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PARABOLIC BUNDLES OVER THE PROJECTIVE LINE AND THE DELIGNE-SIMPSON PROBLEMS

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Abstract. In "Quantization of Hitchin's Integrable System and Hecke Eigensheaves", Beilinson and Drinfeld introduced the "very good" property for a smooth complex equidimensional stack. They prove that for a semisimple group G over $\mathbb{C}$, the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles over a smooth complex projective curve X is "very good", as long as X has genus $g > 1$. In the case of the projective line, when $g = 0$, this is not the case. However, the result can sometimes be extended to the projective line by introducing additional parabolic structure at a collection of marked points and slightly modifying the definition of a "very good" stack. We provide a sufficient condition for the moduli stack of parabolic vector bundles over $\mathbb{P}^1$ to be very good. We then use this property to study the space of solutions to the Deligne-Simpson problem.

1. Introduction

1.1. **The Deligne-Simpson Problem.** Let X be a Riemann surface. Consider the divisor D on X. A *logarithmic connection* or a *connection with regular singularities* (in D) on a vector bundle E over X is a $\mathbb{C}$-linear morphism

$$\nabla : E \to E \otimes \Omega^1_X(D) \text{ such that}$$
$$\nabla(fs) = s \otimes df + f\nabla(s) \text{ for } f \in \mathcal{O}_X, s \in E.$$

Note that the connection ∇ has residues at the points of D, so that there exists $\mathrm{Res}_{x_i}\nabla \in \mathrm{End}(E_{x_i})$, for each fiber E_{x_i} over $x_i \in D$.

Let $C_1, \dots, C_k$ be conjugacy classes of complex, linear endomorphisms of vector spaces of dimension n. We can formulate the following:

The Deligne-Simpson Problem. *Does there exist (for some D and vector bundle E) a connection ∇ on $\mathbb{P}^1$ with regular singularities such that $Res_{x_i}\nabla \in C_i$?*

We will use this formulation of the Deligne-Simpson problem instead of the ones given below, as it is easier to generalize to the case of connections with irregular singularities (see Section 1.5).

The *Riemann-Hilbert correspondence* provides an equivalence between the category of connections ∇ with regular singularities in D on vector bundles over $\mathbb{P}^1$ and the category of representations of the fundamental group of $\mathbb{P}^1 - D$ by way of the monodromy representation of ∇ (see [11]). This provides the following reformulation of the Deligne-Simpson problem:

Multiplicative Deligne-Simpson Problem. *Given conjugacy classes $C_1, \dots, C_k$ of complex matrices in $GL(n, \mathbb{C})$, do there exist $A_1 \in C_1, \dots, A_k \in C_k$ such that $A_1 \cdot A_2 \cdots A_k = Id$?*

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This was the original version of the Deligne-Simpson problem, suggested in a letter from Deligne to Simpson, who considered it in his paper [28].

By considering connections on trivial (and trivialized) vector bundles over $\mathbb{P}^1$ we get another version of the Deligne-Simpson problem:

Additive Deligne-Simpson Problem. *Given conjugacy classes $C_1, \dots, C_k$ of complex matrices in $\mathfrak{gl}_n(\mathbb{C})$, do there exist $A_1 \in C_1, \dots, A_k \in C_k$ such that $A_1 + \dots + A_k = 0$?*

The multiplicative Deligne-Simpson problem and its additive analogue were studied by Crawley-Boevey, Katz, Kostov, and Simpson, among others.

There are several approaches to solving the Deligne-Simpson problem. In [21], Katz describes an algorithm for the existence of rigid local systems, which Kostov applies (see e.g. [22] for summary) to determine when solutions to various cases of the Deligne-Simpson problems exist. The algorithm, called the *middle convolution algorithm*, proceeds by changing the rank of the local system by a number δ, called the *defect*, dependent on $C_1, \dots, C_k$. Solutions exist for the original rank, as long as they exist for the altered rank. This continues until $\delta \geq 0$, in which case there are solutions by a nontrivial existence theorem, or until one arrives at a situation when solutions cannot exist. An overview of the Katz algorithm may be found in [31].

In [6], Crawley-Boevey proposes another approach to the additive version of the Deligne-Simpson problem by examining fibers of the moment map on the cotangent bundle to the space of representations of the star-shaped quiver and the representations of the deformed preprojective algebra associated to this quiver. This gives him a necessary and sufficient condition for the existence of solutions in the additive case. In [10], he and Shaw provide a sufficient condition for the existence of solutions of the multiplicative Deligne-Simpson problem using a multiplicative analogue of the preprojective algebra. This condition is also necessary ([9]). A multiplicative analogue of the moment map approach of [6] may be found in [34].

We approach the Deligne-Simpson problem using at technical condition from Beilinson's and Drinfeld's work on the geometric Langlands program. It provides us with a way of studying the geometry of the varieties of solutions arising from the additive and multiplicative versions of the problem.

1.2. **The Very Good Property.** In [2] Beilinson and Drinfeld introduced the notion of a "very good" stack. They require this property in order to avoid using derived categories in their study of D-modules on the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles over X, where G is a semisimple algebraic group and X is a smooth complex projective curve.

A smooth complex equidimensional stack $\mathcal{Y}$ will be called *very good* if

$$\operatorname{codim}\{y \in \mathcal{Y} | \dim \operatorname{Aut}(y) = n\} > n, \text{for } n > 0,$$

where $\operatorname{Aut}(y)$ is the automorphism group of $y \in \mathcal{Y}$. If $\dim \operatorname{Aut}(y) > 0$ for all $y \in \mathcal{Y}$, then the stack $\mathcal{Y}$ cannot be very good. In this situation, $\mathcal{Y}$ will be called *almost very good* if

$$\operatorname{codim}\{y \in \mathcal{Y} | \dim \operatorname{Aut}(y) - m = n\} > n, \text{for } n > 0,$$

where $m = \min \dim \operatorname{Aut}(y)$. Beilinson and Drinfeld demonstrate that $\mathrm{Bun}_G(X)$ is very good when X has genus $g > 1$. However, in the $g = 0$ case, when $X = \mathbb{P}^1$, this is no longer true.

We approach the very good property in the genus $g = 0$ case, for $G = \mathrm{GL}(n, \mathbb{C})$, by introducing additional parabolic structure at a finite collection of marked points. Since the reductive group $\mathrm{GL}(n, \mathbb{C})$ has a one-dimensional central subgroup $\mathbb{C}^*$ that acts by dilation on the fibers, the automorphism group of any parabolic bundle has a one-dimensional subgroup. It follows that the moduli stack of parabolic bundles can never be very good.

It turns out, however, that a sufficiently elaborate parabolic structure on a vector bundle is enough to make the corresponding moduli stack of parabolic bundles over $\mathbb{P}^1$ almost very good. This is equivalent to showing that the quotient of the moduli stack by the classifying stack of $\mathbb{C}^*$ is very good.

1.3. The Very Good Property for Moduli of Parabolic Bundles.

Seshadri introduced the notion of a parabolic structure on a vector bundle in [27], furnishing parabolic bundles with a stability condition analogous to the usual one for vector bundles. Expanding upon this, Mehta and Seshadri proved the existence of a moduli space of semistable parabolic bundles on a smooth projective curve of genus $g \geq 2$ in [26].

Parabolic bundles over an algebraic curve generalize vector bundles by defining additional structure in the fibers over specified points. Namely, let X be a smooth complex projective curve (in the future, we restrict ourselves to the case when $X = \mathbb{P}^1$). A *parabolic bundle* $\mathbf{E}$ over X consists of a vector bundle E over X, a collection of distinct points $(x_1, \ldots, x_k)$ on X, and a flag $E_{x_i} = E_{i0} \supseteq E_{i1} \supseteq E_{iw_{i-1}} \supseteq E_{iw_i} = 0$ in the fiber over each such point x_i.

If $D = x_1 + \cdots + x_k$ and $w = (w_1, \ldots, w_k)$, we say that the parabolic bundle $\mathbf{E}$ has *weight type* (D, w). If $\alpha_0 = \mathrm{rk}\ E$ and $\alpha_{ij} = \dim\ E_{ij}$, for $1 \leq i \leq k$ and $1 \leq j \leq w_{i-1}$, we say that $\mathbf{E}$ has *dimension vector* $\alpha = (\alpha_0, \alpha_{ij})$.

Note that one possible way of introducing stability and semistability for parabolic bundles, is by defining a parabolic degree. To do this, additional numbers called *weights* are assigned to each subspace in each flag. Since we do not limit ourselves to stable or semistable parabolic bundles, we do not require weights to be part of the definition. Parabolic bundles without weights are sometimes referred to as "quasi-parabolic" bundles.

In order to formulate our main result, we need to specify which dimension vectors give rise to very good parabolic bundles. Let $I = \{0\} \cup \{(i,j) | 1 \leq i \leq k, 1 \leq j \leq w_{i-1}\}$. For a dimension vector $\alpha \in \mathbb{Z}^I$, we define the *Tits quadratic form* as:

$$q(\alpha) = \sum_{i \in I} \alpha_i^2 - \sum_{i \in I} \alpha_i \alpha_{i+1},$$

where $\alpha_{w_i} = 0$. Let $p(\alpha) = 1 - q(\alpha)$. We write: $\delta(\alpha) = -2\alpha_0 + \sum_i \alpha_{i1}$. We say that α is in the *fundamental region* if

$$\delta(\alpha) \geq 0$$
$$-2\alpha_{ij} + \alpha_{ij-1} + \alpha_{ij+1} \geq 0, \text{ for } 1 \leq i \leq k \text{ and } 1 \leq j \leq w_{i-1}$$

(note that we assume $\alpha_{i0} = \alpha_0$, for all i). We now introduce our main result.

Theorem 1.3.1. *The moduli stack $Bun_{D,w,\alpha}(\mathbb{P}^1)$ of parabolic bundles over $\mathbb{P}^1$ of weight type (D, w) and dimension vector α is almost very good if α is in the fundamental region and $\delta(\alpha) > 0$.*

The vector α can be used to define a product of partial flag varieties

$$Fl(\alpha) = \prod_i Fl(\alpha_0, \alpha_{i1}, \dots, \alpha_{iw_i}).$$

That is, α_0 is the dimension of the ambient space $\mathbb{C}^{\alpha_0}$, and for a fixed $1 \le i \le k$, each α_{ij} is the dimension of the j-th subspace in the flag. The group $\mathrm{PGL}(\alpha_0)$ acts diagonally on $Fl(\alpha)$, so it makes sense to discuss the very good property of the resulting quotient stack. Indeed, when the underlying vector bundle is trivial, we can use Theorem 1.3.1 to obtain:

Theorem 1.3.2. *The quotient stack $PGL(\alpha_0)\backslash Fl(\alpha)$ is very good, if α is in the fundamental region and $\delta(\alpha) > 0$.*

Theorem 1.3.2 may also be obtained from Crawley-Boevey's results in [5], after noticing that $Fl(\alpha)$ is the quotient of the space of star-shaped quiver representations of dimension α with injective arrows by the group $H(\alpha) = \prod_{i,j} \mathrm{GL}(\alpha_{ij})$, acting by conjugation on the arrows. In this case, the very good property is equivalent to Crawley-Boevey's inequality $p(\alpha) > \sum_i p(\beta_i)$ (see [5]), for any decomposition $\alpha = \sum_i \beta_i$ into the sum of positive roots corresponding to the star-shaped quiver (see Sections 3.3 and 3.4 below). The condition that α is in the fundamental region and $\delta(\alpha) > 0$ implies this inequality.

1.4. **The Deligne-Simpson problem and the very good property.** Let $\mathbf{E}$ be a parabolic bundle over $\mathbb{P}^1$ of weight type (D, w). Let $\zeta = (\zeta_{ij})_{1\le i\le k, 1\le j\le w_i}$. A *$\zeta$-parabolic connection* on $\mathbf{E}$ is a connection ∇ on the underlying vector bundle E with regular singularities in D, such that

$$(\mathrm{Res}_{x_i}\nabla - \zeta_{ij})(E_{ij-1}) \subset E_{ij},$$

for all $1 \le i \le k$ and $1 \le j \le w_i$.

Given semisimple conjugacy classes $C_1, \dots, C_k$ of n-dimensional complex vector space endomorphisms and an ordering on the eigenvalues of these conjugacy classes, one can write a dimension vector α, where $\alpha_0 = n$ and α_{ij} is the dimension of the direct sum of the first j eigenspaces, for the above ordering on the eigenvalues. One can also obtain a complex vector $\zeta = (\zeta_{ij})$ simply as the vector of eigenvalues for $C_1, \dots, C_k$, counting multiplicity. For these ζ and α, the ζ-parabolic connections on parabolic bundles with dimension vector α over $\mathbb{P}^1$ will have residues in the conjugacy classes $C_1, \dots, C_k$.

Conversely, a ζ-parabolic connection on a parabolic bundle with dimension vector α over $\mathbb{P}^1$ determines semisimple conjugacy classes $C_1, \dots, C_k$, with ζ being the vector of eigenvalues (counting multiplicity), and $\alpha_{ij} - \alpha_{ij+1}$ being the dimension of the eigenspace for ζ_{ij}.

Given the situation described in the previous two paragraphs, it follows that semisimple conjugacy classes may be used to determine (not uniquely) a moduli stack of parabolic bundles $\mathrm{Bun}_{D,w,\alpha}(\mathbb{P}^1)$. Furthermore, the moduli stack of solutions of the Deligne-Simpson problem may be defined as $\mathrm{Conn}_{D,w,\alpha,\zeta}(\mathbb{P}^1)$, the moduli stack of ζ-parabolic connections on parabolic bundles over $\mathbb{P}^1$ of weight type (D, w) and dimension vector α. By presenting $\mathrm{Conn}_{D,w,\alpha,\zeta}(\mathbb{P}^1)$ as a twisted cotangent bundle over the moduli stack of parabolic bundles $\mathrm{Bun}_{D,w,\alpha}(\mathbb{P}^1)$, we prove the following theorem:

Theorem 1.4.1. *If $Bun_{D,w,\alpha}(\mathbb{P}^1)$ is almost very good and $\sum_{i=1}^{k}\sum_{j=1}^{w_i}\zeta_{ij}(\alpha_{ij-1}-\alpha_{ij})$ is an integer, then $Conn_{D,w,\alpha,\zeta}(\mathbb{P}^1)$ is a nonempty, irreducible, locally complete intersection of dimension $2p(\alpha)-1$.*

Theorem 1.4.1 and Theorem 1.3.1 give us the following corollary:

Corollary 1.4.2. *If α is in the fundamental region, $\delta(\alpha)>0$, and we have $\sum_{i=1}^{k}\sum_{j=1}^{w_i}\zeta_{ij}(\alpha_{ij-1}-\alpha_{ij})$ is an integer, then $Conn_{D,w,\alpha,\zeta}(\mathbb{P}^1)$ is a nonempty, irreducible, locally complete intersection of dimension $2p(\alpha)-1$.*

If the vector bundles underlying the parabolic bundles are trivial, then Theorem 1.4.1 may be used to obtain the following:

Theorem 1.4.3. *If the conjugacy classes C_i are semisimple, the corresponding quotient stack $PGL(\alpha_0)\backslash Fl(\alpha)$ is very good and the eigenvalues of all the C_i add up to 0, then the space of solutions of the additive Deligne-Simpson problem for $C_1,\dots,C_k$ is a nonempty, irreducible complete intersection of dimension $2\cdot\dim Fl(\alpha)-\alpha_0^2+1$.*

Applying Theorem 1.3.2 we obtain:

Corollary 1.4.4. *If the conjugacy classes C_i are semisimple, the eigenvalues of all the C_i add up to 0, α is in the fundamental region, and $\delta(\alpha)>0$, then the space of solutions of the additive Deligne-Simpson problem for $C_1,\dots,C_k$ is a nonempty, irreducible complete intersection of dimension $2\cdot dim\ Fl(\alpha)-\alpha_0^2+1$.*

We can obtain results similar to Theorem 1.4.3 and Corollary 1.4.4 for the multiplicative Deligne-Simpson problem. Indeed, let $C_1,\dots,C_k$ be semisimple conjugacy classes in $\mathrm{GL}(n,\mathbb{C})$. The Riemann-Hilbert correspondence provides an analytic isomorphism between the space of solutions to the multiplicative Deligne-Simpson problem for $C_1,\dots,C_k$ and a certain moduli space of ζ-parabolic connections. This is similar to the analytic isomorphism obtained for the moduli space of stable ζ-parabolic connections in [16], [17], or [34]. We get the following:

Theorem 1.4.5. *If we have that the conjugacy classes C_i are semisimple, the corresponding moduli stack $Bun_{D,w,\alpha}(\mathbb{P}^1)$ is almost very good, and the eigenvalues of all the C_i multiply to 1, then the space of solutions of the multiplicative Deligne-Simpson problem for $C_1,\dots,C_k$ is a nonempty, irreducible complete intersection of dimension $2\cdot\dim Fl(\alpha)-\alpha_0^2+1=2p(\alpha)+\alpha_0^2-1$.*

Applying Theorem 1.3.1 we obtain:

Corollary 1.4.6. *If the conjugacy classes C_i are semisimple, the eigenvalues of all the C_i multiply to 1, α is in the fundamental region and $\delta(\alpha)>0$, then the space of solutions of the multiplicative Deligne-Simpson problem for $C_1,\dots,C_k$ is a nonempty, irreducible complete intersection of dimension $2\cdot\dim Fl(\alpha)-\alpha_0^2+1=2p(\alpha)+\alpha_0^2-1$.*

Remark 1.4.7. In the above corollaries, $\delta(\alpha)$ is actually equal to the defect δ that appears in Katz's middle convolution algorithm. Moreover, for the specific ordering on the eigenspaces described above, the condition that α is in the fundamental region reduces to $\delta(\alpha)\geq 0$. Therefore, $\delta(\alpha)>0$ alone is sufficient to obtain the properties for the space of solutions.

