}\rangle \langle p_{i+1}, x\rangle \ge 0\Big\}.$$

Proof. During the proof, we deal only with the points p_{i-1}, p_i, p_{i+1}. We change these points keeping E_{i-1}, F_i, E_i the same. The expression $\big\langle (i-1)i, i(i+1)\big\rangle$ does not change if we substitute p_{i-1} and p_{i+1} respectively by $p_{i-1} + r_1 p_i$ and $p_{i+1} + r_2 p_i$, $r_1, r_2 \in \mathbb{R}$. Also, $\big\langle (i-1)i, i(i+1)\big\rangle \langle x, p_{i-1}\rangle \langle p_{i+1}, x\rangle$ does not change if we alter the sign of p_{i-1}. So, we can assume that $u_{(i-1)i} = u_{i(i+1)} = 0$, $u_{(i-1)(i-1)} =$

$u_{ii}=u_{(i+1)(i+1)}=1$, and $u_{(i-1)(i+1)}\geq 0$. It follows from $\big\langle(i-1)i(i+1),(i-1)i(i+1)\big\rangle<0$ that $u_{(i-1)(i+1)}>1$. The closed 3-ball H_i is fibred over the hyperbolic geodesic $\mathrm{G}_i:=\mathrm{Span}(p_{i-1},p_{i+1})$ by the closed discs $S_p:=\mathrm{Span}(p,p_i)^\perp\cap\overline{\mathrm{B}}V$ called *slices,* $p\in\mathrm{G}_i\setminus\overline{\mathrm{B}}V$. The end slices E_{i-1} and E_i of F_i correspond to $p=p_{i-1}$ and $p=p_{i+1}$. Since $u_{(i-1)(i+1)}>0$, the segment F_i is formed by the slices S_p with $p=(1-t)p_{i-1}+tp_{i+1}$, $t\in[0,1]$. Note that $\mathrm{Span}(p_{i-1},p_{i+1})=\mathrm{Span}({}^{p_{i+1}}p_{i-1},{}^{p_{i-1}}p_{i+1})$ because $u_{(i-1)(i+1)}>1$.

Let $x\in H_i$. Then $x=w-t_1{}^{p_{i+1}}p_{i-1}+t_2{}^{p_{i-1}}p_{i+1}$ for suitable $w\in\mathrm{Span}(p_{i-1},p_{i+1})^\perp$, $t_1,t_2\in\mathbb{R}$, $t_1\geq 0$. We have

$$\big\langle(i-1)i,i(i+1)\big\rangle\langle x,p_{i-1}\rangle\langle p_{i+1},x\rangle=u_{(i-1)(i+1)}(u_{(i-1)(i+1)}^2-1)^2t_1t_2$$

and $\langle t_2p_{i-1}+t_1p_{i+1},x\rangle=0$. It follows from $x\in\overline{\mathrm{B}}V$ that $t_2p_{i-1}+t_1p_{i+1}\notin\overline{\mathrm{B}}V$ and that $t_1\neq 0$ or $t_2\neq 0$. So, $x\in S_{t_2p_0+t_0p_2}$ and the claim easily follows ■

In the sequel, we frequently use the above decomposition of H_i into slices over the hyperbolic geodesic G_i.

The usual convexity is equivalent to the condition $F_i\cap H_j=\varnothing$ for $j\neq i-1,i,i+1$. We fix i and j and express this condition by considering the following cases:

- $\langle ij,ij\rangle<0$. This implies $H_i\cap H_j=\varnothing$, hence, $F_i\cap H_j=\varnothing$.
- $\langle ij,ij\rangle=0$. First, we require $p_j\neq p_i$ (implied by $F_i\cap H_j=\varnothing$). Under these conditions, the isotropic point $u_{ii}p_j-u_{ji}p_i$ is the only point in $\mathrm{Span}(p_i,p_j)^\perp\cap\overline{\mathrm{B}}V$. By Lemma 3.2, the condition $F_i\cap H_j=\varnothing$ is equivalent to

$$\big\langle ij,(i-1)i\big\rangle\big\langle(i-1)i,i(i+1)\big\rangle\big\langle i(i+1),ij\big\rangle>0 \tag{3.3}$$

It obviously implies that $p_j\neq p_i$.

