Tame and wild automorphisms of polynomial algebras and free associative algebras

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Abstract

Let F be an arbitrary field of characteristic 0. We describe a set of defining relations of the group of tame automorphisms of the polynomial algebra $F[x, y, z]$. We also prove that the well-known Anick automorphism (see [3, p. 343]) of the free associative algebra $F < x, y, z >$ is wild.

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1 Introduction

Let $B_n = F < y_1, y_2, \ldots, y_n >$ be the free associative algebra with free generators $y_1, y_2, \ldots, y_n$ over a field F, and let $Aut\, B_n$ be the automorphism group of B_n. Let $\phi = (f_1, f_2, \ldots, f_n)$ denote an automorphism ϕ of B_n such that $\phi(y_i) = f_i$, $1 \leq i \leq n$. An automorphism

$$\sigma(i, \alpha, f) = (y_1, \ldots, y_{i-1}, \alpha y_i + f, y_{i+1}, \ldots, y_n),$$

where $0 \neq \alpha \in F$, $f \in F < y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n >$, is called *elementary*. The subgroup $T Aut\, B_n$ of $Aut\, B_n$ generated by all elementary automorphisms is called the *tame subgroup*, and the elements of this subgroup are called the *tame automorphisms* of B_n. Non-tame automorphisms of B_n are called *wild*.

In 1968 P.Cohn [1] proved that the automorphisms of free Lie algebras with a finite set of generators are tame. Later this result was generalized for the free algebras of Schreier varieties [15]. Remind that the varieties of all nonassociative algebras[14], commutative and anticommutative algebras [25], Lie algebras [25, 31] are Schreier. Another examples of Schreier varieties can be found in [17, 21, 26].

It is well known [5, 12, 13, 16] that the automorphisms of polynomial algebras and free associative algebras in two variables are tame. It was recently proved in [28, 22, 23, 24] that the well-known Nagata automorphism (see [19])

$$\sigma \;=\; (x+(x^2-yz)z,\, y+2(x^2-yz)x+(x^2-yz)^2z,\, z)$$

of the polynomial algebra $F[x,y,z]$ over a field F of characteristic 0 is wild. Moreover, it was shown [29] that the first two coordinates of the Nagata automorphism cannot be a coordinate of any tame automorphism of the algebra $F[x,y,z]$.

In the case of free associative algebras the question about the existence of wild automorphisms was open and was formulated in [8, Problem 1.74] as a problem of P.Cohn. The best known candidate to be nontame was the Anick automorphism (see [3, p. 343])

$$\delta \;=\; (x+z(xz-zy),\, y+(xz-zy)z,\, z)$$

of the algebra $F < x,y,z >$. It is well known [20] that the Nagata and the Anick automorphisms are stably tame.

In combinatorial group theory (see, for example [30]) there are several well-known descriptions of the group of automorphisms of free groups by generators and defining relations. It is well known (see, for example [3]) that the groups of automorphisms of polynomial algebras and free associative algebras in two variables are isomorphic and have a nice representation as a free product of groups. Similar results are well known for two dimensional Cremona groups [11, 33]. Main part of the investigations of the automorphism groups in the case of free linear algebras consist in finding a system of generators.

In this paper we describe a set of defining relations of the group of tame automorphisms $TAut\,F[x,y,z]$ of the polynomial algebra $F[x,y,z]$ over a field F of characteristic 0. Later we use this result to prove that the Anick automorphism δ of the free associative algebra $F < x,y,z >$ is wild.

The paper is organized as follows. Sections 2–4 are devoted to studying the group $TAut\,F[x,y,z]$. In Section 2 we give some preliminaries from [23, 24]. It was proved in [24] that every tame automorphism of the polynomial algebra $F[x,y,z]$ with degree > 3 admits either an elementary reduction or a reduction of types I–IV. In this paper we use some extension of the definition of automorphisms admiting a reduction of type IV in the sence of [24]. Furthermore, in Section 3 we define essential reductions of the tame automorphisms and study the uniqueness of these reductions. In particular, it is proved that every tame automorphism admits only one type of the essential reductions. A set of defining relations of the group of tame automorphisms $TAut\,F[x,y,z]$ is given in Section 4.

Recall that an associative algebra with the identity

$$[x,y][z,t]=0$$

is called *metabelian*. In [27] there were introduced and investigated the Jacobian matrices of endomorphisms of free metabelian associative algebras. In Section 5 we investigate the Jacobian matrices of tame automorphisms of the free metabelian algebra in three variables. It is well known [2] that the matrix

$$\left(\begin{array}{cc} \frac{\partial\delta(x)}{\partial x} & \frac{\partial\delta(y)}{\partial x} \\ \frac{\partial\delta(x)}{\partial y} & \frac{\partial\delta(y)}{\partial y} \end{array} \right) = \left(\begin{array}{cc} 1 + z' \otimes z & 1 \otimes z^2 \\ -(z' \otimes 1)^2 & 1 - z' \otimes z \end{array} \right)$$

is not a product of elementary matrices over the polynomial algebra $F[z' \otimes 1, 1 \otimes z]$. From this fact we deduce that the Anick automorphism δ is wild.

2 Reductions of tame automorphisms

Let F be an arbitrary field of characteristic 0, and let $A = F[x_1, x_2, x_3]$ be the polynomial algebra in the variables x_1, x_2, x_3 over F. As in [23], we will identify the algebra A with the corresponding subspace of the free Poisson algebra $P = PL\langle x_1, x_2, x_3\rangle$. The highest homogeneous part $\overline{f}$ and the degree $\deg f$ can be defined in an ordinary way. Note that

$$\overline{fg} = \overline{f}\,\overline{g},\ \deg(fg) = \deg f + \deg g,\ \deg[f, g] \leq \deg f + \deg g.$$

If $f_1, f_2, \ldots, f_k \in A$, then we denote by $\langle f_1, f_2, \ldots, f_k\rangle$ the subalgebra of A generated by these elements.

We will often use the terminology and results of [24]. Therefore, let us make an agreement that Corollary 2A, Lemma 4A, Proposition 5A, and Theorem 7A mean Corollary 2, Lemma 4, Proposition 5, and Theorem 7 of [24], respectively.

Let us remind now the more necessary definitions and statements from [23, 24].

Lemma 2.1 **[23]** *Let $f, g, h \in A$. Then the following statements are true:*

1) $[f, g] = 0$ *if and only if f, g are algebraically dependent.*

2) *Suppose that $f, g, h \notin F$ and $m = \deg[f, g] + \deg h$, $n = \deg[g, h] + \deg f$, $k = \deg[h, f] + \deg g$. Then $m \leq \max(n, k)$. If $n \neq k$, then $m = \max(n, k)$.*

The following statements are well known (see [3]):

F1) *If a, b are homogeneous algebraically dependent elements of the algebra A, then there exists an element $z \in A$ such that $a = \alpha z^n$, $b = \beta z^m$, $\alpha, \beta \in F$. The subalgebra $\langle a, b\rangle$ is single generated if and only if $m|n$ or $n|m$.*

F2) *Let $f, g \in A$ be such that $\overline{f}, \overline{g}$ are algebraically independent. If $h \in \langle f, g\rangle$, then $\overline{h} \in \langle \overline{f}, \overline{g}\rangle$.*

A pair of elements f, g of the algebra A is called *reduced* if $\overline{f} \notin \langle \overline{g}\rangle$ and $\overline{g} \notin \langle \overline{f}\rangle$. A reduced pair of algebraically independent elements $f, g \in A$ is called $*$*-reduced* if $\overline{f}$ and $\overline{g}$ are algebraically dependent.

Consider a $*$-reduced pair of elements f, g of the algebra A and let $n = \deg f < m = \deg g$. Put $p = \frac{n}{(n,m)}$, $s = \frac{m}{(n,m)}$, and

$$N = N(f,g) = \frac{mn}{(m,n)} - m - n + \deg[f,g] = mp - m - n + \deg[f,g],$$

where (n,m) is the greatest common divisor of n, m. Note that $(p,s) = 1$, and since $\overline{f}, \overline{g}$ are algebraically dependent, there exists an element $a \in A$ such that $\overline{f} = \beta a^p$, $\overline{g} = \gamma a^s$. Sometimes we will call a $*$-reduced pair of elements f, g also a *p-reduced* pair. Assume that $G(x,y) \in F[x,y]$. It was proved in [23] that if $\deg_y(G(x,y)) = pq + r$, $0 \le r < p$, then

$$\deg(G(f,g)) \ge qN + mr, \tag{1}$$

and if $\deg_x(G(x,y)) = sq_1 + r_1$, $0 \le r_1 < s$, then

$$\deg(G(f,g)) \ge q_1 N + nr_1. \tag{2}$$

Corollary 2.1 [24] *Assume that $G(x,y) \in F[x,y]$ and $h = G(f,g)$. Consider the following conditions:*

(i) $\deg h < N(f,g)$;

(ii) $\deg_y(G(x,y)) < p$;

(iii) $h = \sum_{i,j} \alpha_{ij} f^i g^j$, *where* $\alpha_{ij} \in F$ *and* $in + jm \le \deg h$ *for all* i, j;

(iv) $\overline{h} \in \langle \overline{f}, \overline{g} \rangle$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Lemma 2.2 [24] *There exists a polynomial $w(x,y) \in F[x,y]$ of the type*

$$w(x,y) = y^p - \alpha x^s - \sum \alpha_{ij} x^i y^j, \ ni + mj < mp,$$

which satisfies the following conditions:
1) $\deg w(f,g) < pm$;
2) $\overline{w(f,g)} \notin \langle \overline{f}, \overline{g} \rangle$.

A polynomial $w(x,y)$ satisfying the conditions of Lemma 2.2 is called a *derivative polynomial* of the $*$-reduced pair f, g.

Corollary 2.2 [24] *If $h \in \langle f, g \rangle \setminus F$ and $\deg h < n$, then $h = \lambda w(f,g)$, $0 \ne \lambda \in F$, where $w(x,y)$ is a derivative polynomial of the pair f, g.*

Corollary 2.3 [24] *If $w(x, y)$ is a derivative polynomial of the pair f, g, then*

$$\deg(\frac{\partial w}{\partial x}(f, g)) = n(s-1),$$
$$\deg(\frac{\partial w}{\partial y}(f, g)) = m(p-1).$$

Let $\theta = (f_1, f_2, f_3)$ be an arbitrary automorphism of the algebra A. The number

$$\deg\theta = \deg f_1 + \deg f_2 + \deg f_3$$

is called the *degree* of θ.

Recall that an *elementary transformation* of a triple (f_1, f_2, f_3) is, by definition, a transformation that changes only one element f_i to an element of the form $\alpha f_i + g$, where $0 \neq \alpha \in F$ and $g \in \langle\{f_j | j \neq i\}\rangle$. The notation

$$\theta \to \tau$$

means that the triple τ is obtained from θ by a single elementary transformation. An automorphism θ is called *elementarily reducible* or *admits an elementary reduction* if there exists $\tau \in Aut\,A$ such that $\theta \to \tau$ and $\deg\tau < \deg\theta$. The element f_i of the automorphism θ which was changed in τ to an element of less degree is called *reducible* and we will say also that f_i is *reduced in θ by the automorphism τ.*

Definition 2.1 [24] *Let $\theta = (f_1, f_2, f_3)$ be an automorphism of A such that* $\deg f_1 = 2n$, $\deg f_2 = ns$, *$s \geq 3$ is an odd number,* $2n < \deg f_3 \leq ns$, *$\overline{f}_3 \notin \langle\overline{f}_1, \overline{f}_2\rangle$. Suppose that there exists $0 \neq \beta \in F$ such that the elements $g_1 = f_1$, $g_2 = f_2 - \beta f_3$ satisfy the conditions:*

- *i) g_1, g_2 is a 2-reduced pair and* $\deg g_1 = \deg f_1$, $\deg g_2 = \deg f_2$*;*
- *ii) the element f_3 of the automorphism (g_1, g_2, f_3) is reduced by an automorphism (g_1, g_2, g_3) with the condition* $\deg[g_1, g_3] < \deg g_2 + \deg[g_1, g_2]$.

Then we will say that θ admits a reduction of type I and the automorphism (g_1, g_2, g_3) will be called a reduction of θ of type I with an active element f_3.

Definition 2.2 [24] *Let $\theta = (f_1, f_2, f_3)$ be an automorphism of A such that* $\deg f_1 = 2n$, $\deg f_2 = 3n$, $\frac{3n}{2} < \deg f_3 \leq 2n$, *and let $\overline{f}_1, \overline{f}_3$ be linearly independent. Suppose that there exist $\alpha, \beta \in F$, where $(\alpha, \beta) \neq 0$, such that the elements $g_1 = f_1 - \alpha f_3$, $g_2 = f_2 - \beta f_3$ satisfy the conditions i),ii) of Definition 2.1. Then we will say that θ admits a reduction of type II and the automorphism (g_1, g_2, g_3) will be called a reduction of θ of type II with an active element f_3.*

Definition 2.3 **[24]** *Let* $\theta = (f_1, f_2, f_3)$ *be an automorphism of* A *such that* $\deg f_1 = 2n$, *and either* $\deg f_2 = 3n$, $n < \deg f_3 \leq \frac{3n}{2}$, *or* $\frac{5n}{2} < \deg f_2 \leq 3n$, $\deg f_3 = \frac{3n}{2}$. *Suppose that there exist* $\alpha, \beta_1, \beta_2 \in F$, *where* $(\alpha, \beta_1, \beta_2) \neq 0$, *such that the elements* $g_1 = f_1 - \alpha f_3$, $g_2 = f_2 - \beta_1 f_3 - \beta_2 f_3^2$ *satisfy the conditions:*

i) g_1, g_2 *is a* 2*-reduced pair and* $\deg g_1 = 2n$, $\deg g_2 = 3n$*;*

ii) the element f_3 *of the automorphism* (g_1, g_2, f_3) *is reduced by an automorphism* (g_1, g_2, g_3) *with the condition* $\deg g_3 < n + \deg[g_1, g_2]$.

Then we will say that θ *admits a reduction of type III and the automorphism* (g_1, g_2, g_3) *will be called a reduction of* θ *of type III with an active element* f_3.

The next remark can be extracted from the proofs of Propositions 1A, 2A, and 3A.

Remark 2.1 *i)* $\deg [g_1, g_2] \leq 2n$ *in the conditions of Definition 2.1;*

ii) $\deg [g_1, g_2] \leq n$ *in the conditions of Definition 2.2;*

iii) $\deg [g_1, g_2] \leq \frac{n}{2}$ *in the conditions of Definition 2.3.*

Lemma 2.3 *Let* $\theta = (f_1, f_2, f_3)$ *satisfy the conditions of Definition 2.3. Then the following statements are true:*

i) if $a \in < f_2, f_3 >$ *and* $\deg a \leq \frac{5n}{2}$, *then* $a \in < f_3 >$*;*

ii) if $a \in < f_1, f_3 >$ *and* $\deg a \leq \frac{7n}{2}$, *then* $\overline{a} \in < \overline{f_1}, \overline{f_3} >$*;*

iii) if $a \in < f_1, f_2 >$ *and* $\deg a < 2n$, *then* $a = \delta \in F$.

We omit the proof of this lemma, since later in Lemma 2.4 we consider analogous statements for a more complicated case.

The next definition is some extension of the definition of automorphisms admiting a reduction of type IV in the sence of [24].

