The Sobolev Problem in Spaces with Asymptotics

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Abstract

In the paper, we construct the theory of problems of Sobolev type in spaces of functions having the given type of asymptotic expansion near "boundary" manifolds. Such problems arise, for example, in the theory of potentials of zero range in the nuclear physics. Moreover, the corresponding operator algebra is constructed. The finiteness theorems (Fredholm property) are established and the index is calculated.

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Contents

Introduction

1. As it follows from the title, the aim of the paper is to investigate a new class of Sobolev problems[1], that is, Sobolev problems in spaces with asymptotics. To motivate the statements of problems presented below, we first consider some physical situations leading to the necessity of consideration of the Sobolev problems in spaces with asymptotics, namely, the examination the internuclear forces, that is, the theory of Schrödinger equation with potentials of zero range.

2. It seems that[2] it was E. Wigner [3] who first mentioned that the forces of internuclear interaction must act at a very short distance and be very strong. From this remark, the possibility of idealization of the internuclear interaction as

[1]Concerning the "classical" Sobolev problems see, for example, [1], [2] and the bibliography therein.

[2]The short notes presented below do not give any historical priorities as well as the full bibliography. The aim of these notes is to acquaint the reader with some physical papers which served as the starting point for the constructing the mathematical theory presented in the main part of the paper.

an interaction determined by a *potential of zero range* comes in a very natural way. This idea was realized by H. Bethe and R. Peierls [4] as early as in 1935 and have born a lot of exactly solvable models of systems with point interaction (see [5]). The review of these models and a rather full bibliography the reader can find in the book [6].

The further generalization of the short-range interaction theory to N-particle problems meets serious difficulties. L. H. Thomas have noticed that the energy of the base state can tend to $-\infty$ as the interaction radius tends to zero (falling to the center, or collapse, see [7]). The reason for this phenomenon became clear after the appearance of the rigorous mathematical verification of the two-particle problem by F. Berezin and L. Faddeev [8], and by R. Minlos and L. Faddeev [9] appeared soon after the former one. The matter is (as it is shown in the paper by F. Berezin and L. Faddeev) that the description of systems with potentials of zero range is performed with the help of self-adjoint extensions of the Laplace operator with the initial domain consisting of functions vanishing near the center of interaction. However, the paper by R. Minlos and L. Faddeev shows that *all self-adjoint extensions of the Laplace operator corresponding to the pairwise interactions are not bounded from below* and, hence, the energy of the system can be of arbitrary negative value.

One of possible approaches to overcome this difficulty is mentioned in the cited paper by R. Minlos and L. Faddeev. Namely, if one involves into the boundary condition the operator of the convolution type in the impulse variable, then one arrives at semi-bounded self-adjoint extensions. However, the interaction in the system constructed is not more a pairwise one. Later on, such extensions were considered by S. Albeverio, R. Hoegh-Krohn, and L. Streit [10]. The fundamental investigation of the three-particle problems one can find in [11], [12].

The first attempt of constructing a selfadjoint extension of the Laplace operator corresponding to the pairwise interaction is due to Yu. Shondin [13] who has considered an extension in the space $L_2(\mathbf{R}^3) \oplus \mathbf{C}$ instead of $L_2(\mathbf{R}^3)$ (we remark that, physically, such extensions correspond to physical systems with internal degrees of freedom). In future, similar extensions using more wide Hilbert spaces were considered by B. Pavlov [14], [15].

In the recent time, the described field is developed in the works of a lot of mathematicians (see papers by S. Albeverio, K. Makarov, V. Melezhik [16] — [18] and others).

3. Let us show, on the simplest example, how the above considered physical problem leads to the necessity of investigation of Sobolev problems in spaces with asymptotics. Consider the operator Δ on the three-dimensional space as an operator

with the following domain of definition[3]:

$$D = \{u(x) \in H^2(\mathbf{R}^3) \big| \; u \equiv 0 \text{ near } x = 0\}. \tag{1}$$

This operator is a symmetric one, and it possesses self-adjoint extensions. Under the quantum mechanics spirit, each extension of this kind describes some quantum-mechanical system. One of these extensions is the operator Δ with $H^2(\mathbf{R}^3)$ as the domain of definition – this is a self-adjoint operator corresponding to a free one-dimensional particle. However, there are a lot of self-adjoint extensions of the former operator different from the latter one. It is natural to suppose that all these extensions describe a quantum particle in the potential field concentrated at the origin. So, there arises a problem of description of all self-adjoint extensions of the operator Δ with (1) as the domain of definition.

It occurs (see, for example, [17]) that all self-adjoint extensions of the operator in question can be described as the operator Δ considered on subspaces of the space

$$\widetilde{D} = \left\{u(x) = \frac{C}{r} + u_0(x), \;\; u_0(x) \in H^2(\mathbf{R}^3)\right\} \tag{2}$$

defined by homogeneous "boundary conditions" of the type

$$C + \alpha u_0(0) = 0.$$

Spaces of the type (2) are naturally named *spaces with asymptotics* since their elements are sums of the *asymptotics* C/r and the *smooth component* u_0 lying in the space $H^2(\mathbf{R}^3)$ to which the first summand does not belong.

The easiest way of investigating such operators is to consider the corresponding resolvent. This resolvent is the resolving operator for the following problem:

$$\begin{cases} (\Delta - \lambda I) u(x) \equiv f(x), \\ C + \alpha u_0(0) = 0, \end{cases} \tag{3}$$

for $f(x) \in L_2(\mathbf{R}^3)$ if the solution is searched from the class $\widetilde{D}$ determined by relation (2). The comparison $\equiv$ in (3), means that the first equation is fulfilled everywhere except for the origin. The latter problem is the simplest example of the Sobolev problem in *spaces with asymptotics*. At the same time, we have obtained the form of "boundary conditions" to be used in spaces with asymptotics (we emphasize that usual boundary conditions cannot be used here since functions of the form (2) do not admit restrictions to the origin).

[3]By H^2, or, more generally, H^s, we denote usual Sobolev spaces of order s (see, e. g. [19]).

Below we investigate possible statements of Sobolev problems in spaces with asymptotics and construct the following operator calculus – the powerful tool of investigation of such problems.

