EXISTENCE OF HYPERBOLIC MONOPOLES WITH ARBITRARY MASS AT INFINITY

L.M. Sibner* and R.J. Sibner**

*
Polytechnic University
Brooklyn, N.Y. 11201
U.S.A.

**
City University New York
Brooklyn College
Brooklyn, N.Y. 11210
U.S.A.

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

MPI / 95-44

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L.M. SIBNER[1] AND R.J. SIBNER[2]

ABSTRACT. Monopoles on hyperbolic 3-space $\mathbb{H}^3$ arise naturally (by dimensional reduction) from S^1 -invariant instantons on $S^4\backslash S^2$. The hyperbolic monopoles obtained in this way by Atiyah are, however, constrained to have integral mass. The purpose of this article is to show the existence, on $\mathbb{H}^3$, of monopoles with arbitrary mass.

§1. Introduction

Existence theorems for Higgs $SU(2)$ -monopoles (as well as for Yang-Mills instantons) have been obtained in several ways. In the mid 1970's, by making a symmetry ansatz, explicit instanton solutions on $\mathbb{R}^4$ and monopole solutions on $\mathbb{R}^3$ were obtained [t'H, BPST, AHDM, PS]. (Earlier, Dirac [Di] had obtained a $U(1)$ -monopole on $\mathbb{R}^3$.) Later Chakrabarti [C] wrote down the "Prasad-Sommerfield" monopole on hyperbolic 3-space $\mathbb{H}^3$ (see also [B]). Also, Forgacs, Horvath and Palla [FHP1, FHP2] exhibited an instanton on $\mathbb{R}^4\backslash\mathbb{R}^2$ (which did not extend to $\mathbb{R}^4$).

The explicit solutions of the simplest basic type, with charge equal to ± 1, became "building blocks" in the next development. With a procedure that has, by now, become standard, Taubes [JT] constructed monopoles on $\mathbb{R}^3$ (with arbitrary charge) by a "patching" argument - gluing together basic charge one Prasad-Sommerfield monopoles. By modifying this construction, he was able to obtain instantons on $\mathbb{R}^4$, thus verifying the crucial fact that the moduli space of anti-self-dual instantons on (appropriate) 4-manifolds was non-empty (cf. [FU]) and, moreover, obtaining important information about the ends of this moduli space. Other modifications have been used by Floer [F], Ernst [E] and Durenard [Du] in the construction of monopoles on 3-manifolds with a Euclidean end. The basic idea in these constructions is (for example, for monopoles) to construct an "approximate" monopole and then perturb it to a true monopole using some version of the implicit function theorem. This method gives solutions which are (for given charge) absolute minima of the action functional (as are the explicit solutions discussed above).

A third method of showing existence again begins by constructing approximate monopoles (or instantons), but this time a whole loop of such "configurations". One then uses a min-max (i.e. saddle point) argument. (Maximize energy over configurations on the loop, then maximize over homotopic loops.) Here, also, the

1) Research partially supported by NSF Grant DMS-9200576
2) Research partially supported by PSC-CUNY Grants 6-64241 and 6-65363

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX original argument is due to Taubes, who used it to construct non-minimal monopole solutions on $\mathbb{R}^3$ [$T2$]. These are critical points of the Yang-Mills-Higgs functional (with coupling constant zero) and hence solutions of the second order equations, but not solutions of the first order monopole equations which characterize the minima of the functional. In order to settle the "instanton conjecture" concerning the existence of non-minimal instantons on S^4, the authors, together with K. Uhlenbeck [SSU] adapted Taubes construction to obtain, on hyperbolic space $\mathbb{H}^3$, non minimal monopole solutions (with arbitrary mass m). By a construction, due to Manton and Atiyah, which we will review in a moment, these monopoles on H^3 lift to instantons on $S^4 \backslash S^2$. Using the authors codimension two removable singularity theorem [SS1, SS2], when m is an integer, these solutions extend across the "singular set" S^2 to produce, on S^4, a non-minimal critical point of the Yang-Mills functional and hence a solution of the second order Yang-Mills equation but not the first order (anti-) self dual equations.

Before discussing the interplay of these ideas and our specific problem, we recall some basic definitions. In general, a connection on an $su(2)$ vector bundle over a 3-manifold M can be pulled down to an $su(2)$ valued connection 1-form A on M, and gives rise to a covariant derivative $d_A = d + [A, \cdot]$. (The pullback is done via a given trivialization, or "gauge", of the bundle.) A Higgs monopole on M (cf [JT]) is a configuration pair $c = (\Phi, A)$ which satisfies the Bogomoln'yi monopole equations

$$d_A \Phi = *F_A \tag{1.1}$$

where F_A is the curvature of the connection 1-form A, and Φ an $su(2)$ valued function on M. The solutions of the first order equations (1.1) characterize the absolute minima of the Yang-Mills-Higgs action functional

$$\mathcal{YMH}(c) = \frac{1}{2} \int_M |F_A|^2 + |d_A \ \Phi|^2 \tag{1.2}$$

(while, more generally, the critical points are solutions of the second order Euler equations.)

This theory is only interesting if M has an "end" (for example, $M = R^3$ or $\mathbb{H}^3$). Then, natural "boundary conditions" are given by prescribing the mass of the monopole

$$m = \lim_{|x| \to \infty} |\Phi(x)| \tag{1.3}$$

and the magnetic charge

$$k = \frac{1}{4\pi m} \int_M tr(F_A \wedge d_A \Phi) \tag{1.4}$$

On $\mathbb{R}^3$, using scaling techniques, one can assume without loss of generality that $m = 1$. On $\mathbb{H}^3$, even though scaling is not possible, it has often been assumed that

m is an integer. The reason for this is that one can (see [A]), up to conformal equivalence, consider $\mathbb{H}^3$ as $\mathbb{R}^4 \backslash \mathbb{R}^2 (= S^4 \backslash S^2)$ modulo a $U(1)$ action leaving $\mathbb{R}^2$ invariant. Then any $U(1)$ invariant instanton on $\mathbb{R}^4$ produces a "hyperbolic monopole' on $\mathbb{H}^3$ with integral m. (In fact, m is the holonomy around the $\mathbb{R}^2$; of necessity an integer.) Conversely it is precisely the hyperbolic monopoles with integral m which can be lifted to instantons on $\mathbb{R}^4$ [SS1, SS2]. (The holonomy must be integral for the connection to extend.)