Definition 2.4 *Let* $\theta = (f_1, f_2, f_3)$ *be an automorphism of* A *such that* $\deg f_3 \leq \frac{3n}{2}$, $\deg \theta \leq \frac{13n}{2}$. *Suppose that there exist* $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4 \in F$ *such that the elements* $g_1 = f_1 - \alpha_1 f_3 - \alpha_2 f_3^2$, $g_2 = f_2 - \beta_1 f_3 - \beta_2 f_3^2 - \beta_3 f_1 f_3 - \beta_4 f_3^3$ *satisfy the conditions:*

i) g_1, g_2 *is a* 2*-reduced pair and* $\deg g_1 = 2n$, $\deg g_2 = 3n$*;*

ii) there exists an element g_3 *of the form*

$$g_3 = f_3 - \gamma w(g_1, g_2),\ 0 \neq \gamma \in F,$$

where $w(x, y)$ *is a derivative polynomial of the* 2*-reduced pair* g_1, g_2, *such that*

a) $\deg g_3 = \frac{3n}{2}$ *and* $\deg[g_1, g_3] < 3n + \deg[g_1, g_2]$*;*

b) there exists $0 \neq \mu \in F$ *such that* $\deg(g_2 - \mu g_3^2) \leq 2n$.

Then we will say that θ *admits a reduction of type IV and the automorphism* $(g_1, g_2 - \mu g_3^2, g_3)$ *will be called a reduction of* θ *of type IV with an active element* f_3.

Proposition 2.1 *Let* $\theta = (f_1, f_2, f_3)$ *be an automorphism of* A *satisfying the conditions of Definition 2.4. Then the following statements are true:*

1) $\deg\,[g_1, g_2] \leq \frac{n}{2}$, $\deg\, f_3 \geq n + \deg\,[g_1, g_2]$;

2) $2n \leq \deg\, f_1 < \frac{5n}{2}$, $\frac{5n}{2} + \deg\,[g_1, g_2] \leq \deg\, f_2 < \frac{7n}{2}$;

3) $\deg\,[f_1, f_3] = \deg\,[g_1, f_3] > 3n$, $\deg\,[f_2, f_3] > 3n$;

4) $\deg\, f_2 + \deg\, f_3 > 4n$;

5) if $(\beta_3, \beta_4) = 0$, *then either* $\deg\, f_2 = 3n$ *or* $\deg\, f_3 = \frac{3n}{2}$;

6) the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4$ *are uniquely defined;*

7) if $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0$, *then* $\deg\,[f_1, f_2] > 3n$.

Proof. Since $\deg\, g_3, \deg\, f_3 \leq \frac{3n}{2}$, and $\gamma \neq 0$, Definition 2.4 gives $\deg\, w(g_1, g_2) \leq \frac{3n}{2}$. By the definition of a derivative polynomial, applying Corollary 2.1 we get

$$\frac{3n}{2} \geq \deg\, w(g_1, g_2 \geq N(g_1, g_2) = n + \deg\,[g_1, g_2].$$

Consequently, $\deg\,[g_1, g_2] \leq \frac{n}{2}$. According to Corollary 2.3 we have $\deg\, \frac{\partial w}{\partial y}(g_1, g_2) = 3n$. By Definition 2.4, we have also

$$\deg\,[g_1, g_3] < 3n + \deg\,[g_1, g_2] = \deg\,[g_1, g_2]\frac{\partial w}{\partial y}(g_1, g_2).$$

Since

$$[f_1, f_3] = [g_1, f_3] = [g_1, g_3] + [g_1, g_2]\frac{\partial w}{\partial y}(g_1, g_2),$$

comparing the degrees of elements here we find that

$$\deg\,[f_1, f_3] = \deg\,[g_1, f_3] = 3n + \deg\,[g_1, g_2].$$

Since $\deg\, g_1 = 2n$, it follows also that $\deg\, f_3 \geq n + \deg\,[g_1, g_2]$. Consequently, $\deg\, f_3 < \deg\, g_1 = 2n < \deg\, f_3^2$, which gives $\deg\, f_1 \geq 2n$.

Obviously,

$$\deg\,[g_1, g_2] + \deg\, f_3 < \deg\,[g_1, f_3] + \deg\, g_2.$$

Then by the statement 2) of Lemma 2.1 we obtain

$$\deg\,[g_2, f_3] + \deg\, g_1 = \deg\,[g_1, f_3] + \deg\, g_2,$$

i.e.

$$\deg\,[g_2, f_3] = 4n + \deg\,[g_1, g_2].$$

Next we have

$$\deg\,[g_2, f_3] < 4n + 2\deg\,[g_1, g_2] \leq \deg\,[f_1, f_3]f_3.$$

Since

$$[f_2, f_3] = [g_2, f_3] + \beta_3[f_1, f_3]f_3,$$

comparing the degrees of elements we get

$$\deg\,[f_2, f_3] \geq 4n + \deg\,[g_1, g_2].$$

Then

$$\deg\, f_2 \geq 4n + \deg\,[g_1, g_2] - \deg\, f_3 \geq \frac{5n}{2} + \deg\,[g_1, g_2].$$

Assume that $(\beta_3, \beta_4) \neq 0$. Then it is easy to check that $\deg\,(\beta_3 f_1 + \beta_4 f_3^2)f_3 > 3n$. Since $\deg\, f_3 < \deg\, f_3^2 \leq \deg\, g_2 = 3n$, we have $\deg\, f_2 > 3n$. Now if $(\beta_3, \beta_4) = 0$, then we have $\deg\, f_2 \leq 3n$. And the inequality $\deg\, f_2 < 3n$ is possible only if $\deg\, f_3 = \frac{3n}{2}$. It remains to note that the inequality $\deg\, f_2 + \deg\, f_3 > 4n$ is fulfilled in both cases. Since $\deg\, \theta \leq \frac{13n}{2}$, this gives also $\deg\, f_1 < \frac{5n}{2}$. Note that the inequality $\deg\, f_1 + \deg\, f_3 > 3n$ implies that $\deg\, f_2 < \frac{7n}{2}$.

We have

$$\begin{aligned} f_1 &= g_1 + \alpha_1 f_3 + \alpha_2 f_3^2, \\ f_2 &= g_2 + \beta_1 f_3 + \beta_2 f_3^2 + \beta_3 f_1 f_3 + \beta_4 f_3^3 \\ &= g_2 + \beta_1 f_3 + (\beta_2 + \alpha_1\beta_3) f_3^2 + \beta_3 g_1 f_3 + (\beta_4 + \alpha_2\beta_3) f_3^3. \end{aligned}$$

Consequently,

$$\begin{aligned} [f_1, f_2] = [g_1, g_2] &+ \beta_1[g_1, f_3] + \alpha_1[f_3, g_2] + (2\beta_2 + \alpha_1\beta_3)[g_1, f_3]f_3 \\ &+ \beta_3 g_1[g_1, f_3] + 2\alpha_2 f_3[f_3, g_2] + (3\beta_4 + \alpha_2\beta_3)[g_1, f_3]f_3^2. \end{aligned}$$

Since

$$\begin{aligned} \deg\,[g_1, g_2] &< \deg\,[g_1, f_3] < \deg\,[f_3, g_2] < \deg\,[g_1, f_3]f_3 \\ &< \deg\, g_1[g_1, f_3] < \deg\,[f_3, g_2]f_3 < \deg\,[g_1, f_3]f_3^2, \end{aligned}$$

as in the proofs of Propositions 1A, 2A, and 3A we can easily deduce the statements 6), 7) of Proposition 2.1. □

Lemma 2.4 *Let $\theta = (f_1, f_2, f_3)$ satisfy the conditions of Definition 2.4. Then the following statements are true:*

i) if $a \in < f_2, f_3 >$ and $\deg\, a \leq \frac{5n}{2}$, then $a \in < f_3 >$;

ii) if $a \in < f_1, f_3 >$ and $\deg\, a \leq \frac{7n}{2}$, then $\overline{a} \in < \overline{g_1}, \overline{f_3} >$;

iii) if $a \in < f_1, f_2 >$ and $\deg\, a < 2n$, then we have $a = \delta \in F$ if $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0$, while $a = \delta w(g_1, g_2) + \lambda$ otherwise, where $w(x, y)$ is a derivative polynomial of the 2-reduced pair g_1, g_2.

Proof. Assume that $a \in < f_2, f_3 >$. The subalgebra $< f_2, f_3 >$ does not change if we replace f_2 by $g_2 + \beta_3 g_1 f_3$. So, we can assume that $f_2 = g_2 + \beta_3 g_1 f_3$. If $\overline{f_2}$ and $\overline{f_3}$ are algebraically independent, then by F2) we have $\overline{a} \in < \overline{f_2}, \overline{f_3} >$. By Proposition 2.1, we have $\deg\, f_2 > \frac{5n}{2}$. Consequently, $a \in < f_3 >$. Suppose that $\overline{f_2}$ and $\overline{f_3}$ are algebraically dependent. If $\beta_3 \neq 0$, then f_2, f_3 is a $*$-reduced pair. If $\beta_3 = 0$, then either f_2, f_3 is a $*$-reduced pair, or there exists $\mu \in F$ such that $\overline{g_2} = \mu \overline{f_3^2}$. In the last case we can assume that $f_2 = g_2 - \mu f_3^2$. Then again f_2, f_3 is a $*$-reduced pair. Since

$$N(f_2, f_3) > \deg\, [f_2, f_3] > 4n > \frac{5n}{2} \geq \deg\, a,$$

the inequality $\deg\, a \geq N(f_2, f_3)$ leads to a contradiction. Consequently, $\deg\, a < N(f_2, f_3)$. Since $\deg\, f_2 > \frac{5n}{2} \geq \deg\, a$, by Corollary 2.1 we get again $a \in < f_3 >$.

Suppose that $a \in < f_1, f_3 >$. As above we can assume that $f_1 = g_1$. If $\overline{g_1}$ and $\overline{f_3}$ are algebraically dependent, then g_1, f_3 is a p-reduced pair. Consider the case when

$$\frac{5n}{2} \geq \deg\, a \geq N(g_1, f_3) = (p-1) \deg\, g_1 - \deg\, f_3 + \deg[g_1, f_3].$$

If $p \geq 3$, then with regard to Proposition 2.1 we have $N(g_1, f_3) > 5n$. If $p = 2$, then this is possible only if

$$\deg\, g_1 = 2n = 3k,\ \deg\, f_3 = 2k \leq \frac{3n}{2}.$$

Consequently, repeately applying Proposition 2.1 gives

$$N(g_1, f_3) = k + \deg[g_1, f_3] > \frac{2n}{3} + 3n = \frac{11n}{3} > \frac{7n}{2}.$$

Thus $\deg\, a < N(g_1, f_3)$. Then Corollary 2.1 gives $\overline{a} \in < \overline{g_1}, \overline{f_3} >$.

Now suppose that $a \in < f_1, f_2 >$. If $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0$, then by Proposition 2.1 we have $\deg\, [f_1, f_2] > 3n$. From here it is easy to deduce that $\overline{a} \in < \overline{f_1}, \overline{f_2} >$ and $a = \delta \in F$ if $\deg\, a < 2n$. Otherwise Corollary 2.2 gives the statement iii) of the lemma. $\square$

It is easy to deduce from Proposition 2.1 that if $(\alpha_2, \beta_3, \beta_4) = 0$, then Definition 2.4 gives exactly the automorphisms admiting a reduction of type IV in the sence of [24].

Theorem 2.1 **[24]** *Let $\theta = (f_1, f_2, f_3)$ be a tame automorphism of the polynomial algebra $A = F[x_1, x_2, x_3]$ over a field F of characteristic 0. If $\deg \theta > 3$, then θ admits either an elementary reduction or a reduction of one of the types I–IV.*

Further we need the following proposition.

Proposition 2.2 *Let (f_1, f_2, f_3) be a tame automorphism of the algebra A satisfying the following conditions:*

i) f_1, f_2 *is a 2-reduced pair and* $\deg f_1 = 2n$, $\deg f_2 = sn$, *where* $s \geq 3$ *is an odd number;*

ii) $\deg f_3 = m < sn$ *and* f_3 *is an irreducible element of the automorphism* (f_1, f_2, f_3).

Then one of the following statements is true:

1) $m < n(s-2) + \deg[f_1, f_2]$;

2) (f_1, f_2, f_3) *admits a reduction of type IV with the active element* f_3, *and the coefficients* $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4$ *of Definition 2.4 are equal to* 0;

3) $\deg[f_1, f_3] < 3n + \deg[f_1, f_2]$ *and there exists* $\mu \in F$ *such that* $\deg(f_2 - \mu f_3^2) \leq 2n$.

Proof. The conditions of this proposition coincide with the conditions of Propositions 4A and 5A if we take into account the fact that the tame automorphisms of the algebra A are already simple and every elementary reducible element is simple reducible in the sence of [24]. It is easy to check that the proofs of Propositions 4A and 5A give also the proof of Proposition 2.2. □

3 On the uniqueness of reductions

Every tame automorphism θ has a sequence of elementary transformations

$$(x_1, x_2, x_3) = \theta_0 \to \theta_1 \to \ldots \to \theta_{k-1} \to \theta_k = \theta. \tag{3}$$

Put $d = \max\{\deg \theta_i \,|\, 0 \leq i \leq k\}$. The number d will be called the *width of the sequence (3)*. The minimal width of all sequences of the type (3) for θ will be called the *width* of the automorphism θ.

Let $\theta \in TAut\, A$, $\deg \theta > 3$. An automorphism ϕ will be called an *essential reduction* of θ if there exists a sequence of elementary transformations

$$\phi \to \theta_0 \to \theta_1 \to \ldots \to \theta_{t-1} \to \theta_t = \theta \tag{4}$$

such that $d(\phi) < d(\theta)$ and $\deg(\theta_i) \leq d(\theta)$, where $0 \leq i \leq t$.

The minimal number t for which there exists an essential reduction ϕ of θ of the type (4) will be called the *height* of θ, and the corresponding sequence (4) will be called a *minimal essential reduction* of θ. If $t = 0$, then we will say that θ admits an *essential elementary reduction.*

Later we will see that elementary reductions of automorphisms admiting a reduction of types III, IV are not essential.

If $\deg\,\theta = 3$, then we put $t = \infty$. Introduce a lexicographic order on the set of all pairs (d, t), where d is the width and t is the height of some tame automorphism, by putting $(d_1, t_1) < (d_2, t_2)$ if either $d_1 < d_2$ or $d_1 = d_2$, $t_1 < t_2$.

Lemma 3.1 *If (4) is a minimal essential reduction of θ, then* $\deg\,\theta_0 = d(\theta)$ *and*

$$(d(\phi), t(\phi)) < (d(\theta_0), t(\theta_0)) < (d(\theta_1), t(\theta_1)) < \ldots < (d(\theta_t), t(\theta_t)) = (d(\theta), t(\theta)).$$

Proof. If $\deg\,\theta_0 < d(\theta)$, then (4) gives $d(\theta_0) < d(\theta)$. In this case instead of ϕ we can take θ_0, which contradicts the minimality of (4). In addition, by the minimality of (4), ϕ is also a minimal essential reduction of each θ_i, $0 \leq i \leq k$. Consequently,

$$\deg\,\theta_0 = d(\theta_i),\ 0 \leq i \leq k,\ \ t(\theta_{i+1}) = t(\theta_i) + 1,\ 0 \leq i \leq t - 1,$$

which gives the statement of the lemma. □

Let d be the width and t be the height of θ. The sequence (3) will be called a *minimal representation of the automorphism θ* if

$$\theta_{k-t-1} \to \theta_{k-t} \to \theta_{k-t+1} \to \ldots \to \theta_k = \theta$$

is a minimal essential reduction of θ and $\deg\,\theta_i < d(\theta)$, $0 \leq i \leq k - t - 2$.

So, every minimal essential reduction of a tame automorphism can be continued to a minimal representation.