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1 Examples

We begin with the consideration of the simplest examples of a Sobolev problem in spaces with asymptotics. Here we shall not present the exact definitions of spaces with asymptotics and encounter the exact indices of Sobolev spaces. We restrict ourselves only by some intuitive notions sufficient for initial understanding of the problem. All definitions will be refined during the consideration of the examples below and will be finally formulated in the exact manner in the subsequent sections.

So, let M be a smooth n-dimensional manifold and X be its smooth submanifold of codimension ν. Roughly speeking, the element of the space with asymptotics are functions on the pair (M, X) having the form

$$u(x,t) = \sum_k r^{S_k(x)} \sum_{j=0}^{m_k-1} \frac{\ln^j r}{j!} u_j^k(x,\omega) + u_0(x,t) . \qquad (4)$$

near X. Here and below we use the following special coordinate systems[4] near X: $x \in \mathbf{R}_x^{n-\nu}$ are coordinates along the submanifold X, $t \in \mathbf{R}_t^\nu$ are coordinates transversal to X, and (r,ω) are polar coordinates in the plane $\mathbf{R}_t^\nu$, $r \in [0,1]$, $\omega \in S^{\nu-1}$ ($S^{\nu-1}$ being a unit sphere in the Cartesian space $\mathbf{R}_t^\nu$). The function $u_0(x,t)$ must belong to some function space, say, $H^s(M)$, consisting of functions smooth enough so that this space does not include any term

$$r^{S_k(x)} \frac{\ln^j r}{j!} u_j^k(x,\omega)$$

involved into the right-hand part of representation (4).

[4]For the corresponding notions to be global, it is convenient to fix a structure of the bundle over a tube neighborhood of the submanifold X. Then t (or (r,ω)) can be treated as coordinates in a fiber of this bundle.

We remark that, for integer nonnegative $S_k(x)$, functions of the form $r^{S_k(x)}u^k(x,\omega)$ can occur to be of infinite smoothness for some special choice of amplitudes u^k. For example, in the two-dimensional case $r\cos\varphi = x$ and $r\sin\varphi = y$ (where r,φ are polar coordinates) are infinitely smooth functions. However, we require that s is such that the inclusion $r^{S_k(x)}u^k(x,\omega) \in H^s(M)$ is not valid for all smooth amplitude functions $u^k(x,\omega)$.

We suppose that for any given space with asymptotics the following objects are fixed:

i) *Smooth functions* $S_k(x)$, $k = 1,2,\dots N$ on X; subject to the conditions[5]

$$S_k(x) \neq S_j(x) \text{ for } k \neq j;$$

ii) The set of *multiplicities* $m_k \in \mathbf{Z}_+$ given for each $k = 1.2.\dots N$;
iii) The set of *function spaces* for the coefficients $u_j^k(x,\omega)$ on $X \times S^{\nu-1}$ (the exact choice of these spaces will be presented below);
iv) *The number* s characterizing smoothness of the remainder $u_0(x,t)$ in (4) (below we shall use the Sobolev spaces $H^s(M)$ for the description of the smoothness of remainders in (4)).

We denote by

$$T = \{\, S_k(x),\ m_k \mid k = 1.2.\dots N\}$$

the set of degrees (with multiplicities) involved in asymptotic expansions (4) of functions from the considered space with asymptotics. This space will be denoted by $H_T^s(M,X)$.

In the examples below we shall consider mainly the equation (comparison)

$$(-\Delta + 1)\, u(x,t) \equiv f(x,t) \tag{5}$$

which is fulfilled at all points of the manifold M except for the submanifold X. We have tried to choose these examples in such a way that all main features of the general theory can be transparently shown.

1.1 The case of a zero-dimensional submanifold (one-term asymptotics)

1. Our first example is the example of equation (5) considered in the three-dimensional Cartesian space $\mathbf{R}^3 = M$ with the origin as the submanifold X. So, Δ

[5] This requirement means, in essence, that we do not include continuous asymptotics or branching asymptotics into consideration (see [20]).

is here the Laplace operator in $\mathbf{R}^3_t$, $t = (t^1, t^2, t^3)$:

$$\Delta = \frac{\partial^2}{(\partial t^1)^2} + \frac{\partial^2}{(\partial t^2)^2} + \frac{\partial^2}{(\partial t^3)^2} = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + \left(r\frac{\partial}{\partial r}\right) + \Delta_\omega\right]. \qquad (6)$$

Here by Δ_ω we denote the Laplace operator on the sphere S^2 (the "angular part" of the Laplace operator Δ).

We shall consider equation

$$(-\Delta + 1)\, u\,(t) = f\,(t) \qquad (7)$$

in the following function spaces. Suppose that the right-hand part $f(t)$ in (7) is of the form

$$f(t) = r^{\alpha-2} f_{\alpha-2}(\omega) + f_0\,(t)\,, \qquad (8)$$

$f_0\,(t) \in H^{s-2}\,(\mathbf{R}^3)$ with some real α. The smoothness of the function $f_{\alpha-2}\,(\omega)$ will be determined later, now we suppose only that

$$s \geq 1/2 \text{ and } \alpha \leq s - 3/2.$$

The first inequality means that the space $H^{s-2}(\mathbf{R}^3)$ does not contain distributions concentrated at a single point and the second that $r^{\alpha-2} f_{\alpha-2}(\omega) \notin H^{s-2}\,(\mathbf{R}^3)$. The solution to (7) will be searched in the space of function having the form

$$u(t) = r^\alpha u_\alpha(\omega) + u_0\,(t) \qquad (9)$$

with $u_0\,(t) \in H^s\,(\mathbf{R}^3)$. This representation correspond to *one-term* asymptotics of the solution $u(t)$. The latter means that we consider the operator $(-\Delta + 1)$ as an operator in the following spaces with asymptotics:

$$(-\Delta + 1)\ :\ H^s_{(\alpha,0)}\,(M, X)\ \to\ H^{s-2}_{(\alpha-2,0)}\,(M, X)\,.$$

Before investigating the stated problem let us consider some analogy. The matter is that the equation for the operator $-\Delta + 1$ is, in essence, investigated as an equation with right-hand parts of the special form with the help of undefined coefficients method. The similar problem for an ordinary differential equation has the form (cf. (6))

$$P_m\left(x\frac{d}{dx}\right)\, y = f, \qquad (10)$$

where

$$P_m(p) = a_m p^m + \ldots + a_1 p + a_0$$

is a polynomial in p, $x \in \mathbf{R}^1$, the function $f(x)$ has the following special form:

$$f(x) = x^\alpha Q_n(\ln x),$$

and Q_n is an arbitrary polynomial in $\ln x$ of order not more than n. The space of functions of such form is denoted by $\mathcal{F}_{n,\alpha}$.