- $\langle ij,ij\rangle>0$. Define

$$q_1:=\frac{u_{ii}p_{i-1}-u_{(i-1)i}p_i}{\sqrt{u_{ii}\big\langle(i-1)i,(i-1)i\big\rangle}},\quad q_2:=\frac{u_{ii}p_{i+1}-u_{(i+1)i}p_i}{\sqrt{u_{ii}\big\langle(i+1)i,(i+1)i\big\rangle}},\quad q_3:=\frac{u_{ii}p_j-u_{ji}p_i}{\sqrt{u_{ii}\langle ij,ij\rangle}},$$

and $v_{kl}:=\langle q_k,q_l\rangle$. As is easy to see, $q_k\in p_i^\perp$ and $v_{kk}=1$ for all k. The facts that $\mathrm{Span}(q_1,q_2,p_i)=\mathrm{Span}(p_{i-1},p_i,p_{i+1})$ has signature $++-$ and that p_i is positive imply $|v_{12}|>1$. The slices of F_i have the form $S_{q(t)}$, where

$$q(t):=(1-t)q_1+\sigma tq_2,\qquad t\in[0,1],$$

and $\sigma:=\dfrac{v_{12}}{|v_{12}|}$. The condition $F_i\cap H_j=\varnothing$ is equivalent to the requirement that $\mathrm{Span}\big(q(t),q_3\big)$ has signature $+-$ for all $t\in[0,1]$. It can be written as

$$f(t):=t^2\big((v_{13}-\sigma v_{23})^2+2|v_{12}|-2\big)-2t\big(v_{13}^2-\sigma v_{13}v_{23}+|v_{12}|-1\big)+v_{13}^2-1>0$$

by Sylvester's criterion.

Writing $f(t)=t^2a-2tb+c$, we have $a>0$, $f(0)=c=v_{13}^2-1$, and $f(1)=v_{23}^2-1$. The polynomial $f(t)$ attains its minimum at $t=b/a$. Clearly, $f(b/a)>0$ if and only if $ac>b^2$. Hence, the condition $F_i\cap H_j=\varnothing$ is equivalent to $v_{13}^2,v_{23}^2>1$ and $0<b<a\Longrightarrow ac>b^2$. One readily verifies that

$$ac-b^2=1+2v_{12}v_{23}v_{31}-v_{12}^2-v_{23}^2-v_{31}^2=\det U(q_1,q_2,q_3)$$

and that $0<b<a$ is equivalent to

$$v_{13}^2-1>\sigma(v_{13}v_{32}-v_{12}),\qquad v_{23}^2-1>\sigma(v_{13}v_{32}-v_{12}).$$

The inequality $ac>b^2$ is impossible because $\operatorname{Span}(q_1,q_2,q_3)$ contains a negative point belonging to $\mathrm{G}_i=\operatorname{Span}(q_1,q_2)$. Therefore, $F_i\cap H_j=\varnothing$ is equivalent to $v_{13}^2,v_{23}^2>1$ and $v_{13}^2-1\le\sigma(v_{13}v_{32}-v_{12})$ or $v_{23}^2-1\le\sigma(v_{13}v_{32}-v_{12})$. Either of the last two inequalities implies $\sigma(v_{13}v_{32}-v_{12})>0$, that is, $\sigma v_{13}v_{32}>|v_{12}|$, i.e.,

$$v_{12}v_{23}v_{31}>v_{12}^2.\tag{3.4}$$

Clearly, (3.4) implies $\sigma(v_{13}v_{32}-v_{12})>0$. Assuming that (3.4) is true, we can rewrite the condition $v_{13}^2-1\le\sigma(v_{13}v_{32}-v_{12})$ or $v_{23}^2-1\le\sigma(v_{13}v_{32}-v_{12})$ in the form

$$(v_{13}^2-1)^2\le(v_{13}v_{32}-v_{12})^2\qquad\text{or}\qquad(v_{23}^2-1)^2\le(v_{13}v_{32}-v_{12})^2.\tag{3.5}$$