Proposition 3.1 *Let $\theta \in TAut\,A$, $\deg\theta > 3$, let d be the width and t be the height of θ, and let (4) be a minimal essential reduction of θ. Then one of the following statements is true:*

a) If θ admits a reduction of type I, then $d = \deg\,\theta$, $t = 1$, *and in the conditions of Definition 2.1 we have*

$$\theta_{t-1} = (f_1, \eta g_2 + \kappa g_1 + \nu, f_3),\ \ \eta \neq 0. \tag{5}$$

b) If θ admits a reduction of type II, then $d = \deg\,\theta$ *and in the conditions of Definition 2.2 the automorphism θ_{t-1} and the height t will be calculated by the following way:*

1) if $\alpha\beta \neq 0$, then $t = 2$ and θ_{t-1} can be written down simultaneously in the form (5) and in the form

$$\theta_{t-1} = (\rho g_1 + \sigma, f_2, f_3),\ \ \rho \neq 0; \tag{6}$$

2) if $\alpha = 0$, then $t = 1$ and θ_{t-1} has the form (5);

3) if $\beta = 0$, then $t = 1$ and θ_{t-1} has the form (6).

c) If θ admits a reduction of type III, then in the conditions of Definition 2.3 we have $d = 5n + \deg f_3$ and the automorphism θ_{t-1} and the height t will be calculated by the following way:

1) if $\alpha \neq 0$ and $(\beta_1, \beta_2) \neq 0$, then $t = 2$ and θ_{t-1} can be written down simultaneously in the form (5) and (6);

2) if $\alpha = 0$, then $t = 1$ and θ_{t-1} has the form (5);

3) if $(\beta_1, \beta_2) = 0$, then $t = 1$ and θ_{t-1} has the form (6).

d) If θ admits a reduction of type IV, then in the conditions of Definition 2.4 we have $d = \frac{13n}{2}$ and the automorphism θ_{t-1} and the height t will be calculated by the following way:

1) if $(\alpha_1, \alpha_2) \neq 0$ and $(\beta_1, \beta_2, \beta_3, \beta_4) \neq 0$, then $t = 3$ and θ_{t-1} can be written down simultaneously in the form (5) and (6);

2) if $(\alpha_1, \alpha_2) = 0$ and $(\beta_1, \beta_2, \beta_3, \beta_4) \neq 0$, then $t = 2$ and θ_{t-1} has the form (5);

3) if $(\alpha_1, \alpha_2) \neq 0$ and $(\beta_1, \beta_2, \beta_3, \beta_4) = 0$, then $t = 2$ and θ_{t-1} has the form (6);

4) if $(\alpha_1, \alpha_2) = 0$ and $(\beta_1, \beta_2, \beta_3, \beta_4) = 0$, then $t = 1$ and $\theta_{t-1} = (g_1, g_2, \rho g_3 + \sigma)$.

e) In the other cases θ admits an essential elementary reduction, i.e. $d = \deg \theta$ and $t = 0$.

Proof. Let $\theta \in T\mathit{Aut}\, A$ and let d be the width and t be the height of θ. Denote by (d_1, t_1) the value of (d, t) indicated in Proposition 3.1. We first show that

$$(d, t) \leq (d_1, t_1) \tag{7}$$

by induction on $\deg \theta$. So, we can assume that the inequality (7) is true for automorphisms with less degree.

Consider the case d). In this case θ admits a reduction of type IV, and we adopt all the conditions and notation of Definition 2.4. Consider the sequence of elementary transformations

$$\phi = (g_1, g_2 - \mu g_3^2, g_3) \longrightarrow (g_1, g_2, g_3) \longrightarrow (g_1, g_2, f_3) \longrightarrow (f_1, g_2, f_3) \longrightarrow (f_1, f_2, f_3). \tag{8}$$

Since $\deg \phi < \deg \theta$, it follows that the inequality (7) is valid for ϕ.

We will show that

$$d(\phi) < \deg \theta. \tag{9}$$

First of all we will give a standart argument that deduces (7) from (9). In the sequence (8) the automorphism (g_1, g_2, g_3) has the maximal degree and $\deg (g_1, g_2, g_3) = \frac{13n}{2} = d_1$.

If (9) is fulfilled, then we get $d \leq d_1$. If $d < d_1$, then (7) is fulfilled. Suppose that $d = d_1$. Then the sequence (8) gives that ϕ is an essential reduction of θ. Consequently, $t \leq 3$. We have $f_1 = g_1$ if $(\alpha_1, \alpha_2) = 0$, and $f_2 = g_2$ if $(\beta_1, \beta_2, \beta_3, \beta_4) = 0$. In these cases, excluding (f_1, g_2, f_3) from (8), we get $t \leq 2$. If $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) = 0$, then excluding also (g_1, g_2, f_3) from (8), we get $t \leq 1$.

If ϕ satisfies one of the conditions a), b), and e) of the proposition, then by (7) we have $d(\phi) \leq \deg(\phi)$, which gives (9).

We have

$$\deg\, g_1 = 2n,\ \deg\,(g_2 - \mu g_3^2) \leq 2n,\ \deg\, g_3 = \frac{3n}{2}. \tag{10}$$

Suppose that ϕ admits a reduction of types III, IV. According to (10), the element g_1 has the highest degree among the components of ϕ. Therefore $g_2 - \mu g_3^2$ or g_3 can be an active element of the reduction. If g_3 is an active element of the reduction, then there exists m such that

$$m < \deg\, g_3 \leq \frac{3m}{2},\ \ 2m \leq \deg\,(g_2 - \mu g_3^2) < \frac{5m}{2},\ \ \frac{5m}{2} < \deg\, g_1 < \frac{7m}{2}.$$

Taking account of (10), from here we deduce the inequalties

$$2m \leq 2n,\ \ \frac{3n}{2} \leq \frac{3m}{2},\ \ \frac{5m}{2} < \deg\, g_1 = 2m,$$

which are in a contradiction.

If $g_2 - \mu g_3^2$ is an active element of the reduction of ϕ, then there exists m such that

$$m < \deg\,(g_2 - \mu g_3^2) \leq \frac{3m}{2},\ \ 2m \leq \deg\, g_3 < \frac{5m}{2},\ \ \frac{5m}{2} < \deg\, g_1 < \frac{7m}{2}.$$

Consequently, $2m \leq \frac{3n}{2}$, i.e. $m \leq \frac{3n}{4}$. By (7), we have $d(\phi) \leq \frac{13m}{2} \leq \frac{39n}{8} < 5n$. From Proposition 2.1 we have $\deg\, \theta > 6n$, i.e. the inequality (9) is fulfilled.

The case when θ satisfies one of the conditions a), b), and c), can be considered similarly.

Consider the case e). If θ does not admit a reduction of types I–IV, then according to Theorem 2.1 θ admits an elementary reduction. Let ϕ be an elementary reduction of θ. If ϕ satisfies one of the conditions a), b), and e), then by (7) we have $d(\phi) \leq \deg\, \phi < \deg\, \theta$, i.e. the inequality (9) is fulfilled.

Assume that ϕ admits a reduction of type IV. Temporarily we assume that $\phi = (f_1, f_2, f_3)$ and that (f_1, f_2, f_3) satisfies the conditions of Definition 2.4. According to (7), we have $d(\phi) \leq \frac{13n}{2}$. Put also $\theta = (q_1, q_2, q_3)$. If $\deg\, \theta > \frac{13n}{2}$, then (9) is fulfilled. Therefore we may assume that

$$\deg\, \theta \leq \frac{13n}{2}. \tag{11}$$

We first consider the case in which

$$q_1 = \rho f_1 + a,\ \ q_2 = f_2,\ \ q_3 = f_3,\ \ a \in < f_2, f_3 > . \tag{12}$$

Changing ρg_1 to g_1, we may assume that $\rho = 1$. Proposition 2.1 gives deg f_2+deg $f_3 > 6n$ and deg $f_1 < \frac{5n}{2}$. Then from (11) we obtain deg $g_1 < \frac{5n}{2}$. Consequently, deg $a < \frac{5n}{2}$. By Lemma 2.4 we have $a \in < f_3 >$ and

$$a = \alpha'_0 + \alpha'_1 f_3 + \alpha'_2 f_3^2.$$

Consequently,

$$q_1 = g_1 + \alpha'_0 + (\alpha_1 + \alpha'_1) f_3 + (\alpha_2 + \alpha'_2) f_3^2.$$

Changing $g_1 + \alpha'_0$ to g_1, we may assume that $\alpha'_0 = 0$. Then θ admits a reduction of type IV, which contradicts the condition e).

Let

$$q_1 = f_1,\ q_2 = \eta f_2 + a,\ q_3 = f_3,\ a \in < f_1, f_3 > . \tag{13}$$

As above we can take $\eta = 1$. By Proposition 2.1 we have deg f_1 + deg $f_3 > 3n$ and $\frac{5n}{2} <$ deg $f_2 < \frac{7n}{2}$. Applying (11) we find that deg $q_2 < \frac{7n}{2}$, i.e. deg $a < \frac{7n}{2}$. By Lemma 2.4 we obtain $\overline{a} \in < \overline{g_1}, \overline{f_3} >$. Now it is easy to deduce that

$$a = \gamma_0 + \gamma_1 g_1 + \beta'_1 f_3 + \beta'_2 f_3^2 + \beta'_3 f_1 f_3 + \beta'_4 f_3^3.$$

Consequently,

$$q_2 = g_2 + \gamma_0 + \gamma_1 g_1 + (\beta_1 + \beta'_1) f_3 + (\beta_2 + \beta'_2) f_3^2 + (\beta_3 + \beta'_3) f_1 f_3 + (\beta_4 + \beta'_4) f_3^3.$$

Changing $g_2 + \gamma_0 + \gamma_1 g_1$ to g_2, we may assume that $\gamma_0 = \gamma_1 = 0$. Then θ admits a reduction of type IV.

Now we consider the case in which

$$q_1 = f_1,\ q_2 = f_2,\ q_3 = \rho f_3 + a,\ a \in < f_1, f_2 > . \tag{14}$$

Changing ρg_3 to g_3, we may assume that $\rho = 1$. Proposition 2.1 gives deg f_1+deg $f_2 > \frac{9n}{2}$. By (11), from here we get deg $q_3 < 2n$, i.e. deg $a < 2n$. If $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0$, then according to Lemma 2.4 we obtain $a = \delta \in F$, i.e. θ admits a reduction of type IV. If $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) = 0$, then $f_1 = g_1,\ f_2 = g_2$, and by Lemma 2.4 we have

$$a = \delta w(g_1, g_2) + \sigma.$$

Then

$$q_3 = g_3 + (\gamma + \delta) w(g_1, g_2) + \sigma.$$

Changing $g_3 + \sigma$ to g_3, we can take $\sigma = 0$. If $\gamma + \delta \neq 0$, then θ admits a reduction of type IV. If $\gamma + \delta = 0$, then $\theta = (g_1, g_2, g_3)$ and instead of ϕ we can take

$$\phi = (g_1, g_2 - \mu q_3^2, q_3).$$

Recall that, considering the case when θ admits a reduction of type IV, we have simultaneously proved that

$$d(\phi) < \frac{13n}{2}.$$

Since $\deg(\theta) = \frac{13n}{2}$, the inequality (9) is also fulfilled.

If ϕ admits a reduction of type III, then as above we can assume that $\phi = (f_1, f_2, f_3)$ and that (f_1, f_2, f_3) satisfies the conditions of Definition 2.3. This case can be settled by analogy to the case when ϕ admits a reduction of type IV. We only note that instead of (11) in this case we have a stronger inequality

$$\deg\ \theta \leq 5n + \deg\ f_3.$$

The inequality (7) is thus proved if θ satisfies one of the conditions d), e). The case when θ satisfies one of the conditions a), b), and c) can be considered similarly. Therefore we will assume that the inequality (7) is proved.

Now we begin a full proof of Proposition 3.1. Assume that the statement of the proposition is not true. Let θ be an automorphism with minimal (d, t), which does not satisfy the statement of the proposition. Note that if θ satisfies the condition e), then we have nothing to prove.

Restrict ourselves to the case when θ admits a reduction of type IV.
Let θ satisfy the conditions of Definition 2.4. Put $\theta_{t-1} = (q_1, q_2, q_3)$. Since

$$(d(\theta_{t-1}), t(\theta_{t-1})) < (d, t) \tag{15}$$

by Lemma 3.1, it follows that Proposition 3.1 is valid for θ_{t-1}. By (7) we have $d = d(\theta) \leq \frac{13n}{2}$. Then

$$d(\theta_{t-1}) \leq \frac{13n}{2}. \tag{16}$$

Suppose that (q_1, q_2, q_3) has the form (12). Repeating the same arguments which were given after (12), we can assume that

$$q_1 = g_1 + (\alpha_1 + \alpha_1')f_3 + (\alpha_2 + \alpha_2')f_3^2,$$

i.e. θ_{t-1} admits a reduction of type IV.

Let $(\beta_1, \beta_2, \beta_3, \beta_4) \neq 0$. If $(\alpha_1 + \alpha_1', \alpha_2 + \alpha_2') \neq 0$, then according to the statement d) of Proposition 3.1 we have $d(\theta_{t-1}) = \frac{13n}{2}$ and $t(\theta_{t-1}) = 3$. Since $\deg\ \theta \leq \frac{13n}{2}$, from here we get $d = \frac{13n}{2}$ and $t = 4$, which contradicts (7). Consequently, $(\alpha_1 + \alpha_1', \alpha_2 + \alpha_2') = 0$. In this case Proposition 3.1 gives $d(\theta_{t-1}) = \frac{13n}{2}$ and $t(\theta_{t-1}) = 2$. Hence, $d = \frac{13n}{2}$ and $t = 3$. If $(\alpha_1, \alpha_2) \neq 0$, then Proposition 3.1 is valid. If $(\alpha_1, \alpha_2) = 0$, then the inequality (7) is not fulfilled. It means that θ_{t-1} is obtained from θ by another elementary transformation!

Now let $(\beta_1, \beta_2, \beta_3, \beta_4) = 0$. As above we have $(\alpha_1 + \alpha_1', \alpha_2 + \alpha_2') = 0$. Then $d(\theta_{t-1}) = \frac{13n}{2}$ and $t(\theta_{t-1}) = 1$. Consequently, $d = \frac{13n}{2}$ and $t = 2$. If $(\alpha_1, \alpha_2) \neq 0$, then Proposition 3.1 is valid, and if $(\alpha_1, \alpha_2) = 0$, then the inequality (7) is not fulfilled.

The above discussion is standart and the other cases can be examined similarly. $\square$

Corollary 3.1 *Reductions of types I–IV are essential minimal reductions.*

Corollary 3.2 *If $\theta \in TAut\,A$ admits a reduction of types I–IV, then the type of this reduction is uniquely defined.*

Proof. Let $\theta = (f_1, f_2, f_3)$ admit a reduction of types I–IV. By Proposition 3.1, an active element f_3 of this reduction is uniquely characterized as a component of θ which does not change at a minimal reduction before appearing an automorphism (g_1, g_2, f_3) with the property $\deg\,[g_1, g_2] \leq \deg\, f_3$. The last inequality is a generalization of the estimaties of the degree of $[g_1, g_2]$ in Remark 2.1 and in Proposition 2.1. In addition, if $\deg\, f_1 \leq \deg\, f_2$, then the roles of elements in Definitions 2.1–2.4 are uniquely defined.

Now reductions of types I, II, and III can be easiely distinguished among themselves by the degree of $\deg\, f_3$.