Let us search for a solution to (10) in the space $\mathcal{F}_{n,\alpha}$. The operator $P_m(xd/dx)$ can be considered as a linear operator in finite-dimensional spaces:

$$P_m\left(x\frac{d}{dx}\right) : \mathcal{F}_{n,\alpha} \to \mathcal{F}_{n,\alpha}, \tag{11}$$

and, hence, it can be written in a matrix form with the help of some base in $\mathcal{F}_{n,\alpha}$. The most convenient base is

$$e_k(x) = x^\alpha \frac{\ln^k x}{k!},\ k = 0, \dots, n.$$

Then the matrix of the operator (11) is an upper triangle matrix with $P_m(\alpha)$ on the diagonal:

$$\begin{pmatrix} P_m(\alpha) & * & \dots & * \\ 0 & P_m(\alpha) & \dots & * \\ & & \dots & \\ 0 & 0 & \dots & P_m(\alpha) \end{pmatrix}.$$

Now we see that:

1) If $P_m(\alpha) \neq 0$ (*non-resonance case*), then the operator (11) is an isomorphism, that is, equation (10) is uniquely solvable in the space $\mathcal{F}_{n,\alpha}$.

2) If $P_m(\alpha) = 0$ (*resonance case*), then the homogeneous equation corresponding to (10), has nontrivial solutions, and non-homogeneous equation (10) is solvable not for any right-hand part $f(x)$, that is, the operator in question has nontrivial kernel and cokernel.

The more detailed analysis shows that the kernel and cokernel of operator (11) have dimension equal to the multiplicity m_0 of the root α of equation $P_m(p) = 0$. Hence, for the unique solvability it is necessary, first, to add to (11) m_α additional conditions (for example, of the Cauchy type) as well as pose m_α additional requirements on the right-hand part $f(x)$ for the equation to be solvable.

Now let us turn our mind to the consideration of problem (7). In this case the function space (8) (of special right-hand parts) can be represented as a direct sum[6]

$$H^{s-2}(S^2) \oplus H^{s-2}(\mathbf{R}^3),$$

$$f(t) \leftrightarrow (f_{\alpha-2}(\omega), f_0(t)),$$

and space (9), of the left-hand parts is

$$H^s(S^2) \oplus H^s(\mathbf{R}^3),$$

$$u(t) \leftrightarrow (u_\alpha(\omega), u_0(t)).$$

[6]For simplicity we suppose that $\alpha \notin \mathbf{N}$ (see the footnote 18 on page 29 below).

Let us calculate the representation of the operator $-\Delta + 1$ corresponding to the splitting of the considered spaces into the direct sums. To do this, we substitute relation (9) into equation (7) thus obtaining

$$-r^{\alpha-2}[\Delta_\omega+\alpha(\alpha+1)]\,u_\alpha(\omega)+r^\alpha u_\alpha(\omega)+(-\Delta+1)\,u_0(t) = r^{\alpha-2}f_{\alpha-2}(\omega)+f_0(t). \quad (12)$$

One can notice that the right-hand part of the latter equation does not contain terms of the form $r^\alpha\varphi(\omega)$, whereas the left-hand part does contain such a term. So, if we want to rewrite this equation in the matrix form, it is necessary to choose an index s of the Sobolev space in such a way that $r^\alpha\varphi(\omega) \in H^{s-2}(\mathbf{R}^3)$ for sufficiently smooth function $\varphi(\omega)$. This last requirement will be fulfilled if $s < \alpha + 7/2$. Thus, in the case considered the number s must belong to the half-interval

$$[\max(1/2, \alpha + 3/2), \alpha + 7/2)\,.$$

Under this requirements, there are two kind of terms in both sides of equation (12). These are, first, terms from the space $H^{s-2}(\mathbf{R}^3)$, and, second, all the rest terms. Equating terms of these kinds in both parts of (12) we arrive at the following system of equations for the functions $u_\alpha(\omega)$ and $u_0(t)$:

$$\begin{cases} (-\Delta + 1)u_0(t) + r^\alpha u_\alpha(\omega) \equiv f_0(t), \\ -[\Delta_\omega + \alpha(\alpha+1)]\,u_\alpha(\omega) = f_{\alpha-2}(\omega), \end{cases} \quad (13)$$

or, in the "matrix" form

$$\begin{pmatrix} -\Delta+1 & r^\alpha \\ 0 & -\Delta_\omega + \alpha(\alpha+1) \end{pmatrix} \begin{pmatrix} u_0(t) \\ u_\alpha(\omega) \end{pmatrix} = \begin{pmatrix} f_0(t) \\ f_{\alpha-2}(\omega) \end{pmatrix}.$$

Clearly, the latter matrix is invertible iff both operators on the diagonal are invertible. What concerns the operator $-\Delta + 1$, it is invertible since for $s \geq 1/2$ the comparison in (13) becomes an equality, and the operator $-\Delta + 1$ is invertible in $H^s(\mathbf{R}^2)$ for any s. The different situation takes place for the operator $-\Delta_\omega+\alpha(\alpha+1)$. This operator is uniquely invertible on the sphere S^2 for $\alpha \neq -1, 0, 1, 2, \ldots$ (non-resonance case), and has nontrivial kernel and cokernel for $\alpha = -1, 0, \ldots$ (resonance case).

The unique invertibility of the operator $-\Delta_\omega + \alpha(\alpha - 1)$ for $\alpha \neq -1, 0, 1, \ldots$ follows from the well- known fact that the eigenvalues for the Laplace operator on the sphere equal $\alpha(\alpha - 1)$ for $\alpha \in \mathbf{Z}_+$.

So, we see that equation (7) is uniquely solvable in spaces (8), (9) with asymptotics for the *non-resonance* values of α.