On the other hand, from the point of view of the 3-manifold $\mathbb{H}^3$ the restriction to integral m seems clearly artificial. This is supported by the example [FHP1] of an instanton on $\mathbb{R}^4 \backslash \mathbb{R}^2$ which does not extend and also by the results in [SSU] producing non-minimal solutions on $\mathbb{H}^3$ for arbitrary mass. Indeed, on $\mathbb{H}^3$ it makes analytic sense to prescribe the mass as any non-negative real number. Our main result is the following

Theorem. *There exists a smooth configuration $c = (\Phi, A)$ on $\mathbb{H}^3$, having prescribed magnetic charge $k \in \mathbb{Z}$, prescribed mass $m > 0$ at infinity, and satisfying the Bogomoln'yi equations (1.1).*

It should be noted that this result has been obtained for large mass by McAllister[McA]. His approach is different, but related to some of the ideas described above. To understand better this relationship, we briefly outline some developments since [SS1]. In that paper, we introduced the idea of a singular connection and its asymptotic holonomy around a codimension two singular set. Our estimates showed that finite action connections have a well-defined limit holonomy which is constant along the singular set. Moreover, the L^2 connections are classified by the space of flat connections (their asymptotic limits). In the case of $S^4 \backslash S^2$, the limit holonomy corresponds (via Atiyah's construction) to the mass of the corresponding Higgs configuration on $\mathbb{H}^3$, with integral mass corresponding to a connection on $S^4 \backslash S^2$ that extends across the S^2. McAllister considers the $\mathbb{H}^3$ monopole problem by converting it to an instanton problem on $S^4 \backslash S^2$. Here, he uses our result mentioned above about the existence of limit holonomy. His proof depends upon estimates of Räde [R]. He requires that the mass m be large.

To prove the theorem, we follow a program similar to the patching argument developed by Taubes and discussed above. We work directly on the space $\mathbb{H}^3$ and do not require any assumption on the magnitude of m.

In section 2 we construct an approximate solution and in section 3 the perturbation problem is derived and a lower bound estimate is obtained for the linearized equation. The continuity argument which proves the theorem is then described.

We remark again, that this method of proof is, by now, standard and has been exploited successfully in many situations in which the lower bound can be established. However, in our case, the *a* priori bound is false in the usual L^2_k Sobolev spaces over $\mathbb{H}^3$ and, in order to work directly on $\mathbb{H}^3$, one must use weighted Sobolev spaces. The use of weighted Sobolev spaces directly in $\mathbb{H}^3$ and the elimination of any restriction on the mass are our main contributions to the problem.

We note also that our proof works equally well over $\mathbb{R}^3$ and as such, yields a slightly different proof of Taubes original theorem. It is also somewhat different than the recent proof over $\mathbb{R}^3$ given in [E].

§2. The Approximate monopole

As discussed above, the basic (charge $k = 1$) $SU(2)$ m-monopole can be written down explicitly. To obtain an approximate monopole of given charge k, first choose k points $x_1, \dots, x_k$ as centers of charge one monopoles, requiring that $d = \min |x_i - x_j| > 4R$ where R is a constant to be chosen later. Denoting by B^i_ρ a ball of radius ρ about x_i, we can choose geodesic spherical coordinates (r_i, θ_i, χ_i) centered at x_i with $r_i = x - x_i$ and θ_i so choosen that the half rays $\theta = 0$ and $\theta = \pi$ do not intersect the closure of any of the sets $U^k_R = B^k_{2R} \backslash B^k_R$ for $k \neq i$.

Let $c_i = (\Phi_i, A_i)$ be the basic monopole centered at x_i;

$$
\text{(2.1a)} \qquad \begin{cases} \Phi_i = (\alpha \coth \alpha r_i - \coth r_i)\hat{i} \\ A_i = \dfrac{\alpha \sinh r_i}{\sinh \alpha r_i}(d\theta_i \hat{j} + \sin \theta_i d\chi_i \hat{k}) + (1 - \cos \theta_i) d\chi_i \hat{i} \end{cases}
$$

where we have written $\alpha = m + 1$.

In the neighborhood of infinity $N_\infty = \mathbb{H}^3 \setminus \cup_{i=1}^k B^i_R$, we take a $\mathsf{U}(1)$- Dirac monopole $c_\infty = (\Phi_\infty, A_\infty)$ where

$$
\text{(2.1b)} \qquad \begin{aligned} \Phi_\infty &= \{(\alpha - 1) + (1 - \coth r_1) + \cdots + (1 - \coth r_k)\}\,\hat{i} \\ A_\infty &= \{(1 - \cos\theta_1) d\chi_1 = \cdots + (1 - \cos \theta_k) d\chi_k\}\,\hat{i} \end{aligned}
$$

In any system of geodesic polar coordinates (r, θ, χ) the metric is given by

$$
\begin{aligned} ds^2 &= dr^2 + \sinh^2 r d\Omega^2 \\ &= dr^2 + \sinh^2 r d\theta^2 + \sinh^2 r \sin^2\theta d\chi^2 \end{aligned}
$$

so that the volume element is given by $d(\text{vol}) = \sinh^2 r \sin\theta dr d\theta d\chi$. In the metric induced on the cotangent space

$$
\text{(2.2)} \qquad |dr| = 1,\ |d\theta| = (\sinh r)^{-1} \text{ and } |d\chi| = (\sinh r \sin \theta)^{-1}.
$$

While the Higgs action of $c_i, i = 1, \cdots, k$, is finite. the action of c_∞ on $\mathbb{H}^3$ is <u>not</u> finite because of the singular behavior of F_{A_∞} at the points $x_1, \dots, x_k$. However the restriction of F_{A_∞} to N_∞ does have finite action. In the gauge of (2.1a) c_i has a Dirac string singularity along the half ray $\theta_i = \pi$. (Note that $|1 - \cos\theta_i||d\chi_i| \to \infty$ as $\theta_i \to \pi$). However, since the holonomy around the string is integral, the codimension two removable singularity theorem [SS1, SS2] ensures that c_i is gauge equivalent to a smooth configuration. The same is true of c_∞ in N_∞

We emphasize that c_∞ and each of the c_i are solutions of the Bogomoln'yi equations:

$$
\text{(2.3)} \qquad d_A \Phi = *F_A.
$$

More precisely, this is true for c_i on B^i_{2R} and c_∞ on N_∞, so that if we glue them together by a partition of unity $\{\lambda_1, \ldots, \lambda_k, \lambda_\infty\}$ subordinate to the covering of $\mathbb{H}^3$ by N_∞ and the B^i_{2R}, $i = 1, \ldots, k$, $1 \le i \le k$, we obtain an "approximate monopole" $c_0 = (\Phi_0, A_0)$:

$$\tag{2.4} \begin{cases} \Phi_0 = \lambda_\infty \Phi_\infty + \sum_{i=1}^{k} \lambda_i \Phi_i \\ A_0 = \lambda_\infty A_\infty + \sum_{i=1}^{k} \lambda_i A_i . \end{cases}$$