1.5. **Further Discussion.** In our formulation, the Deligne-Simpson problem asks whether there exist connections on $\mathbb{P}^1$ with simple poles such that the residues lie in prescribed conjugacy classes. It is also possible to ask a similar question for connections with poles of higher order.

We replace the idea of a logarithmic connection on $\mathbb{P}^1$ that has residues in prescribed conjugacy classes with the more general one of a connection with irregular singularities that has prescribed *formal types*. The notion of formal type (see e.g. [1]) allows one to classify connections with irregular singularities based on their restrictions to formal neighborhoods of points. Using this notion it is possible to formulate a more general version of the Deligne-Simpson problem by asking whether there exist connections with irregular singularities on $\mathbb{P}^1$ with prescribed formal types at a fixed collection of points D on $\mathbb{P}^1$.

Hiroe in [15] solves the "additive" version of this problem (when the connections are on trivial vector bundles) by using Boalch's quiver construction from [3]. This approach, similar to what Crawley-Boevey does in [6] for the case of regular singularities, suggests that it is possible to apply the very good condition to obtain certain geometric properties for the space of solutions to the irregular version of the additive Deligne-Simpson problem. Moreover, it may be possible to generalize representations of squids, in order to study the space of solutions to the general version of the irregular Deligne-Simpson problem.

It would also be interesting to extend the result of Theorem 1.3.1 to other reductive groups. By analogy with flag varieties, it is possible to define a parabolic structure on G-bundles, when G is not $GL(n, \mathbb{C})$, by specifying parabolic subgroups P_i at each marked point $x_i \in \mathbb{P}^1$. Although there is no correspondence with quiver representations for a general G, it may be possible to modify Beilinson and Drinfeld's original proof of the very good property for Bun_G. A key part of their argument consists of showing that the *global nilpotent cone* $\mathrm{Nilp}(G)$ (introduced in [23] and [24]), the fiber over 0 in the Hitchin system, is Lagrangian (see [14]). One can consider the parabolic analogue of the Hitchin system, which has its own global nilpotent cone. It has been proved to be Lagrangian in specific instances, such as for complete flags ([12], [32]) or rank 3 ([13]). However, the author is unaware of a proof for the case of partial flags.

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2. Very Good Property

2.1. **Definitions.** Let $\mathcal{Y}$ be an equidimensional algebraic stack over $\mathbb{C}$, and denote by $\mathrm{Aut}(y)$ the automorphism group of $y \in \mathcal{Y}$. Let $\mathcal{Y}^n = \{y \in \mathcal{Y} | \dim \mathrm{Aut}(y) = n\}$, which gives rise to a reduced locally closed substack of $\mathcal{Y}$. The following two definitions come from [2]. We call $\mathcal{Y}$ *good* when:

$$\mathrm{codim}\ \mathcal{Y}^n \geq n \quad \forall n > 0,$$

and we call it *very good* when:

$$\text{codim } \mathcal{Y}^n > n \quad \forall n > 0.$$

In the case when $\mathcal{Y}$ is smooth, being good is equivalent to the condition that $\dim T^*\mathcal{Y} = 2 \dim \mathcal{Y}$, where $T^*\mathcal{Y}$ is the cotangent stack to $\mathcal{Y}$ (see [2]). Furthermore, $\mathcal{Y}$ is very good if and only if $T^*\mathcal{Y}^0$ is dense in $T^*\mathcal{Y}$. Now, suppose there exists an integer $m > 0$ such that for all $y \in \mathcal{Y}$ we have $\dim \text{Aut}(y) \geq m$. In this case, we can see that $\mathcal{Y}$ cannot be very good.

Let $m = \min \dim \text{Aut}(y)$ over all $y \in \mathcal{Y}$. We say $\mathcal{Y}$ is *almost good* if:

$$\text{codim } \mathcal{Y}^{n+m} \geq n \quad \forall n > 0,$$

and we say it is *almost very good* if:

$$\text{codim } \mathcal{Y}^{n+m} > n \quad \forall n > 0.$$

2.2. **The very good property and the inertia stack.** In order to prove our Theorem 1.3.1, we will need to reformulate the very good property in terms of the inertia stack. Let $\mathcal{I}_\mathcal{Y}$ be the inertia stack associated with the stack $\mathcal{Y}$, which consists of pairs (y, f), such that $y \in \mathcal{Y}$ and $f \in \text{Aut}(y)$. We will be using the following lemma (see Properties of Algebraic Stacks in [33]):

Lemma 2.2.1. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a flat morphism of stacks of finite type and let $x \in \mathcal{X}_1$. We have:*

$$\dim_x (\mathcal{X}_1)_{f(x)} = \dim_x \mathcal{X}_1 - \dim_{f(x)} \mathcal{X}_2,$$

where $(\mathcal{X}_1)_{f(x)}$ is the fiber over $f(x)$.

Now, we can obtain:

Theorem 2.2.2. *The stack $\mathcal{Y}$ is good if and only if $\dim \mathcal{I}_\mathcal{Y} \leq \dim \mathcal{Y}$.*

Proof. Let $\mathcal{I}^n$ be the locally closed, reduced substack of $\mathcal{I}_\mathcal{Y}$ consisting of objects (y, g) such that $\dim \text{Aut}(y) = n$. Furthermore, let $f : \mathcal{I}_\mathcal{Y} \to \mathcal{Y}$ be the canonical morphism and let $f_n : \mathcal{I}^n \to \mathcal{Y}^n$ be its restriction to $\mathcal{I}^n$. By Lemma 2.2.1, we have that:

$$\dim \mathcal{I}^n = n + \dim \mathcal{Y}^n.$$

Note that $\dim \mathcal{I}^0 = \dim \mathcal{Y}^0$. Now, suppose $\mathcal{Y}$ is good. This implies $\dim \mathcal{I}^n \leq \dim \mathcal{Y}$ for $n > 0$. By the definition of dimension, there exists an $n \geq 0$ such that $\dim \mathcal{I}^n = \dim \mathcal{I}_\mathcal{Y}$. It follows that $\dim \mathcal{I}_\mathcal{Y} \leq \dim \mathcal{Y}$.

Now, suppose $\dim \mathcal{I}_\mathcal{Y} \leq \dim \mathcal{Y}$. We have that:

$$n + \dim \mathcal{Y}^n = \dim \mathcal{I}^n \leq \dim \mathcal{Y},$$

for all $n \geq 0$. Therefore, we obtain that $\text{codim } \mathcal{Y}^n \geq n$ for all $n > 0$, and $\mathcal{Y}$ is good. □

From this theorem we can then obtain:

Corollary 2.2.3. *The stack $\mathcal{Y}$ is very good if and only if $\dim(\mathcal{I}_\mathcal{Y} - \mathcal{I}^0) < \dim \mathcal{Y}$.*

Similarly, we have:

Corollary 2.2.4. *Let m and I^n be as before. The stack $\mathcal{Y}$ is almost very good if and only if $\dim(\mathcal{I}_\mathcal{Y} - \coprod_{i=0}^m \mathcal{I}^i) < \dim \mathcal{Y}$.*

Let X be a variety over $\mathbb{C}$, and let G be an algebraic group over $\mathbb{C}$, acting on X. Consider the quotient stack $\mathcal{Y} = g\backslash X$. For $y \in Y$, we have that dim $\mathrm{Aut}(y) =$ dim G_x, where G_x is the stabilizer subgroup of a point $x \in X$ corresponding to y.

If $Y \subset X$ is a G-stable constructible subset, then we define the *number of parameters* (see e.g. [4] or [5]) of G on Y as

$$\dim_G Y = \max_s\{\dim Y \cap X_s + s - \dim G\},$$

where $X_s = \{x \in X | \dim G_x = s\}$.

We can easily see that the number of parameters for $Y = X$ is simply the dimension of the inertia stack associated to the quotient stack $\mathcal{Y}$. Therefore, by Theorem 2.2.2, the good condition on $G\backslash Y$ is equivalent to

$$\dim_G X \leq \dim X - \dim G.$$

Similarly, we can apply Corollary 2.2.3 in order to obtain that $\mathcal{Y}$ is very good if and only if

$$\dim_G X_n < \dim X - \dim G \text{ for all } n > 0.$$

2.3. **The very good property and the moment map.** Let X be a smooth algebraic variety over $\mathbb{C}$ with a semisimple complex group G acting on it. This gives rise to a natural Hamiltonian G-action on the cotangent bundle T^*X equipped with the standard symplectic form. There is a moment map $\mu : T^*X \to \mathfrak{g}^*$, defined by:

$$\mu(y)(\xi) = y(\xi_X(x)),$$

where $\mathfrak{g}$ is the Lie algebra of G, $y \in T^*_x X$, and ξ_X is the vector field on X induced by $\xi \in \mathfrak{g}$. It is clear from the above description that μ is linear on each cotangent space $T^*_x X$. Therefore, the image is a vector subspace of $\mathfrak{g}^*$.

Lemma 2.3.1. *The image $\mu(T^*_x X)$ is the annihilator of $\mathfrak{g}_x$, where $\mathfrak{g}_x$ is the Lie algebra of the stabilizer of $x \in X$ under the action of G.*

Proof. Let $\mathfrak{g}_x^\perp$ be the annihilator of $\mathfrak{g}_x$ and consider $\xi \in \mathfrak{g}_x$. For $y \in X$, let $f_y : G \to X$ be the map that takes $g \in G$ to $g \cdot y$. By definition,

$$\xi_X(x) = (df_x)_e(\xi),$$

where e is the identity element of G, so we have that $\xi_X(x) = 0$. Therefore, $\mu(T^*_x X) \subset \mathfrak{g}_x^\perp$. We can compute the dimension of $\mu(T^*_x X)$ as

$$\dim \mu(T^*_x X) = \dim T^*_x X - \dim \ker \mu|_{T^*_x X}\,.$$

Let $V \subset T_x X$ be the vector subspace spanned by $\xi_X(x)$ for all $\xi \in \mathfrak{g}$. By definition, $\ker \mu|_{T^*_x X}$ is the annihilator of V. Therefore, we have that

$$\dim \ker \mu|_{T^*_x X} = \dim T_x X - \dim \mathrm{V}.$$

Note that $\mathfrak{g}_x$ contains all $\xi \in \mathfrak{g}$ such that $\xi_X(x) = 0$. It follows that dim V $=$ dim $\mathfrak{g} -$ dim $\mathfrak{g}_x$. Thus:

$$\dim \mu(T^*_x X) = \dim \mathfrak{g} - \dim \mathfrak{g}_x = \dim \mathfrak{g}_x^\perp,$$

and $\mu(T^*_x X) = \mathfrak{g}_x^\perp$. □

Note that the moment map is algebraic, so the fiber $\mu^{-1}(\theta)$ is a closed algebraic subvariety of T^*X for any $\theta \in \mathfrak{g}^*$. We are now ready to prove the following theorem:

Theorem 2.3.2. *If the quotient stack $G\backslash X$ is very good, then for any $\theta \in \mathfrak{g}^*$ we have that $\mu^{-1}(\theta)$ is a nonempty, equidimensional complete intersection of dimension $2\dim X - \dim G$. Moreover, there is a bijective correspondence between the irreducible components of $\mu^{-1}(\theta)$ and the irreducible components of X.*

Proof. Let $x \in X$ and let $\pi : T^*X \to X$ be the natural projection. By Lemma 2.3.1 we have that

$$\dim \mu(\pi^{-1}(x)) = \dim \mathfrak{g}^* - \dim G_x.$$

Let $X_0 = \{x \in X | \dim G_x = 0\}$. We have that $\pi^{-1}(X_0) \cap \mu^{-1}(\theta)$ is nonempty. Since $G\backslash X$ is very good, then X_0 is nonempty. Consequently, μ is surjective, and we have:

$$\dim \mu^{-1}(\theta) \geq= 2\dim X - \dim G.$$

In fact, for every irreducible component I of $\mu^{-1}(\theta)$ we have that $\dim I \geq 2\dim X - \dim G$.

Let p be the restriction of π to $\mu^{-1}(\theta)$ and let I be an irreducible component of $\mu^{-1}(\theta)$, as above. Since X is stratified by the dimension of the stabilizer of the G-action, there exists an $m \geq 0$ such that

$$\dim X - \dim G + m = \dim I - \dim p(I).$$

If $m > 0$, by the very good property for the quotient stack $G\backslash X$ we have the following:

$$2\dim X - \dim G > \dim X - \dim G + m + \dim p(I) = \dim I,$$

which is impossible by our previous estimate from below. In that case $m = 0$, and $\dim I = 2\dim X - \dim G$. It follows that $\mu^{-1}(\theta)$ is an equidimensional complete intersection of dimension $2\dim X - \dim G$.

Let $Z \subset X$ be an irreducible component of X. Since $G\backslash X$ is very good, then X_0 intersects Z. Moreover, X_0 is open in X, so $Y := Z \cap X_0$ is irreducible and open.

We have that $p^{-1}(Y)$ is irreducible in $\mu^{-1}(\theta)$, since Y is irreducible and the fibers of p are isomorphic to $\mathbb{C}^{\dim X}$. It follows that $p^{-1}(Y)$ must be contained entirely in some irreducible component of $\mu^{-1}(\theta)$.

This means there is a correspondence between the irreducible components of X and the irreducible components of $\mu^{-1}(\theta)$. Since X is smooth, its irreducible components are disjoint, and therefore the correspondence is injective. It is also surjective, because the above computation implies $p^{-1}(X_0)$ intersects each irreducible component of $\mu^{-1}(\theta)$. □

We immediately obtain the following corollary:

Corollary 2.3.3. *If X is irreducible and the quotient stack $G\backslash X$ is very good, then $\mu^{-1}(\theta)$ is a nonempty, irreducible, complete intersection of dimension $2\dim X - \dim G$.*

Remark 2.3.4. If we assume the quotient stack $G\backslash X$ merely to be good, then the result that $\mu^{-1}(\theta)$ is an equidimensional complete intersection of dimension $2\dim X - \dim G$ still holds.