Let us consider now the *resonance case*, e. g. $\alpha = -1$. In this case $s \in [1/2, 5/2)$ and equation (12) becomes

$$-\frac{\Delta_\omega u_{-1}(\omega)}{r^3} + \frac{u_{-1}(\omega)}{r} + (-\Delta + 1)\, u_0(t) = \frac{f_{-3}(\omega)}{r^3} + f_0(t), \tag{14}$$

and the corresponding system of equations for $u_{-1}(\omega)$ and $u_0(t)$ is

$$\begin{cases} (-\Delta + 1)\, u_0(t) + \frac{u_{-1}(\omega)}{r} = f_0(t), \\ \Delta_\omega u_{-1}(\omega) = -f_{-3}(\omega). \end{cases} \tag{15}$$

Let us analyze the obtained system. First of all, one can see that the second equation in (15) has nonzero kernel and cokernel. The reason for this phenomenon is that (as we have already mentioned above) we are considering the resonance case. This means that the homogeneous equation corresponding to equation (7)

$$(-\Delta + 1)\, u^{(0)}(t) \equiv 0$$

(we recall that the equation must be fulfilled outside the origin) has a nontrivial solution

$$u^{(0)}(t) = C\frac{e^{-r}}{r}$$

with an arbitrary constant C which admits the following asymptotic expansion

$$u^{(0)}(t) = \frac{C}{r} + u_0^{(0)}(t),$$

where $u_0^{(0)}(t) \in H^s(\mathbf{R}^3)$ (for the above chosen values of s). The set of constants form exactly the kernel of the second equation in (15).

Below, we shall show how one can get rid of the kernel of the equation in question, and now we concentrate our attention at dealing with its cokernel. We shall consider the two different methods.

2. From the first glance, to eliminate the cokernel of the equation in question one should use the following procedure. For simplicity, let us consider an equation of Fuchsian type with constant coefficients and a special right-hand part

$$\left(x\frac{d}{dx}\right)^2 y(x) - y(x) = x^\alpha$$

on the real line $\mathbf{R}^1_x$. It is not hard to see that for any α except for $\alpha = \pm 1$ there exist a particular solution of this equation of the form Cx^α with some coefficient C. However, say, for $\alpha = 1$ the particular solution of the equation in question must be searched in the form

$$y^*(x) = x \ln x.$$

So one can see that *the multiplicity of solution to an equation of the Fuchsian type is increased by one in the (simple) resonance case.*

Similar to the case considered, one can search for the solution to equation (5) for $\alpha = -1$ with right-hand part (8) in the form

$$u(t) = \frac{1}{r}\left[u_{-1}^{1}(\omega)\ln r + u_{-1}^{0}(\omega)\right] + u_0(t). \tag{16}$$

As above, substituting this expression into equation (5) one obtains

$$-\frac{\ln r}{r^3}\Delta_\omega u_{-1}^{1}(\omega) + \frac{1}{r^3}\left[-\Delta_\omega u_{-1}^{0}(\omega) + u_{-1}^{1}(\omega)\right]$$
$$+\frac{\ln r}{r}u_{-1}^{1}(\omega) + \frac{1}{r}u_{-1}^{0}(\omega) + (-\Delta + 1)\,u_0(t) = \frac{f_{-3}(\omega)}{r^3} + f_0(t). \tag{17}$$

Equating terms with equal smoothness, we arrive at the following system of equations for the functions $u_{-1}^{1}(\omega)$, $u_{-1}^{0}(\omega)$, and $u_0(t)$:

$$\begin{cases} (-\Delta + 1)\,u_0(t) = f_0(t) - \frac{\ln r}{r}u_{-1}^{1}(\omega) - \frac{1}{r}u_{-1}^{0}(\omega), \\ \Delta_\omega u_{-1}^{1}(\omega) = 0, \\ -\Delta_\omega u_{-1}^{0}(\omega) = f_{-3}(\omega) - u_{-1}^{1}(\omega), \end{cases}$$

From the second equation of the last system, one easily finds that $u_{-1}^{1}(\omega) = \text{const}$, and the exact value of this constant can be determined from the third equation (more exactly, from the solvability condition for this equation). Since the compatibility conditions for the equation $-\Delta_\omega u = f$ is the orthogonality of its right-hand part to the space of constants, we obtain

$$u_{-1}^{1}(\omega) = \frac{1}{V_2}\int\limits_{S^2} f_{-3}(\omega)\, ds_\omega.$$

(ds_ω being the volume element on the unit sphere). The functions $u_{-1}^{0}(\omega)$, and $u_0(t)$ can now be easily found.

In spite of the fact that with the help of this method one can guarantee the existence of solution of the given form for any right-hand part $f(t)$, this method has a serious defect. Namely, the operator $(-\Delta + 1)$ cannot be considered as an operator from functions having the form (16) to the space of functions having the form (8). Actually, expression (17) shows that the result of application of the operator $-\Delta + 1$ to the function of the form (16) is a function of the form

$$\frac{1}{r^3}[f_{-3}^{1}(\omega)\ln r + f_{-3}^{0}(\omega)] + f_0(t), \tag{18}$$

instead of (8) for $\alpha = -1$. Thus, to consider $-\Delta + 1$ as an operator in the corresponding function spaces one has to allow the functions with the multiplicity 1 on the right in equation (5). In this case, however, we are again in the resonance situation, and the multiplicity must be enlarged once more. It is easy to see that the process of enlarging the multiplicity will not stop at any stage. This is the reason why it is better to use another method of eliminating the cokernel of the operator in question.[7]

[7] Of course, one can consider spaces of functions of the form (16) with u_{-1}^{1} from the kernel of the operator Δ_ω. However, such spaces will depend on the operator under consideration.

3. Namely, to eliminate the cokernel of the operator considered, we shall slightly modify the initial problem by including the so-called *coboundary terms* into the right-hand part of the equation. More precisely, we shall consider the comparison

$$(-\Delta + 1)\, u\,(t) \equiv f\,(t) + \frac{c}{r^3}. \tag{19}$$

instead of (7). Here c is a new unknown number which must be found in the process of solving the equation, and the function $f\,(x,t)$ has, as above, the form (8).