By its construction, c_0 satisfies the monopole equation (2.3) in each B^i_R and in $N_\infty \setminus \cup_{i=1}^k B^i_{2R}$. We need to estimate the deviation of c_0 from a solution in the intersections $U^i_R = B^i_{2R} \backslash B^i_R$. Note that, in the partition of unity construction, at most two λ's can be non-zero simultaneously. In particular, in U^i_R we have $\lambda_i + \lambda_\infty = 1$. Moreover, for $x \in U^i_R$ and $k \neq i$, one has $r_k(x) > d - 2R > 2R$. In U^i_R,

$$\tag{2.5} \begin{aligned} d_{A_0}\Phi_0 - *F_{A_0} = {} & \lambda_\infty(1-\lambda_\infty)\left\{[A_i, \Phi_\infty - \Phi_i] + *[A_i, A_i]\right\} \\ & + d\lambda_\infty(\Phi_\infty - \Phi_i) - *(d\lambda_\infty \wedge (A_\infty - A_i)) \end{aligned}$$

The terms in (2.5) can be estimated, using (2.2), to obtain the pointwise bound in U^i_R:

$$\tag{2.6} |d_{A_0}\Phi_0 - *F_{A_0}| \le K\left(e^{-\alpha r_i} + e^{-2R}\right)$$

This can be done in the subdomain of U^i_R where $0 \le \theta_i < \frac{3\pi}{4}$ in the gauge in which the configuration is represented by (2.1ab). In the overlapping region where $\pi/4 < \theta_i \le \pi$ one should choose a gauge in which the string is given by $\theta_i = 0$. See [JT] for a discussion of inverting strings and also for the fact that the existence of local smoothing gauges implies the existence of a global smoothing gauge. (The estimate (2.6) is gauge invariant.) In computing bounds for each term of (2.5) in the region U^i_R, one finds that it is the term $|A_\infty - A_i|$ that decays most slowly and gives the upper bound (2.6).

We define the weighted spaces L^p_β on q-forms, as the completion of $C^\infty_0(\wedge^q)$ in the norm:

$$\|\omega\|_{p,\beta} = \left(\int_{\mathbb{H}^3} e^{2\beta r}|\omega \wedge *\omega|^{p/2} dV\right)^{1/p} .$$

We assume $\beta < 1$ which ensures that our approximate monopole c_0 has finite weighted action; namely,

$$\mathcal{YMH}_\beta(c_0) = \frac{1}{2}\int_{\mathbb{H}^3} e^{2\beta r}(|F_{A_0}|^2 + |d_{A_0}\Phi_0|^2) dV < \infty .$$

We can now easily show

Proposition 2.7. *If $\beta < m$ and $p \geq 2$, there is a constant $c > 0$ depending on m and p, but not on R, such that*

$$\|d_{A_0}\Phi_0 - *F_{A_0}\|_{p,\beta} \leq Ke^{-cR}$$

Proof. Outside $\cup_{i=1}^k U_R^i$, $d_{A_0}\Phi_0 - *F_{A_0}$ vanishes. Using (2.6) in U_R^i, we see that

$$\begin{aligned}\|d_{A_0}\Phi_0 - *F_{A_0}\|_{p,\beta} &\leq K\left\{\left(\int_R^{2R} e^{(-\alpha p+2\beta+2)r_i}dr_i\right)^{1/p} + e^{-2R}\left(\int_R^{2R} \sinh^2 r_i dr_i\right)^{1/p}\right\} \\ &\leq K\left\{e^{-2R+2R/p} + e^{(-2(\alpha-1)+2\beta)R/2}\right\}.\end{aligned}$$

The constant c will have the right sign if $-m+\beta < 0$. This proves the proposition.

§3. The Perturbation Problem

We now look for solutions of (1.1) of the form $c = c_0 + \zeta = (\Phi_0 + \varphi, A_0 + a)$ with c_0 the approximate monopole of §2. Following Floer [F2], we expand (1.1) in a Taylor expansion to obtain

$$Lc = d_A\Phi - *F_A = Lc_0 + D_L\zeta + \sigma(\zeta,\zeta). \tag{3.1}$$

Here, D_L is the linearization of L at c_0; i.e.,

$$D_L\zeta = D_L(\varphi, a) = - * d_{A_0}a + d_{A_0}\varphi - [\Phi_0, a]. \tag{3.2}$$

The quadratic term $\sigma(\zeta,\zeta)$ is defined by a bilinear map of the tensor bundle $\wedge^0 \oplus \wedge^1$ into $\wedge^1$.

We work in the weighted Sobolev spaces described in §2. Throughout this section, the mass is a fixed arbitrary positive number and the constant β appearing in the weight factor satisfies $0 < \beta < m$. In the weighted Sobolev space, the adjoint of the operator d_{A_0} on forms $\omega \in \wedge^q$ is :

$$d_{A_0}^\dagger\omega = e^{-2\beta r}d_{A_0}^*(e^{2\beta r}\omega),$$

where $d_{A_0}^*$ is the ordinary L^2 adjoint.

For the sake of ellipticity, we must add a "slice" condition:

$$D_S\zeta = D_S(\varphi, a) = d_{A_0}^\dagger a - [\Phi_0, \varphi] = 0. \tag{3.3}$$

This gives an elliptic operator on pairs

$$\delta = (D_S, D_L) : \wedge^0 \oplus \wedge^1 \to \wedge^0 \oplus \wedge^1$$

defined by

$$
\begin{aligned}
\delta\zeta &= (\delta_{A_0} - ad\Phi_0)(\varphi, a) \\
&:= (d^{\dagger}_{A_0} a, d_{A_0}\varphi - *d_{A_0} a) - ([\Phi_0, \varphi], [\Phi_0, a])\,.
\end{aligned}
\tag{3.4}
$$

A computation shows that the L^2_β adjoint of δ_{A_0} is:

$$
\begin{aligned}
\delta^{\dagger}_{A_0}\eta &= \delta^{\dagger}_{A_0}(\psi, b) \\
&:= \left(d^{\dagger}_{A_0} b, d_{A_0}\psi - *e^{-2\beta r} d_{A_0}(e^{2\beta r} b)\right)
\end{aligned}
\tag{3.5}
$$

It follows that

$$
\delta_{A_0}\delta^{\dagger}_{A_0} = (\Delta_{A_0}\psi, \Delta_{A_0} b) + T_{A_0}\eta = \Delta_{A_0}\eta + T_{A_0}\eta \tag{3.6}
$$

where

$$
T_{A_0}\eta = \left(*[F_{A_0}, b], *[F_{A_0}, \psi] - 2\beta\left\{*(dr \wedge *d_{A_0} b) + d^{*}_{A_0}(b \wedge dr)\right\}\right). \tag{3.7}
$$

Since $ad\Phi_0$ is skew adjoint,

$$
\delta^{\dagger} = \delta^{\dagger}_{A_0} + ad\Phi_0. \tag{3.8}
$$

To solve (1.1), we want to find solutions ζ of

$$
\delta\zeta + \zeta\#\zeta = G_0 \tag{3.9}
$$

where $\zeta\#\zeta = (0, \sigma(\zeta,\zeta))$ and $G_0 = (0, -Lc_0)$ is sufficiently small in appropriate norms.