Remark 2.3.5. Note that even if G is not assumed to be semisimple, then Lemma 2.3.1 still holds. Let $X_s = \{x \in X | \dim G_x = s\}$. If the quotient stack $G\backslash X$ is only almost very good for a given m, then Theorem 2.3.2 and Remark 2.3.4 still hold, as long as $\mu(\pi^{-1}(X_m))$ contains θ, with the exception that

$$\dim \mu^{-1}(\theta) = 2\dim X - \dim G + m.$$

3. Quivers and their Representations

3.1. **Preliminaries.** Before proceeding with the proof of Theorem 1.3.1, we will consider the very good property for the quotient stack of quiver representations (in coordinate spaces) by the change of basis action at each vertex. This example is related to the special case of Theorem 1.3.1, when the vector bundle underlying the parabolic bundles is trivial. We will largely follow the arguments outlined in Section 6 of [4] and Sections 1-4 of [5], since his results imply ours.

Let Q be a finite, loop-free quiver, with vertices I_Q and arrows A_Q. Let $\mathrm{Rep}(Q,\alpha) = \bigoplus_{a\in A_Q} \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, K)$ be the complex variety of its representations in the standard coordinate spaces over an algebraically closed field K. The dimensions of these coordinate spaces can be encoded as the *dimension vector* $\alpha = (\alpha_i)_{i\in I_Q}$. The elements of $\mathrm{Rep}(Q,\alpha)$ may be thought of as left modules of the path algebra $R(Q)$. The group $G(\alpha) = \prod_{i\in I_Q} \mathrm{GL}(\alpha_i, K)/K^*$ acts on $\mathrm{Rep}(Q,\alpha)$ by change of basis at each vertex $i \in I_Q$.

The *Euler-Ringel form* is defined as follows:

$$\langle \alpha, \beta\rangle = \sum_{i\in I_Q} \alpha_i\beta_i - \sum_{a\in A_Q} \alpha_{t(a)}\beta_{h(a)}.$$

Let $q(\alpha) = \langle\alpha,\alpha\rangle$ be the associated Tits quadratic form, and set $p(\alpha) = 1 - q(\alpha)$ following [4].

Recall that a dimension vector α is in the *fundamental region* if it is nonzero, has connected support, and satisfies the following inequalities:

$$2\alpha_i - \sum_{a:i\to j} \alpha_j - \sum_{a:l\to i} \alpha_l \le 0 \quad \forall i \in I_Q,$$

where the sums are taken over all arrows going into i and coming out of i.

The symmetrized Euler-Ringel form $(\cdot,\cdot)$ defines a generalized Cartan matrix (see [20] for details). Therefore, we can associate to Q a Kac-Moody Lie algebra and consider a subset of the dimension vectors as roots of this algebra. The fundamental region consists of integer points of $-C^\vee$, where $C^\vee$ is the dual of the fundamental chamber of the Weyl group associated with the Kac-Moody algebra.

3.2. **The very good property for quiver representations.** The contents of this section largely follow Crawley-Boevey in [4] and [5]. Let Q be a finite loop-free quiver, and fix $\alpha \in \mathbb{Z}_{\ge 0}^{I_Q}$.

Let $\mathrm{Ind}(Q,\beta^{(1)},\dots,\beta^{(l)})$ be the $G(\alpha)$-stable constructible set consisting of all quiver representations that can be written as the sum of indecomposable representations of dimension types $\beta^{(1)},\dots,\beta^{(l)}$, where $\alpha = \sum_i \beta^{(i)}$. Since $\mathrm{Rep}(Q,\alpha)$ is the union of all the $\mathrm{Ind}(Q,\beta^{(1)},\dots,\beta^{(l)})$, the following lemma holds.

Lemma 3.2.1. *We have* $\dim_{G(\alpha)} Rep(Q,\alpha) = max\,\{\dim_{G(\alpha)} Ind(Q,\beta^{(1)},..,\beta^{(l)})\}$, *where the maximum is taken over all decompositions into indecomposables of dimensions* $\beta^{(1)},\dots,\beta^{(l)}$.

Note that by the Kac Theorem ([18] and [19]) the dimension vectors $\beta^{(1)} \dots \beta^{(l)}$ are actually positive roots of the Kac-Moody algebra corresponding to Q. We can now prove the following result (compare with Lemma 4.3 in [5]):

Theorem 3.2.2. *Let one of the following hold:*

(1) *The maximum in Lemma 3.2.1 is achieved for $l = 1$.*
(2) *The maximum in Lemma 3.2.1 is achieved for $l \geq 2$, and for the corresponding collection $\beta^{(1)}, ..., \beta^{(l)}$ we have $p(\alpha) > \sum_i p(\beta^{(i)})$.*

Then the stack $G(\alpha)\backslash Rep(Q, \alpha)$ is very good.

Proof. The case when $l = 1$ is discussed below. Assume the second case holds. We have $\dim_{G(\alpha)}\mathrm{Rep}(Q, \alpha) = \dim_{G(\alpha)}\mathrm{Ind}(Q, \beta^{(1)}, \dots, \beta^{(l)})$, for some $\beta^{(1)}, \dots, \beta^{(l)}$ with $l \geq 2$. By Lemma 4.3 in [5], we have

$$\dim_{G(\alpha)}\mathrm{Ind}(Q, \beta^{(1)}, \dots, \beta^{(l)}) = \sum_i p(\beta^{(i)}).$$

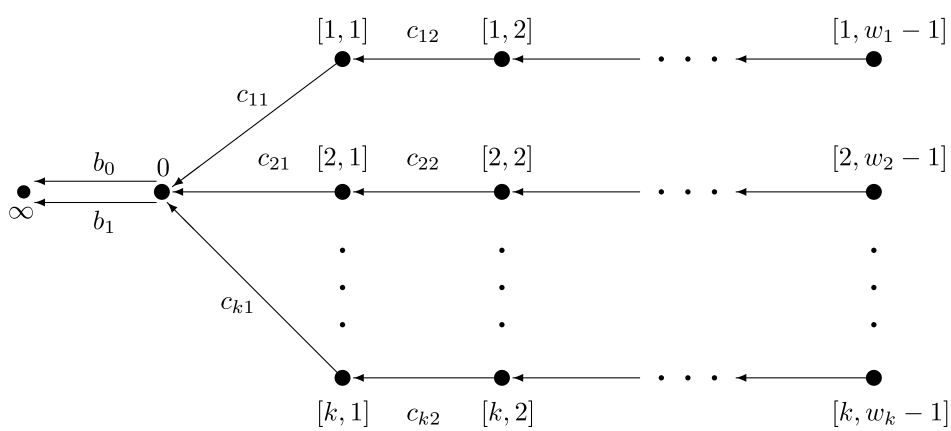
Now, $\dim \mathrm{Rep}(Q, \alpha) = \dim G(\alpha) + p(\alpha)$ and Corollary 2.2.3 imply that the very good condition on $\mathrm{Rep}(Q, \alpha)$ holds if $\dim_{G(\alpha)}\mathrm{Rep}(Q, \alpha) < p(\alpha)$. This, however, is clearly true since $\dim_{G(\alpha)}\mathrm{Rep}(Q, \alpha) = \sum_i p(\beta^{(i)})$, for the decomposition $\alpha = \sum_i \beta^{(i)}$. □

Note that a key argument in Lemma 4.3 from [5] is the Kac Theorem, which computes the number of parameters $\dim_{G(\alpha)} \mathrm{Ind}(Q, \alpha) = p(\alpha)$. If $l = 1$ in Theorem 3.2.2, then the fact that $G(\alpha)\backslash\mathrm{Rep}(Q, \alpha)$ is very good follows from Lemma 4 in Section 6 in [4]. Thus, we obtain:

Theorem 3.2.3. *Suppose α is in the fundamental region and $p(\alpha) > \sum_i p(\beta^{(i)})$ for any decomposition $\alpha = \sum_i \beta^{(i)}$ into the sum of two or more dimension vectors, then the quotient stack $G(\alpha)\backslash Rep(Q, \alpha)$ is very good.*

Note that in the statement of the theorem it suffices for $\beta^{(i)}$ to be roots of the Kac-Moody algebra associated with Q.

3.3. **Squids and Star-shaped Quivers.** Let $D = (x_1, \dots, x_k)$ be a collection of points of $\mathbb{P}^1$, and let $w = (w_1, \dots, w_k)$ be a collection of positive integers. Consider the following quiver $Q_{D,w}$:



Recall that $R(Q_{D,w})$ denotes the path algebra corresponding to the above quiver. A *squid* (see e.g. [7]) is the following algebra:

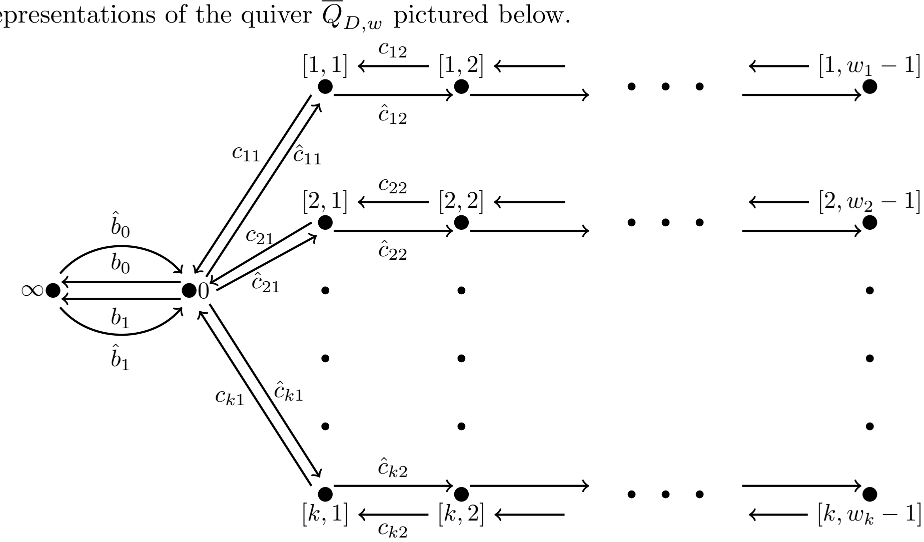
$$S_{D,w} = R(Q_{D,w})/\{(\lambda_{i0}b_0 + \lambda_{i1}b_1)c_{i1}\},$$

where $x_i = (\lambda_{i0} : \lambda_{i1})$.

- The part of $Q_{D,w}$ consisting of the vertices $\{0, \infty\}$ and the arrows $\{b_0, b_1\}$ is called the *Kronecker quiver*.
- The quiver $Q^{st}_{D,w}$ with vertex set $I_{Q_{D,w}} - \{\infty\}$ and arrow set $A_{Q_{D,w}} - \{b_0, b_1\}$ is called a *star-shaped quiver*.

Note that we can identify representations of a star-shaped quiver with representations of the corresponding $Q_{D,w}$ that have $\alpha_\infty = 0$. A representation of the Kronecker quiver is called *preinjective* if $\lambda_0 b_0 + \lambda_1 b_1$ is surjective for all $(\lambda_0 : \lambda_1) \in \mathbb{P}^1$. A representation of $Q_{D,w}$ is called *Kronecker-preinjective*, if the corresponding Kronecker quiver representation is preinjective.

3.4. **The cotangent bundle for squids.** The cotangent bundle $T^*\mathrm{Rep}(Q_{D,w}, \alpha)$ to the space of representations $\mathrm{Rep}(Q_{D,w}, \alpha)$ may be identified with the space of representations of the quiver $\overline{Q}_{D,w}$ pictured below.



Recall from Section 3.3 that a squid representation is a representation of $Q_{D,w}$ such that $(\lambda_{i0}b_0 + \lambda_{i1}b_1)c_{i1} = 0$. Further recall that $KS(D, w, \alpha)$ is the space of Kronecker-preinjective squid representations, such that the arrows c_{ij} are injective (see Section 3.3 for details).

Squid representations form a closed subvariety of representations of $Q_{D,w}$. Therefore, it follows $T^*KS(D, w, \alpha)$ may be identified with the quotient of $\mathrm{Rep}(\overline{Q}_{D,w}, \alpha)$ such that:

- The maps $\hat{b}_0 \in \mathrm{Hom}(\mathbb{C}^{\alpha_\infty}, \mathbb{C}^{\alpha_0})$ are taken modulo the relations $\lambda_{0i}c_{1i}A_i = 0$, where $A_i : \mathbb{C}^{\alpha_\infty} \to \mathbb{C}^{\alpha_{ij}}$ are linear maps.
- The maps $\hat{b}_1 \in \mathrm{Hom}(\mathbb{C}^{\alpha_\infty}, \mathbb{C}^{\alpha_0})$ are taken modulo the relations $\lambda_{1i}c_{1i}A_i = 0$.
- The maps $\hat{c}_{1i} \in \mathrm{Hom}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_{i1}})$ modulo the relations $A_i(\lambda_{0i}b_0 + \lambda_{1i}b_1) = 0$.

Recall from Section 3.1 that the group

$$G(\alpha) = \mathrm{GL}(\alpha_\infty, \mathbb{C}) \times \mathrm{GL}(\alpha_0, \mathbb{C}) \times \prod \mathrm{GL}(\alpha_{ij}, \mathbb{C})/\mathbb{C}^*$$

acts on $\mathrm{Rep}(Q_{D,w}, \alpha)$ by change of basis. This action induces a canonical Hamiltonian action of $G(\alpha)$ on $T^*\mathrm{Rep}(Q_{D,w}, \alpha)$. Identifying $\mathrm{Rep}(\overline{Q}_{D,w}, \alpha)$ with its tangent space at a point, the standard symplectic form on $T^*\mathrm{Rep}(Q_{D,w}, \alpha)$ may be written as:

$$\omega(X, X') = \sum_{l=0,1} \mathrm{tr}(b_l \hat{b}'_l) - \mathrm{tr}(b'_l \hat{b}_l) + \sum_{\substack{1\leq i\leq k \\ 1\leq j\leq w_i - 1}} \mathrm{tr}(c_{ij}\hat{c}'_{ij}) - \mathrm{tr}(c'_{ij}\hat{c}_{ij}),$$

where $X = (b_0, b_1, c_{ij}, \hat{b}_0, \hat{b}_1, \hat{c}_{ij})$ and $X' = (b'_0, b'_1, c'_{ij}, \hat{b}'_0, \hat{b}'_1, \hat{c}'_{ij})$ are cotangent vectors. Recall that

$$\mathrm{Mat}(\alpha) = \mathrm{Mat}(\alpha_\infty, \mathbb{C}) \times \mathrm{Mat}(\alpha_0, \mathbb{C}) \times \prod_{ij} \mathrm{Mat}(\alpha_{ij}, \mathbb{C}).$$

Using the trace pairing, we can identify $\mathrm{Lie}(G(\alpha))^*$ with

$$\mathrm{Mat}(\alpha)_0 = \{(A_i) \in \mathrm{Mat}(\alpha) | \sum_i \mathrm{tr}(A_i) = 0\}.$$

Note that $KS(D, w, \alpha)$ is invariant under the $G(\alpha)$ action, and the symplectic form defined above descends to the cotangent bundle $T^*KS(D, w, \alpha)$. Therefore, we can write the corresponding moment map as:

$$\mu_{G(\alpha)}(X)_\infty = b_0\hat{b}_0 + b_1\hat{b}_1$$
$$\mu_{G(\alpha)}(X)_0 = \sum_{1\leq i\leq k} c_{i1}\hat{c}_{i1} - (\hat{b}_0 b_0 + \hat{b}_1 b_1)$$
$$\mu_{G(\alpha)}(X)_{ij} = c_{ij+1}\hat{c}_{ij+1} - \hat{c}_{ij}c_{ij} \text{ where } 1 \leq i \leq k \text{ and } 1 \leq j \leq w_i - 1 \ ,$$

at the vertices ∞, 0, and $[i, j]$, respectively.