Searching for a solution to equation (19) in the form (9) with $\alpha = -1$, we arrive at the following system of equations for the unknowns $u_{-1}\,(\omega)$, $u_0\,(t)$, and c:

$$\begin{cases} (-\Delta + 1)\, u_0\,(t) + \dfrac{u_{-1}\,(\omega)}{r} = f_0\,(t)\,, \\ \Delta_\omega u_{-1}\,(\omega) + c = -f_{-3}\,(\omega)\,, \end{cases} \tag{20}$$

or, in the matrix form,

$$\begin{pmatrix} -\Delta + 1 & r^{-1} & 0 \\ 0 & -\Delta_\omega & -1 \end{pmatrix} \begin{pmatrix} u_0(t) \\ u_{-1}(\omega) \\ c \end{pmatrix} = \begin{pmatrix} f_0(t) \\ f_{-3}(\omega) \end{pmatrix}.$$

Now the constant c must be chosen in such a way that the second equation in the latter system is solvable for the given right-hand part $f_{-3}\,(\omega)$. Rewriting this equation in the form

$$\Delta_\omega u_{-1}\,(\omega) = -f_{-3}\,(\omega) - c, \tag{21}$$

and taking into account the fact that the image of the operator Δ_ω consists of functions orthogonal to constants, we have

$$c = -\frac{1}{V_2} \int\limits_{S_2} f_{-3}\,(\omega)\; ds_\omega.$$

The solution of the second equation in (20) is

$$u_{-1}\,(\omega) = -\Delta_\omega^{-1}\,(f_{-3}\,(\omega) + c) + d,$$

where by Δ_ω^{-1} we have denoted the resolving operator for equation (21) defined on the image of the operator Δ_ω, and the constant d is an arbitrary constant which appears as an element of the kernel of the operator Δ_ω. Now due to the invertibility of the operator $-\Delta + 1$ on the whole space $\mathbf{R}^3$, the first equation of (20) gives

$$u_0\,(t) = (-\Delta + 1)^{-1} \left[f_0\,(t) - \frac{u_{-1}\,(\omega)}{r} \right].$$

The latter formula completes the process of solving equation (19). Now the only thing rest is to eliminate the kernel of the operator, that is, to pose a condition for determining the arbitrary constant d involved into the obtained solution of equation (19). This will be done in the next point.

4. The usual way of stating boundary conditions is to prescribe some concrete values of the unknown ($u(t)$ in our case) or, more general, of the result of application of some differential operator to this function, at the boundary point. Unfortunately, in the above considered case this is not possible: the unknown, in general, does not admit the restriction to the origin (we recall that the origin plays role of the only boundary point in our example). However, it is possible to consider the following two functions to be used in the future boundary conditions (we remark that these functions are involved into the boundary conditions of the problem which was constructed in the Introduction in the process of investigation of zero-range potentials):

- the coefficient $u_{-1}(\omega)$ involved into the asymptotic expansion of the unknown function $u(t)$ near the origin (see formula (9));
- the restriction $u_0(0)$ of the regular part $u_0(t)$ to the origin (here we suppose that the number s belongs to the interval $(1/2, 5/2)$.

What for the second quantity mentioned above, it can be used in the "boundary condition" without any additional transformation. Quite another situation takes place for the first of these two unknowns. The matter is that the kernel of the considered equation is one-dimensional whereas the set of coefficients $u_{-1}(\omega)$ of the asymptotics form an infinite-dimensional function space. Therefore, before using these coefficients in our future boundary condition, one must extract some "component" from the coefficient $u_{-1}(\omega)$ in such a way that the set of these "components" form some one-dimensional space. Below, we shall use the constant component

$$P_1 u_{-1}(\omega) \stackrel{\text{def}}{=} \int\limits_{S^2} u_{-1}(\omega)\, ds_\omega,$$

which is, in essence, the projection of the function $u_{-1}(\omega)$ to the subspace of constant functions[8] (this motivates the name "projector" for the operator P_1 which we shall use below).

As a result of the considerations above, we arrive at the following "boundary condition" which can eliminate the kernel of the equation in question:

$$P_1 u_{-1}(\omega) + \gamma i^* u_0(t) = g. \tag{22}$$

[8] We recall that the space of constants form a kernel of the operator Δ_ω.

Here γ is some constant, i^* is a restriction to the origin, and g plays the role of the right-hand part of the introduced boundary conditions. The considerations above show that equation (19) with boundary condition (22)

$$\begin{cases} (-\Delta+1)\,u\,(t) = f\,(t) + \frac{c}{r^3}, \\ P_1 u_{-1}\,(\omega) + \gamma i^* u_0\,(t) = g, \end{cases} \tag{23}$$

is equivalent to the following system of equations with respect to the unknowns $u_{-1}\,(\omega)$, $u_0\,(t)$, and c:

$$\begin{cases} (-\Delta+1)\,u_0\,(t) + \frac{u_{-1}\,(\omega)}{r} = f_0\,(t)\,, \\ \Delta_\omega u_{-1}\,(\omega) + c = -f_{-3}\,(\omega)\,, \\ P_1 u_{-1}\,(\omega) + \gamma i^* u_0\,(t) = g, \end{cases} \tag{24}$$

or, in the matrix form

$$\begin{pmatrix} -\Delta+1 & r^{-1} & 0 \\ 0 & -\Delta_\omega & -1 \\ \gamma_i^* & P_1 & 0 \end{pmatrix} \begin{pmatrix} u_0(t) \\ u_{-1}(\omega) \\ c \end{pmatrix} = \begin{pmatrix} f_0(t) \\ f_{-3}(\omega) \\ g \end{pmatrix}. \tag{25}$$

Now the solution of problem (23) can be constructed within the following four steps:

1) Determination of the constant c so that the second equation in (24) is solvable with respect to $u_{-1}\,(\omega)$:

$$c = -\frac{1}{V_2} \int\limits_{S_2} f_{-3}\,(\omega)\; ds_\omega.$$

2) Determination of the function $u_{-1}\,(\omega)$ as a particular solution to the second equation in (24):

$$u_{-1}\,(\omega) = -\Delta_\omega^{-1}\,[f_{-3}\,(\omega) + c] + d,$$

where, as above, d is an arbitrary constant, and the application of the operator Δ_ω^{-1} is well-defined since the function $f_{-3}(\omega) + c$ belongs to the image of the operator Δ_ω.

3) Determination of the $u_0\,(t)$ as the function depending on the arbitrary constant d from the first equation in (24):

$$u_0\,(t) = (-\Delta+1)^{-1}\left\{ f_0\,(t) + \frac{1}{r}\Delta_\omega^{-1}\,[f_{-3}\,(\omega) + c]\right\} - d\,(-\Delta+1)^{-1}\,\frac{1}{r}$$

(we have used here the fact that the operator $-\Delta+1$ is invertible in $\mathbf{R}^3$).