As in [FU], [T1], we look for a solution perpendicular to the kernel of δ by setting $\zeta = \delta^{\dagger}\eta$ and solving for $\eta = (\psi, b), \psi \in \wedge^0$ and $b \in \wedge^1$,

$$
\delta\delta^{\dagger}\eta + \delta^{\dagger}\eta\#\delta^{\dagger}\eta = G_0. \tag{3.10}
$$

We next define the Hilbert space in which this problem will be solved. First, extend the L^2_β norm to pairs in $\wedge^0 \oplus \wedge^1$. Now, let ∇_μ denote the covariant derivative with respect to a connection μ. (In practice, μ will either be A_0 or the zero connection.) Then, take $\mathcal{H}_\mu$ to be the completion of C_0^∞ pairs with respect to the norm

$$\|\eta\|_{\mathcal{H}_\mu} = \|\eta\|_{2,\beta} + \| \nabla_\mu \eta\|_{2,\beta} + \| \nabla^2_\mu \eta\|_{2,\beta}. \tag{3.11}$$

Note that the norms, $\|\eta\|_{\mathcal{H}_{A_0}}$ and $\|\eta\|_{\mathcal{H}_0}$ are equivalent and that $\mathcal{H}_0 = L^{2,2}_\beta(\mathbb{H}^3)$.

Our objective is to solve (3.10) for $\eta \in \mathcal{H}_{A_0}$. We will use a continuity method to solve, for $0 \leq t \leq 1$,

$$\mathcal{L}_t\eta_t = \delta\delta^\dagger\eta_t + \delta^\dagger\eta_t\#\delta^\dagger\eta_t = tG_0 \tag{3.12}$$

for $\eta_t \in \mathcal{H}_{A_0}$.

The main step in the proof is to obtain an L^2-lower bound for $\delta\delta^\dagger$. It is here that the weighted norms are required. As shown in Donnelly [D], the scalar Laplacian Δ_1 is not invertible on L^2 one forms over $\mathbb{H}^3$. In fact, the spectrum, $\sigma(\Delta_1)$ consists only of essential spectrum and is the interval $[0, \infty)$.

The weight factor, $e^{2\beta r}$, which we have introduced in the norm, shifts the spectrum of the Laplacian to the right. Our operator $\delta\delta^\dagger$ then differs from the weighted Laplacian by the addition of a first order partial differential operator. A close examination of this operator reveals that it consists of two kinds of contributions. The first are non-negative and hence, do not decrease the spectrum. The second are relatively compact perturbations, so that an extension of the theorem of Weyl [RS] then shows that the essential spectrum is unchanged. Hence the spectrum lies in a positive closed subinterval of R. Note that for all operators considered, $\sigma_{ess} = \sigma$, since there is no point spectrum.

The idea of the proof is first to determine the spectra of the scalar versions of our operators and show a positive lower bound. Here, we use a modification of Donnelly's approach which incorporates the weight factor. We will then compare the Lie algebra valued operators with the scalar operators.

To this end, let $\Delta_p = d^\dagger d + dd^\dagger$ ($p = 0$ or 1) denote the self-adjoint Laplacian on scalar p-forms in $L^2_\beta(\mathbb{H}^3)$, with the domain of the operator $D(\Delta_p) = L^{2,2}_\beta(\mathbb{H}^3)$. Also, let $\delta_0\delta_0^\dagger$ be the Floer operator (3.6) at (0,0) acting on scalar pairs. Then,

Proposition 3.13.

(a) $\sigma(\Delta_0) = [(1+\beta)^2, \infty)$
(b) $\sigma(\Delta_1) = [\beta^2, \infty)$
(c) $\sigma(\delta_0\delta_0^\dagger) = [\beta^2, \infty)$.

The proof proceeds by showing that the Laplacian and the Floer operator can be decomposed, using separation of variables, as a sum of ordinary differential operators which are unitarily equivalent, up to compact perturbation, to multiplication operators on $L^2(R^+, dx)$.

We make extensive use of the following proposition which tells us that, as long as the coefficients tend to zero at infinity, a smooth first order operator C is relatively compact with respect to a self-adjoint second order elliptic operator L; its addition to L does not change the essential spectrum. We state this in the form in which it will be applied.

Proposition 3.14. *Let L be an elliptic, second order, self-adjoint operator on $L^2_\beta(M)$, where M is either the non-negative reals R^+, or $\mathbb{H}^3$. Assume that the domain of L is $\mathcal{H}_\mu = L^{2,2}_{\beta,\mu}$. Let $C = \Sigma a_i \frac{\partial}{\partial x_i} + b$ where the coefficients are smooth functions and $\tau(r) = \max_{|x|\leq r}(\mid a_i \mid, \mid b \mid)$. If $\tau(r)$ tends to zero as r tends to infinity then $\sigma_{ess}(L) = \sigma_{ess}(L+C)$.*

Proof. For some (and hence every) z in the resolvent of L, $R = (L - zI)^{-1}$ is a bounded operator from L^2_β to $\mathcal{H}_\mu$ and, hence,

$$\|Rf\|_{\mathcal{H}_\mu} \leq k\|f\|_{L^2_\beta}.$$

Choose an exhaustion $\{B_n\}$ of M and cutoff functions u_n with $\operatorname{supp} u_n \subset B_n$, $u_n \equiv 1$ on B_{n-1} and $\mid \nabla u_n \mid \to 0$ as $n \to \infty$. By Rellich's lemma, $D_n = u_n CR$ is compact on $L^2_\beta(M)$.

Claim. C is relatively compact with respect to L; i.e., $D = CR$ is compact on $L^2_\beta(M)$. This follows from the inequality:

$$\begin{aligned}
\|(D - D_n)f\|^2_{L^2_\beta(M)} &\leq \|(1-u_n)CRf\|^2_{2,\beta} \\
&\leq \int_{M\setminus B_{n-1}} \mid (1-u_n)CRf \mid^2 e^{2\beta r} dvol \\
&\leq \tau(r_n)\|Rf\|^2_{L^{1,2}_\beta} \\
&\leq \tau(r_n)\|f\|^2_{L^2_\beta(M)},
\end{aligned}$$

which shows that D_n converges to D in norm.

The result now follows from

Theorem. (cf. [RS] Corollary 2, IV p. 113) *If L is self-adjoint and $\exists z \in \mathbb{C}$ such that $C(L-zI)^{-1}$ is compact, then $\sigma_{ess}(L) = \sigma_{ess}(L+C)$.*

We shall also make frequent use of the

Raleigh Quotient Theorem. (cf. [Da] Theorem 4.3.1, p. 78) *If L is self-adjoint, then $(Lf, f) \geq c\|f\|^2$ for all $f \in D(L)$, if and only if $\sigma(L) \subseteq [c, \infty)$.*

The prototype of the ordinary differential operators which arise in the Donnelly decomposition is

$$\mathcal{D}f = -\frac{d^2 f}{dx^2} - 2\gamma\frac{df}{dx} + c(x)f \tag{3.15}$$

where $c(x)$ is rapidly decaying, $\gamma > 0$ is constant, and $f \in L^2(R^+, e^{2\gamma x}dx)$.