In fact, the meaning of the inequalities $v_{13}^2,v_{23}^2>1$ is that $\operatorname{Span}(p_{i-1},p_i,p_j)$ and $\operatorname{Span}(p_{i+1},p_i,p_j)$ have signature $++-$, that is,

$$\big\langle(i-1)ij,(i-1)ij\big\rangle<0,\qquad\big\langle i(i+1)j,i(i+1)j\big\rangle<0.$$

Under these conditions, (3.5) takes the form

$$\min(v_{13}^2-1,v_{23}^2-1)\le|v_{13}v_{32}-v_{12}|\tag{3.6}$$

By straightforward calculus, we have

$$v_{12}=-\frac{\big\langle(i-1)i,i(i+1)\big\rangle}{\sqrt{\big\langle(i-1)i,(i-1)i\big\rangle\big\langle i(i+1),i(i+1)\big\rangle}},\qquad v_{23}=\frac{\big\langle i(i+1),ij\big\rangle}{\sqrt{\big\langle i(i+1),i(i+1)\big\rangle\langle ij,ij\rangle}},$$

$$v_{13}=-\frac{\big\langle(i-1)i,ij\big\rangle}{\sqrt{\big\langle(i-1)i,(i-1)i\big\rangle\langle ij,ij\rangle}}.$$

Hence, (3.4) takes the form

$$\frac{\big\langle(i-1)i,ij\big\rangle\big\langle ij,i(i+1)\big\rangle}{\big\langle(i-1)i,i(i+1)\big\rangle}>\langle ij,ij\rangle.\tag{3.7}$$

Note that (3.7) is equivalent to (3.3) in the case of $\langle ij,ij\rangle=0$ because $\big\langle(i-1)i,i(i+1)\big\rangle=0$ would imply $v_{12}=0$, that is, $\langle{}^{p_i}p_{i-1},{}^{p_i}p_{i+1}\rangle=0$, contradicting $E_{i-1}\cap E_i=\varnothing$.

Since

$$v_{13}v_{32}-v_{12}=\frac{\big\langle(i-1)i,i(i+1)\big\rangle\langle ij,ij\rangle-\big\langle(i-1)i,ij\big\rangle\big\langle i(i+1),ij\big\rangle}{\langle ij,ij\rangle\sqrt{\big\langle(i-1)i,(i-1)i\big\rangle\big\langle i(i+1),i(i+1)\big\rangle}}=$$

$$=\frac{u_{ii}\big\langle(i-1)ij,i(i+1)j\big\rangle}{\langle ij,ij\rangle\sqrt{\big\langle(i-1)i,(i-1)i\big\rangle\big\langle i(i+1),i(i+1)\big\rangle}},$$

$$v_{13}^2-1=\frac{\big\langle(i-1)i,ij\big\rangle\big\langle(i-1)i,ij\big\rangle-\big\langle(i-1)i,(i-1)i\big\rangle\langle ij,ij\rangle}{\big\langle(i-1)i,(i-1)i\big\rangle\langle ij,ij\rangle}=-\frac{u_{ii}\big\langle(i-1)ij,(i-1)ij\big\rangle}{\langle ij,ij\rangle\big\langle(i-1)i,(i-1)i\big\rangle},$$

$$v_{23}^2-1=\frac{\big\langle i(i+1),ij\big\rangle\big\langle i(i+1),ij\big\rangle-\big\langle i(i+1),i(i+1)\big\rangle\langle ij,ij\rangle}{\big\langle i(i+1),i(i+1)\big\rangle\langle ij,ij\rangle}=-\frac{u_{ii}\big\langle i(i+1)j,i(i+1)j\big\rangle}{\langle ij,ij\rangle\big\langle i(i+1),i(i+1)\big\rangle},$$

$u_{ii}>0$, and $\langle ij,ij\rangle>0$, (3.6) takes the form

$$\textbf{(3.8)}\quad \frac{\Big|\big\langle(i-1)ij,i(i+1)j\big\rangle\Big|}{\sqrt{\big\langle(i-1)i,(i-1)i\big\rangle\big\langle i(i+1),i(i+1)\big\rangle}}+\max\left(\frac{\big\langle(i-1)ij,(i-1)ij\big\rangle}{\big\langle(i-1)i,(i-1)i\big\rangle},\frac{\big\langle i(i+1)j,i(i+1)j\big\rangle}{\big\langle i(i+1),i(i+1)\big\rangle}\right)\geq 0.$$