4) Determination of the arbitrary constant d. For this one substitutes the latter expression into the "boundary conditions" (the last equation in (24)), thus obtaining

$$-P_1\left[\Delta_\omega^{-1}\left[f_{-3}(\omega)+c\right]\right]+V_2 d$$
$$+\gamma i^*(-\Delta+1)^{-1}\left\{f_0(t)+\tfrac{1}{r}\Delta_\omega^{-1}\left[f_{-3}(\omega)+c\right]\right\}-d\gamma i^*(-\Delta+1)^{-1}\tfrac{1}{r}=g,$$

or

$$\left\{V_2-\gamma i^*(-\Delta+1)^{-1}\tfrac{1}{r}\right\}d=P_1\left\{\Delta_\omega^{-1}\left[f_{-3}(\omega)+c\right]\right\}$$
$$-\gamma i^*(-\Delta+1)^{-1}\left\{f_0(t)+\tfrac{1}{r}\Delta_\omega^{-1}\left[f_{-3}(\omega)+c\right]\right\}.$$

Hence, one has

$$d=\frac{P_1\left[\Delta_\omega^{-1}\left[f_{-3}(\omega)+c\right]\right]-\gamma i^*(-\Delta+1)^{-1}\left\{f_0(t)+\frac{1}{r}\Delta_\omega^{-1}\left[f_{-3}(\omega)+c\right]\right\}}{V_2-\gamma i^*(-\Delta+1)^{-1}\frac{1}{r}}$$

for each γ such that the denominator does not vanish.

This completes the determination of the solution to problem (23).

This method of constructing a solution to problem (23) allows one to choose the *exact function spaces* for this problem. As it was already told, the functions $u_0(t)$ and $f_0(t)$ belong to the spaces $H^s(M)$ and $H^{s-2}(M)$, respectively, where s is chosen as it was described above. Now we can see that the functions $u_{-1}(\omega)$ and $f_{-3}(\omega)$ can be treated as elements from $H^s(S^2)$ and $H^{s-2}(S^2)$, respectively, since the functions $u(t)$ and $f(t)$, being elements of spaces (8), (9) with asymptotics must belong to the space H^s outside X (in our case, everywhere except for the point $t=0$).

5. Let us now *interpret* the obtained technique of solving Sobolev problems in spaces with asymptotics. The matter is that, in essence, for solving these problems we have used the equivalent statements in the operator form. Namely, system (24) of equations for the unknowns $u_{-1}(\omega)$, $u_0(t)$, and c can be rewritten in the following form[9]:

$$\begin{pmatrix}-\Delta+1 & A & 0\\ 0 & -\Delta_\omega & -B\\ i^* & P_1 & 0\end{pmatrix}\begin{pmatrix}u_0(t)\\ u_{-1}(\omega)\\ c\end{pmatrix}=\begin{pmatrix}f_0(t)\\ f_{-3}(\omega)\\ g\end{pmatrix},$$

where the operators A and B are defined as follows:

$$Au_{-1}(\omega)=\frac{1}{r}u_{-1}(\omega) \tag{26}$$

[9]The corresponding matrix was already written down above (see formula (25). However, the operators r^{-1} and -1, involved into (25) must be interpreted more precisely since they change the number of arguments of the corresponding function.

(the latter function is understood as a function on the manifold $M = \mathbf{R}^3$), and

$$Bc = c \cdot 1(\omega), \tag{27}$$

that is, the action of the operator B gives the constant function c on the unit sphere. Let us first interpret the operator B. It is easy to check that the operator (27) is exactly the adjoint for the operator

$$P_1 : H^s(S^2) \to \mathbf{C}, \quad P_1 u_{-1}(\omega) = \int_{S^2} u_{-1}(\omega)\, ds_\omega$$

for any value of s. In other words,

$$B = P_1^* : \mathbf{C} \to H^{-s}(S^2).$$

Actually, one has

$$\langle P_1 u_{-1}(\omega), c\rangle = c \int_{S^2} u_{-1}(\omega)\, ds_\omega = \int_{S^2} c \cdot 1(\omega)\, u_{-1}(\omega)\, ds_\omega = \langle u_{-1}(\omega), Bc\rangle.$$

Similar, the operator (26) can be interpreted as an adjoint to the operator

$$\mathcal{P}_{1/r} : H^s(M) \to H^s(S^2), \quad \mathcal{P}_{1/r} u(t) = \int_M \frac{1}{r} u(r,\omega)\, r^2\, dr ds_\omega$$

for $s > -1/2$, that is

$$A = \mathcal{P}_{1/r}^* : H^{-s}(S^2) \to H^{-s}(M)$$

for $s > -1/2$. Now the operator equation corresponding to problem (23) becomes

$$\begin{pmatrix} -\Delta + 1 & \mathcal{P}_{1/r}^* & 0 \\ 0 & -\Delta_\omega & P_1^* \\ i^* & P_1 & 0 \end{pmatrix} \begin{pmatrix} u_0(t) \\ u_{-1}(\omega) \\ c \end{pmatrix} = \begin{pmatrix} f_0(t) \\ f_{-3}(\omega) \\ g \end{pmatrix}.$$

The above obtained matrix will be called a *matrix operator* or simply operator when the meaning of this word is clear from the context. Below, we shall use more general operators of this kind, chosen in such a way that they form an operator algebra closed with respect to the conjugation operation. However, to see the main features of these operators, we need two more examples.

1.2 The case of a zero-dimensional submanifold (multi-term asymptotics)

Here we consider the problem for the operator $(-\Delta + 1)$ in spaces with asymptotics *including more than one term.* We shall consider here only the resonance case; all the considerations for the non-resonance case the reader can carry out by himself or herself. Since all the effects can be seen in the case when two terms of asymptotics is taken into account, we consider the comparison

$$(-\Delta + 1)\, u\,(t) \equiv f\,(t) \tag{28}$$

with functions $u\,(t)$ and $f\,(t)$ having the form

$$u\,(t) = \frac{u_{-1}\,(\omega)}{r} + r u_1\,(\omega) + u_0\,(t)\,, \tag{29}$$

$$f\,(t) = \frac{f_{-3}\,(\omega)}{r^3} + \frac{f_{-1}\,(\omega)}{r} + f_0\,(t)\,. \tag{30}$$

The principle of choosing the functional spaces are just the same as in the above considered example and we omit deriving the exact values of the indices of the corresponding Sobolev spaces, postponing these considerations until the end of this subsection.