Lemma 3.16. *Up to compact perturbation, $\mathcal{D}$ as defined in (3.15) is unitarily equivalent to the multiplication operator, $(\mathcal{M}f)(x) = (x^2+\gamma^2)f(x)$ acting on $L^2(R^+, dx)$. Therefore, $\sigma_{ess}(\mathcal{D}) = \sigma(\mathcal{D}) = [\gamma^2, \infty)$.*

To prove this, recall that the change of dependent variable, $f = e^{-\gamma x}k = Uk$, defines a unitary transformation U from $L^2(R^+, dx)$ to $L^2(R^+, e^{2\gamma x}dx)$ under which

$$\mathcal{D}_1 k := (U^{-1}\mathcal{D}U)k = -\frac{d^2 k}{dx^2} + (\gamma^2 + c)k,$$

so that $\mathcal{D}$ is unitarily equivalent to $\mathcal{D}_1$ acting on ordinary $L^2(R^+, dx)$. From proposition 3.14, $\mathcal{D}_1$ has the same essential spectrum as

$$\mathcal{D}_2 k = -\frac{d^2 k}{dx^2} + \gamma^2 k$$

and, by Fourier transformation, $\mathcal{D}_2$ is unitarily equivalent to the multiplication operator $\mathcal{M}$, in the lemma. The conclusion about the spectrum is immediate.

We next turn to the proof of Proposition 3.13. We use d_s and d_s^* to denote exterior differentation and its L^2 adjoint on S^2. Let Δ_s denote the Laplacian on S^2. For notational simplicity, we write $g = \sinh r$ and $w = e^{2\beta r}$. Sometimes, for clarity, we use the subscript 0 or 1 on the Laplacian to distinguish the domain as functions or 1-forms.

On $\mathbb{H}^3$, the formulas for the weighted Laplacian are: for $\varphi \in \wedge^0$,

$$\Delta_0\varphi = g^{-2}\Delta_s\varphi - g^{-2}w^{-1}\frac{\partial}{\partial r}\left(g^2 w\frac{\partial\varphi}{\partial r}\right)$$

and for $a = a_1 + a_2 dr \in \wedge^1$,

$$\Delta_1 a = g^{-2}\Delta_s a - w^{-1}\frac{\partial}{\partial r}\left(w\frac{\partial a_1}{\partial r}\right) - \frac{\partial}{\partial r}\left(g^{-2}w^{-1}\frac{\partial}{\partial r}(g^2 w a_2)\right)\wedge dr$$
$$- 2g^{-1}\frac{\partial g}{\partial r}(d_s a_2 + g^{-2}d_s^* a_1 \wedge dr).$$

To prove the proposition, we separate variables and expand any p-form in eigen-expansions on S^2. Then, every $\varphi \in \wedge^0$ is a sum of terms of the form $h_0\tau_0$ (with τ_0 an eigenfunction on S^2). Using a Hodge decomposition, one sees that every 1-form on $\mathbb{H}^3$ is a sum of three terms (corresponding to the eigenvalue λ of Δ_s) of the form:

$$h_1(r)\tau_1 + h_2(r)\tau_2 dr + (h_3(r)d_s\tau_3 + h_4(r)\tau_3 dr) \tag{3.17}$$

where τ_1 is a co-closed eigen 1-form, and τ_2 and τ_3 are eigenfunctions on S^2 (with τ_2 = constant corresponding to $\lambda = 0$ and τ_3 occuring only if $\lambda \neq 0$). This decomposition into three types is orthogonal and is preserved both by the Laplacian and the Floer operator.

Note that the forms under consideration will be in $L^2_\beta(\mathbb{H}^3)$ if and only if

$$\|h_i\|^2_{2,\gamma_i} = \int_0^\infty h_i^2(r)\gamma_i(r)dr < \infty \qquad i = 0,1,2,3,4 \tag{3.18}$$

where $\gamma_0 = \gamma_2 = \gamma_4 = g^2 w$ and $\gamma_1 = \gamma_3 = w$.

In this context, the Laplacian defines ordinary differential operators on the spaces in (3.18) as follows:

(i) $(\mathcal{D}_0 h_0)\tau_0 = \Delta_0(h_0\tau_0) = \left(-g^{-2}w^{-1}\frac{d}{dr}\left(g^2 w\frac{dh_0}{dr}\right) + \lambda g^{-2}h_0\right)\tau_0$

(ii) $(\mathcal{D}_1 h_1)\tau_1 = \Delta_1(h_1\tau_1) = \left(-w^{-1}\frac{d}{dr}\left(w\frac{dh_1}{dr}\right) + \lambda g^{-2}h_1\right)\tau_1$

(iii) $(\mathcal{D}_2 h_2)\tau_2 dr = \Delta_1(h_2\tau_2 dr) = \left(-\frac{d}{dr}\left(g^{-2}w^{-1}\frac{d}{dr}\left(g^2 w h_2\right)\right) + \lambda g^{-2}h_2\right)\tau_2 dr$

(iv) $\mathcal{D}_3(h_3, h_4) = \left(\mathcal{D}_1 h_3 + 2g^{-1}\frac{dg}{dr}h_4, \mathcal{D}_2 h_4 + 2g^{-3}\frac{dg}{dr}h_3\right)$.

The explanation for (iv) is that

$$\Delta_1(h_3 d_s\tau_3 + h_4\tau_3 dr) = \left(\mathcal{D}_1 h_3 + 2g^{-1}\frac{dg}{dr}h_4\right)d_s\tau_3 + \left(\mathcal{D}_2 h_4 + 2g^{-3}\frac{dg}{dr}h_3\right)\tau_3 dr.$$

Note that $\mathcal{D}_0$ and $\mathcal{D}_2$ are operators on $L^2(R^+, g^2 w dr)$, $\mathcal{D}_1$ is an operator on $L^2(R^+, w dr)$ and $\mathcal{D}_3$ acts on $L^2(R^+, w dr) \times L^2(R^+, g^2 w dr)$.

It follows immediately that $\mathcal{D}_1$ is of the form (3.15) with $\gamma = \beta$. Hence, $\sigma(\mathcal{D}_1) = [\beta^2, \infty)$.

Using Proposition 3.14, we may replace g^2 by e^{2r} in $\mathcal{D}_0$ and $\mathcal{D}_2$ without changing the spectra. This gives us an operator $\tilde{\mathcal{D}}$ acting on $h \in L^2(R^+, e^{2(1+\beta)r}dr)$ given by:

$$\tilde{\mathcal{D}}h = -\frac{d}{dr}\left(e^{-2(1+\beta)r}\frac{d}{dr}\left(e^{2(1+\beta)r}h\right)\right) + \lambda e^{-2r}h,$$

which is of the form (3.15) with $\gamma = 1+\beta$. Hence, $\sigma(\mathcal{D}_0) = \sigma(\mathcal{D}_2) = [(1+\beta)^2, \infty)$.