It follows from $E_{i-1}\subset F_i$ and $F_i\cap H_j=\varnothing$ that $E_{i-1}\cap H_j=\varnothing$ for $j\neq i-1,i,i+1$. In other words, $H_{i-1}\cap H_i\cap H_j=\varnothing$, that is, $\big\langle(i-1)ij,(i-1)ij\big\rangle<0$.

Summarizing, we arrive at the

3.9. Criterion of convexity. *The polyhedron formed by segments of hyperplanes given by* $p_1,\dots,p_n\in V$ *is convex* (*hence, simple*) *if and only if the following conditions written in the terms of* (3.1), *where* $u_{ij}:=\langle p_i,p_j\rangle$, *hold* (*the indices are modulo* n):

- *The inequalities* $u_{ii}>0$ *are valid for all* i.
- *The inequalities* $\big\langle(i-1)i,(i-1)i\big\rangle>0$ *and* $\big\langle(i-1)ij,(i-1)ij\big\rangle<0$ *are valid for all* $j\neq i-1,i$.
- *The inequalities* (3.7) *are valid for all* $j\neq i-1,i,i+1$ *such that* $\langle ij,ij\rangle\geq 0$.
- *The inequalities* (3.8) *are valid for all* $j\neq i-1,i,i+1$ *such that* $\langle ij,ij\rangle>0$ ■

Note that

$$\langle i_1i_2,j_1j_2\rangle=\langle g_{i_1i_2},g_{j_1j_2}\rangle,\qquad\langle i_1i_2i_3,j_1j_2j_3\rangle=\langle g_{i_1i_2i_3},g_{j_1j_2j_3}\rangle,$$

where $g_{i_1i_2}:=p_{i_1}\wedge p_{i_2}$ and $g_{i_1i_2i_3}:=p_{i_1}\wedge p_{i_2}\wedge p_{i_3}$ represent respectively $\bigwedge^2\operatorname{Span}(p_{i_1},p_{i_2})\in\mathbb{P}_{\mathbb{R}}\bigwedge^2V$ and $\bigwedge^3\operatorname{Span}(p_{i_1},p_{i_2},p_{i_3})\in\mathbb{P}_{\mathbb{R}}\bigwedge^3V$. So, Criterion 3.9 deals with the usual projective invariants.

4. References

[AGr] S. Anan′in, Carlos H. Grossi, *Coordinate-free Classic Geometries.* I. *Projective Case,* preprint http://arxiv.org/abs/math/0702714

[AGoG] S. Anan′in, Eduardo C. Bento Gonçalves, Carlos H. Grossi, *Coordinate-free Classic Geometries.* II. *Conformal Structure,* in preparation

[BoN] A. A. Borisenko, Yu. A. Nikolaevskii, *Grassmann manifolds and the Grassmann image of submanifolds,* Russian Math. Surveys, No. 46 (1991), 45–94.

[dCa] M. P. do Carmo, *Riemannian Geometry,* Mathematics: Theory & Applications, Birkhäuser, Boston, 1992. vi+300 pp.

[GuK] B. Guilfoyle, W. Klingenberg, *Proof of the Caratheodory Conjecture by Mean Curvature Flow in the Space of Oriented Affine Lines,* preprint http://arxiv.org/abs/0808.0851

[KNo] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry,* Volume II, Interscience Tracts in Pure and Applied Mathematics. Interscience Publishers. John Wiley & Sons, 1969. xv+470 pp.

[Kui] N. H. Kuiper, *Hyperbolic* 4*-manifolds and tessellations,* Inst. Hautes Études Sci. Publ. Math., No. 68 (1988), 47–76.

[Luo] F. Luo, *Constructing conformally flat structures on some Seifert fibred* 3*-manifolds,* Math. Ann. **294** (1992), No. 3, 449–458.

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