3.5. **The very good property for star-shaped quivers.** We can simplify the statement of Theorem 3.2.3 if the quiver we are considering is a star-shaped quiver $Q^{st}_{D,w}$, described above in Section 3.3. The indexing set for the vertices of $Q^{st}_{D,w}$ is $I_{Q^{st}_{D,w}} = \{0\} \cup \{(i, j) | 1 \leq i \leq k, 1 \leq j \leq w_{i-1}\}$. This means a dimension vector of a representation of $Q^{st}_{D,w}$ has the form $\alpha = (\alpha_0, \alpha_{ij})$.

Recall that $\delta(\alpha) = -2\alpha_0 + \sum_j \alpha_{ij}$. In the case of a star-shaped quiver, the condition that a dimension vector α is in the fundamental region is equivalent to the following inequalities:

$$\delta(\alpha) \geq 0$$
$$-2\alpha_{ij} + \alpha_{ij-1} + \alpha_{ij+1} \geq 0, \text{ for } 1 \leq i \leq k \text{ and } 1 \leq j \leq w_{i-1}$$

(note that we assume $\alpha_{i0} = \alpha_0$, for all i). We wish to prove:

Theorem 3.5.1. *Suppose $\delta(\alpha) > 0$ and α is in the fundamental region, then the quotient stack $G(\alpha)\backslash Rep(Q^{st}_{D,w}, \alpha)$ is very good.*

Recall from Section 3.1 that we can symmetrize the Euler-Ringel form, in order to define a bilinear symmetric form on dimension vectors of quiver representations.

For the quiver $Q^{st}_{D,w}$ this form can be written as:

$$(\alpha,\beta)=2\alpha_0\beta_0-\sum_{i=1}^{k}\beta_0\alpha_{i1}+\sum_{i=1}^{k}\sum_{j=1}^{w_i-1}2\beta_{ij}\alpha_{ij}-\beta_{ij}\alpha_{ij-1}-\beta_{ij}\alpha_{ij+1},$$

where $\alpha_{iw_i}=0$, $\alpha_{i0}=\alpha_0$ and where $\beta_{iw_i}=0$, $\beta_{i0}=\beta_0$. The associated Tits quadratic can be expressed as:

$$q(\alpha)=\alpha_0^2-\sum_{1\le i\le k}\alpha_0\alpha_{i1}+\frac{1}{2}\sum_{1\le i\le k}\alpha_{i1}^2+\sum_{1\le i\le k}\sum_{1\le j\le w_i-1}\frac{1}{2}(\alpha_{ij}-\alpha_{ij+1})^2,$$

where $\alpha_{iw_i}=0$ and $\alpha_{i0}=\alpha_0$. Recall that $p(\alpha)=1-q(\alpha)$. Note that the Tits form can be defined on real vectors instead of integer vectors. We distinguish the real version from the integer version by writing $q(x)$, instead of $q(\alpha)$, where $x=(x_0,x_{ij})$ is indexed by $I_{Q^{st}_{D,w}}$.

To prove the theorem, it suffices to show that $\delta(\alpha)>0$ and α in the fundamental region imply that $p(\alpha)>\sum_i p(\beta^{(i)})$ for any decomposition $\alpha=\sum_i\beta^{(i)}$ into the sum of nonzero dimension vectors. However, before proving the inequality on $p(\alpha)$, we need several facts about the signature of $q(x)$. Note that the signature will consist of a triple (n_+,n_-,n_0), corresponding to the positive index of inertia, the negative index of inertia, and the nullity, respectively. It is easy to see that:

Proposition 3.5.2. *Assume $q(x)$ has rank n. On the $(n-1)$-dimensional subspace defined by $x_0=0$, we have that $q(x)$ is positive definite.*

Corollary 3.5.3. *Assume $q(x)$ has rank n. The signature of $q(x)$ can be $(n,0,0)$, $(n-1,0,1)$, or $(n-1,1,0)$.*

The ordering on the elements of α, such that $\alpha_{ij-1}-\alpha_{ij}\ge\alpha_{ij}-\alpha_{ij+1}$, together with $\delta(\alpha)>0$, imply that α is in the fundamental region.

Proposition 3.5.4. *Suppose $\delta(\alpha)>0$ and α is in the fundamental region, then $p(\alpha)>\sum_i p(\beta^{(i)})$, for any decomposition $\alpha=\sum_i\beta^{(i)}$ into the sum of two or more vectors in $\mathbb{Z}_{\ge 0}^{I_{Q^{st}}}$.*

Proof. Note that the necessary inequality may be rewritten as

$$\sum_i q(\beta^{(i)})-q(\alpha)>l-1.$$

We proceed by induction on l. Consider the base case when $l=2$. In this case, we prove that the inequality holds for $\alpha=\beta+\gamma$. We can directly compute

$$(\alpha,\beta)=\beta_0(2\alpha_0-\sum_{i=1}^{k}\alpha_{i1})+\sum_{i=1}^{k}\sum_{j=1}^{w_i-1}\beta_{ij}(2\alpha_{ij}-\alpha_{ij-1}-\alpha_{ij+1})\le 0.$$

Similarly, we obtain $(\alpha,\gamma)\le 0$. By Corollary 3.5.3, signature of $q(x)$ can be $(n,0,0)$, $(n-1,0,1)$, or $(n-1,1,0)$. Since $q(\alpha)<0$ it is $(n-1,1,0)$. Restrict $q(x)$ to the subspace spanned by α and β. On this space the signature of $q(x)$ is $(1,1,0)$. By the Gram-Schmidt process there is an orthogonal basis for this space containing α. That means we can write

$$\beta=a_1\alpha+\delta_1$$
$$\gamma=a_2\alpha+\delta_2,$$

where a_i are nonnegative with $a_1 + a_2 = 1$, $\delta_1 + \delta_2 = 0$, $(\alpha, \delta_i) = 0$ and $q(\delta_i) \geq 0$ ($q(\delta_i) = 0$ only if $\delta_i = 0$), for all i. It follows that

$$q(\beta) + q(\gamma) - q(\alpha) = -(\beta, \gamma) = -a_1 a_2(\alpha, \alpha) - (\delta_1, \delta_2) \geq 1,$$

since the last sum is positive and $-(\beta, \gamma)$ is an integer. Therefore, we have $(\beta, \beta) > (\alpha, \beta) \geq (\alpha, \alpha)$ hence $q(\beta) - q(\alpha) > 0$. Similarly, we also have $q(\gamma) - q(\alpha) > 0$.

We proceed by considering cases. Let us first suppose that $q(\beta) \neq 0$ and $q(\gamma) \neq 0$. We can assume without loss of generality that $(\alpha, \beta) \leq (\alpha, \gamma)$. We will suppose $(\beta, \gamma) = -1$ and arrive at a contradiction. From the previous decomposition in the orthogonal basis, we obtain that $a_1 \geq a_2$. Therefore,

$$(\gamma, \gamma) = a_2^2(\alpha, \alpha) - (\delta_1, \delta_1) \geq a_1 a_2(\alpha, \alpha) + (\delta_1, \delta_1) = -1,$$

and it follows that $q(\gamma) \geq -\frac{1}{2}$. Since $q(\gamma)$ is an integer we have $q(\gamma) > 0$. Together with $q(\beta) - q(\alpha) > 0$ this gives us $q(\beta) + q(\gamma) - q(\alpha) > 1$, which is what we need. Now suppose $q(\beta) = 0$. We have that

$$(\beta, \gamma) = (\beta, \alpha) = \beta_0(2\alpha_0 - \sum_i \alpha_{i1}) + \sum_{ij} \beta_{ij}(2\alpha_{ij} - \alpha_{ij-1} - \alpha_{ij+1}).$$

Since $\delta(\alpha) > 0$, we have that $2\alpha_0 - \sum_i \alpha_{i1} \leq -1$. Thus, for $\beta_0 \geq 2$ and α in the fundamental region we have $-(\beta, \gamma) > 1$, contradicting our assumption that $(\beta, \gamma) = -1$. If $\beta_0 = 0$, then we have

$$q(\beta) = \frac{1}{2} \sum_{1 \leq i \leq k} \beta_{i1}^2 + \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq w_i - 1} \frac{1}{2}(\beta_{ij} - \beta_{ij+1})^2 > 0,$$

for nontrivial β. This contradicts the original assumption that $q(\beta) = 0$. If $\beta_0 = 1$, then we can show

$$q(\beta) = 1 - \sum_{1 \leq i \leq k} \beta_{i1} + \frac{1}{2} \sum_{1 \leq i \leq k} \beta_{i1}^2 + \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq v_i - 1} \frac{1}{2}(\beta_{ij} - \beta_{ij+1})^2 + \frac{1}{2} \sum_{1 \leq i \leq k} \beta_{iv_i}^2 > 0,$$

where v_i is the maximal entry with $\beta_{iv_i} \neq 0$. Indeed, the inequality is valid since $\frac{1}{2}\beta_{i1}^2 + \frac{1}{2}\beta_{iv_i}^2 - \beta_{i1} \geq 0$. Again this contradicts the assumption that $q(\beta) = 0$. This covers all of the possibilities for β. A similar argument works if $q(\gamma) = 0$. Hence, in all cases $q(\beta) + q(\gamma) - q(\alpha) > 1$.

By induction we may assume that:

$$\begin{aligned}
&q(\beta^{(1)}) + \cdots + q(\beta^{(l)}) - q(\alpha) \\
&= q(\beta^{(1)}) + \cdots + q(\beta^{(i)} + \beta^{(j)}) - (\beta^{(i)}, \beta^{(j)}) + \cdots + q(\beta^{(l)}) - q(\alpha) \\
&> l - 2 - (\beta^{(i)}, \beta^{(j)}),
\end{aligned}$$

for any choice $i \neq j$. Therefore, it suffices to prove that there exist differing $1 \leq i, j \leq l$ such that $(\beta^{(i)}, \beta^{(j)}) < 0$. Consider the the subspaces spanned by $\alpha, \beta^{(i)}$. As in the $l = 2$ case, each such space has an orthogonal basis consisting of α and a vector on which $q(x)$ is positive. It follows that for each i we have $\beta^{(i)} = a_i\alpha + \delta_i$, with nonnegative a_i such that $a_1 + \cdots + a_l = 1$, $\delta_1 + \cdots + \delta_l = 0$, $(\alpha, \delta_i) = 0$, and $q(\delta_i) \geq 0$. Note that $q(\delta_i) = 0$ only when $\delta_i = 0$. Now fix $\beta^{(i_0)}$. If $\delta_{i_0} = 0$, then $1 > a_{i_0} > 0$. There is a $j_0 \neq i_0$ such that

$$(\beta^{(i_0)}, \beta^{(j_0)}) = a_{i_0} a_{j_0}(\alpha, \alpha) < 0.$$

Otherwise, we have: $\sum_i (\delta_i, \delta_{i_0}) = 0$, so for some $\beta^{(j_0)}$ it is true that $(\delta_{i_0}, \delta_{j_0}) < 0$, because $(\delta_{i_0}, \delta_{i_0}) > 0$. It follows that

$$(\beta^{(i_0)}, \beta^{(j_0)}) = a_{i_0} a_{j_0} (\alpha, \alpha) + (\delta_{i_0}, \delta_{j_0}) < 0.$$

So, Proposition 3.5.4 is proven. □

Proof of Theorem 3.5.1. The theorem follows from Theorem 3.2.3 and Proposition 3.5.4. □

4. Moduli of Parabolic Bundles

4.1. **Parabolic Bundles.** In this section we will that the moduli stack of parabolic bundles over $\mathbb{P}^1$ is almost very good under some restrictions on the parabolic structure. Our proof resembles Crawley-Boevey's arguments in [4] and [5]. However, Kac's theorem is inapplicable, and we replace it with an algebro-geometric result that works in the case of nontrivial parabolic bundles.

Let X be a smooth complex projective curve, $D = x_1 + \cdots + x_k$, and $w = (w_1, \dots, w_k)$ be a collection of positive integers. It is possible to generalize the notions such as "subbundle" or "morphism" from vector bundles to parabolic bundles on X with fixed weight type (D, w) (e.g. [26]). We denote the subsheaf of morphisms of parabolic bundles between $\mathbf{F}$ and $\mathbf{E}$ by $\mathscr{H}om_{\mathrm{Par}}(\mathbf{F}, \mathbf{E}) \subset \mathscr{H}om(F, E)$ and the subsheaf of endomorphisms by $\mathscr{E}nd_{\mathrm{Par}}(\mathbf{E}) \subset \mathscr{E}nd(E)$.

Note that $\mathscr{H}om_{\mathrm{Par}}(\mathbf{F}, \mathbf{E})$ and $\mathscr{E}nd_{\mathrm{Par}}(\mathbf{E})$ are both vector bundles. Therefore, we can compute the Euler characteristic of $\mathscr{H}om_{\mathrm{Par}}(\mathbf{F}, \mathbf{E})$ by applying the Riemann-Roch theorem. Specifically, let $\mathbf{E}$ have dimension vector α and let $\mathbf{F}$ have dimension vector β. We obtain:

$$\chi(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F}, \mathbf{E})) = \beta_0 \cdot \deg(E) - \alpha_0 \cdot \deg(F) - g\alpha_0\beta_0 + \langle \beta, \alpha \rangle,$$

where $\langle \beta, \alpha \rangle$ is as in Section 3.1. Note that in the case when $g = 0$ and $\mathbf{F} = \mathbf{E}$, we obtain that $\chi(\mathscr{E}nd_{\mathrm{Par}}(\mathbf{E})) = q(\alpha)$.

4.2. **The moduli stack of parabolic bundles over $\mathbb{P}^1$.** Definitions and general properties of algebraic stacks are given in Laumon and Moret-Bailly's book [25]. We will view a stack as a sheaf of groupoids in the fppf-topology and an algebraic stack as a stack with a smooth presentation by a scheme. We will use $\langle \quad \rangle$ to denote a category in which the objects are enclosed by the brackets and the morphisms are all isomorphisms.

As before, let X be the smooth complex projective curve. Fix the weight type (D, w) as in Section 4.1. Let $I = \{0\} \cup \{(i,j) | 1 \leq i \leq k, 1 \leq j \leq w_i - 1\}$, let $d \in \mathbb{Z}$, and fix $\alpha \in \mathbb{Z}^I_{\geq 0}$, such that $\alpha_0 \geq \alpha_{i1} \geq \cdots \geq \alpha_{iw_i}$, for all i.

The stack of parabolic bundles of weight type (D, w), degree d, dimension type α, over X is a functor that associates to a test scheme T the groupoid $\mathrm{Bun}^d_{D,w,\alpha}(T) = \left\langle (E, E^{i,j})_{1 \leq i \leq k} \right\rangle$, where

- E is a vector bundle on $T \times X$,
- $E|_{T \times \{x_i\}} \supset E^{i,1} \supset \cdots \supset E^{i,w_i - 1} \supset E^{i,w_i} = 0$ is a filtration by vector bundles,
- $\mathrm{rk}(E) = \alpha_0$ and $\mathrm{rk}(E^{i,j}) = \alpha_{ij}$,
- $\deg E|_{\{y\} \times \mathbb{P}^1} = d$ for all $y \in T$.