Finally, in the third case, again using Proposition 3.14, it suffices to consider the spectrum of

$$\mathcal{D}_3'(h_3, h_4) = (\mathcal{D}_1 h_3 + 2h_4, \mathcal{D}_2 h_4 + e^{-2r}h_3)$$

acting on pairs $(h_3, h_4) \in L^2(R^+, e^{2\beta r}dr) \times L^2(R^+, e^{2(1+\beta)r}dr)$. Making the unitary change of variable $(h_3, h_4) = (e^{-\beta r}k_3, e^{-(1+\beta)r}k_4)$ gives the operator $\mathcal{D}_3'' = U^{-1}\mathcal{D}_3' U$ where

$$\mathcal{D}_3''(k_3, k_4) = \left(\frac{-d^2k_3}{dr^2} + \beta^2 k_3 + 2e^{-r}k_4, \frac{-d^2k_4}{dr^2} + (1+\beta)^2 k_4 + 2e^{-r}k_3\right)$$

on pairs $(k_3, k_4) \in L^2(R^+, dr) \times L^2(R^+, dr)$. Another application of Proposition 3.14 shows that the spectrum of $\mathcal{D}_3$ is

$$\sigma(\mathcal{D}_3) = [\beta^2, \infty) \cup [(1+\beta^2), \infty) = [\beta^2, \infty).$$

Since up to compact perturbation, Δ_p is unitarily equivalent to sums of the above operators, we have demonstrated (a) and (b) of proposition 3.13.

To prove (c), we show that $\delta_0\delta_0^\dagger$ differs from the Laplacian on pairs by a compact perturbation. From (3.6) and (3.7) we have with $A_0 = 0$ and $d_{A_0} = d$,

$$\delta_0\delta_0^\dagger(\psi, b) = (\Delta_0\psi, \Delta_1 b) + (0, T_1 b) \tag{3.19}$$

where

$$T_1 b = -2\beta\{*(dr \wedge *db) + d^*(b \wedge dr)\} \tag{3.20}$$

Evaluating T_1 on each of the three types of one forms occuring in (3.17), we find that $T_1 b = 0$ on the subspaces spanned by τ_1 and $\tau_2 dr$. For b of the third kind, T_1 gives rise to the operator

$$\mathcal{T}(h_3, h_4) = -2\beta(h_4, \lambda g^{-2}h_3)$$

since $T_1(h_3 d_s\tau_3 + h_4\tau_3 dr) = -2\beta(h_4 d_s\tau_3 + \lambda g^{-2}h_3\tau_3 dr)$. As before, $(h_3, h_4) \in L^2(R^+, e^{2\beta r}dr) \times L^2(R^+, e^{2(1+\beta)r}dr)$. The unitary change of dependent variable $(h_3, h_4) = \left(e^{-\beta r}k_3, e^{-(1+\beta)r}k_4\right)$ gives a unitary equivalence of $\mathcal{T}$ with

$$\hat{\mathcal{T}}(k_3, k_4) = (U^{-1}\mathcal{T}U)(k_3, k_4) = (e^{-r}k_4, \lambda e^r g^{-2}k_3)$$

acting on $L^2(R^+, dr) \times L^2(R^+, dr)$.

It follows that T_1 contributes a compact perturbation to Δ_1, and hence, by Prop 3.14 does not change the spectrum, from which (c) of proposition 3.13 follows.

Now, letting Δ denote the Laplacian on pairs, the information in proposition 3.13 may be translated (using the Raleigh-Quotient Theorem) into the inequalities:

$$\begin{aligned} &(b') \quad \beta^2\|\eta\|_{2,\beta}^2 \le (\Delta\eta, \eta)_\beta \\ &(c') \quad \beta^2\|\eta\|_{2,\beta}^2 \le (\delta_0\delta_0^\dagger\eta, \eta)_\beta = (\delta_0^\dagger\eta, \delta_0^\dagger\eta)_\beta. \end{aligned} \tag{3.21}$$

It is now relatively standard to show

Proposition 3.22. *There is a constant $\kappa > 0$ such that*

$$\kappa\|\eta\|_{\mathcal{H}_0} \leq \|\delta_0\delta_0^\dagger\eta\|_{2,\beta}$$

and $\delta_0\delta_0^\dagger$ is invertible.

Proof. First, we note that since $\delta_0\delta_0^\dagger$ is a self-adjoint operator defined on $\mathcal{H}_0$, the inequality implies that the co-kernel is zero and hence, $\delta_0\delta_0^\dagger$ is not only injective but surjective with closed range, and hence, invertible.

To prove the inequality, we recall the Weitzenbock formula at a point, for the standard Laplacian $\Delta_p^0 = dd^* + d^*d$ ([FU]):

$$-\nabla^2\omega = \Delta_p^0\omega + \mathrm{Ric}_p(\omega,\cdot), \quad \omega \in \wedge^p, p = 0, 1,$$

and $\mathrm{Ric}_0 = 0$.

Letting $\nabla^\dagger\omega = e^{-2\beta r}\nabla(e^{2\beta r}\omega)$, and, recalling that $\Delta_p\omega = (dd^\dagger + d^\dagger d)\omega = \Delta_p^0\omega + 2\beta(d\omega_r - *(dr \wedge *d\omega)$, we obtain

$$-\nabla^\dagger\nabla\omega = \Delta_p\omega + \mathrm{Ric}_p(\omega,\cdot) - 2\beta\nabla r\cdot\nabla\omega - 2\beta(d\omega_r - *(dr\wedge *d\omega)). \tag{3.24}$$

An integration by parts and use of $(3.21b')$ gives

$$\begin{aligned}\|\nabla\omega\|_{2,\beta}^2 &\leq (\Delta_p\omega,\omega) + c\|\nabla\omega\|_{2,\beta}\|\omega\|_{2,\beta}\\ &\leq (1 + C(\epsilon))(\Delta_p\omega,\omega) + \epsilon\|\nabla\omega\|_{2,\beta}^2\end{aligned}$$

from which it follows that

$$\|\omega\|_{L_\beta^{1,2}} \leq C\|\Delta_p\omega\|_{2,\beta}. \tag{3.25}$$

Using the Weitzenbock formula once again,

$$\|\nabla^2\omega\|_{2,\beta}^2 \leq \|\Delta_p\omega\|_{2,\beta}^2 + C'\|\omega\|_{L_\beta^{1,2}}^2 \leq C''\|\Delta_p\omega\|_{2,\beta}^2$$

which gives

$$\|\omega\|_{L_\beta^{2,2}} \leq C\|\Delta_p\omega\|_{2,\beta} \qquad \omega\in\wedge^p \quad p = 0, 1. \tag{3.26}$$

Next, recall that on pairs,

$$\delta_0\delta_0^\dagger\eta = \Delta\eta + T\eta \text{ where } \quad T = (0, T_1)$$

and T_1 is defined by (3.20).