Assume that θ admits a reduction of type IV and satisfies the conditions of Definition 2.4. If $\alpha_2 \neq 0$, then $\overline{f_1} = \alpha_2 \overline{f_3}^2$. Consequently, f_1 is an elementarily reducible element of θ. Note that if θ admits a reduction of types I–III, then θ does not admit such an elementary reduction. If $\alpha_2 = 0$, then $\deg\, f_1 = 2n$ and $\deg\, f_3 \leq \frac{3n}{2}$. Consequently, θ does not admit a reduction of type I or II. If θ admits a reduction of type III, then the elements g_1, g_2 are uniquely defined for both reductions. We have $g_3 = f_3 - \gamma w(g_1, g_2)$. Since $\deg\,[g_1, g_3] < 3n + \deg\,[g_1, g_2]$ and $\deg\,[g_1, w(g_1, g_2)] = 3n + \deg\,[g_1, g_2]$, it follows that γ is also uniquely defined from the equality $\overline{[g_1, f_3]} = \gamma \overline{[g_1, w(g_1, g_2)]}$. Now if $\deg\, g_3 < \frac{3n}{2}$, then θ admits a reduction of type III, and if $\deg\, g_3 = \frac{3n}{2}$, then θ admits a reduction of type IV. So, θ does not admit reductions of types III, IV simultaneously. $\square$

As we can see from the proof of Corollary 3.2, not only the type of reduction of θ but also the elements g_1, g_2 are uniquely defined. The element g_3 in Definitions 2.3, 2.4 is uniquely defined up to a summand from the field F, since it is with this exactness that the element $w(g_1, g_2)$ is defined. In Definition 2.1 instead of g_3 we can take any element of the form $\rho g_3 + F(g_1)$, where $0 \neq \rho \in F$ and $\deg\, F(g_1) \leq \deg\, g_3$. Furthermore, in Definition 2.2 instead of g_3 we can take any element of the form $\rho g_3 + \gamma$, where $0 \neq \rho \in F$.

Proposition 3.1 and Corollary 3.2 immediately give the next stronger form of Theorem 2.1.

Theorem 3.1 *Let $\theta \in TAut\,A$, $\deg\, \theta > 3$. Then θ admits only one type of the following reductions: an essential elementary reduction or a reduction of types I–IV.*

Besides, from the proof of Proposition 3.1 we can extract the next corollary.

Corollary 3.3 *Let $\theta \in TAut\,A$, $\deg\, \theta > 3$. If θ admits a reduction of types I–IV, then every elementary reduction of θ again admits a reduction of the same type.*

In fact, the elementary reductions considered in this corollary are not essential.

Note that if θ admits a reduction of types I, II, then θ does not admit any elementary reduction. If θ admits a reduction of type III, then in the conditions of Definition 2.3 θ admits an elementary reduction if and only if $\overline{f_2} = \varsigma \overline{f_3^2}$, $\deg\, f_3 = \frac{3n}{2}$, and $\deg\, f_2 = 3n$.

Assume that θ admits a reduction of type IV. If $(\alpha_2, \beta_3, \beta_4) = 0$ in the conditions of Definition 2.4, then θ also admits an elementary reduction if and only if $\overline{f_2} = \varsigma \overline{f_3^2}$, $\deg\, f_3 = \frac{3n}{2}$, and $\deg\, f_2 = 3n$. If $(\alpha_2, \beta_3, \beta_4) \neq 0$, then θ admits obvious elementary reductions.

4 Defining relations of the group of tame automorphisms

Let $A_n = F[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over a field F with the set of free generators $X = \{x_1, x_2, \ldots, x_n\}$. Then the group $TAut\, A_n$ is generated by all elementary automorphisms

$$\sigma(i, \alpha, f) = (x_1, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n), \tag{17}$$

where $0 \neq \alpha \in F,\ f \in < X \setminus \{x_i\} >$.

Our aim in this section is to describe defining relations of the group $TAut\, A_n$ with respect to the generators (17). It is easy to check that

$$\sigma(i, \alpha, f)\sigma(i, \beta, g) = \sigma(i, \alpha\beta, \beta f + g). \tag{18}$$

Note that from here we obtain trivial relations

$$\sigma(i, 1, 0) = id, \quad 1 \leq i \leq n.$$

If $i \neq j$ and $f \in < X \setminus \{x_i, x_j\} >$, then we have also

$$\sigma(i, \alpha, f)^{-1}\sigma(j, \beta, g)\sigma(i, \alpha, f) = \sigma(j, \beta, \sigma(i, \alpha, f)^{-1}(g)). \tag{19}$$

Consequently, if $i \neq j$ and $f, g \in < X \setminus \{x_i, x_j\} >$, then the automorphisms $\sigma(i, \alpha, f)$, $\sigma(j, \beta, g)$ commute.

For every pair k, s, where $1 \leq k \neq s \leq n$, we define a tame automorphism (ks) by putting

$$(ks) = \sigma(s, -1, x_k)\sigma(k, 1, -x_s)\sigma(s, 1, x_k).$$

Note that the automorphism (ks) of the algebra A_n changes only the positions of the variables x_k and x_s. Now it is easy to see that

$$\sigma(i, \alpha, f)^{(ks)} = \sigma(j, \alpha, (ks)(f)), \tag{20}$$

where $j = (ks)(x_i)$.

Lemma 4.1 *From the relations (18)–(20) it follows that the subgroup generated by all automorphisms (ks), where $1 \leq k \neq s \leq n$, is isomorphic to the symmetric group S_n.*

Proof. By (18) and (19), we have

$$\begin{aligned}(ks)^2 &= \sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,-1,0)\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,-1,x_k)\sigma(s,-1,0)\sigma(k,1,-x_s)^{\sigma(s,-1,0)}\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,1,-x_k)\sigma(k,1,x_s)\sigma(k,1,-x_s)\sigma(s,1,x_k) = \sigma(s,1,-x_k)\sigma(s,1,x_k) = id.\end{aligned}$$

Then (20) gives

$$\begin{aligned}(ks)^{(sk)} &= \sigma(s,-1,x_k)^{(sk)}\sigma(k,1,-x_s)^{(sk)}\sigma(s,1,x_k)^{(sk)}\\ &= \sigma(k,-1,x_s)\sigma(s,1,-x_k)\sigma(k,1,x_s) = (sk),\end{aligned}$$

i.e. $(ks) = (sk)$. Now it is not difficult to deduce from (18)–(20) that

$$[(ij),(ks)] = id,\ \ (ik)^{(is)} = (ks),$$

where i,j,k,s are distinct. It is immediate that the given relations imply the defining relations of the group S_n with respect to the system of generators $(i\ i+1)$, where $1 \leq i \leq n+1$, which are indicated in [4]. □

By Lemma 4.1, elements of the symmetric group S_n can be identified with automorphisms of the algebra A_n if we put

$$\pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$$

for every $\pi \in S_n$. We note that (20) can be rewritten as

$$\sigma(i,\alpha,f)^\pi = \sigma(\pi^{-1}(i),\alpha,\pi^{-1}(f)).$$

It is well known that the group of affine automorphisms $GA_n(F)$ of the algebra A_n is generated by all affine elementary automorphisms.

Lemma 4.2 *The relations (18)–(20) for elementary affine automorphisms are defining relations of the group $GA_n(F)$.*

Proof. Let φ be a product of elementary affine automorphisms. Suppose that $\varphi = id$. By (18), (19), we can represent φ in the form

$$\varphi = \sigma(1,1,\alpha_1)\sigma(2,1,\alpha_2)\ldots\sigma(n,1,\alpha_n)\varphi',$$

where φ' is a product of elementary linear automorphisms. Obviously, $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. Therefore we can assume that φ is a product of elementary linear automorphisms. By (18), (19), we can easily represent φ in the form

$$\varphi = \sigma(1,\alpha_1,0)\sigma(2,\alpha_2,0)\ldots\sigma(n,\alpha_n,0)\varphi',$$

where φ' is a product of elementary automorphisms of the type $\sigma(i,1,f)$. By (18)–(20), we have

$$\begin{aligned}
&\sigma(k,\alpha,0) = \sigma(s,\alpha,0)^{(ks)} \\
&= \sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\sigma(s,\alpha,0)\sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k) \\
&= \sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,-\alpha,0)\sigma(s,1,(1-\alpha)x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k) \\
&= \sigma(s,-1,x_k)\sigma(s,-\alpha,0)\sigma(k,1,\alpha^{-1}x_s)\sigma(s,1,(1-\alpha)x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k) \\
&= \sigma(s,\alpha,0)\sigma(s,1,-\alpha x_k)\sigma(k,1,\alpha^{-1}x_s)\sigma(s,1,(1-\alpha)x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k).
\end{aligned}$$

By this relation, we can represent φ in the form

$$\varphi = \sigma(n,\alpha_n,0)\varphi',$$

where φ' is a product of elementary linear automorphisms of the form $\sigma(i,1,f)$. Hence $\alpha_n = 1$. Note that $\sigma(i,1,f)$ can be represented as a product of automorphisms

$$X_{ij}(\lambda) = \sigma(j,1,\lambda x_i), \quad \lambda \in F, \quad i \neq j. \tag{21}$$

Thus, we can assume that φ is a product of automorphisms of the form (21).

Let G be the subgroup of $TAut\, A_n$ generated by all automorphisms of the form (21). We define a map

$$J : G \longrightarrow SL_n(F),$$

where $J(\psi)$ is the Jacobian matrix of $\psi \in G$. It is easy to check that

$$J(X_{ij}(\lambda)) = E_{ij}(\lambda),$$

and that J is an isomorphism of the groups.

Now it is sufficient to prove that every relation of the group $SL_n(F)$ is a corollary of (18)–(20). Obviously, (18)–(19) cover the Steinberg relations (see, for example [18]). Besides, according to [18], we need to check the relation

$$\{u,v\} = id, \quad 0 \neq u,v \in F,$$

where

$$\begin{aligned}
\{u,v\} &= h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1}, \\
h_{ij}(u) &= w_{ij}(u)w_{ij}(-1), \\
w_{ij}(u) &= X_{ij}(u)X_{ji}(-u^{-1})X_{ij}(u).
\end{aligned}$$

Applying (18)–(20) we have

$$\begin{aligned}
&w_{ij}(u) = \sigma(j,1,ux_i)\sigma(i,1,-u^{-1}x_j)\sigma(j,1,ux_i) \\
&= \sigma(j,1,ux_i)\sigma(i,u,0)\sigma(i,1,-x_j)\sigma(i,u^{-1},0)\sigma(j,1,ux_i) \\
&= \sigma(i,u,0)\sigma(j,1,ux_i)^{\sigma(i,u,0)}\sigma(i,1,-x_j)\sigma(j,1,ux_i)^{\sigma(i,u,0)}\sigma(i,u^{-1},0) \\
&= \sigma(i,u,0)\sigma(j,1,x_i)\sigma(i,1,-x_j)\sigma(j,1,x_i)\sigma(i,u^{-1},0) \\
&= \sigma(i,u,0)\sigma(j,-1,0)\sigma(j,-1,x_i)\sigma(i,1,-x_j)\sigma(j,1,x_i)\sigma(i,u^{-1},0) \\
&= \sigma(i,u,0)\sigma(j,-1,0)(ij)\sigma(i,u^{-1},0) = (ij)\sigma(i,u,0)^{(ij)}\sigma(j,-1,0)^{(ij)}\sigma(i,u^{-1},0) \\
&= (ij)\sigma(j,u,0)\sigma(i,-1,0)\sigma(i,u^{-1},0) = (ij)\sigma(j,u,0)\sigma(i,-u^{-1},0).
\end{aligned}$$

Consequently,

$$\begin{aligned}h_{ij}(u) = w_{ij}(u)w_{ij}(-1) = (ij)\sigma(j,u,0)\sigma(i,-u^{-1},0)(ij)\sigma(j,-1,0)\sigma(i,1,0)\\ = \sigma(j,u,0)^{(ij)}\sigma(i,-u^{-1},0)^{(ij)}\sigma(j,-1,0)\\ = \sigma(i,u,0)\sigma(j,-u^{-1},0)\sigma(j,-1,0) = \sigma(i,u,0)\sigma(j,u^{-1},0).\end{aligned}$$

Hence

$$\begin{aligned}\{u,v\} = h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1}\\ = \sigma(i,uv,0)\sigma(j,(uv)^{-1},0)\sigma(i,u,0)\sigma(j,u^{-1},0)\sigma(i,v,0)\sigma(j,v^{-1},0) = id.\end{aligned}$$

Thus we can say that every relation of the group $SL_n(F)$ follows from (18)–(20). $\Box$

Assume that

$$\theta = \phi_1\phi_2\ldots\phi_r \in TAut\,A_n, \tag{22}$$

where ϕ_i, $1 \leq i \leq r$, are elementary automorphisms. Put

$$\psi_i = \phi_1\phi_2\ldots\phi_i,\ \ 0 \leq i \leq r.$$

In particular, we have

$$\psi_r = \theta,\ \ \psi_0 = id.$$

To (22) corresponds the sequence of elementary transformations

$$id = \psi_0 \xrightarrow{\phi_1} \psi_1 \xrightarrow{\phi_2} \psi_2 \xrightarrow{\phi_3} \ldots \xrightarrow{\phi_r} \psi_r = \theta. \tag{23}$$

Note that the representations (22) and (23) of the automorphism θ are equivalent. If (23) is a minimal representation of θ, then the representation (22) will be also called a *minimal representation of* θ.

Theorem 4.1 *Let F be a field of characteristic 0 and $A = F[x_1, x_2, x_3]$. Then the relations (18), (19), (20) are defining relations of the group $TAut\,A$ with respect to the generators (17).*

Plan of the proof. Assume that

$$\varphi_1\varphi_2\ldots\varphi_k = id = (x_1, x_2, x_3), \tag{24}$$

where φ_i, $1 \leq i \leq k$, are elementary automorphisms. Put

$$\theta_i = \varphi_1\varphi_2\ldots\varphi_i,\ \ 0 \leq i \leq k.$$

In particular, we have $\theta_0 = \theta_k = (x_1, x_2, x_3)$. To (24) corresponds the sequence of elementary transformations

$$id = \theta_0 \xrightarrow{\varphi_1} \theta_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_k} \theta_k = id. \tag{25}$$

Denote by $d = max\{\deg \theta_i | 0 \le i \le k\}$ the *width* of the sequence (25). Let i_1 be the minimal number and i_2 be the maximal number which satisfy the equations $\deg \theta_{i_1} = d$ and $\deg \theta_{i_2} = d$. Put $q = i_2 - i_1$. The pair (d, q) will be called the *exponent of the relation (24).*

To prove the theorem, we show that (24) follows from (18)–(20). Assume that our theorem is not true. Call a relation of the form (24) *trivial* if it follows from (18)–(20). We choose a nontrivial relation (24) with the minimal exponent (d, q) with respect to the lexicographical order. To arrive at a contradiction, we show that (24) is also trivial.

If $d = 3$, then Lemma 4.2 gives the triviality of the relation (24). Therefore we can assume that $d > 3$.

Our plan is to change the product (24) using (18)–(20) and to obtain a new sequence (25) with the exponent strictly less than (d, q). The proof of the theorem will be completed by Lemmas 4.3–4.10. □

Denote by $t = [\frac{q}{2}]$ the integral part of $\frac{q}{2}$. Put also

$$\phi = \theta_{i_1+t-1}, \ \theta = \theta_{i_1+t}, \ \tau = \theta_{i_1+t+1}.$$

Then we have

$$\phi \xrightarrow{\varphi_{i_1+t}} \theta \xrightarrow{\varphi_{i_1+t+1}} \tau. \tag{26}$$

Lemma 4.3 *The following statements are true:*

1) d is the width of θ, t is the height of θ, and

$$\theta = \varphi_1 \varphi_2 \dots \varphi_{i_1+t} \tag{27}$$

is a minimal representation of θ.