In the case when $X = \mathbb{P}^1$, we see that $\mathrm{Bun}^d_{D,w,\alpha}$ admits the following presentation as an algebraic stack: $U = \coprod_{N \in \mathbb{Z}_{\geq 0}} \langle (\mathbf{E}, s_i, t_j) \rangle$, where

- $\mathbf{E}$ is a parabolic bundle on X,
- $\deg(E) = d$ and $\mathbf{E}$ has dimension vector α,
- $H^0(E^* \otimes \mathcal{O}(N))$ is generated by global sections,
- s_i is a basis for $H^0(E^* \otimes \mathcal{O}(N))^*$,
- t_j is a basis for $H^0(E^* \otimes \mathcal{O}(N-1))^*$.
- r_{ij} is a basis for E_{ij}

For $X = \mathbb{P}^1$, we will give a more detailed description of U in Section 5. Let $\mathcal{B} := \mathrm{Bun}_{D,w,\alpha}(X) = \coprod_{d\in\mathbb{Z}} \mathrm{Bun}^d_{D,w,\alpha}$ be the moduli stack of parabolic bundles of weight type (D, w) and with dimension vector α. We can use the presentation above to turn this stack into an algebraic stack.

Note that $\mathrm{Bun}_{D,w,\alpha}(X)$ is smooth, and by Lemma 2.2.1 we can compute its dimension as: dim $\mathrm{Bun}_{D,w,\alpha}(X) = (g-1)\alpha_0^2 + \alpha_0^2 - q(\alpha) = g\alpha_0^2 - q(\alpha)$.

From now on, let $X = \mathbb{P}^1$. This means $g = 0$, and therefore $\dim \mathrm{Bun}_{D,w,\alpha}(X) = -q(\alpha)$. Before proceeding with the proof of Theorem 1.3.1, we introduce serveral other stacks, which we will refer to in the proof. The descriptions of these as algebraic stacks may be easily obtained from the description of $\mathrm{Bun}_{D,w,\alpha}(X)$. Let $\mathcal{P}_\mathcal{B} := \mathcal{P}_{\mathrm{Bun}_{D,w,\alpha}(X)}$ be the stack of pairs $(\mathbf{E}, f)$, where $\mathbf{E}$ is in $\mathrm{Bun}_{D,w,\alpha}(X)$ and f is its endomorphism.

Let $\mathcal{I}_\mathcal{B} := \mathcal{I}_{\mathrm{Bun}_{D,w,\alpha}(X)}$ be the inertia stack corresponding to $\mathrm{Bun}_{D,w,\alpha}(X)$. Note that $\mathcal{P}_{\mathrm{Bun}_{D,w,\alpha}(X)}$ contains the inertia stack associated to $\mathrm{Bun}_{D,w,\alpha}(X)$ as an open substack. That is, it contains the stack $\mathcal{I}_{\mathrm{Bun}_{D,w,\alpha}(X)}$, consisting of pairs $(\mathbf{E}, f)$, where $\mathbf{E}$ is in $\mathrm{Bun}_{D,w,\alpha}(X)$ and f is its automorphism.

Similarly, it contains the reduced closed substack $\mathcal{N}(D, w, \alpha)$, consisting of pairs $(\mathbf{E}, f)$, where $\mathbf{E}$ is in $\mathrm{Bun}_{D,w,\alpha}(X)$ and f is its nilpotent endomorphism.

4.3. **Proof of Theorems 1.3.1 and 1.3.2.** Let us define

$$\tilde{q}(\alpha) = \min \sum q(\gamma_i),$$

where the minimum is taken over all positive, finite decompositions $\alpha = \sum_i \gamma_i$. We can summarize the properties of $\tilde{q}(\alpha)$ in the following obvious proposition:

Proposition 4.3.1. *Let α and β be dimension vectors. For $\tilde{q}(\alpha)$, we have:*
a) $\tilde{q}(\alpha) \le q(\alpha)$
b) $\tilde{q}(\alpha + \beta) \le \tilde{q}(\alpha) + \tilde{q}(\beta)$
c) $\tilde{q}(\alpha) = q(\alpha)$, if α is in the fundamental region.

Consider the two-element complex

$$C^\bullet : \mathscr{E}nd_{\mathrm{Par}}(\mathbf{W}) \to \mathscr{H}om_{\mathrm{Par}}(\mathbf{V}, \mathbf{W}),$$

induced by the inclusion of parabolic bundles $i : \mathbf{V} \hookrightarrow \mathbf{W}$. This complex arises when we consider first-order deformations of pairs $(\mathbf{W}, i)$, for a fixed $\mathbf{V}$. It follows that we can study the deformations of the pairs $(\mathbf{W}, i)$ by studying the hypercohomology groups of $C^\bullet$.

Lemma 4.3.2. *We have that $\mathbb{H}^2(C^\bullet) = 0$.*

Proof. Consider the chain complexes

$$A^\bullet : 0 \to \mathscr{E}nd_{\mathrm{Par}}(\mathbf{W})$$
$$B^\bullet : 0 \to \mathscr{H}om_{\mathrm{Par}}(\mathbf{V}, \mathbf{W}),$$

which are nontrivial only in degree 1. Since i induces the obvious chain map, we have an exact triangle $A^\bullet \to B^\bullet \to C^\bullet$, which gives rise to the long exact sequence for hypercohomology

$$\cdots \to \mathbb{H}^2(\mathbb{P}^1, A^\bullet) \to \mathbb{H}^2(\mathbb{P}^1, B^\bullet) \to \mathbb{H}^2(\mathbb{P}^1, C^\bullet) \to \mathbb{H}^3(\mathbb{P}^1, A^\bullet) \to \cdots .$$

Since $A^\bullet$ and $B^\bullet$ are only nontrivial in degree 1, we have both that $\mathbb{H}^2(\mathbb{P}^1, A^\bullet) = H^1(\mathbb{P}^1, \mathscr{E}nd_{\mathrm{Par}}(\mathbf{W}))$ and $\mathbb{H}^2(\mathbb{P}^1, B^\bullet) = H^1(\mathscr{H}om_{\mathrm{Par}}(\mathbf{V}, \mathbf{W}))$. We also obtain that $\mathbb{H}^3(\mathbb{P}^1, A^\bullet) = 0$. Hence, it follows that we have the exact sequence

$$H^1(\mathbb{P}^1, \mathscr{E}nd_{\mathrm{Par}}(\mathbf{W})) \to H^1(\mathscr{H}om_{\mathrm{Par}}(\mathbf{V}, \mathbf{W})) \to \mathbb{H}^2(\mathbb{P}^1, C^\bullet) \to 0.$$

Therefore, it follows $\mathbb{H}^2(\mathbb{P}^1, C^\bullet)$ is the cokernel of $i^* : H^1(\mathbb{P}^1, \mathscr{E}nd_{\mathrm{Par}}(\mathbf{W})) \to H^1(\mathscr{H}om_{\mathrm{Par}}(\mathbf{V}, \mathbf{W}))$. Applying Serre Duality, we obtain that $\mathbb{H}^2(\mathbb{P}^1, C^\bullet)$ is isomorphic to the dual of the kernel of

$$H^0(\mathscr{H}om_{\mathrm{Par}}(\mathbf{W}, \mathbf{V}) \otimes \Omega^1) \to H^0(\mathscr{E}nd_{\mathrm{Par}}(\mathbf{W}) \otimes \Omega^1).$$

However, this map comes from the inclusion of $\mathscr{H}om_{\mathrm{Par}}(\mathbf{W}, \mathbf{V}) \hookrightarrow \mathscr{E}nd_{\mathrm{Par}}(\mathbf{W})$, which is induced by i. Therefore, the map is injective, so the kernel is trivial. Thus, $\mathbb{H}^2(\mathbb{P}^1, C^\bullet) = 0$. □

Let $\mathbf{V}$ be a parabolic bundle over $\mathbb{P}^1$ and let $\mathcal{P}_{\mathbf{V}} = \mathcal{P}_{\mathbf{V}}(D, w, \alpha)$ be the algebraic stack consisting of pairs $\{\mathbf{W}, i : \mathbf{V} \hookrightarrow \mathbf{W}\}$, where i is an inclusion of parabolic bundles and $\mathbf{W}$ is a parabolic bundle of weight type (D, w) and dimension vector α.

Lemma 4.3.3. *Either $\mathcal{P}_{\mathbf{V}}(D, w, \alpha)$ is empty or we have*

$$\dim \mathcal{P}_{\mathbf{V}}(D, w, \alpha) = \chi(\mathscr{H}om_{Par}(\mathbf{V}, \mathbf{W})) - \chi(\mathscr{E}nd_{Par}(\mathbf{W})).$$

Proof. Assume that $\mathcal{P}_{\mathbf{V}}$ is nonempty. The dimension of $\mathcal{P}_{\mathbf{V}}$ is equal to the dimension of the corresponding tangent complex. We compute its dimension by considering the deformations of $(\mathbf{W}, i) \in \mathcal{P}_{\mathbf{V}}$. These deformations are governed by the hypercohomology of the complex $C^\bullet$, defined above. It follows that

$$\dim \mathcal{P}_{\mathbf{V}} = \dim \mathbb{H}^1(\mathbb{P}^1, C^\bullet) - \dim \mathbb{H}^0(\mathbb{P}^1, C^\bullet),$$

since $\mathbb{H}^2(C^\bullet) = 0$ by Lemma 4.3.2.

Let $\chi(D^\bullet)$ denote the Euler characteristic of the hypercohomology of a complex of sheaves $D^\bullet$ and let $A^\bullet, B^\bullet$ be as in Lemma 4.3.2. Since $\chi(D^\bullet)$ additive on exact triangles, we have that

$$\chi(C^\bullet) = \chi(B^\bullet) - \chi(A^\bullet).$$

Moreover, because $\chi(B^\bullet) = -\chi(\mathscr{H}om_{\mathrm{Par}}(\mathbf{V}, \mathbf{W}))$ and $\chi(A^\bullet) = -\chi(\mathscr{E}nd_{\mathrm{Par}}(\mathbf{W}))$, we can simplify this to

$$\chi(C^\bullet) = \chi(\mathscr{E}nd_{\mathrm{Par}}(\mathbf{W})) - \chi(\mathscr{H}om_{\mathrm{Par}}(\mathbf{V}, \mathbf{W})).$$

By Lemma 4.3.2, $\dim \mathcal{P}_{\mathbf{V}} = -\chi(C^\bullet)$. Thus,

$$\dim \mathcal{P}_{\mathbf{V}} = \chi(\mathscr{H}om_{\mathrm{Par}}(\mathbf{V}, \mathbf{W})) - \chi(\mathscr{E}nd_{\mathrm{Par}}(\mathbf{W})).$$

□

Let $\mathbf{F}, \mathbf{G}$ be parabolic bundles over $\mathbb{P}^1$, and let g be an endomorphism of $\mathbf{G}$. Let $D^\bullet$ be the following chain complex:

$$\mathscr{H}om_{\mathrm{Par}}(\mathbf{G}, \mathbf{F}) \to \mathscr{H}om_{\mathrm{Par}}(\mathbf{G}, \mathbf{F}),$$

where the connecting map is induced by g.

Lemma 4.3.4. *We can compute the following:* $\dim \mathbb{H}^1(\mathbb{P}^1, D^\bullet) - \dim \mathbb{H}^0(\mathbb{P}^1, D^\bullet) = \dim H^1(\mathscr{H}om_{Par}(\textbf{\textit{ker}}\ g, \mathbf{F}))$.

Proof. Since $D^\bullet$ consists of two copies of $\mathscr{H}om_{\mathrm{Par}}(\mathbf{G}, \mathbf{F})$ we can see (by the argument from Lemma 4.3.2) that the Euler characteristic for hypercohomology is 0. That is, we have:

$$\dim \mathbb{H}^1(\mathbb{P}^1, D^\bullet) - \dim \mathbb{H}^0(\mathbb{P}^1, D^\bullet) = \dim \mathbb{H}^2(\mathbb{P}^1, D^\bullet).$$

By Serre duality, $\mathbb{H}^2(\mathbb{P}^1, D^\bullet)$ is isomorphic to $\mathbb{H}^0$ for the complex

$$\mathscr{H}om_{\mathrm{Par}}(\mathbf{F}, \mathbf{G} \otimes \Omega^1_{\mathbb{P}^1}) \to \mathscr{H}om_{\mathrm{Par}}(\mathbf{F}, \mathbf{G} \otimes \Omega^1_{\mathbb{P}^1}),$$

where the connecting map is induced by $g \otimes \mathrm{Id}$. However, by definition, this is just:

$$H^0(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F}, (\mathbf{ker}\ g) \otimes \Omega^1_{\mathbb{P}^1})) \cong H^0(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F}, \mathbf{ker}\ g) \otimes \Omega^1_{\mathbb{P}^1}).$$

Applying Serre duality, we get:

$$\dim \mathbb{H}^1(\mathbb{P}^1, D^\bullet) - \dim \mathbb{H}^0(\mathbb{P}^1, D^\bullet) = \dim \mathbb{H}^2(\mathbb{P}^1, D^\bullet) = \dim H^1(\mathscr{H}om_{\mathrm{Par}}(\mathbf{ker}\ g, \mathbf{F})).$$

□

We need the following key argument:

Theorem 4.3.5. *We have the inequality* $\dim \mathcal{N}(D, w, \alpha) \leq -\tilde{q}(\alpha)$.

Proof. Let $(\mathbf{E}, f)$ be a point of $\mathcal{N}(D, w, \alpha)$. Let $\mathbf{F} = \mathbf{ker}\ f$ and $\mathbf{G} = \mathbf{E}/\mathbf{F}$. We wish to prove this theorem by induction on the rank of the vector bundle E (note that this is α_0 in our notation). To that end, it suffices to prove that for all β we have:

$$\dim \mathcal{N}_\beta(D, w, \alpha) \leq -\tilde{q}(\alpha),$$

where $\mathcal{N}_\beta(D, w, \alpha)$ is a substack consisting of objects $(\mathbf{E}, f)$ of $\mathcal{N}(D, w, \alpha)$ such that the corresponding $\mathbf{F}$ belongs to $\mathrm{Bun}_{D,w,\beta}(X)$. In order to accomplish this, consider the morphism

$$\phi : \mathcal{N}_\beta(D, w, \alpha) \to \mathcal{N}(D, w, \alpha - \beta),$$

which is defined by sending $(\mathbf{E}, f)$ to $(\mathbf{G}, f|_{\mathbf{G}}) \in \mathcal{N}(D, w, \alpha - \beta)$, with corresponding restrictions on the arrows. In this case, after applying the induction hypothesis, we get

$$\dim \mathcal{N}_\beta(D, w, \alpha) \leq \dim \mathcal{N}_\beta(D, w, \alpha)_x - \tilde{q}(\alpha - \beta),$$

for some $x = (\mathbf{G}, g) \in \mathcal{N}(D, w, \alpha - \beta)$. Now, we wish to compute the dimension of the fiber $\mathcal{X} = \mathcal{N}_\beta(D, w, \alpha)_x$. Let $\mathbf{F_1} = \mathbf{ker}\ g$ and let $\mathcal{X}' = \mathcal{P}_{\mathbf{F_1}}(D, w, \beta)$. In this case, we have two morphisms $\psi_1 : \mathcal{X} \to \mathrm{Bun}_{D,w,\beta}(X)$ and $\psi_2 : \mathcal{X}' \to \mathrm{Bun}_{D,w,\beta}(X)$, where ψ_1 sends the pair $(\mathbf{E}, f)$ to $\mathbf{ker}\ f$ and likewise ψ_2 sends $(\mathbf{F}, i)$ to $\mathbf{F}$.