Using (3.26) and (3.21c'),

$$\begin{aligned}
\|\eta\|_{\mathcal{H}_0} &\le C\|\Delta\eta\|_{2,\beta} \le C\left(\|\delta_0\delta_0^\dagger\eta\|_{2,\beta} + \|T_0 b\|_{2,\beta}\right)\\
&\le C\|\delta_0\delta_0^\dagger\eta\|_{2,\beta} + C'\|b\|_{L^{1,2}_\beta}\\
&\le C\|\delta_0\delta_0^\dagger\eta\|_{2,\beta} + C(\epsilon)\|\eta\|_{2,\beta} + \epsilon\|\eta\|_{\mathcal{H}_0}\\
&\le C'\|\delta_0\delta_0^\dagger\eta\|_{2,\beta} + \epsilon\|\eta\|_{\mathcal{H}_0}. \qquad (3.27)
\end{aligned}$$

Absorbing the term $\epsilon\|\eta\|_{\mathcal{H}_0}$, proves the inequality of Proposition 3.22.

Finally, letting $\mathcal{L} = \delta\delta^\dagger$, (cf. (3.4) and (3.8)) we are ready to prove the main estimate of this paper.

Theorem 3.28. *There is a constant $\alpha_1 > 0$ such that*

$$\alpha_1\|\eta\|_{\mathcal{H}_{A_0}} \le \|\mathcal{L}\eta\|_{2,\beta}$$

and $\mathcal{L}$ is invertible.

We notice that, as before, $\mathcal{L}$ is self-adjoint on its domain $\mathcal{H}_{A_0}$ and hence, the inequality shows invertibility.

To prove the theorem, we compare the various operators evaluated at A_0 with their scalar analogues to show that the lower bounds on spectra do not decrease.

Recall, from (2.1b) that near infinity, $c_0 = (\Phi_0, A_0) = (\Phi_\infty, A_\infty)$. This configuration decays exponentially in the sense that

$$|d\Phi_0|, |A_0|, |F_{A_0}| \le e^{-c|x|} \text{ for large } |x|. \qquad (3.29)$$

First, note that for the configuration $\eta = (\psi, b)$,

$$\Delta_{A_0}\eta = \Delta\eta + S\eta$$

where $S\eta = (S_0\psi, S_1 b)$ with

$$S_0\psi = d^\dagger[A_0, \psi] + *[A_0, *(d\psi + [A_0, \psi])]$$

and

$$\begin{aligned} S_1 b = &\, d^\dagger[A_0, b] + d(*[A_0, *b]) \\ &+ *[A_0, *(db + [A_0, b]] + [A_0, d^\dagger b + *[A_0, *b]]. \end{aligned}$$

The bounds in (3.29) and Proposition 3.14, imply

$$\sigma_{ess}(\Delta_{A_0}) = \sigma(\Delta_{A_0}) = [\beta^2, \infty). \tag{3.30}$$

Recall from (3.6) and (3.7) that

$$\delta_{A_0}\delta_{A_0}^\dagger \eta = (\Delta_{A_0} + T_{A_0})\eta.$$

But $T_{A_0} = T + R$ where $T = (0, T_1)$ is the operator in (3.20) and

$$R\eta = (*[F_{A_0}, b], *[F_{A_0}, \psi] + *2\beta\{[A_0, *(dr \wedge b)] - dr \wedge *[A_0, b]\})$$

is rapidly decreasing by (3.29).

Comparing,

$$\delta_{A_0}\delta_{A_0}^\dagger \eta = (\Delta_{A_0} + T_{A_0})\eta = (\Delta + T)\eta + (R + S)\eta = (\delta_0\delta_0^\dagger + (R + S))\eta \tag{3.31}$$

where R and S are rapidly decaying at infinity. By Proposition 3.14,

$$\sigma_{ess}(\delta_{A_0}\delta_{A_0}^\dagger) = \sigma(\delta_{A_0}\delta_{A_0}^\dagger) = [\beta^2, \infty). \tag{3.32}$$

A final computation shows that

$$\begin{aligned} \delta\delta^\dagger\eta &= (\delta_{A_0} - ad\Phi_0)(\delta_{A_0}^\dagger + ad\Phi_0) \\ &= \delta_{A_0}\delta_{A_0}^\dagger - (ad\Phi_0)^2 + E \end{aligned} \tag{3.33}$$

where $E\eta = (- * [d_{A_0}\Phi_0, *b], [d_{A_0}\Phi_0, \psi] - *[d_{A_0}\Phi_0, b])$ is rapidly decaying. Since $-(ad\Phi_0)^2$ is non-negative, we obtain

$$\sigma(\delta\delta^\dagger) \subseteq [\beta^2, \infty)$$

and, using the Raleigh-Quotient Theorem,

$$\kappa\|\eta\|_{2,\beta}^2 \leq (\delta\delta^\dagger\eta,\eta)_\beta = \|\delta^\dagger\eta\|_{2,\beta}^2 \tag{3.35}$$

for some $\kappa \geq \beta^2$.

To prove the inequality in the theorem, we use the ordinary Weitzenbock formula (cf [E]) at a point:

$$\delta\delta^* = -\nabla_{A_0}^2 - \{G_0,\cdot\} + \{\text{ Ric },\cdot\} - (ad\Phi_0)^2 \tag{3.36}$$

which translates for the weighted operators as

$$\delta\delta^\dagger = -\nabla_{A_0}^\dagger \nabla_{A_0} + Q \tag{3.37}$$

where $|Q\eta| \leq c(|\eta| + |\nabla_{A_0}\eta|)$.

As before,

$$\|\nabla_{A_0}\eta\|_{2,\beta}^2 \leq (\delta\delta^\dagger\eta,\eta)_\beta + (Q\eta,\eta)_\beta \leq (\delta\delta^\dagger\eta,\eta)_\beta + \epsilon\|\nabla_{A_0}\eta\|_{2,\beta}^2 + C(\epsilon)\|\eta\|_{2,\beta}^2 \tag{3.37}$$

and therefore,

$$\|\eta\|_{2,\beta} + \|\nabla_{A_0}\eta\|_{2,\beta} \leq C'\|\delta\delta^\dagger\eta\|_{2,\beta}. \tag{3.38}$$

Using the Weitzenbock formula again gives the bound on second derivatives and proves the theorem.

Note that because of the explicit knowledge we have about the Chakrabarti monopole, the decay of the approximate monopole is known for $\mathbb{H}^3$ and also that the basic estimate, Theorem 3.28, holds without any assumption that $Lc_0 = d_{A_0}\Phi_0 - *F_{A_0}$ is small. However, this condition will be needed later in applying the Implicit Function Theorem.

We have also made no assumptions on the magnitude β of the weight factor. The restriction $\beta < m$, required to make Lc_0 small, will also be used later.