2) If q is an even number, then

$$\theta = \varphi_k^{-1} \varphi_{k-1}^{-1} \dots \varphi_{i_1+t+1}^{-1}$$

is also a minimal representation of θ.

3) If q is an odd number, then $(d(\tau), t(\tau)) = (d, t)$ and

$$\tau = \varphi_k^{-1} \varphi_{k-1}^{-1} \dots \varphi_{i_1+t+2}^{-1}$$

is a minimal representation of τ. Moreover, in (24) the product (27) can be replaced by an arbitrary minimal representation of θ.

Proof. Assume that $(d(\theta), t(\theta)) < (d, t)$ and let (22) be a minimal representation of θ. Then (24) is a consequence of the equalities

$$\varphi_1 \varphi_2 \dots \varphi_{i_1+t} \phi_r^{-1} \dots \phi_2^{-1} \phi_1^{-1} = id, \tag{28}$$

$$\phi_1\phi_2\dots\phi_r\varphi_{i_1+t+1}\dots\varphi_{k-1}\varphi_k = id. \tag{29}$$

To (28) corresponds the sequence of elementary transformations

$$(x_1,x_2,x_3)\to\theta_1\to\dots\to\theta_{i_1+t}=\theta=\psi_r\to\psi_{r-1}\dots\to\psi_1\to(x_1,x_2,x_3),$$

and to (29) corresponds

$$(x_1,x_2,x_3)\to\psi_1\to\dots\to\psi_r=\theta=\theta_{i_1+t}\to\theta_{i_1+t+1}\to\dots\to\theta_{k-1}\to(x_1,x_2,x_3).$$

Since $(d(\theta),t(\theta))<(d,t)$, it follows that (28) and (29) have the exponents strictly less than (d,q). This gives the first statement of the lemma.

It is obvious that

$$\varphi_k^{-1}\varphi_{k-1}^{-1}\dots\varphi_1^{-1}=id$$

has the same exponent (d,q). Applying the first statement of the lemma to this relation, we get the second statement of the lemma as well as the minimality of the representation of τ if q is an odd number. If q is an odd number, then (28) has the exponent strictly less than (d,q), and (29) has the exponent (d,q). Consequently, (24) and (29) are equivalent modulo (18)–(20). Thus θ can be changed by an arbitrary minimal representation in (29). $\square$

Lemma 4.4 *If $t\geq 1$, then the relation (24) is trivial.*

Proof. Our aim is to change the product $\varphi_{i_1+t}\varphi_{i_1+t+1}$ by (18)–(20) and to get a new sequence (25) with the exponent strictly less than (d,q), i.e. to show that (24) is trivial.

Since $t\geq 1$, according to Proposition 3.1 θ admits a reduction of types I–IV. We restrict ourselves only to the case when θ admits a reduction of type IV. The other cases can be considered similarly.

Let $\theta=(f_1,f_2,f_3)$ satisfy the conditions of Definition 2.4. According to Lemma 4.3, the representation (27) of θ is minimal. Then ϕ can be calculated by using Proposition 3.1.

Case I: q is an even, $q=2t$. According to Lemma 4.3 τ can also be calculated by using Proposition 3.1.

Assume that $t=2$. By Proposition 3.1 the automorphisms ϕ and τ have the same form, i.e. either

$$\phi=(f_1,\eta_1g_2+\kappa_1g_1+\nu_1,f_3),\ \ \tau=(f_1,\eta_2g_2+\kappa_2g_1+\nu_2,f_3),$$

or

$$\phi=(\rho_1g_1+\sigma_1,f_2,f_3),\ \ \tau=(\rho_2g_1+\sigma_2,f_2,f_3).$$

By (18), in both cases $\varphi_{i_1+t}\varphi_{i_1+t+1}$ can be replaced by an elementary automorphism. Obviously, (24) will then be replaced by a relation with the exponent strictly less than (d,q).

Assume that $t = 3$. According to Proposition 3.1, the automorphisms ϕ, τ have the forms (5), (6). If ϕ, τ have the same form (5) (or (6)), then as above, by (18), we obtain the triviality of (24). Suppose that ϕ has the form (5), and τ has the form (6). It is immediate that

$$\varphi_{i_1+t} = \sigma(2, \eta^{-1}, F(x_1, x_3)), \quad \varphi_{i_1+t+1} = \sigma(1, \rho, G(x_3)).$$

By (19) we get

$$\varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(1, \rho, G(x_3))\sigma(2, \eta^{-1}, F_1(x_1, x_3)).$$

We replace $\varphi_{i_1+t}\varphi_{i_1+t+1}$ in (24) according to this equality,. Then θ in (25) can be changed to

$$\theta' = (\rho g_1 + \alpha, \eta g_2 + \kappa g_1 + \nu, f_3).$$

Note that $\deg\theta' = \frac{13n}{2}$ and after such replacement the exponent (d, q) of the sequence (25) does not change. As before θ' admits a reduction of type IV. But according to Proposition 3.1, in this case we have $t(\theta') = 1$. This contradicts Lemma 4.3, since $t(\theta') < t$.

If $t = 1$, then according to Proposition 3.1 we have

$$\phi = (g_1, g_2, \rho_1 g_3 + \alpha_1), \quad \tau = (g_1, g_2, \rho_2 g_3 + \alpha_2).$$

By (18), we obtain the triviality of (24) as above.

Case II: q is odd, $q = 2t+1$. If $(\alpha_1, \alpha_2) = 0$ and $(\beta_1, \beta_2, \beta_3, \beta_4) = 0$ in the conditions of Definition 2.4, then $\theta = (g_1, g_2, f_3)$. Moreover, according to Proposition 3.1, we have $t = 1$ and

$$\phi = (g_1, g_2, \rho g_3 + \alpha).$$

Assume that

$$\tau = (\lambda f_1 + a, f_2, f_3), \quad a \in < f_2, f_3 > . \tag{30}$$

By Lemma 4.3 we have $(d(\tau), t(\tau)) = (d, t)$. Since $d = d(\theta) = \frac{13n}{2}$, from here we get $\deg\tau \le \frac{13n}{2}$. With regard to the inequalities of Proposition 2.1, we have $\deg(\lambda f_1 + a) < \frac{5n}{2}$ and $\deg a < \frac{5n}{2}$. Applying Lemma 2.4 gives also

$$a = \alpha'_0 + \alpha'_1 f_3 + \alpha'_2 f_3^2.$$

Consequently,

$$\lambda f_1 + a = \lambda g_1 + \alpha'_0 + \alpha'_1 f_3 + \alpha'_2 f_3^2.$$

If $(\alpha'_1, \alpha'_2) \neq 0$, then τ admits a reduction of type IV. Moreover, $t(\tau) > 1$, which contradicts the equality $(d(\tau), t(\tau)) = (d, t)$. Consequently,

$$\tau = (\lambda g_1 + \alpha'_0, g_2, f_3).$$

We have

$$\varphi_{i_1+t} = \sigma(3, \rho^{-1}, F(x_1, x_2)), \;\; \varphi_{i_1+t+1} = \sigma(1, \lambda, \alpha_0').$$

According to (19), we obtain

$$\varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(1, \lambda, \alpha_0')\sigma(3, \rho^{-1}, F_1(x_1, x_2)).$$

After the corresponding replacement, instead of θ we have

$$\theta' = (\lambda g_1 + \alpha_0', g_2, \rho g_3 + \alpha).$$

Note that θ' admits an essential elementary reduction, i.e. $t(\theta') = 0$.

Now assume that

$$\tau = (f_1, \eta f_2 + a, f_3), \;\; a \in < f_1, f_3 > . \tag{31}$$

Using the same arguments as above we get

$$\tau = (g_1, \eta g_2 + \kappa g_1 + \nu, f_3).$$

We have

$$\varphi_{i_1+t} = \sigma(3, \rho^{-1}, F(x_1, x_2)), \;\; \varphi_{i_1+t+1} = \sigma(2, \eta, \kappa x_1 + \nu).$$

The relation (19) gives

$$\varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(2, \eta, \kappa x_1 + \nu)\sigma(3, \rho^{-1}, F_1(x_1, x_2)).$$

After such replacement, instead of θ we have

$$\theta' = (g_1, \eta g_2 + \kappa g_1 + \nu, \rho g_3 + \alpha),$$

and this automorphism also admits an essential elementary reduction.

If the elementary reduction $\theta \to \tau$ replaces the element f_3 of the automorphism θ, then applying (18) also gives the triviality of (24).

We now consider the case when $(\alpha_1, \alpha_2) = 0$ and $(\beta_1, \beta_2, \beta_3, \beta_4) \neq 0$. Then according to Proposition 3.1 we have $t = 2$ and ϕ has the form (5), i.e.

$$\phi = (g_1, \eta g_2 + \kappa g_1 + \nu, f_3).$$

If τ has the form (31), then (18) gives the triviality of (24). Assume that τ has the form (30). Then by the same discussion as above we get

$$\tau = (\rho g_1 + \alpha, f_2, f_3).$$

We have

$$\varphi_{i_1+t} = \sigma(2, \eta^{-1}, F(x_1, x_2)), \;\; \varphi_{i_1+t+1} = \sigma(1, \rho, \alpha).$$

By (19), we get

$$\varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(1, \rho, \alpha)\sigma(3, \eta^{-1}, F_1(x_1, x_2)).$$

After the corresponding replacement, instead of θ we obtain

$$\theta' = (\rho g_1 + \alpha, \eta g_2 + \kappa g_1 + \nu, f_3).$$

Proposition 3.1 gives $t(\theta') = 1 < t$; a contradiction.

Assume that

$$\tau = (f_1, f_2, \rho f_3 + a), \;\; a \in < f_1, f_2 > . \tag{32}$$

By Proposition 2.1 we have deg f_1 + deg $f_2 > \frac{9n}{2}$. Since $d(\theta) = \frac{13n}{2} \geq$ deg τ, we have deg $(\rho f_3 + a) < 2n$ and deg $a < 2n$. Lemma 2.4 gives $a = \alpha \in F$. Then $\varphi_{i_1+t+1} = \sigma(3, \rho, \alpha)$. Hence

$$\varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(3, \rho, \alpha)\sigma(2, \eta^{-1}, F_1(x_1, x_2)).$$

After this replacement, θ will be changed to

$$\theta' = (g_1, \eta g_2 + \kappa g_1 + \nu, \rho f_3 + a).$$

Proposition 3.1 gives $t(\theta') = 1$, which also leads to a contradiction.

The case when $(\alpha_1, \alpha_2) \neq 0$ and $(\beta_1, \beta_2, \beta_3, \beta_4) = 0$ can be considered analogously.

Assume that $(\alpha_1, \alpha_2) \neq 0$ and $(\beta_1, \beta_2, \beta_3, \beta_4) \neq 0$. According to Proposition 3.1, we have $t = 3$. Now we use the statement 3) of Lemma 4.3 about the arbitrariness of the minimal representation of θ in (24). If τ has the form (30), then we can assume that ϕ has the form (6), and if τ has the form (31), then we can assume that ϕ has the form (5). By (18), in both cases we can reduce the exponent of (24). If τ has the form (32), then using the same arguments we get

$$\tau = (f_1, f_2, \rho f_3 + \alpha).$$

Assume that $\phi = (\lambda g_1 + \beta, f_2, f_3)$. Applying (19) to $\varphi_{i_1+t}\varphi_{i_1+t+1}$ changes θ to

$$\theta' = (g_1, g_2, \rho f_3 + a).$$

Then we have $t(\theta') = 2$, which also leads to a contradiction. $\square$

Now we begin to consider the most complicated case when $t = 0$, i.e. $q = 0, 1$. Put $\theta = (f_1, f_2, f_3)$. According to Lemma 4.3, $t = 0$ is the height of θ, i.e. θ admits an essential elementary reduction. Moreover, ϕ is an essential elementary reduction of θ. If $q = 0$, then τ is also an esssential elementary reduction of θ. If $q = 1$, then we have $(d(\tau), t(\tau)) = (d, t)$. Consequently, deg $\tau \leq$ deg $\theta = d$. Thus we can assume that

$$\tau = (f_1, f_2, f), \quad f = \alpha f_3 + a, \quad a = a(f_1, f_2) \in \langle f_1, f_2 \rangle, \quad \deg a \leq \deg f_3. \tag{33}$$

Lemma 4.5 *If ϕ reduces the element f_3 of θ, then the relation (24) is trivial.*

Proof. Applying (18) we can replace $\varphi_{i_1+t}\varphi_{i_1+t+1}$ by an elementary automorphism. Obviously, this replacement also reduces the exponent of (25). $\Box$

By Lemma 4.5, we can assume that ϕ reduces one of the elements f_1, f_2 of θ. Without loss of generality, later on we consider that ϕ reduces the element f_2 of θ, i.e. $\phi = (f_1, g_2, f_3)$, deg $g_2 <$ deg f_2.

Lemma 4.6 *If ϕ' reduces the element f_2 of θ, then in (26) the automorphism ϕ can be replaced by ϕ'.*

Proof. According to (18), in this case the elementary transformation $\phi \to \theta$ can be changed to $\phi \to \phi' \to \theta$. Since deg $\phi' <$ deg $\theta = d$, the exponent (d, q) of the sequence (25) does not change after this replacement. But in the new sequence (25) we have ϕ' instead of ϕ. $\Box$

Taking this lemma into account, we can assume that

$$\phi = (f_1, g_2, f_3), \quad g_2 = f_2 - b, \quad b \in \langle f_1, f_3 \rangle, \quad \deg g_2 < \deg f_2. \tag{34}$$

Lemma 4.7 *If one of the following conditions is fulfilled, then (24) is trivial:*
1) $\overline{f_2} \in \langle \overline{f_1} \rangle$;
2) $\overline{f_3} \in \langle \overline{f_1} \rangle$;
3) *a does not depend on f_2;*
4) $\overline{f_2} = \beta \overline{f_3} + \gamma \overline{f_1^k}$;
5) *$\overline{f_1}, \overline{f_3}$ are algebraically independent.*

Proof. Assume that $\overline{f_2} \in \langle \overline{f_1} \rangle$ and $\overline{f_2} = \beta \overline{f_1}^l$. According to Lemma 4.6, we can suppose that $g_2 = f_2 - \beta f_1^l$. Then

$$\varphi_{i_1+t} = \sigma(2, 1, \beta x_1^l), \quad \varphi_{i_1+t+1} = \sigma(3, \alpha, a(x_1, x_2)).$$

By (19), we have

$$\varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(3, \alpha, a_1(x_1, x_2))\sigma(2, 1, \beta x_1^l).$$

After the corresponding replacement in (24), θ will be replaced by $\theta' = (f_1, g_2, f)$ in (25). Since deg $f_1 +$ deg $g_2 +$ deg $f < d$, the exponent of (24) is decreased.

Assume that $\overline{f_3} \in \langle \overline{f_1} \rangle$ and $\overline{f_3} = T(\overline{f_1})$. Put $g_3 = f_3 - T(f_1)$. According to (19), we have

$$\varphi_{i_1+t} = \sigma(2, 1, b(x_1, x_3)) = \sigma(3, 1, -T(x_1))\sigma(2, 1, b_1(x_1, x_3))\sigma(3, 1, T(x_1)).$$

After the corresponding replacement in (24), the elementary transformation $\phi \to \theta$ will be replaced by the sequence elementary transformations

$$\phi = (f_1, g_2, f_3) \to (f_1, g_2, g_3) \to (f_1, f_2, g_3) \to (f_1, f_2, f_3) = \theta.$$

Since $d(\phi), \deg(f_1, g_2, g_3), \deg(f_1, f_2, g_3) < d = \deg\theta$, the new sequence (25) has the same exponent (d, q). However, instead of ϕ we have (f_1, f_2, g_3). By Lemma 4.5, we obtain the triviality of (24).