Now, substituting expressions (29) and (30) into equation (28) and equating terms of one and the same equal smoothness, we obtain the system of equations for the unknowns $u_{-1}\,(\omega)$, $u_1\,(\omega)$, and $u_0\,(t)$:

$$\begin{cases} (-\Delta + 1)\, u_0\,(t) = f_0\,(t) - r u_1\,(\omega)\,. \\ -\Delta_\omega u_{-1}\,(\omega) = f_{-3}\,(\omega)\,, \\ -\left(\Delta_\omega + 2\right) u_1\,(\omega) = f_{-1}\,(\omega) - u_{-1}\,(\omega)\,, \end{cases} \tag{31}$$

Again, we see that the last two equations in system (31) have nontrivial kernels and cokernels. The first equation was considered in the previous subsection.

It is easy to see that the kernel of the third equation in (31) consists of the angular parts of linear functions, that is,

$$\operatorname{Ker}\,(\Delta_\omega + 2) = \left\{C^1\omega_1 + C^2\omega_2 + C^3\omega_3\right\},$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is a point of the two-dimensional unit sphere. Consequently (since the operator Δ_ω is a self-adjoint one), the image of the operator $\Delta_\omega + 2$ consists of functions orthogonal to the kernel of this operator. Now the considerations similar to those of the previous subsection lead us to the following statement of the problem for the operator $-\Delta+1$, involving boundary and coboundary conditions:

$$\begin{cases} (-\Delta+1)\,u\,(t) = f\,(t) + \frac{c_{-3}}{r^3} + r\left(c_1^1\omega_1 + c_1^2\omega_2 + c_1^3\omega_3\right), \\ P_1 u_{-1}\,(\omega) + \alpha_{-1} i^* u_0\,(t) = g_{-1}, \\ P_{\omega_j} u_1\,(\omega) + \alpha_1^j i^* u_0\,(t) = g_1^j,\ j=1,2,3, \end{cases} \tag{32}$$

(to be short, we do not use here boundary conditions of the most general form). One can check that the corresponding equation for the matrix operator is

$$\begin{pmatrix} -\Delta+1 & 0 & \mathcal{P}_r^* & 0 & 0 & 0 & 0 \\ 0 & -\Delta_\omega & 0 & -P_1^* & 0 & 0 & 0 \\ 0 & 1 & -(\Delta_\omega+2) & 0 & P_{\omega_1}^* & P_{\omega_2}^* & P_{\omega_3}^* \\ \alpha_{-1} i^* & P_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1^1 i^* & 0 & P_{\omega_1} & 0 & 0 & 0 & 0 \\ \alpha_1^2 i^* & 0 & P_{\omega_2} & 0 & 0 & 0 & 0 \\ \alpha_1^3 i^* & 0 & P_{\omega_3} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_{-1} \\ u_1 \\ c_{-3} \\ c_1^1 \\ c_1^2 \\ c_1^3 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_{-3} \\ f_1 \\ g_{-1} \\ g_1^1 \\ g_1^2 \\ g_1^3 \end{pmatrix}.$$

Now one can derive that:

1. The function $u\,(t)$ must belong to the Sobolev space $H^s\,(M)$ with an s such that any function of the form $r\varphi\,(\omega)$ belongs to $H^{s-2}\,(M)$ but not to $H^s\,(M)$. This means that $s \in [5/2, 9/2)$.

2. The functions $u_{-1}\,(\omega)$ and $u_1\,(\omega)$ must belong to $H^s\,(S^2)$.

3. The functions $f_{-3}\,(\omega)$ and $f_{-1}\,(\omega)$ must belong to $H^{s-2}\,(S^2)$.

Under these requirements there exists a unique solution to problem (32). The simple verification of this fact is left to the reader.

1.3 The case of a higher dimensional submanifold

In this subsection, we consider Sobolev problems in spaces with asymptotics in the case when *the submanifold X has a non-zero dimension.* In this case, new effects arise connected with the appearance of *operator families* parameterized by points of X in the matrix operators in question.

1. Let us consider first the Sobolev problem for the equation

$$(-\Delta+1)\,u\,(t,x) = f\,(t,x) \tag{33}$$

in the space $M = \mathbf{R}^4$ with $X = \mathbf{R}^1$. Here $x \in \mathbf{R}^1$ is a coordinate along X, whereas $t \in \mathbf{R}^3$ are transversal coordinates. For simplicity, we shall consider here a one-term asymptotic expansions. So, suppose that the right-hand part of equation (33) has the form[10]

$$f(t,x) = \frac{f_{-3}(\omega,x)}{r^3} + f_0(t,x),$$

and let us search for solutions to (33) in the form

$$u(t,x) = \frac{u_{-1}(\omega,x)}{r} + u_0(t,x),$$

where the functions $f_{-3}(\omega,x)$, $f_0(t,x)$, $u_{-1}(\omega,x)$, and $u_0(t,x)$ belong to the corresponding Sobolev spaces (the exact values of indices of these spaces will be determined below). As above, one must pose the corresponding boundary and coboundary conditions, thus arriving at the following problem[11]:

$$\begin{cases} (-\Delta+1)\,u(t,x) = f(t,x) + \dfrac{c(x)}{r^3}, \\ P_1 u_{-1}(\omega,x) = g(x). \end{cases} \tag{34}$$

Since in the space $\mathbf{R}^4$ the operator Δ has the form

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_\omega + \frac{\partial^2}{\partial x^2},$$

we arrive at the following system of equations for the unknowns $u_{-1}(\omega,x)$, $u_0(t,x)$, and $c(x)$:

$$\begin{cases} (-\Delta+1)\,u_0(t,x) + \frac{1}{r}u_{-1}(\omega,x) + \frac{1}{r}\dfrac{\partial^2 u_{-1}}{\partial x^2}(\omega,x) = f_0(t,x), \\ -\Delta_\omega u_{-1}(\omega,x) - c(x) = f_{-3}(\omega,x), \\ P_1 u_{-1}(\omega,x) = g(x). \end{cases} \tag{35}$$

The procedure of solving this system goes through the following four steps:

1. Determination of the function $c(x)$ so that the solution to the second equation in (35) does exist for each fixed x:

$$c(x) = -\frac{1}{V_2}\int_{S^2} f_{-3}(\omega,x)\, ds_\omega. \tag{36}$$

We remark (this is very important) that the variable x is just a parameter in the second equation in (35).