Corollary 3.39. If $\|\nu\|_{6,\beta}$ is sufficiently small, then

$$\mathcal{L}_\nu = \mathcal{L} + \nu\#\delta^\dagger$$

is invertible.

Proof. We use a weighted version of Sobolev's inequality which says that, for $2 < p \leq 6$,

$$\|\eta\|_{p,\beta} \leq C(\| \nabla_{A_0} \eta\|_{2,\beta} + \|\eta\|_{2,\beta}). \tag{3.40}$$

Using Holder's inequality,

$$\begin{aligned} \|\nu \# \delta^\dagger \eta\|_{2,\beta} &\leq \|\nu\|_{6,\beta} \|\delta^\dagger \eta\|_{3,\beta} \\ &\leq C\|\nu\|_{6,\beta}(\| \nabla_{A_0} \delta^\dagger \eta\|_{2,\beta} + \|\delta^\dagger \eta\|_{2,\beta}) \\ &\leq \alpha_2 \|\nu\|_{6,\beta} \| \|\eta\|_{\mathcal{H}_{A_0}}. \end{aligned} \tag{3.41}$$

Using Theorem 3.28,

$$\alpha_1 \|\eta\|_{\mathcal{H}_{A_0}} \leq \|\mathcal{L}\eta\|_{2,\beta} \leq \|\mathcal{L}_\nu \eta\|_{2,\beta} + \|\nu \# \delta^\dagger \eta\|_{2,\beta}, \tag{3.42}$$

which combined with (3.41) gives, for $\|\nu\|_{6,\beta}$ sufficiently small,

$$\alpha' \|\eta\|_{\mathcal{H}_{A_0}} \leq \|\mathcal{L}_\nu \nu\|_{2,\beta}. \tag{3.43}$$

Invertibility, for $\|\nu\|_{6,\beta}$ small, follows from the fact that $\mathcal{L}$ is invertible and $\|\mathcal{L} - \mathcal{L}_\nu\|_{2,\beta} \leq C\|\nu\|_{6,\beta}$.

We now (following [FU]) apply the continuity method to the equation

$$\mathcal{L}_t \eta_t = \delta\delta^\dagger \eta_t + \delta^\dagger \eta_t \# \delta^\dagger \eta_t = tG_0, \tag{3.12}$$

where $0 \leq t \leq 1$.

To that end, let $\lambda < \frac{\alpha_1}{4\alpha_2^2}$ where α_1 and α_2 are the constants occuring in the inequalities (3.28) and (3.41). Also, assume that $\|G_0\|_{2,\beta} \leq \frac{\alpha_1 \lambda}{4}$.

Let

$$\Omega = \{\eta \in \mathcal{H}_{A_0} \mid \|\eta\|_{\mathcal{H}_{A_0}} \leq \lambda\}$$

and

$$J = \{t \in [0,1] \mid \text{equation (3.12) has a solution in } \Omega\}$$

We show that J is non-empty, open and closed. Clearly, $t = 0$ belongs to J since $\eta \equiv 0$ is the unique solution.

To show that J is open, let $t_0 \in J$, with η_0 the solution of (3.12) belonging to Ω. The linearized operator at η_0 is $\mathcal{L}_\nu$ with $\nu = 2\delta^\dagger\eta_0$. From (3.40) and (3.41) at $2\delta^\dagger\eta_0$ and the choice of λ above, one has $\|\nu\#\delta^\dagger\eta\|_{2,\beta} \leq \frac{\alpha_1}{2}\|\eta\|_{\mathcal{H}_{A_0}}$. It follows from (3.42) that (3.43) holds with $\alpha' = \frac{\alpha_1}{2}$ and $\mathcal{L}_\nu$ is invertible.

From the Implicit Function Theorem, we conclude that (3.12) has a solution η_t for t sufficiently close to t_0, and $\|\eta_t - \eta_0\|_{\mathcal{H}_{A_0}} < \epsilon$, for ϵ sufficiently small. Estimating again (as in 3.41), and using the fact that $\eta_0 \in \Omega$, we find, from (3.28)

$$\begin{aligned} \alpha_1\|\eta_0\|_{\mathcal{H}_{A_0}} &\leq \|t_0G_0\|_{2,\beta} + \|\delta^\dagger\eta_0\#\delta^\dagger\eta_0\|_{2,\beta} \\ &\leq \|t_0G_0\|_{2,\beta} + \frac{\alpha_1}{4}\|\eta_0\|_{\mathcal{H}_{A_0}}. \end{aligned} \tag{3.44}$$

Using the bound on G_0, we find

$$\|\eta_0\|_{\mathcal{H}_{A_0}} \leq \frac{1}{3}\lambda \tag{3.45}$$

so that for η_t sufficiently close to η_0, which will be the case if t is close to t_0, we see that

$$\|\eta_t\|_{\mathcal{H}_{A_0}} \leq \lambda. \tag{3.46}$$

Therefore, J is open.

To prove that J is closed, let $t_n \in J$ and converge to t_0. Then, for each n, let η_n be a solution of (3.12) corresponding to t_n. Since $\|\eta_n\|_{\mathcal{H}_{A_0}} \leq \lambda$, a subsequence converges weakly in $\mathcal{H}_{A_0}$ to η_0 and by lower semi-continuity with respect to weak convergence, $\|\eta_0\|_{\mathcal{H}_{A_0}} \leq \lambda$. We claim that $\mathcal{L}_{t_0}\eta_0 = t_0\,G_0$. It suffices to show this on any compact subdomain. The linear term $\delta\delta^\dagger\eta_n$ converges weakly to $\delta\delta^\dagger\eta_0$. By Sobolev embedding $\delta^\dagger\eta_n$ converges strongly to $\delta^\dagger\eta_0$ in L^p_β for $p < 6$ and therefore, $\delta^\dagger\eta_n\#\delta^\dagger\eta_n$ converges strongly to $\delta^\dagger\eta_0\#\delta^\dagger\eta_0$ in L^2_β, since $\|\delta^\dagger\eta_0\#\delta^\dagger\eta_0\|_{2,\beta} \leq \|\delta^\dagger\eta_0\|^2_{4,\beta}$ and Lebesgue dominated convergence is applicable. It follows that $\mathcal{L}_{t_n}\eta_n$ converges weakly to $\mathcal{L}_{t_0}\eta_0$ which is a solution of the equation, as desired. This shows that J is closed, and completes the proof of our main theorem.

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May 1995

Lesley M. Sibner, Polytechnic University, Brooklyn, N.Y. 11201, U.S.A.

Current address: Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, 53225 Bonn, Germany

E-mail address: lsibner@ magnus.poly.edu

Robert J. Sibner, City University New York, Brooklyn College, Brooklyn, N.Y. 11210, U.S.A.

Current address: Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, 53225 Bonn, Germany

E-mail address: rjsgc@ cunyvm.cuny.edu