Assume that a does not depend on f_2. By (19) we have

$$\varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(2,1,b(x_1,x_3))\sigma(3,\alpha,a(x_1)) = \sigma(3,\alpha,a(x_1))\sigma(2,1,b_1(x_1,x_3)).$$

After the corresponding replacement in (24), instead of θ we obtain $\theta' = (f_1, g_2, f)$. Since $\deg(f_1, g_2, f) < d$, this replacement also decreases the exponent of (24).

Consider the case when $\overline{f_2} = \beta\overline{f_3} + \gamma\overline{f_1^k}$. By the statement 1) we can assume that $\beta \neq 0$. By Lemma 4.6 we can also assume that $b = \beta f_3 + \gamma f_1^k$. Consequently,

$$\begin{array}{rcl} g_2 &=& f_2 - \beta f_3 - \gamma f_1^k, \\ f_2 &=& g_2 + \beta f_3 + \gamma f_1^k, \\ f_3 &=& -\dfrac{1}{\beta}g_2 + \dfrac{1}{\beta}f_2 - \dfrac{\gamma}{\beta}f_1^k. \end{array}$$

These equalities justify the sequence of elementary transformations

$$(f_1, f_3, g_2) \to (f_1, f_2, g_2) \to (f_1, f_2, f_3) = \theta.$$

We have

$$\varphi_{i_1+t} = \sigma(2,1,\beta x_3 + \gamma x_1^k)).$$

Applying (18) and (19) we get

$$\begin{aligned} \varphi_{i_1+t} &= \sigma(2,1,\beta x_3)\sigma(2,1,\gamma x_1^k) \\ &= \sigma(2,1,\beta x_3)\sigma(3,-\beta,x_2)\sigma(3,-\frac{1}{\beta},\frac{1}{\beta}x_2)\sigma(2,1,\gamma x_1^k) \\ &= \sigma(2,1,\beta x_3)\sigma(3,-\beta,x_2)\sigma(2,1,\gamma x_1^k)\sigma(3,-\frac{1}{\beta},\frac{1}{\beta}x_2)^{\sigma(2,1,\gamma x_1^k)} \\ &= \sigma(2,1,\beta x_3)\sigma(3,-\beta,x_2)\sigma(2,1,\gamma x_1^k)\sigma(3,-\frac{1}{\beta},\frac{1}{\beta}(x_2-\gamma x_1^k)) \\ &= \sigma(2,1,\beta x_3)\sigma(3,-\beta,x_2)\sigma(2,\frac{1}{\beta},-\frac{1}{\beta}x_3)\sigma(2,\beta,x_3+\gamma x_1^k)\sigma(3,-\frac{1}{\beta},\frac{1}{\beta}(x_2-\gamma x_1^k))). \end{aligned}$$

Since the transposition $(23) \in S_3$ has a linear representation

$$(23) = \sigma(2,1,\beta x_3)\sigma(3,-\beta,x_2)\sigma(2,\frac{1}{\beta},-\frac{1}{\beta}x_3),$$

we obtain

$$\varphi_{i_1+t} = (23)\sigma(2,\beta,x_3+\gamma x_1^k))\sigma(3,-\frac{1}{\beta},\frac{1}{\beta}(x_2-\gamma x_1^k)).$$

Then

$$\theta = (23)\varphi_1^{(23)}\varphi_2^{(23)}\ldots\varphi_{i_1+t-1}^{(23)}\sigma(2,\beta,x_3+\gamma x_1^k))\sigma(3,-\frac{1}{\beta},\frac{1}{\beta}(x_2-\gamma x_1^k)), \qquad (35)$$

where $\varphi_i^{(23)}$ are elementary automorphisms, according to (20). To (35) corresponds the sequence of elementary transformations

$$\begin{aligned}(x_1,x_2,x_3) \mapsto (x_1,x_3,x_2) \to \theta_1' \to \theta_2' \to \ldots \to \theta_{i_1+t-1}' \\ = (f_1,f_3,g_2) \to (f_1,f_2,g_2) \to (f_1,f_2,f_3) = \theta,\end{aligned}$$

where θ_i' is obtained from θ_i only by the permutation of the second and the third components, and $(x_1,x_2,x_3) \mapsto (x_1,x_3,x_2)$ denotes a composition of elementary linear transformations corresponding to the definition of the automorphism $(23) \in S_3$.

If in (24) we replace θ by (35), then the exponent of (25) remains the same. But instead of ϕ we have (f_1,f_2,g_2), and Lemma 4.5 gives the triviality of (24).

We now consider the case when $\overline{f_1},\overline{f_3}$ are algebraically independent. Then $\overline{f_2} = \overline{b} \in \langle\overline{f_1},\overline{f_3}\rangle$. By 1) we can assume that $\overline{f_2} \notin \langle\overline{f_1}\rangle$, i.e. $\overline{f_2}$ depends on $\overline{f_3}$. Consequently, $\deg f_3 \leq \deg f_2$. If $\overline{f_1},\overline{f_2}$ are algebraically dependent, then it follows that $\overline{f_1},\overline{f_3}$ are algebraically dependent. Consequently, $\overline{f_1},\overline{f_2}$ are algebraically independent. Then $\overline{a} \in \langle\overline{f_1},\overline{f_2}\rangle$. By 3) we can assume that a necessarily contains f_2. Then $\deg\ f_2 \leq \deg\ a \leq \deg\ f_3$, i.e. $\deg\ f_2 = \deg\ f_3$. Hence

$$\overline{b} = \overline{f_2} = \beta\overline{f_3} + \gamma\overline{f_1^k}.$$

From the statement 4) of the lemma we obtain that (24) is trivial. □

So, by Lemma 4.7, we can assume that $\overline{f_1},\overline{f_3}$ are algebraically dependent and that $\overline{f_3} \notin \langle\overline{f_1}\rangle$. It remains to consider the following three cases separately:

1) f_1,f_3 is a $*$-reduced pair and $\deg f_1 < \deg f_3$;

2) f_1,f_3 is a $*$-reduced pair and $\deg f_3 < \deg f_1$;

3) $\overline{f_1} \in \langle\overline{f_3}\rangle$ and $\deg f_1 > \deg f_3$.

Lemma 4.8 *If f_1,f_3 is a $*$-reduced pair and $\deg f_1 < \deg f_3$, then (24) is trivial.*

Proof. We first consider the case when $\deg f_2 > \deg f_3$. If $\overline{f_1},\overline{f_2}$ are algebraically independent, then $\overline{a} \in <\overline{f_1},\overline{f_2}>$. Since $\deg\ a \leq \deg\ f_3 < \deg\ f_2$, we have $a \in \langle f_1\rangle$. Lemma 4.7 gives the triviality of (24).

Suppose that $\overline{f_1},\overline{f_2}$ are algebraically dependent. By Lemma 4.7, we can assume that $\overline{f_2} \notin \langle\overline{f_1}\rangle$. Since $\deg\ f_1 < \deg\ f_3 < \deg\ f_2$, therefore f_1,f_2 is a $*$-reduced pair.

Assume that $\deg\ a < N = N(f_1,f_2)$. Since $\deg\ a \leq \deg\ f_3 < \deg\ f_2$, Corollary 2.1 gives $a \in < f_1 >$. By Lemma 4.7, the relation (24) is trivial.

Therefore we can assume that $\deg\ a \geq N$. Then $\deg\ f_2 > N = N(f_1,f_2)$. By the definition of $N = N(f_1,f_2)$, this is possible only if f_1,f_2 is a 2-reduced pair. Consequently, $\deg\ f_1 = 2n$, $\deg\ f_2 = sn$, where $s \geq 3$ is an odd number, and moreover

$$N = n(s-2) + \deg\,[f_1,f_2] \leq \deg\ a \leq \deg\ f_3.$$

Let (f_1, f_2, g_3) be an elementary reduction of θ such that g_3 is an irreducible element of (f_1, f_2, g_3). If f_3 is an irreducible element of θ, then we put $g_3 = f_3$. Then (f_1, f_2, g_3) satisfies the conditions of Proposition 2.2. Assume that the statement 1) of Proposition 2.2 is valid for (f_1, f_2, g_3), i.e.

$$\deg\, g_3 < n(s-2) + \deg\,[f_1, f_2] = N.$$

Since $\deg\, f_3 \geq N$, it means that

$$f_3 = g_3 + c(f_1, f_2), \quad \overline{c(f_1, f_2)} = \overline{f_3}.$$

By Lemma 4.7 we can assume that $\overline{c(f_1, f_2)} = \overline{f_3} \notin < f_1 >$. Since $\deg\, c(f_1, f_2) = \deg\, f_3 < \deg\, f_2$, we have $\overline{c(f_1, f_2)} \notin < \overline{f_1}, \overline{f_2} >$. It is easy to deduce from (1) that $\deg_y c(x, y) \in \{2, 4\}$. Then $\deg_y \frac{\partial c}{\partial y}(x, y) \in \{1, 3\}$, and (1) gives $\frac{\partial c}{\partial y}(f_1, f_2) \geq \deg\, f_2$. We have

$$\deg\,[f_1, g_3] \leq \deg\, f_1 + \deg\, g_3 < ns + \deg\,[f_1, f_2] = \deg\, f_2 + \deg\,[f_1, f_2].$$

Then

$$[f_1, f_3] = [f_1, g_3] + [f_1, f_2]\frac{\partial c}{\partial y}(f_1, f_2)$$

gives that

$$\deg\,[f_1, f_3] = \deg\,[f_1, f_2]\frac{\partial c}{\partial y}(f_1, f_2) \geq \deg\,[f_1, f_2] + \deg\, f_2.$$

Consequently, $N(f_1, f_3) > \deg\, f_2$. By (34) we have $\deg\, b = \deg\, f_2 < N(f_1, f_3)$. Corollary 2.1 gives $\overline{f_2} \in < \overline{f_1}, \overline{f_3} >$. Since $\deg\, f_1 f_3, \deg\, f_3^2 > \deg\, f_2$, we obtain $\overline{f_2} \in < \overline{f_1} >$; a contradiction.

We now assume that one of the statements 2), 3) of Proposition 2.2 is valid for (f_1, f_2, g_3). Combining both the cases we can say that

$$f_3 = h_3 + r(f_1, f_2),\ \deg\, f_2 = 3n,\ \deg\,(f_2 - \beta h_3^2) \leq 2n,\ \deg\,[f_1, h_3] < n + \deg\,[f_1, f_2].$$

Since $\deg\, f_3 < 2n$ and $\deg\, h_3 = \frac{3n}{2}$, it follows that $\deg\, r(f_1, f_2) < 2n$. By Corollary 2.2, we have $r(f_1, f_2) = \lambda w(f_1, f_2) + \mu$, where $w(x, y)$ is a derivative polynomail of the pair f_1, f_2. If $\lambda \neq 0$, then an immediate calculation gives $\deg\,[f_1, f_3] > \deg\, f_2$. This leads to a contradiction, as above. Then $\lambda = 0$, and we can assume that $f_3 = h_3$. Combining (33) and Corollary 2.2 we also conclude that

$$a = \lambda w(f_1, f_2) + \mu, \quad f = \alpha f_3 + \lambda w(f_1, f_2) + \mu.$$

If $\lambda \neq 0$, then τ admits a reduction of type IV. Otherwise $a = \mu \in F$ and we can apply Lemma 4.7.

Now, suppose that $\deg f_2 \leq \deg f_3$. If $\overline{f_2} \in < \overline{f_1}, \overline{f_3} >$, then $\overline{f_2} = \beta\overline{f_3} + \gamma\overline{f_1^k}$. Hence Lemma 4.7 gives the triviality of (24).

Let $\overline{b} = \overline{f_2} \notin < \overline{f_1}, \overline{f_3} >$. Since deg $b =$ deg $f_2 \leq$ deg f_3, applying Corollary 2.1 gives deg $f_3 \geq N(f_1, f_3)$. By the definition of $N(f_1, f_3)$, we see that f_1, f_3 is a 2-reduced pair, i.e. deg $f_1 = 2n$ and deg $f_3 = ns$, where $s \geq 3$ is an odd number. According to Lemma 4.6, we can assume that g_2 is an irreducible element of $\phi = (f_1, g_2, f_3)$. Then ϕ satisfies the conditions of Proposition 2.2.

If $\deg_y(b(x,y)) = k$, then (1) gives $k = 2$ if $s > 3$, and $k = 2, 4$ if $s = 3$. Consequently, $\deg_y(\frac{\partial b}{\partial y}(x,y)) = k - 1 \in \{1, 3\}$, and (1) gives also $\deg(\frac{\partial b}{\partial y}(f_1, f_3)) \geq ns$. We have

$$[f_1, f_2] = [f_1, g_2] + [f_1, f_3]\frac{\partial b}{\partial y}(f_1, f_3). \tag{36}$$

Assume that the statement 1) of Proposition 2.2 is valid for ϕ, i.e. $\deg g_2 < n(s-2) + \deg[f_1, f_3]$. Comparing the degrees of elements in (36) we obtain

$$\deg[f_1, f_2] \geq \deg[f_1, f_3] + ns. \tag{37}$$

Consequently, $\deg f_2 \geq n(s-2) + \deg[f_1, f_3] > n$ and $\overline{f_1} \notin \langle \overline{f_2} \rangle$. Then either $\overline{f_1}$ and $\overline{f_2}$ are algebraically independent or f_1, f_2 is a $*$-reduced pair. Note that $N(f_1, f_2) > \deg[f_1, f_2] > ns \geq$ deg a. By Corollary 2.1, in both cases we have $\overline{a} \in \langle \overline{f_1}, \overline{f_2} \rangle$. Since deg $f_2^3 >$ deg $f_3 \geq$ deg a and deg $f_1 f_2 >$ deg f_3, it follows that

$$a = \beta f_2^2 + \gamma f_2 + G(f_1), \quad \deg(G(f_1)) \leq \deg f_3;$$

moreover, we can have $\beta \neq 0$ only if deg $f_3 \geq 2$ deg f_2. Note that the equality $\deg(G(f_1)) =$ deg f_3 is impossible. Consequently, $\deg(G(f_1)) <$ deg f_3. Since f_1, f_3 is a 2-reduced pair, it follows that $f_1, \alpha f_3 + G(f_1)$ is also a 2-reduced pair. We have

$$\tau = (f_1, f_2, \alpha f_3 + \beta f_2^2 + \gamma f_2 + G(f_1)).$$

If $(\beta, \gamma) \neq (0, 0)$, then it is easy to check that $(f_1, g_2, \alpha f_3 + G(f_1))$ is a reduction of types I–III of the automorphism τ with the active element f_2. Proposition 3.1 gives $d(\tau) = n(s+2) +$ deg $f_2 = d =$ deg θ and $t(\tau) \geq 1$. This contradicts the inequality $(d(\tau), t(\tau)) \leq (d, t)$. If $\beta = \gamma = 0$, then we apply Lemma 4.7.

We now consider the case when the statement 3) of Proposition 2.2 is valid for ϕ. Then (36) again gives (37). Besides, in this case $s = 3$ and $\deg f_2 > \deg g_2 = \frac{3n}{2}$. Consequently, the same argument as above gives $a = \beta f_2 + \gamma f_1 + \delta$. By Lemma 4.7, we can assume that $\beta \neq 0$. Then $(f_1, g_2, \alpha f_3 + \gamma f_1 + \delta)$ is a reduction of type I or II of τ with the active element f_2. The value of $(d(\tau), t(\tau))$, which can be calculated by Proposition 3.1, again contradicts the inequality $(d(\tau), t(\tau)) \leq (d, t)$.