The deformations of elements of the fiber $\mathcal{X}_\mathbf{F}$ are governed by the hypercohomology of the complex

$$\mathscr{H}om_{\mathrm{Par}}(\mathbf{G}, \mathbf{F}) \xrightarrow{g} \mathscr{H}om_{\mathrm{Par}}(\mathbf{G}, \mathbf{F}),$$

defined in Lemma 4.3.4. Therefore, by Lemma 4.3.4, we get that:

$$\dim \mathcal{X}_\mathbf{F} = \dim H^1(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F_1}, \mathbf{F})).$$

Furthermore, since f induces an injective morphism $\mathbf{ker}\ f^2/\mathbf{ker}\ f \to \mathbf{ker}\ f$, then the fiber $\mathcal{X}'_\mathbf{F}$ is nonempty. Therefore,

$$\dim \mathcal{X}'_\mathbf{F} = \dim H^0(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F_1}, \mathbf{F})).$$

Thus, $\dim \mathcal{X}_{\mathbf{F}} = \dim \mathcal{X}'_{\mathbf{F}} - \chi(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F_1}, \mathbf{F}))$. We have $\dim \mathcal{X} = \dim \mathcal{X}' - \chi(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F_1}, \mathbf{F}))$. So, we obtain

$$\dim \mathcal{N}_\beta(D, w, \alpha) \leq \dim \mathcal{X}' - \tilde{q}(\alpha - \beta) - \chi(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F_1}, \mathbf{F})).$$

It follows from Lemma 4.3.3 that $\dim \mathcal{X}' = \chi(\mathscr{H}om_{\mathrm{Par}}(\mathbf{F_1}, \mathbf{F})) - \chi(\mathscr{E}nd_{\mathrm{Par}}(\mathbf{F}))$, which means

$$\dim \mathcal{N}_\beta(D, w, \alpha) \leq -\chi(\mathscr{E}nd_{\mathrm{Par}}(\mathbf{F})) - \tilde{q}(\alpha - \beta).$$

Since $\chi(\mathscr{E}nd_{\mathrm{Par}}(\mathbf{F})) = q(\beta)$ and $\tilde{q}(\alpha) \leq \tilde{q}(\alpha - \beta) + \tilde{q}(\beta)$ (by Proposition 4.3.1 b)), we can reduce this to

$$\dim \mathcal{N}_\beta(D, w, \alpha) \leq -q(\beta) + \tilde{q}(\beta) - \tilde{q}(\alpha).$$

The result follows from Proposition 4.3.1 a). □

Corollary 4.3.6. *For α lying in the fundamental region, we have $\dim \mathcal{N}(D, w, \alpha) \leq -q(\alpha)$. If, in addition, $\delta(\alpha) > 0$, then $\dim(\mathcal{N}(D, w, \alpha) - \mathcal{N}_\alpha(D, w, \alpha)) < -q(\alpha)$.*

Proof. The first statement clearly follows from Theorem 4.3.5 and Proposition 4.3.1 c). Now, let α be in the fundamental region and $\delta(\alpha) > 0$. By the proof of Theorem 4.3.5,

$$\dim \mathcal{N}_\beta(D, w, \alpha) \leq -q(\beta) - \tilde{q}(\alpha - \beta),$$

for all nonnegative $\beta \leq \alpha$. If $\alpha \neq \beta$, then by Proposition 3.5.4, $\dim \mathcal{N}_\beta(D, w, \alpha) < -q(\alpha)$. □

Let $c : \mathcal{P}_{\mathcal{B}} \to \mathbb{A}^{\alpha_0}$ be the morphism defined by sending the pair $(\mathbf{E}, f)$ to the coefficients of the characteristic polynomial $\mathrm{char}(f)$ of f. We will need the following lemma:

Lemma 4.3.7. *There exists a decomposition into nonnegative dimension vectors $\alpha = \sum_{i=1}^{r} \beta^{(i)}$ such that $\dim \mathcal{P}_{\mathcal{B}} = r + \sum_{i=1}^{r} \dim \mathcal{N}(D, w, \beta^{(i)})$.*

Proof. Fix a point of $x \in \mathbb{A}^{\alpha_0}$. This defines some characteristic polynomial $x(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \cdots (t-\lambda_r)^{m_r}$. Consider $(\mathcal{P}_{\mathcal{B}})_x$, the fiber of c over x. The points of $(\mathcal{P}_{\mathcal{B}})_x$ may be identified with pairs $(\mathbf{E}, f)$, such that f is an endomorphism of the parabolic bundle $\mathbf{E}$ with $\mathrm{char}(f) = x(t)$. Therefore, $\mathbf{E}$ decomposes as

$$\mathbf{E} = \bigoplus_i \mathbf{ker}(f - \lambda_i)^{m_i},$$

and the fiber $(\mathcal{P}_{\mathcal{B}})_x$ is isomorphic to $\prod_i \mathcal{P}_i$. Here $\mathcal{P}_i$ is the substack of pairs $(\mathbf{E}_i, f_i)$, where $\mathbf{E}_i$ is a parabolic bundle and f_i is its endomorphism such that $\mathrm{char}(f_i) = (t - \lambda_i)^{m_i}$. Since $f_i - \lambda_i$ is nilpotent, we can compute

$$\dim \mathcal{P}_i = \dim \mathcal{N}(D, w, \beta^{(i)}),$$

for some dimension vector $\beta^{(i)} \leq \alpha$. Note that $\alpha = \beta_1 + \cdots + \beta_r$. Since c maps $(\mathcal{P}_{\mathcal{B}})_x$ to the subvariety consisting of polynomials with r distinct roots, we can compute:

$$\dim \mathcal{P}_{\mathcal{B}} = r + \sum_{i=1}^{r} \dim \mathcal{N}(D, w, \beta^{(i)}),$$

for some decomposition $\alpha = \sum_{i=1}^{r} \beta^{(i)}$ into nonnegative dimension vectors. □

Proof of Theorem 1.2.1. Suppose $r = 1$ in Lemma 4.3.7. That is, the decomposition of α contains only one summand. Since $\mathcal{N}_\alpha(D, w, \alpha)$ may be interpreted as pairs $(\mathbf{E}, f)$, where $\mathbf{E}$ is a parabolic bundle and f is the zero endomorphism, we have by the proof of Lemma 4.3.7:

$$\dim(\mathcal{I}_{\mathcal{B}} - \coprod_{i=0}^{1} \mathcal{I}^i) \leq \dim(\mathcal{N}(D, w, \alpha) - \mathcal{N}_\alpha(D, w, \alpha)).$$

Therefore, by Corollary 4.3.6 and Corollary 2.2.4, $\mathrm{Bun}_{D,w,\alpha}(X)$ is almost very good.

Now, suppose $r \geq 2$ in Lemma 4.3.7. In this case, by Proposition 3.5.4, Lemma 4.3.7, and Corollary 4.3.6, we have that:

$$\dim \mathcal{I}_{\mathcal{B}} = \dim \mathcal{P}_{\mathcal{B}} \leq \sum_{i=1}^{r} p(\beta^{(i)}) < p(\alpha).$$

Therefore, $\dim \mathcal{I}_{\mathcal{B}} - 1 < \dim \mathrm{Bun}_{D,w,\alpha}(X)$. It follows from Corollary 2.2.4 that $\mathrm{Bun}_{D,w,\alpha}(X)$ is almost very good. □

Proof of Theorem 1.3.2. This is an obvious consequence of Theorem 1.3.1. □

5. Quivers and Parabolic Bundles

5.1. **Moduli functor: parabolic bundles and squids.** In this and the next section, we will use $\langle \quad \rangle$ to denote the isomorphism class of the collection of enclosed objects. All the schemes we consider from now on will be schemes of finite type. Let

$$p : T \times_K \mathbb{P}^1 \to T \qquad\qquad \pi : T \times_K \mathbb{P}^1 \to \mathbb{P}^1$$

be the two natural projections. Let (D, w) be a parabolic bundle weight type (see Section 1.3). Let $\mathscr{P}(D, w)$ be the category of vector bundles E over $T \times_K \mathbb{P}^1$ such that $E^*|_{\{x\}\times\mathbb{P}^1}$ is generated by global sections for all $x \in T$, together with filtrations

$$E|_{T\times\{x_i\}} = E^{i,0} \supset E^{i,1} \supset \cdots \supset E^{i,w_i} = 0,$$

for $1 \leq i \leq k$. The morphisms of $\mathscr{P}(D, w)$ are vector bundle morphisms such that map filtrations to each other. We can think of $\mathscr{P}(D, w)$ as the category of families over T of parabolic bundles of weight type (D, w) over $\mathbb{P}^1$, such that the dual to the underlying bundle is generated by global sections.

Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundle over T, and let Ψ_0, Ψ_1 be morphisms of vector bundles from $\mathcal{V}$ to $\mathcal{W}$ such that on every fiber over $x \in T$ all linear combinations $\lambda_0\Psi_0(x) + \lambda_1\Psi_1(x)$ for $(\lambda_0 : \lambda_1) \in \mathbb{P}^1$ are surjective. Let $\mathcal{V}_{ij}$ be vector bundles over T, for $1 \leq i \leq k$ and $1 \leq j \leq w_i - 1$. Let $C_{ij} : \mathcal{V}_{ij} \to \mathcal{V}_{ij-1}$ be injective morphisms of vector bundles such that $(\lambda_{i0}\Psi_0(x) + \lambda_{i1}\Psi_1(x))C_{i1}(x) = 0$ in the fiber over each $x \in T$, where $x_i = (\lambda_{i0} : \lambda_{i1})$ and $\mathcal{V}_{i0} = \mathcal{V}$.

Let $\mathscr{S}(D, w)$ be a category where objects are collections $(\mathcal{V}, \mathcal{W}, \mathcal{V}_{ij}, \Psi_0, \Psi_1, C_{ij})$. Morphisms between $(\mathcal{V}, \mathcal{W}, \mathcal{V}_{ij}, \Psi_0, \Psi_1, C_{ij})$ and $(\mathcal{V}', \mathcal{W}', \mathcal{V}'_{ij}, \Psi'_0, \Psi'_1, C'_{ij})$ consist of collections (f, g, h_{ij}) of vector bundle morphisms

$$f : \mathcal{V} \to \mathcal{V}'$$
$$g : \mathcal{W} \to \mathcal{W}'$$
$$h_{ij} : \mathcal{V}_{ij} \to \mathcal{V}'_{ij}.$$

such that:

$$g \circ \Psi_0 = \Psi'_0 \circ f$$

$g \circ \Psi_1 = \Psi_1^{'} \circ f$
$h_{ij-1} \circ C_{ij} = C_{ij}^{'} \circ h_{ij}$ for $1 \leq i \leq k$ and $2 \leq j \leq w_i - 1$
$f \circ C_{i1} = C_{i1}^{'} \circ h_{i1}$ for $1 \leq i \leq k$.

Note that the objects of $\mathscr{S}(D, w)$ are families of Kronecker-preinjective squid representations but not necessarily in coordinate spaces. By an argument analogous to Lemma 5.5 in [7] we obtain:

Theorem 5.1.1. *The categories $\mathscr{P}(D, w)$ and $\mathscr{S}(D, w)$ are equivalent.*

Let us define a functor $\tilde{F}(T)$, from the category of schemes over K to the category of sets as $\tilde{F}(T) = \langle (E, E^{i,j}, s, t, r_{ij}) \rangle$, where

- E is a vector bundle on $T \times \mathbb{P}^1$,
- $p_*(E^*(N))$ and $p_*(E^*(N-1))$ are trivial vector bundles,
- $s : \mathcal{O}_T^{(N+1)\alpha_0 + \alpha_\infty} \simeq p_*(E^*(N))$,
- $t : \mathcal{O}_T^{N\alpha_0 + \alpha_\infty} \simeq p_*(E^*(N-1))$,
- $E|_{T \times \{x_i\}} \supset E^{i,1} \supset \cdots \supset E^{i,w_i - 1} \supset E^{i,w_i} = 0$ are filtrations by trivial vector subbundles of fixed ranks rk $E^{i,j} = \alpha_{ij}$,
- $r_{ij} : \mathcal{O}_T^{\alpha_{ij}} \simeq E^{i,j}$.

Here, $E^*(N) = E^* \otimes \pi(\mathcal{O}(N))$. We have the following:

Theorem 5.1.2. *Let $\alpha^N = (N\alpha_0 + \alpha_\infty, (N+1)\alpha_0 + \alpha_\infty, \alpha_{ij})$. The functor $\tilde{F}$ is representable by the scheme $KS(D, w, \alpha^N)$.*

Proof. Fix a test scheme T and let $x_i = (\lambda_{i0} : \lambda_{i1})$. By Theorem 5.1.1, $\tilde{F}(T)$ defines a family of elements of $KS(D, w, \alpha)$ over T. Therefore, we have a morphism $T \to KS(D, w, \alpha)$. Conversely, given a morphism $T \to KS(D, w, \alpha)$, by Theorem 5.1.1 we have an element of $\tilde{F}(T)$.

We can now define a pair of natural transformations:

$$\eta_T : \tilde{F}(T) \to \mathrm{Hom}(T, KS(\alpha))$$

$$\rho_T : \mathrm{Hom}(T, KS(\alpha)) \to \tilde{F}(T),$$

between the functor $\tilde{F}$ and the functor of points $\underline{KS(D, w, \alpha)}$ corresponding to $KS(D, w, \alpha)$. It follows from construction that η_T and ρ_T are mutually inverse. Therefore, the functors are isomorphic, and $KS(D, w, \alpha)$ represents $\tilde{F}$. □

6. Application to the Deligne-Simpson Problem

In this section, we wish to relate the almost very good property for the moduli of parabolic bundles to the space of solutions to the Deligne-Simpson problem.

6.1. Logarithmic Connections and Squid Representations.

Let $X = \mathbb{P}^1$, let $D = x_1 + \cdots + x_k$, and $w = (w_1, \ldots, w_k)$ be a collection of positive integers. Recall from Section 5.1 that $KS(D, w, \alpha)$ parametrizes parabolic bundles over $\mathbb{P}^1$ together with some rigidity conditions. Let $\mu_{G(\alpha)}^{-1}(\theta^N)$ be the fiber of the moment map described in Section 3.4 over

$$\theta^N = (N + 1 + \sum_{1 \leq i \leq k} \zeta_{i1}, -N - \sum_{1 \leq i \leq k} \zeta_{i1}, \zeta_{i1} - \zeta_{i2}, \ldots, \zeta_{iw_i - 1}) \in \mathrm{Mat}(\alpha^N)_0.$$

Note this is well-defined for ζ coming from a parabolic connection with vector bundle of degree $-\alpha_\infty$, since $\mathrm{tr}(\theta^N) = \alpha_\infty - \sum_{i=1}^k \sum_{j=1}^{w_i} \zeta_{ij}(\alpha_{ij-1} - \alpha_{ij}) = 0$ by Remark 6.1.3.

Let us define a functor $L_\zeta(T)$, from the category of schemes over $\mathbb{C}$ to the category of sets as $L_\zeta(T) = \big\langle (E, E^{i,j}, s, t, r_{ij}, \nabla) \big\rangle$, where

- E is a vector bundle on $T \times \mathbb{P}^1$,
- $p_*(E^*(N))$ and $p_*(E^*(N-1))$ are trivial vector bundles,
- $s : \mathcal{O}_T^{(N+1)\alpha_0 + \alpha_\infty} \simeq p_*(E^*(N))$,
- $t : \mathcal{O}_T^{N\alpha_0 + \alpha_\infty} \simeq p_*(E^*(N-1))$,
- $E|_{T\times\{x_i\}} \supset E^{i,1} \supset \cdots \supset E^{i,w_i-1} \supset E^{i,w_i} = 0$ are filtrations by trivial vector subbundles of fixed ranks rk $E^{i,j} = \alpha_{ij}$,
- $r_{ij} : \mathcal{O}_T^{\alpha_{ij}} \simeq E^{i,j}$,
- $\nabla : E \to E \otimes \pi^*\Omega^1_{\mathbb{P}^1}(\log D)$ is a $\mathbb{C}$-linear morphism of sheaves,
- $\nabla(fs) = s \otimes df + f\nabla(s)$ for s a section of E and f a section of $\pi^*(\mathcal{O}_{\mathbb{P}^1}) \subset \mathcal{O}_{T\times\mathbb{P}^1}$,
- $(\mathrm{Res}_{x_i}\nabla - \zeta_{ij}\cdot \mathrm{Id})(E^{i,j-1}) \subset E^{i,j}$, where $E^{i,0} = E|_{T\times\{x_i\}}$, and $\mathrm{Res}_{x_i}\nabla := \nabla|_{T\times\{x_i\}}$.