At last we consider the case when the statement 2) of Proposition 3.1 is valid for ϕ. Let $(f_1, h_2, f_3 - \delta h_2^2)$ be a reduction of type IV of ϕ, where

$$g_2 = h_2 + g, \; g \in \langle f_1, f_3 \rangle \setminus F, \; \deg(f_3 - \delta h_2^2) \leq 2n.$$

Since $\deg g_2 \leq \deg h_2 = \frac{3n}{2}$, we have $\deg g \leq \frac{3n}{2}$. By Definition 2.4, the statement 3) of Proposition 2.2 is valid for (f_1, h_2, f_3).

If $\deg f_2 > \frac{3n}{2}$, then by Lemma 4.6 we can assume that $\phi = (f_1, h_2, f_3)$ and this case reduces to the preceding case.

Assume that $\deg f_2 \leq \frac{3n}{2}$. By Proposition 3.1, we have $d(\phi) = \frac{13n}{2}$, $t(\phi) = 1$. Since $d = \deg\theta \leq \frac{13n}{2}$, this contradicts the inequality $(d(\phi), t(\phi)) < (d, t)$. $\square$

Lemma 4.9 *If f_1, f_3 is a $*$-reduced pair and* $\deg f_3 < \deg f_1$*, then (24) is trivial.*

Proof. We first consider the case when $\deg f_1 < \deg f_2$. Since $\deg a \leq \deg f_3 < \deg f_3 < \deg f_1$, applying Lemma 4.7 we conclude that $\overline{a} \notin \langle \overline{f_1}, \overline{f_2} \rangle$. Consequently, $\overline{f_1}, \overline{f_2}$ are algebraically dependent. By Lemma 4.7, we assume that $\overline{f_2} \notin \langle \overline{f_1} \rangle$. Then f_1, f_2 is a $*$-reduced pair. Corollary 2.1 gives also $\deg a \geq N(f_1, f_2)$. Combining these inequalities and the definition of $N(f_1, f_2)$ we conclude that $\deg f_1 = 2n$ and $\deg f_2 = 3n$. Consider an elementary reduction (f_1, f_2, g_3) of θ with an irreducible g_3 (assume that $g_3 = f_3$ if f_3 is an irreducible element of θ). Then (f_1, f_2, g_3) satisfies the conditions of Proposition 2.2. By the same discussions related to the automorphism (f_1, f_2, g_3) as in the proof of Lemma 4.8, we obtain the triviality of (24).

If $\deg f_1 = \deg f_2$, then by Lemma 4.7 we may assume that $\overline{f_1}, \overline{f_2}$ are linearly independent. Then $\overline{f_1}, \overline{f_2}$ are algebraically independent. Consequently, $a \in F$.

It remains to consider the case when $\deg f_2 < \deg f_1$. Assume that $\overline{f_2} \in \langle \overline{f_3} \rangle$. By Lemma 4.7 we can assume that $\overline{f_2}, \overline{f_3}$ are linearly independent. Then $\deg f_2 = t \cdot \deg f_3$, $t \geq 2$. In this case $\overline{f_1} \notin \langle \overline{f_2} \rangle$, since f_1, f_3 is a $*$-reduced pair. Consequently, f_1, f_2 is also a $*$-reduced pair. Since $\deg a \leq \deg f_3 < \min\{\deg f_1, \deg f_2\}$, applying Corollary 2.1 and Lemma 4.7 we obtain $\deg f_2 = 2n$, $\deg f_1 = 3n$, and

$$\deg f_3 \geq \deg a \geq N(f_1, f_2) = n + \deg [f_1, f_2] > \frac{2n}{t} = \deg f_3,$$

which gives a contradiction.

Therefore we can assume that $\overline{f_2} \notin \langle \overline{f_3} \rangle$, i.e. $\overline{b} = \overline{f_2} \notin \langle \overline{f_1}, \overline{f_3} \rangle$. Then Corollary 2.1 gives $N(f_1, f_3) \leq \deg f_2 < \deg f_1$. By the definition of $N(f_1, f_3)$ this is possible only if f_1, f_3 is a 2-reduced pair, and $\deg f_3 = 2n$, $\deg f_1 = ns$, where $s \geq 3$ is an odd number. By Lemma 4.6 we can assume that g_2 is an irreducible element of $\phi = (f_1, g_2, f_3)$. Then ϕ satisfies the conditions of Proposition 2.2. Applying the same part of discussions related to the automorphism ϕ as in the proof of Lemma 4.8, we obtain the triviality of (24). $\square$

Lemma 4.10 *If $\overline{f_1} \in \langle \overline{f_3} \rangle$ and* $\deg f_1 > \deg f_3$*, then the relation (24) is trivial.*

Proof. Assume that $n = \deg f_3$ and $\overline{f_1} = \beta \overline{f_3}^k$, $k \geq 2$. Then $\deg f_1 = nk$. By (19) we have

$$\varphi_{i_1+t} = \sigma(2, 1, b(x_1, x_2)) = \sigma(1, 1, -\beta x_3^k)\sigma(2, 1, b(x_1, x_2))\sigma(1, 1, \beta x_3^k).$$

After such replacement, instead of $\phi \to \theta$ we obtain

$$\phi \to (f_1 - \beta f_3^k, g_2, f_3) \to (f_1 - \beta f_3^k, f_2, f_3) \to \theta.$$

Since $\deg(f_1 - \beta f_3^k, g_2, f_3), \deg(f_1 - \beta f_3^k, f_2, f_3) < d = \deg\theta$, the new sequence (25) has the same exponent. Then instead of ϕ we can take $(f_1 - \beta f_3^k, f_2, f_3)$, i.e. we can assume that ϕ reduces the element f_1 of θ. Interchanging the elements f_1, f_2 and applying Lemmas 4.7, 4.8, and 4.9 we can restrict ourselves to the case when $\overline{f_2} \in \langle\overline{f_3}\rangle$, $\deg f_2 > \deg f_3$. Then $\deg f_2 = nr$, $r \geq 2$. By Lemma 4.7 we may assume that f_1, f_2 is a $*$-reduced pair, i.e. $\deg f_1 = mp$, $\deg f_2 = ms$, $(p, s) = 1$, $m \geq n$. Hence $N(f_1, f_2) > m(ps - p - s) > n \geq \deg a$. Corollary 2.1 gives $\overline{a} \in \langle\overline{f_1}, \overline{f_2}\rangle$, i.e. $a \in F$. $\Box$

This finishes the proof of Theorem 4.1.

5 Free metabelian associative algebras

Let us remind that the variety of associative algebras defined by the identity

$$[x, y][z, t] = 0$$

is called *metabelian* [27]. We denote by C the free metabelian algebra (with unit 1) with free generators z_1, z_2, z_3. Then the generators (17) of the group of tame automorphisms $TAut\,C$ satisfy the relations (18)–(20). However these relations may fail to be a full system of defining relations of the group $TAut\,C$. Consider the homomorphism

$$\tau : C \longrightarrow A = F[x_1, x_2, x_3], \quad \tau(z_i) = x_i, \quad 1 \leq i \leq 3.$$

If $\theta = (f_1, f_2, f_3) \in Aut\,C$, then obviously

$$\tau^*(\theta) = (\tau(f_1), \tau(f_2), \tau(f_3)) \in Aut\,A.$$

It is easy to check that the map

$$\tau^* : Aut\,C \longrightarrow Aut\,A$$

is a homomorphism of the groups. We denote by T the restriction of the homomorphism τ^* to $TAut\,C$, i.e.

$$T : TAut\,C \longrightarrow TAut\,A, \quad T(\theta) = \tau^*(\theta), \quad \theta \in TAut\,C.$$

Note that the kernel $I = Ker\,\tau$ of the homomorphism τ is the commutator ideal of C. Obviously, any automorphism of the form

$$\sigma(i, \alpha, f), \quad \alpha = 1, \ f \in I, \tag{38}$$

is contained in $Ker\,T$.

Theorem 5.1 *Let F be a field of characteristic 0. Then the kernel of the homomorphism*

$$T : TAut\, C \longrightarrow TAut\, A,$$

as a normal subgroup of the group $TAut\, C$, is generated by all automorphisms of the form (38).

Proof. Let N be a normal subgroup of $TAut\, C$ generated by all elements of the form (38). Since $N \subseteq Ker\, T$, the homomorphism T induces the group homomorphism

$$\mu : G = TAut\, C/N \longrightarrow TAut\, A.$$

We show that μ is an isomorphism. Every element $f \in C$ has a unique representation in the form $f = f_0 + f_1$, where f_0 is a linear combination of the words $z_1^i z_2^j z_3^k$ and $f_1 \in I$. We have

$$\sigma(i, \alpha, f) = \sigma(i, \alpha, f_0)\sigma(i, 1, f_1).$$

Since $\sigma(i, 1, f_1) \in N$, the group G is generated by all elements of the form

$$\sigma(i, \alpha, f_0)N, \quad f \in C. \tag{39}$$

Consequently, μ establishes a one to one correspondence between the set of generators (39) of the group G and the set of generators (17) of the group $TAut\, A$. Of course, the relations (18)–(20) are also true for elementary automorphisms of C. They induce the same relations for elements (39). By Theorem 4.1, there exists a homomorphism

$$\nu : TAut\, A \longrightarrow G$$

which is innverse to μ on the set of generators. Consequently, μ is an isomorphism. □

It was proved in [27] that every automorphism of the algebra A can be lifted to an automorphism of the algebra C. Thus, the algebra C has wild automorphisms. Now our aim is to find another set of wild automorphisms of the algebra C which can be lifted to automorphisms of the free associative algebra $B = F < y_1, y_2, y_3 >$.

The Fox derivatives in free associative algebras can be found in [9]. We recall the definition of Fox derivatives of elements of the algebra C, following [27].

Let C' be the algebra antiisomorphic to C, and A' be the algebra antiisomorphic to A. We will denote the corresponding antiisomorphisms by $' : C \to C'$, $' : A \to A'$. Put also $U(C) = C' \otimes_F C$, $U(A) = A' \otimes_F A$. Given homomorphisms

$$\varphi : C \to C, \quad \varphi : A \to A, \quad \varphi : C \to A,$$

we will denote by the same symbol the corresponding homomorphims

$$\varphi : U(C) \to U(C), \quad \varphi : U(A) \to U(A), \quad \varphi : U(C) \to U(A),$$

defined as $\varphi(f'\otimes g)=\varphi(f)'\otimes\varphi(g)$. If φ is an endomorphism of the algebra C, then we will denote the induced endomorphism of the algebra A by $\overline{\varphi}$.

A linear map

$$\frac{\partial}{\partial z_i}: C\to U(A),\quad 1\leq i\leq 3,$$

is uniquely defined by the rules

$$\frac{\partial z_j}{\partial z_i}=\delta_{ij},\quad \frac{\partial fg}{\partial z_i}=\frac{\partial f}{\partial z_i}\tau(1\otimes g)+\frac{\partial g}{\partial z_i}\tau(f'\otimes 1),$$

where δ_{ij} is Kronecker symbol. For every $f\in C$ we put

$$\partial(f)=(\frac{\partial f}{\partial z_1},\frac{\partial f}{\partial z_2},\frac{\partial f}{\partial z_3})^t,$$

where t means transposition. If φ is an endomorphism of C, then the matrix

$$J(\varphi)=(\partial(\varphi(z_1)),\partial(\varphi(z_2)),\partial(\varphi(z_3)))$$

is called the Jacobian matrix of φ. For arbitrary endomorphisms φ,ψ we have

$$J(\varphi\psi)=J(\varphi)\overline{\varphi}(J(\psi)). \tag{40}$$

Consequently, $J(\varphi)$ is invertible over $U(A)$ if φ is an automorphism. The converse statement is also true [27].

The ideal I has a right $U(C)$-module structure with respect to the action

$$m\cdot(f'\otimes g)=fmg,\quad m\in I, f,g\in C.$$

Since $I^2=0$, this action induces a right $U(A)$-module structure on I. If $f\in I$, then

$$f=[z_1,z_2]\cdot f_{12}+[z_1,z_3]\cdot f_{13}+[z_2,z_3]\cdot f_{23},\quad f_{ij}\in U(A_n). \tag{41}$$

Put $\Delta(x_i)=x_i'\otimes 1-1\otimes x_i$, $1\leq i\leq 3$. It is easy to check that if f has the form (41), then

$$\frac{\partial f}{\partial z_3}=\Delta(x_1)f_{13}+\Delta(x_2)f_{23}. \tag{42}$$

We denote by e_i the vector-row of length 3 whose ith component is 1 and the others are 0, $1\leq i\leq 3$.

Lemma 5.1 *Let φ be an automorphism of the algebra C of the type (38) and let $\psi\in Aut\, C$. Then*

$$J(\psi\varphi\psi^{-1})=E+u^t\cdot v,$$

where $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)$, $u_i,v_i\in U(A_n)$.

Proof. Assume that $\varphi = \sigma(i,1,f)$, where $f \in I$. Then $J(\varphi) = E + \partial(f)\cdot e_i$. We have

$$\begin{aligned} J(\psi\varphi\psi^{-1}) &= J(\psi)\overline{\psi}(J(\varphi)\overline{\varphi}(J(\psi))) = J(\psi)\overline{\psi}(J(\varphi))\overline{\psi}(J(\psi)) \\ &= J(\psi)\overline{\psi}(J(\varphi))J(\psi)^{-1} = E + J(\psi)\overline{\psi}(\partial(f))\cdot\overline{\psi}(e_i)J(\psi)^{-1}. \end{aligned}$$

Put $u^t = J(\psi)\overline{\psi}(\partial(f))$, $v = \overline{\psi}(e_i)J(\psi)^{-1}$. $\square$

For an arbitrary automorphism φ of C we put

$$J_2(\varphi) = \begin{pmatrix} \frac{\partial\varphi(z_1)}{\partial z_1} & \frac{\partial\varphi(z_2)}{\partial z_1} \\ \frac{\partial\varphi(z_1)}{\partial z_2} & \frac{\partial\varphi(z_2)}{\partial z_2} \end{pmatrix}.$$

We now consider the homomorphism

$$\eta : A \longrightarrow F[x_3], \quad \eta(x_1) = \eta(x_2) = 0,\ \eta(x_3) = x_3.$$

Lemma 5.2 *Let $\varphi \in Aut\, C$ and $\overline{\varphi} = id$. Then*

$$\eta(J(\varphi)) = \begin{pmatrix} \eta(J_2(\varphi)) & * \\ 0 & 1 \end{pmatrix}$$

Proof. We have $\varphi = (z_1 + f_1, z_2 + f_2, z_3 + f_3)$, where $f_i \in I$. From (42) it follows that $\eta(\frac{\partial f_i}{\partial z_3}) = 0$. $\square$

Denote the standart matrix units by e_{ij}. A matrix $E_{ij}(a) = E + ae_{ij}$, where $i \neq j$, $a \in U(A)$, is called an *elementary matrix* over $U(A)$. We denote the subgroup of $SL_2(U(A))$ generated by all elementary matrices by $E_2(U(A))$. We also denote by $diag(\alpha,\beta)$ a diagonal matrix with the diagonal entries α,β.

For every $0 \neq \lambda \in U(A)$ we will denote by $U_\lambda(A)$ the subring of the ring of quotients $Q(U(A))$ generated by $U(A) \cup \{\lambda^{-1}\}$.

Now we put $S_\lambda = E_2(U_\lambda(A))\cdot diag(\lambda,1)$. It is easy to check that

$$\begin{gathered} E_{12}(a)\cdot diag(\lambda,1) = diag(\lambda,1)\cdot E_{12}(\frac{a}{\lambda}), \quad E_{21}(a)\cdot diag(\lambda,1) = diag(\lambda,1)\cdot E_{21}(\lambda a), \\ diag(\lambda,1) = E_{12}(1)\cdot E_{21}(-1)\cdot E_{12}(1)\cdot diag(1,\lambda)\cdot E_{12}(-1)\cdot E_{21}(1)\cdot E_{12}(-1). \end{gathered}$$

From here we get

$$S_\lambda = E_2(U_\lambda(A))\cdot diag(1,\lambda) = diag(\lambda,1)\cdot E_2(U_\lambda(A)) = diag(1,\lambda)\cdot E_2(U_\lambda(A)). \quad (43)$$

Lemma 5.3 *Let φ be an automorphism of C of the form (38) and let $\psi \in TAut\, C$. Then*

$$\lambda = \det(J_2(\psi\varphi\psi^{-1})) \neq 0, \qquad J_2(\psi\varphi\psi^{-1}) \in S_\lambda.$$

Proof. Since $\overline{\psi\varphi\psi^{-1}} = id$, it follows that $\det(J(\psi\varphi\psi^{-1})) = 1$. By Lemma 5.2, we have $\eta(\lambda) = 1$, i.e. $\lambda \neq 0$.

Assume that $\varphi = \sigma(i, 1, f)$, where $f \in I$. Then

$$J(\varphi) = E + \partial(f) \cdot e_i.$$

If $i = 3$, then $J_2(\varphi) = E$. Since f does not contain z_i, the ith component of $\partial(f)$ is 0. Consequently, if $i \leq 2$, then the matrix $J_2(\varphi)$ is elementary. Since $\det(J_2(\varphi)) = 1$, the statement of the lemma is valid for φ.

We have

$$\sigma(i, \alpha, g) = (1i)\sigma(1, \alpha, (1i)(g))(1i)$$

if $i > 1$. Since

$$(1i) = \sigma(1, -1, 0)\sigma(1, 1, z_i)\sigma(i, 1, -z_1)\sigma(1, 1, z_i),$$

the relations (18), (19) give

$$\sigma(i, \alpha, g) = \sigma(1, 1, z_i)\sigma(i, 1, z_1)\sigma(1, \alpha, (1 - \alpha)z_i - (1i)(g))\sigma(i, 1, -z_1)\sigma(1, 1, z_i).$$

Therefore we can assume that ψ is a product of the automorphisms

$$\sigma(1, \alpha, g), \quad X_{ij}(\beta) = \sigma(j, 1, \beta x_i), \quad 1 < j, \quad i \neq j. \tag{44}$$

Consider $\psi = \psi_1\psi_2 \ldots \psi_k$, where ψ_i, $1 \leq i \leq k$, are automorphisms of the form (44). We now begin an induction on k. Above we have checked that the lemma is true if $k = 0$.

Assume that the statement of the lemma is valid for

$$\varphi_1 = \psi_2 \ldots \psi_k\varphi(\psi_2 \ldots \psi_k)^{-1},$$

i.e. $J_2(\varphi_1) \in S_\lambda$, where $\lambda = \det(J_2(\varphi_1))$. By Lemma 5.3, we have

$$J(\varphi_1) = E + u^t \cdot v,$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$. Then

$$J_2(\varphi_1) = E + (u_1, u_2)^t \cdot (v_1, v_2) \in S_\lambda.$$

Consequently, $\overline{\psi}_1(J_2(\varphi_1)) \in S_\mu$, where $\mu = \overline{\psi}_1(\lambda) = \det(\overline{\psi}_1(J_2(\varphi_1)))$.

Put

$$\overline{\psi}_1(u) = (b_1, b_2, b_3) = b, \quad \overline{\psi}_1(v) = (c_1, c_2, c_3) = c,$$

$$P = \overline{\psi}_1(J_2(\varphi_1)), \quad Q = J_2(\psi_1\varphi_1\psi_1^{-1}).$$

Then

$$P = E + (b_1, b_2)^t \cdot (c_1, c_2).$$

Since $\overline{\varphi}_1 = id$, it follows that

$$J(\psi_1\varphi_1\psi_1^{-1}) = J(\psi_1)\overline{\psi}_1(J(\varphi_1))J(\psi_1)^{-1} = E + J(\psi_1)b^t \cdot cJ(\psi_1)^{-1}.$$

We first consider the case when $\psi_1 = \sigma(1, \alpha, g)$. We have

$$\partial(\psi_1(z_1)) = (\alpha, a_2, a_3), \quad J(\psi_1) = (\partial(\psi_1(z_1)), e_2^t, e_3^t).$$

An immediate calculation gives

$$Q = E + (\alpha b_1, a_2 b_1 + b_2)^t \cdot (\alpha^{-1}(c_1 - c_2 a_2), c_2) + (\alpha b_1, a_2 b_1 + b_2)^t \cdot (-\alpha^{-1} c_3 a_3, 0).$$

Hence

$$\begin{aligned} Q_1 &= E + (\alpha b_1, a_2 b_1 + b_2)^t \cdot (\alpha^{-1}(c_1 - c_2 a_2), c_2) \\ &= diag(\alpha, 1) \cdot E_{21}(a_2) \cdot P \cdot E_{21}(-a_2) \cdot diag(\alpha^{-1}, 1). \end{aligned}$$

It follows from (43) that $Q_1 \in S_\mu$. By (43), we can also assume that $Q_1 = P_1 D$, where $P_1 \in E_2(U_\mu(A))$, $D = diag(1, \mu)$. Then

$$Q = P_1(D + P_1^{-1}(\alpha b_1, a_2 b_1 + b_2)^t \cdot (-\alpha^{-1} c_3 a_3, 0)).$$

We have

$$D + P_1^{-1}(\alpha b_1, a_2 b_1 + b_2)^t \cdot (-\alpha^{-1} c_3 a_3, 0) = \begin{pmatrix} \lambda_1 & 0 \\ \lambda_2 & \mu \end{pmatrix}.$$

It follows from (43) that

$$D + P_1^{-1}(\alpha b_1, a_2 b_1 + b_2)^t \cdot (-\alpha^{-1} c_3 a_3, 0) \in S_\nu,$$

where $\nu = \lambda_1 \mu = \det(Q)$. Since $\mu | \nu$, we have $U_\mu(A) \subseteq U_\nu(A)$ and $E_2(U_\mu(A)) \subseteq E_2(U_\nu(A))$. Consequently, $Q \in S_\nu$.

We now assume that $\psi_1 = X_{ij}(\beta)$, $j > 1$, $i \neq j$. We have $J(\psi_1) = E_{ij}(\beta)$. Suppose that $i = 1$. If $j = 2$, then

$$Q = E + (b_1 + \beta b_2, b_2)^t \cdot (c_1, c_2 - \beta c_1) = E_{12}(\beta) \cdot P \cdot E_{12}(-\beta) \in S_\mu.$$

If $j = 3$, then

$$Q = E + (b_1 + \beta b_3, b_2)^t \cdot (c_1, c_2) = P + (\beta b_3, 0)^t \cdot (c_1, c_2).$$

By (43), we can assume that $P = DP_1$, where $P_1 \in E_2(U_\mu(A))$, $D = diag(1, \mu)$. Then

$$Q = (D + (\beta b_3, 0)^t \cdot (c_1, c_2)P^{-1})P \in S_\nu, \quad \mu | \nu, \quad \nu = \det(Q).$$

Suppose that $i = 2$. Then $j = 3$ and

$$Q = E + (b_1, b_2 + \beta b_3)^t \cdot (c_1, c_2) = P + (0, \beta b_3)^t \cdot (c_1, c_2).$$

By (43), in this case we can assume that $P = DP_1$, where $P_1 \in E_2(U_\mu(A))$, $D = diag(\mu, 1)$. Hence

$$Q = (D + (0, \beta b_3)^t \cdot (c_1, c_2) P_1^{-1}) P_1 \in S_\nu, \quad \mu | \nu, \quad \nu = \det(Q).$$

Now suppose that $i = 3$. Then $j = 2$ and

$$Q = E + (b_1, b_2)^t \cdot (c_1, c_2 - \beta c_3) = P + (b_1, b_2)^t \cdot (0, -\beta c_3).$$

Put $P = P_1 D$, where $P_1 \in E_2(U_\mu(A))$, $D = diag(\mu, 1)$. Then we have

$$Q = P_1(D + P_1^{-1}(b_1, b_2)^t \cdot (0, -\beta c_3)) \in S_\nu, \quad \mu | \nu, \quad \nu = \det(Q).$$

Thus $J_2(\psi\varphi\psi^{-1}) = Q \in S_\nu$. $\square$

Lemma 5.4 *Let φ be an automorphism of C of the form (38) and let $\psi \in T Aut\, C$. Then*

$$\eta(J_2(\psi\varphi\psi^{-1})) \in E_2(U(F[x_3])).$$

Proof. Assume that $\det(J_2(\psi\varphi\psi^{-1})) = \lambda$. By Lemma 5.2, we have $\eta(\lambda) = 1$. Consequently, the homomorphism η may be extended to the homomorphism of the algebra $U_\lambda(A)$ to $U(F[x_3])$. Moreover, we have $\eta(S_\lambda) \subseteq E_2(U(F[x_3]))$. Now Lemma 5.3 gives our statement. $\square$

Theorem 5.2 *Assume that $\varphi \in T Aut\, C$ and $\overline{\varphi} = id$. Then $\eta(J_2(\varphi)) \in E_2(U(F[x_3]))$.*

Proof. According to Theorem 5.1, we have

$$\varphi = \theta_1 \theta_2 \ldots \theta_s, \quad \theta_i = \psi_i \varphi_i \psi_i^{-1},$$

where φ_i is an automorphism of the form (38) and ψ_i is a tame automorphism. Since $\overline{\theta}_i = id$, applying (40) we have

$$J(\varphi) = J(\theta_1) J(\theta_2) \ldots J(\theta_s).$$

Lemma 5.2 implies that

$$\eta(J_2(\varphi)) = \eta(J_2(\theta_1)) \eta(J_2(\theta_2)) \ldots \eta(J_2(\theta_s)).$$

By Lemma 5.4, we have $\eta(J_2(\theta_i)) \in E_2(U(F[x_3]))$. Hence $\eta(J_2(\varphi)) \in E_2(U(F[x_3]))$. $\square$

Corollary 5.1 *The automorphism*

$$\varphi \;\; = \;\; (z_1 + z_3(z_1 z_3 - z_3 z_2),\, z_2 + (z_1 z_3 - z_3 z_2) z_3,\, z_3)$$

of the algebra C over a field F of characteristic 0 is wild.

Proof. We have

$$\eta(J_2(\varphi)) = \left(\begin{array}{cc} 1 + z_3' \otimes z_3 & 1 \otimes z_3^2 \\ -(z_3' \otimes 1)^2 & 1 - z_3' \otimes z_3 \end{array} \right).$$

It is well known [2] that $\eta(J_2(\varphi)) \notin E_2(F[z_3' \otimes 1, 1 \otimes z_3])$. Put $\varphi\sigma = \psi$, where

$$\sigma = \sigma(1, 1, -z_2)\sigma(2, 1, -z_1 z_3^2)\sigma(1, 1, z_2).$$

We can immediately check that

$$\overline{\psi} = id, \quad \eta(J_2(\psi)) = \eta(J_2(\varphi))\eta(\overline{\varphi}J_2(\sigma)), \quad \eta(\overline{\varphi}J_2(\sigma))) \in E_2(F[z_3' \otimes 1, 1 \otimes z_3]).$$

Consequently, $\eta(J_2(\psi)) \notin E_2(F[z_3' \otimes 1, 1 \otimes z_3])$. Then Theorem 5.2 gives that $\psi \notin TAut\, C$. Therefore $\varphi \notin TAut\, C$, i.e. φ is wild. $\square$

Thus the Anick automorphism

$$\delta \ = \ (x + z(xz - zy),\, y + (xz - zy)z,\, z)$$

of the free associative algebra $F < x, y, z >$ induces a wild automorphism of the algebra C.

Theorem 5.3 *The Anick automorphism δ of the free associative algebra $F < x, y, z >$ over a field F of characteristic 0 is wild.*

Combining the results of [32, 6] and [24], we conclude that an automorphism $\varphi = (f, g, z)$ of the polynomial algebra $F[x, y, z]$ is tame if and only if

$$J_2(\varphi) \in GL_2(F)E_2(F[x, y, z]).$$

Theorem 5.4 *Let $\varphi = (f, g, z)$ be an automorphism of the free associative algebra $F < x, y, z >$ over a field F of characteristic 0 such that $\deg_{x,y} f, \deg_{x,y} g \leq 1$. Then φ is tame if and only if*

$$J_2(\varphi) \in GL_2(F)E_2(F[z' \otimes 1, 1 \otimes z]).$$

Proof. Assume that the matrix $J_2(\varphi)$ satisfies the condition of the theorem. Without loss of generality we can suppose that

$$J_2(\varphi) \in E_2(F[x, y, z]).$$

Since $\deg_{x,y} f, \deg_{x,y} g \leq 1$, to each elementary matrix of $E_2(F[x, y, z])$ corresponds an elementary transformation of the first two coordinates of φ. Applying elementary transformations corresponding to a decomposition of $J_2(\varphi)$, we can get $(x + a(z), y + b(z), z)$ from φ. Consequently, φ is tame.

Now, assume that φ is tame. Let $\tilde{\varphi}$ be an automorphism of the free metabelian associative algebra induced by φ. Since $\deg_{x,y} f, \deg_{x,y} g \leq 1$, it is easy to check that

$$J_2(\varphi) = \nu(J_2(\tilde{\varphi})).$$

If $J_2(\varphi) \notin E_2(F[x, y, z])$, then as in the proof of Theorem 5.3 we can obtain a contradiction. $\square$

Automorphisms of the form considered in Theorem 5.4 were investigated in [7].

Now we formulate some problems related to the relations (18)–(20) and discuss why it is impotant to study them.

Let us denote by $G(A_n)$ the group defined by the system of generators (17) and the system of relations (18)–(20).

Problem 5.1 *Find a canonical form of elements of the group $G(A_n)$. In particular, is the word problem solvable in this group?*

This problem is also interesting for the group $G(B_n)$, where B_n is a free associative algebra in n variables, which can be defined similarly $G(A_n)$. Note that the group $G(A_2) \cong G(B_2) \cong Aut\, A_2 \cong Aut\, B_2$ has a nice representation (see [3]). By Theorem 4.1, if $n = 3$ and F is a field of characteristic 0, then the group $G(A_3)$ has the solvable word problem. But we have no canonical form of elements of $G(A_3)$.

Let L_n be a free Lie algebra (or a free algebra of some Schreier variety of algebras) with n free generators. Certainly, we can define the group $G(L_n)$ also. Applying only general ideas of the proof of Theorem 4.1 it is not difficult to prove that $G(L_n) \cong Aut\, L_n$.

Now consider the homomorphism

$$s_n : G(A_n) \longrightarrow Aut\, A_n$$

which sends the generators (17) to the corresponding automorphisms of A_n.

Problem 5.2 *Find the kernel of s_n.*

We know that the kernel of s_2 is trivial. By Theorem 4.1 the kernel of s_3 is also trivial if $char(F) = 0$.

A solution of Problem 5.2 gives a description of the group $TAut\, A_n$ by generators and relations. If $n \geq 4$, then it is possible also that $Aut\, A_n = TAut\, A_n$.

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