Here, $E^*(N) = E^* \otimes \pi^*(\mathcal{O}(N))$, and π and p are defined at the beginning of Section 5.1.

Theorem 6.1.1. *The functor L_ζ is represented by $\mu^{-1}_{G(\alpha)}(\theta^N)$.*

Proof. Let $L_\zeta(T) = (E, E^{i,j}, s, t, r_{ij}, \nabla)$. We know that by Theorem 5.1.2 the functor $\tilde{F}$ is representable by the variety $KS(D, w, \alpha)$. Let $\tilde{F}(T) = (E, E^{i,j}, s, t, r_{ij})$. Note that the natural pairing with the vector field $\frac{d}{dz}$ on $\mathbb{P}^1$ defines the $\mathbb{C}$-linear morphism

$$\nabla^*_{\frac{d}{dz}} : E^* \to E^*(D),$$

satisfying the Leibniz rule, where $E^*(D) = E^* \otimes \pi^*\mathcal{O}(D)$ (we regard D as the divisor $x_1 + \cdots + x_k$). Further note that this morphism uniquely determines ∇. We have that $\nabla_{\frac{d}{dz}}$ induces the morphism $E^*(N) \to E^*(N)(D)$. In fact, it induces a morphism $B : E^*(N) \to E^*(N-1)(D)$. From B we obtain a $\mathbb{C}$-linear morphism

$$\tilde{B} : p_*(E^*(N)) \to p_*(E^*(N-1)(D)).$$

Similarly, from $\nabla_{z\frac{d}{dz}} : E \to E(D)$, we obtain

$$\tilde{B}' : p_*(E^*(N)) \to p_*(E^*(N)(D)).$$

Let $\Psi_0, \Psi_1 : p_*(E^*(N))^* \to p_*(E^*(N-1))^*$ be the morphisms induced by the two inclusions $E^*(N-1) \hookrightarrow E^*(N)$ (corresponding to multiplication by the two global sections 1 and $-z$ of $\pi^*(\mathcal{O}(1))$). Analogous to Section 5 in [7], $\ker(\lambda_{0i}\Psi_0 + \lambda_{1i}\Psi_1) \simeq E|_{T\times\{x_i\}}$. Therefore, $\mathrm{Res}_{x_i}\nabla$ defines the maps

$$\tilde{C}_{i1} := (\mathrm{Res}_{x_i}\nabla - \zeta_{i1}\cdot \mathrm{Id}) : \ker(\lambda_{0i}\Psi_0 + \lambda_{1i}\Psi_1) \to E^{i,1}$$

$$\hat{C}_{ij} := (\mathrm{Res}_{x_i}\nabla - \zeta_{ij}\cdot \mathrm{Id})|_{E^{i,j-1}} : E^{i,j-1} \to E^{i,j} \text{ for } 1 \le i \le k \text{ and } 2 \le j \le w_i - 1.$$

We can extend $\tilde{C}_{i1}$ to $p_*(E^*(N))^*$. Note that any two such extensions differ by a morphism that sends $\ker(\lambda_{0i}\Psi_0 + \lambda_{1i}\Psi_1)$ to 0. Therefore, it has the form $A_i(\lambda_{0i}\Psi_0 + \lambda_{1i}\Psi_1)$ for some $A_i : p_*(E^*(N-1))^* \to E^{i,1}$. Fix such an extension $\hat{C}_{i1}$ for each $1 \leq i \leq k$. We can now define two morphisms of vector bundles: $\hat{B}_0, \hat{B}_1 : p_*(E^*(N-1))^* \to p_*(E^*(N))^*$ in the following way:

$$\hat{B}_0^* = N \cdot \mathrm{Id} - \tilde{B}' - \sum_{1 \leq i \leq k} \frac{x_i}{z - x_i}(\hat{C}_{i1}^* C_{i1}^* + \zeta_{i1} \cdot \mathrm{Id})$$

$$\hat{B}_1^* = -\tilde{B} - \sum_{1 \leq i \leq k} \frac{1}{z - x_i}(\hat{C}_{i1}^* C_{i1}^* + \zeta_{i1} \cdot \mathrm{Id}),$$

where $C_{ij} : E^{i,j} \to E^{i,j+1}$ are as defined in Theorem 5.1.1, and z is the standard coordinate on $\mathbb{P}^1$. Note that $\hat{B}_0, \hat{B}_1$ are well-defined by the construction of $\hat{C}_{ij}$.

We can see that $\hat{B}_1$ (respectively $\hat{B}_0$) depends on the choice of extension in the construction of $\hat{C}_{i1}$. However, any two such choices differ by $A_i(\lambda_{0i}\Psi_0 + \lambda_{1i}\Psi_1)$, so any two $\hat{B}_1$ (respectively $\hat{B}_0$) obtained in this way differ by

$$\sum_{1 \leq i \leq k} C_{i1} A_i(\lambda_{0i}\Psi_0 + \lambda_{1i}\Psi_1).$$

By the Leibniz rule we have $[\tilde{B}, \Psi_0^*] = 0$ and $[\tilde{B}, \Psi_1^*] = -\mathrm{Id}$. Also, note $\tilde{B}' = -\Psi_1^* \tilde{B}$ and $C_{i1}^*(x_i \Psi_0^* + \Psi_1^*) = 0$. Therefore, we have:

$$\begin{aligned}(\Psi_0 \hat{B}_0 + \Psi_1 \hat{B}_1)^* &= (N \cdot \mathrm{Id} - \tilde{B}')\Psi_0^* - \tilde{B}\Psi_1^* - \sum_{1 \leq i \leq k} (\frac{1}{z - x_i} \hat{C}_{i1}^* C_{i1}^*)(x_i \Psi_0^* + \Psi_1^*) \\ &- \sum_{1 \leq i \leq k} \frac{x_i}{z - x_i} \zeta_{i1} \cdot \mathrm{Id} + \sum_{1 \leq i \leq k} \frac{z}{z - x_i} \zeta_{i1} \cdot \mathrm{Id} \\ &= (N + 1 + \sum_{1 \leq i \leq k} \zeta_{i1} \cdot \mathrm{Id}),\end{aligned}$$

and

$$\begin{aligned}\sum_{1 \leq i \leq k} \hat{C}_{i1}^* C_{i1}^* - (\hat{B}_0 \Psi_0 + \hat{B}_1 \Psi_1)^* &= \sum_{1 \leq i \leq k} \hat{C}_{i1}^* C_{i1}^* - \sum_{1 \leq i \leq k} (\hat{C}_{i1}^* C_{i1}^* + \zeta_{i1} \cdot \mathrm{Id}) \\ &- \Psi_0^*(N \cdot \mathrm{Id} - \tilde{B}') + \Psi_1^* \tilde{B} = (-N - \sum_{1 \leq i \leq k} \zeta_{i1}) \cdot \mathrm{Id}.\end{aligned}$$

Furthermore, we have:

$$C_{ij+1}\hat{C}_{ij+1} - \hat{C}_{ij}C_{ij} = (\zeta_{ij} - \zeta_{ij+1}) \cdot \mathrm{Id}, \text{ where } 1 \leq i \leq k \text{ and } 1 \leq j \leq w_i - 1.$$

Since $\hat{B}_0, \hat{B}_1, \hat{C}_{ij}$ vary algebraically with the points of T, then the family $L_\zeta(T)$ defines a morphism $f : T \to \mu_{G(\alpha)}^{-1}(\theta^N)$ by construction.

Conversely, given a morphism $f : T \to \mu_{G(\alpha)}^{-1}(\theta^N)$, we get the corresponding morphism $T \to KS(D, w, \alpha)$. Therefore, from the proof of Theorem 5.1.1 (see Section 5.1) we get a collection $\tilde{F}(T) = (E, E^{i,j}, s, t, r_{ij})$. Moreover, f defines the family $(\mathcal{V}, \mathcal{W}, \mathcal{V}_{ij}, \Psi_0, \Psi_1, C_{ij})$ of elements of $KS(D, w, \alpha)$, as well as families of morphisms $\hat{B}_0, \hat{B}_1 : \mathcal{W} \to \mathcal{V}$ and $\hat{C}_{ij} : \mathcal{V}_{ij} \to \mathcal{V}_{ij+1}$. Note that we have that $\mathcal{V} \simeq p_*(E^*(N))^*$ and $\mathcal{W} \simeq p_*(E^*(N-1))^*$. From the construction of $\hat{B}_1$ above, we obtain:

$$\tilde{B} : p_*(E^*(N)) \to p_*(E^*(N-1)(D)) \hookrightarrow p_*(E^*(N)(D)).$$

Since $E^*(N)$ is generated by global sections, we can use the Leibniz rule to extend $\tilde{B}$ to a $\mathbb{C}$-linear morphism of vector bundles $B : E^*(N) \to E^*(N-1)(D)$ that satisfies

$$B(fs) = s \otimes \frac{df}{dz} + f\nabla(s),$$

for s a section of $E^*(N)$ and f a section of $\pi^*(\mathcal{O}_{\mathbb{P}^1})$. We can further obtain a $\mathbb{C}$-linear morphism $\nabla_{\frac{d}{dz}} : E \to E(D)$ that satisfies the Leibniz rule. This is the same as defining the $\mathbb{C}$-linear morphism

$$\nabla : E \to E \otimes \pi^*\Omega^1_{\mathbb{P}^1}(\log D),$$

which satisfies the Leibniz rule.

Note that we have $\hat{C}_{i1}|_{\ker(\lambda_{0i}\Psi_0+\lambda_{1i}\Psi_1)} = \nabla|_{T\times\{x_i\}} - \zeta_{i1}\cdot \mathrm{Id}$. By Theorem 5.1.1 we have $\mathcal{V}_{ij} = E^{i,j}$. Therefore, $C_{ij+1}\hat{C}_{ij+1} - \hat{C}_{ij}C_{ij} = (\zeta_{ij} - \zeta_{ij+1})\cdot \mathrm{Id}$ implies that $(\mathrm{Res}_{x_i}\nabla - \zeta_{ij}\cdot \mathrm{Id})(E^{i,j-1}) \subset E^{i,j}$. Thus $f : T \to \mu^{-1}_{G(\alpha)}(\theta^N)$ defines the family $L_\zeta(T) = (E, E^{i,j}, s, t, r_{ij}, \nabla)$.

The above constructions define a pair of natural transformations:

$$\eta_T : L_\zeta(T) \to \mathrm{Hom}(T, \mu^{-1}_{G(\alpha)}(\theta^N))$$
$$\rho_T : \mathrm{Hom}(T, \mu^{-1}_{G(\alpha)}(\theta^N)) \to L_\zeta(T),$$

between the functor L_ζ and the functor of points $\underline{\mu^{-1}_{G(\alpha)}(\theta^N)}$ corresponding to $\mu^{-1}_{G(\alpha)}(\theta^N)$. It follows by construction and Theorem 5.1.1 that η_T and ρ_T are mutual inverse. Therefore, the functors are isomorphic, and $\mu^{-1}_{G(\alpha)}(\theta^N)$ represents L_ζ. $\square$

Remark 6.1.2. We can follow the proof of Theorem 6.1.1 in order to obtain that $\mu^{-1}_{G(\alpha)}(0)$ is a moduli space parameterizing parabolic Higgs bundles over $\mathbb{P}^1$ together with rigidity. This is natural, as parabolic Higgs bundles constitute the cotangent stack to the moduli stack of parabolic bundles, and ζ-parabolic connections constitute the twisted cotangent stack to the moduli stack of parabolic bundles.

Assuming the conventions from Section 4.2, we have the following: the stack of ζ-parabolic connections on parabolic bundles of weight type (D, w) and of dimension type α over X is a functor that associates to a test scheme T the groupoid $\mathrm{Conn}_{D,w,\alpha,\zeta}(T) = \left\langle (E, E^{i,j}, \nabla)_{1\le i\le k}\right\rangle$, where

- E is a vector bundle on $T \times X$,
- $E|_{T\times\{x_i\}} \supset E^{i,1} \supset \cdots \supset E^{i,w_i-1} \supset E^{i,w_i} = 0$ is a filtration by vector bundles,
- $\mathrm{rk}(E) = \alpha_0$ and $\mathrm{rk}(E^{i,j}) = \alpha_{ij}$,
- $\nabla : E \to E \otimes \pi^*\Omega^1_{\mathbb{P}^1}(\log D)$ is a $\mathbb{C}$-linear morphism of sheaves,
- $\nabla(fs) = s \otimes df + f\nabla(s)$ for s a section of E and f a section of $\pi^*(\mathcal{O}_{\mathbb{P}^1}) \subset \mathcal{O}_{T\times\mathbb{P}^1}$,
- $(\mathrm{Res}_{x_i}\nabla - \zeta_{ij}\cdot \mathrm{Id})(E^{i,j-1}) \subset E^{i,j}$, where $E^{i,0} = E|_{T\times\{x_i\}}$, and $\mathrm{Res}_{x_i}\nabla := \nabla|_{T\times\{x_i\}}$.

Remark 6.1.3. Note that if a ζ-parabolic connection ∇ exists on a parabolic bundle $\mathbf{E}$ of weight type (D, w) and dimension vector α over X, then

$$\sum_{i=1}^{k} \mathrm{tr}(\mathrm{Res}_{x_i}\nabla) = \sum_{i=1}^{k}\sum_{j=1}^{w_i} \zeta_{ij}(\alpha_{ij-1} - \alpha_{ij}) = -\mathrm{deg}\ E.$$

Therefore, fixing ζ automatically fixes $d = \mathrm{deg}\ E$.

Set $d = -\alpha_\infty$, $\alpha^N = (\alpha_\infty + N, \alpha_\infty + \alpha_0 + N, \alpha_{ij})$, and θ^N as in Theorem 6.1.1. By Theorem 6.1.1, we have that $U = \coprod_{N \in \mathbb{Z}_{\geq 0}} \mu^{-1}_{G(\alpha)}(\theta^N)$ is a presentation for the algebraic stack $\mathrm{Conn}_{D,w,\alpha,\zeta}(X)$. In fact, there exists an $N \in \mathbb{Z}_{\geq 0}$ such that $\mu^{-1}_{G(\alpha)}(\theta^N)$ is a presentation for $\mathrm{Conn}_{D,w,\alpha,\zeta}(X)$. Indeed, if a ζ-parabolic connection exists on parabolic bundle $\mathbf{E}$, then the width (the difference between the maximal and minimal line bundle degrees in the Grothedieck Theorem decomposition of E) of E is bounded (this follows, for example, from Theorem 7.1 in [7] and Lemma 1 in [8]). Therefore, for a fixed ζ, there is a single N such that $E^*(N)$ is generated by global sections. This implies the statement we need.

Proof of Theorem 1.4.1. Let $\alpha_\infty = -d = \sum_{i=1}^k \sum_{j=1}^{w_i} \zeta_{ij}(\alpha_{ij-1} - \alpha_{ij})$. Note that the stack $\mathrm{Bun}^d_{D,w,\alpha}(X)$ admits the presentation $U = \coprod_{N \in \mathbb{Z}_{\geq 0}} KS(D, w, \alpha^N)$, where $\alpha^N = (\alpha_\infty + N, \alpha_\infty + \alpha_0 + N, \alpha_{ij})$. Since $KS(D, w, \alpha^N)$ is irreducible for each N and the fibers are products of general linear groups, then $\mathrm{Bun}^d_{D,w,\alpha}(X)$ is irreducible. It follows that the irreducible components of $\mathrm{Bun}_{D,w,\alpha}(X)$ are the $\mathrm{Bun}^d_{D,w,\alpha}(X)$.

Let θ^N be as in Theorem 6.1.1. Fix $N \geq 0$ such that $\mu^{-1}_{G(\alpha)}(\theta^N)$ is a presentation for $\mathrm{Conn}_{D,w,\alpha,\zeta}(X)$. If $\mathrm{Bun}_{D,w,\alpha}(X)$ is almost very good, then $\mathrm{Bun}^d_{D,w,\alpha}(X)$ is almost very good for each d. Consequently, we have that the quotient stack $G(\alpha^N)\backslash KS(D, w, \alpha^N)$ is very good.

By Corollary 2.3.3, we have that $\mu^{-1}_{G(\alpha)}(\theta^N)$ is nonempty, irreducible, complete intersection of dimension

$$\dim 2(G(\alpha) + p(\alpha)) - \dim G(\alpha) = 2p(\alpha) + \dim G(\alpha).$$

It follows that $\mathrm{Conn}_{D,w,\alpha,\zeta}(X)$ is a nonempty, irreducible, locally complete intersection of dimension $2p(\alpha) - 1$. □

Proof of Corollary 1.4.2. This instantly follows from Theorems 1.3.1 and 1.4.1. □

Remark 6.1.4. Let $C_1, \ldots, C_k$ be semisimple conjugacy classes of endomorphisms of $E_{x_1}, \ldots, E_{x_k}$, respectively. We may interpret $\mathrm{Conn}_{D,w,\alpha,\zeta}(X)$ as the moduli stack of solutions to the Deligne-Simpson problem.

Indeed, a solution of the Deligne-Simpson problem is a connection ∇ on a vector bundle E over $\mathbb{P}^1$, with regular singularities in D such that $\mathrm{Res}_{x_i} \nabla \in C_i$ for all $x_i \in D$. This determines a dimension vector $\alpha = (\alpha_0, \alpha_{ij})$, where $\alpha_0 = \mathrm{rk}\ E$ and $\alpha_{ij} = \mathrm{rk}\ (\mathrm{Res}_{x_i} \nabla - \zeta_{ij} \cdot \mathrm{Id})$ is the dimension of the direct sum of the first $w_i - j$ eigenspaces of C_i ordered from least to greatest, and a vector of eigenvalues ζ (accounting for multiplicity). Therefore, ∇ is a ζ-parabolic connection on a parabolic bundle with underlying vector bundle E, weight type (D, w), and dimension type α.

Conversely, any parabolic ζ-connection in $\mathrm{Conn}_{D,w,\alpha,\zeta}(X)$ has residues lying in the conjugacy classes C_i with eigenvalues in ζ (accounting for multiplicity), and eigenspaces ordered from least to greatest of dimensions α_{ij}.

Remark 6.1.5. Note that the above remark is a special case of Theorem 2.1 in [7]. In general, this theorem implies that a regular singular connection ∇ on $\mathbb{P}^1$ is a ζ-parabolic connection if and only if its residues lie in the closures of conjugacy classes defined by ζ.

Therefore, if we relax the conditions in the statement of the Deligne-Simpson problem to allow solutions to lie in conjugacy class closures (rather than the conjugacy classes themselves), we may interpret $\mathrm{Conn}_{D,w,\alpha,\zeta}(X)$ as the moduli stack of solutions.

6.2. **The very good property and the additive Deligne-Simpson problem.** Recall from the Introduction that the additive Deligne-Simpson problem asks whether there exist matrices $A_1, \dots, A_k$ in prescribed conjugacy classes $C_1, \dots, C_k$ such that $A_1 + \cdots + A_k = 0$.

Let $C_1, \dots, C_k$ be conjugacy classes of matrices in ${}_n(\mathbb{C})$. We denote by

$$ADS(C_1, \dots, C_k) := \{(A_1, \dots, A_k) \in C_1 \times \cdots \times C_k | A_1 + \cdots + A_k = 0\}$$

the algebraic subvariety of solutions of the additive Deligne-Simpson problem in $C_1 \times \cdots \times C_k$.

Recall from section 3.3 that $\mathrm{Rep}(Q^{st}_{D,w}, \alpha)$ is the space of star-shaped quiver representations. Let $RI(Q^{st}_{D,w}, \alpha) \subset \mathrm{Rep}(Q^{st}_{D,w}, \alpha)$ consist of representations for which the maps associated to c_{ij} are injective. The group $G(\alpha)$ acts on both $RI(Q^{st}_{D,w}, \alpha)$ in the usual way.

Lemma 6.2.1. *The quotient stack $G(\alpha)\backslash RI(Q^{st}_{D,w}, \alpha)$ is very good if and only if $PGL(\alpha_0, \mathbb{C})\backslash Fl(\alpha)$ is very good.*

Proof. Note that there is an action of the subgroup $H(\alpha) = \prod_{\alpha_{i,j}} \mathrm{GL}(\alpha_{i,j}) \subset G(\alpha)$ on $RI(Q^{st}_{D,w}, \alpha)$ induced by the action of $G(\alpha)$. Furthermore, the space $Fl(\alpha)$ is obtained as a quotient of $RI(Q^{st}_{D,w}, \alpha)$ by this action. Clearly $\mathrm{PGL}(\alpha_0, \mathbb{C}) = G(\alpha)/H(\alpha)$. The lemma statement follows. □

We are now ready to prove that the very good property for the quotient stack $\mathrm{PGL}(\alpha_0, \mathbb{C})\backslash Fl(\alpha)$ implies that $ADS(C_1, \cdots, C_k)$ is nonempty, irreducible, and a complete intersection of dimension $2\dim Fl(\alpha) - \alpha_0^2 + 1$, as long as $\mathrm{tr}(A_1 + \cdots + A_k) = 0$ for $A_i \in C_i$.

Proof of Theorem 1.4.3. By Lemma 6.2.1, $G(\alpha)\backslash RI(Q^{st}_{D,w}, \alpha)$ is very good. Therefore, by Corollary 2.3.3, we have that $\mu^{-1}_{G(\alpha)}(\theta^N)$ is a nonempty, irreducible, complete intersection of dimension $2\dim RI(Q^{st}_{D,w}, \alpha) - \dim G(\alpha)$.

From the proof of Lemma 6.2.1 we have that $Fl(\alpha)$ is the locally trivial quotient of $RI(Q^{st}_{D,w}, \alpha)$ by the group $H(\alpha)$. Moreover, by Theorem 6.1.1 we see that the locally trivial quotient $\mu^{-1}_{G(\alpha)}(\theta^N)/H(\alpha)$ is isomorphic to $ADS(C_1, \dots, C_k)$. It follows that $ADS(C_1, \dots, C_k)$ is a nonempty, irreducible, complete intersection of dimension

$$2\dim RI(Q^{st}_{D,w}, \alpha) - \dim G(\alpha) - H(\alpha) = 2p(\alpha) + \alpha_0^2 - 1 = 2\dim Fl(\alpha) - \alpha_0^2 + 1.$$

□

Remark 6.2.2. Note that the dimension formula $\dim ADS(C_1, \dots, C_k) = 2p(\alpha) + \alpha_0^2 - 1$ is similar to the formula given in Theorem 1.2 of [5].

If $\delta(\alpha) > 0$ and we assume that the eigenvalues of $C_1, \dots, C_k$ are ordered as in Section 6.1, then we obtain that α is in the fundamental region.

Proof of Corollary 1.4.4. This follows from Theorem 1.3.2 and Theorem 1.4.3. □

6.3. **The very good property and the multiplicative Deligne-Simpson problem.** The multiplicative Deligne-Simpson asks whether there exist matrices $A_1, \ldots, A_k$ in prescribed conjugacy classes $C_1, \ldots, C_k$ such that $A_1 \cdots A_k = \mathrm{Id}$.

Let $C_1, \ldots, C_k$ be conjugacy classes of matrices in $\mathrm{GL}(n, \mathbb{C})$. We denote by

$$MDS(C_1, \ldots, C_k) := \{(A_1, \ldots, A_k) \in C_1 \times \cdots \times C_k | A_1 \cdot A_2 \cdots A_k = \mathrm{Id}\}$$

the algebraic subvariety of solutions of the multiplicative Deligne-Simpson problem in $C_1 \times \cdots \times C_k$.

Instead of using the moduli space of ζ-parabolic connections defined in Section 6.1, we will introduce a different moduli space, representing the following functor:

Let E be as in Section 6.1, and let $y \in \mathbb{P}^1$. Let us define a functor $\tilde{L}_\zeta(T)$, from the category of schemes over $\mathbb{C}$ to the category of sets as $\tilde{L}_\zeta(T) = \langle (E, E^{i,j}, r, \nabla) \rangle$, where

- E is a vector bundle on $T \times \mathbb{P}^1$,
- $E|_{T \times \{y\}}$ is a trivial vector bundles,
- $r : E|_{T \times \{y\}} \simeq \mathcal{O}_T^{\alpha_0}$,
- $E|_{T \times \{x_i\}} \supset E^{i,1} \supset \cdots \supset E^{i,w_i-1} \supset E^{i,w_i} = 0$ are filtrations by vector subbundles of fixed ranks rk $E^{i,j} = \alpha_{ij}$,
- $\nabla : E \to E \otimes \pi^*\Omega^1_{\mathbb{P}^1}(\log D)$ is a $\mathbb{C}$-linear morphism of sheaves,
- $\nabla(fs) = s \otimes df + f\nabla(s)$ for s a section of E and f a section of $\pi^*(\mathcal{O}_{\mathbb{P}^1}) \subset \mathcal{O}_{T \times \mathbb{P}^1}$,
- $(\mathrm{Res}_{x_i}\nabla - \zeta_{ij} \cdot \mathrm{Id})(E^{i,j-1}) \subset E^{i,j}$, where $E^{i,0} = E|_{T \times \{x_i\}}$, and $\mathrm{Res}_{x_i}\nabla := \nabla|_{T \times \{x_i\}}$.

Similar to Theorem 6.13 in [30] and Section 4 in [29], it follows that the functor $\tilde{L}_\zeta$ is representable by a quasiprojective scheme. We will denote this scheme by $R_{DR}(D, w, y, \alpha, \zeta)$.

We need one more concept, in order for the Riemann-Hilbert correspondence to establish a well-defined analytic isomorphism between $R_{DR}(D, w, y, \alpha, \zeta)$ and the space $MDS(C_1, \ldots, C_k)$. A *transversal* to $\mathbb{Z}$ in $\mathbb{C}$ is a subset $T \subset \mathbb{C}$ such that $t \mapsto \exp(-2\pi\sqrt{-1}t)$ bijectively maps T to $\mathbb{C}^*$ (see e.g. [7]). We will henceforth denote $T = (T_1, \ldots, T_k)$ is a collection of transversals.

Assume that $C_1, \ldots, C_k$ are semisimple. Let $\tau = (\tau_{ij})$ be the vector of eigenvalues (counting multiplicity) for the conjugacy classes $C_1, \ldots, C_k$. Fix a collection of transversals T, and let ζ be defined by $\tau_{ij} = \exp(-2\pi\sqrt{-1}\zeta_{ij})$ such that $\zeta_{ij} \in T_i$. The multiplicities of the eigenvalues τ define a dimension vector α as in Remark 6.1.4. Fix some $D = (x_1, \ldots, x_k)$ and $y \in \mathbb{P}^1$ such that $y \notin D$.

Theorem 6.3.1. *The Riemann-Hilbert correspondence establishes an isomorphism of analytic spaces between $R_{DR}(D, w, y, \alpha, \zeta)$ and $MDS(C_1, \ldots, C_k)$.*

Proof. Let $(\mathbf{E}, r, \nabla) \in R_{DR}(D, w, y, \alpha, \zeta)$ be a triple consisting of a parabolic bundle $\mathbf{E}$, a ζ-parabolic connection on $\mathbf{E}$, and a trivialization r of the fiber E_y. We have the following map:

$$RH : R_{DR}(D, w, y, \alpha, \zeta) \to MDS(C_1, \ldots, C_k)$$
$$(\mathbf{E}, r, \nabla) \mapsto (\rho_y(a_1), \ldots, \rho_y(a_k)),$$

where $\rho_y : \pi_1(\mathbb{P}^1 - D, y) \to E_y \simeq \mathbb{C}^{\alpha_0}$ is the monodromy representation defined by the pair $(\mathbf{E}, \nabla)$ under the Riemann-Hilbert correspondence, and $a_1, \ldots, a_k$ are the loops at base point y around the punctures x_i. This map is well-defined.

Indeed, $\pi_1(\mathbb{P}^1 - D, y)$ is the group freely generated by the loops a_i, satisfying the relation $a_1 \cdots a_k = 1$. Therefore, for the corresponding monodromy operators satisfy $\rho_y(a_1) \cdots \rho_y(a_k) = \mathrm{Id}$. Furthermore, it is a well-known fact (see e.g. Lemma 6.2 in [7]) that $\rho_y(a_i)$ is conjugate to $\exp(-2\pi\sqrt{-1}\mathrm{Res}_{x_i}\nabla)$ if ∇ is a ζ-parabolic connection with ζ as defined above. Therefore, by construction, $\rho_y(a_i) \in C_i$. Since $\sum_{ij} \zeta_{ij} = -\deg E$ is an integer, then $\prod_{ij} \tau_{ij} = 1$. If the pair $(\mathbf{E}, \nabla)$ is defined by complex analytic parameters, then the local system corresponding to this pair, and the monodromy operators ρ_{a_i} depend analytically on these parameters. It follows that RH is analytic.

The Riemann-Hilbert correspondence provides the map RH with a well-defined inverse, sending the k-tuple of monodromy operators $(\rho_y(a_1), \ldots, \rho_y(a_k))$ to the corresponding triple $(\mathbf{E}, r, \nabla)$. As above, we can see that the inverse is complex analytic. Therefore, RH is an analytic isomorphism between $R_{DR}(D, w, y, \alpha, \zeta)$ and $MDS(C_1, \ldots, C_k)$. □

Proof of Theorem 1.4.5. There is a smooth, representable morphism

$$R_{DR}(D, w, y, \alpha, \zeta) \to \mathrm{Conn}_{D,w,\alpha,\zeta}(X),$$

defined by forgetting the rigidity condition on $R_{DR}(D, w, y, \alpha, \zeta)$. It is therefore easy to see that $R_{DR}(D, w, y, \alpha, \zeta)$ is an irreducible, complete intersection of dimension $2p(\alpha) + \alpha_0^2 - 1$. By Theorem 6.3.1 there is an analytic isomorphism between $R_{DR}(D, w, y, \alpha, \zeta)$ and $MDS(C_1, \ldots, C_k)$. It follows that $MDS(C_1, \ldots, C_k)$ is a complete intersection of dimension $2p(\alpha) + \alpha_0^2 - 1$. Since the smooth locus of $R_{DR}(D, w, y, \alpha, \zeta)$ is irreducible, it is connected. Therefore, the smooth locus of $MDS(C_1, \ldots, C_k)$ is also connected. Thus, $MDS(C_1, \ldots, C_k)$ is irreducible. □

As before, if we assume an appropriate ordering on the eigenvalues of $C_1, \ldots, C_k$, then α is automatically in the fundamental region.

Proof of Corollary 1.4.6. This follows immediately from Theorems 1.3.1 and 1.4.5. □

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