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[10]Here, as above, (r,ω) are the polar coordinates in the plane $\mathbf{R}^3_t$.

[11]To be short, we consider here the boundary condition which does not involve the operator i^*. The general case can be considered similar to the preceding subsection.

2. Determination of the function $u_{-1}(\omega, x)$:

$$u_{-1}(\omega,x) = -\Delta_\omega^{-1}\left[f_{-3}(\omega,x) + c(x)\right] + d(x), \tag{37}$$

where, as above, Δ_ω^{-1} is an operator defined on the image of the operator Δ_ω, and $d(x)$ is an arbitrary function.

3. Determination of the function $u_0(t)$ as the function depending on the arbitrary constant d from the first equation in (24):

$$\begin{aligned} u_0(t,x) = (-\Delta+1)^{-1}\left\{f_0(t,x) + \tfrac{1}{r}\left(1+\frac{\partial^2}{\partial x^2}\right)\Delta_\omega^{-1}\left[f_{-3}(\omega,x)+c(x)\right]\right\} \\ -(-\Delta+1)^{-1}\tfrac{1}{r}\left(1+\frac{\partial^2}{\partial x^2}\right)d(x) \end{aligned} \tag{38}$$

(we have used here the fact that the operator $-\Delta+1$ is invertible in $\mathbf{R}^4$).

4. Determination of the arbitrary function $d(x)$. For this one substitutes the latter expression into the "boundary conditions" (the last equation in (24)), thus obtaining

$$-P_1\left[\Delta_\omega^{-1}\left[f_{-3}(\omega,x)+c(x)\right]\right] + V_2 d(x) = g(x),$$

or

$$d(x) = \frac{1}{V_2}\left\{g(x) + P_1\left[\Delta_\omega^{-1}\left(f_{-3}(\omega,x)+c(x)\right)\right]\right\}. \tag{39}$$

This completes the construction of solution to problem (34).

The above scheme of solving system (35) shows that problem (34) is well-posed and uniquely solvable in the following function spaces:

1) $f_{-3}(\omega,x) \in H^{s,s-2}(X\times S^2)$, $f_0(t,x) \in H^{s-2}(\mathbf{R}^4)$;

2) $u_{-1}(\omega,x) \in H^s(X\times S^2)$, $u_0(t,x) \in H^s(\mathbf{R}^4)$;

3) $c(x) \in H^s(X)$;

4) $g(x) \in H^s(X)$.

Actually, if $f_{-3}(\omega,x) \in H^{s,s-2}(X\times S^2)$, then formula (36) determines $c(x)$ as an element from $H^{s-2}(X)$. Later on, formula (37) shows that $u_{-1}(\omega,x) \in H^s(X\times S^2)$, provided that $d(x) \in H^s(X)$. Under the same assumption, formula (38) determines $u_0(x,t)$ as an element from $H^3(\mathbf{R}^4)$. Finally, the function $d(x)$, determined from (39) belongs to $H^3(\mathbf{R}^1)$, as required.

The corresponding operator equation has the form

$$\begin{pmatrix} -\Delta+1 & \mathcal{P}^*_{1/r}\left(1+\dfrac{\partial^2}{\partial x^2}\right) & 0 \\ 0 & -\Delta_\omega & -P_1^* \\ 0 & P_1 & 0 \end{pmatrix} \begin{pmatrix} u_0(t,x) \\ u_{-1}(\omega,x) \\ c(x) \end{pmatrix} = \begin{pmatrix} f_0(t,x) \\ f_{-3}(\omega,x) \\ g(x) \end{pmatrix}.$$

2. We shall consider here one more example of "one-dimensional" situation (that is, the situation when the manifold X is one-dimensional). The necessity of consideration of this example can be understood from the following reasons.

As it was already mentioned, one of the important features of the theory is that the equation for the main term of the asymptotic expansion ($u_{-1}(\omega,x)$ in the above considered case) contains the tangent variable x only as a parameter. From the other hand, this equation determines the powers of r which are involved into the asymptotic expansions of solutions to homogeneous equation. These powers determine, in turn, for which types of spaces with asymptotics the resonance phenomenon does occur. However, if the mentioned equation involves the parameter x, it is clear that these powers can be functions of x. Here we shall try to understand, what changes must be done in our considerations in order to include this phenomenon.

Let

$$\widehat{a} = a\left(t, x, -i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}\right)$$

be an elliptic differential operator of the second (for simplicity) order. Using polar coordinates (r,ω) in the t-plane, it is convenient to write down this operator as a differential operator on the manifold X with operator-valued coefficients in t-plane as a fiber. One has

$$\widehat{a} = \widehat{a}_2\left(\frac{\partial}{\partial x}\right)^2 + \widehat{a}_1\frac{\partial}{\partial x} + \widehat{a}_0,$$

where

$$\widehat{a}_2 = A_2^0(r,\omega,x)$$

is a function (a differential operator of zero order),

$$\widehat{a}_1 = \frac{1}{r}\left[A_1^0(r,\omega,x)\left(r\frac{\partial}{\partial r}\right) + A_1^1(r,\omega,x,D_\omega)\right]$$

is a differential operator of order 1, and

$$\widehat{a}_0 = \frac{1}{r^2}\left[A_0^0(r,\omega,x)\left(r\frac{\partial}{\partial r}\right)^2 + A_0^1(r,\omega,x,D_\omega)\left(r\frac{\partial}{\partial r}\right) + A_0^2(r,\omega,x,D_\omega)\right]$$

is a differential operator of order 2. All operators $A_l^k(r,\omega,x,D_\omega)$ are supposed to be differential operators of order k with smooth coefficients and, therefore, admit the expansion in powers of r:

$$A_l^k(r,\omega,x,D_\omega) = A_l^k(0,\omega,x,D_\omega) + rB_l^k(r,\omega,x,D_\omega).$$

First of all, let us examine conormal solutions to the homogeneous equation

$$a\left(t,x,-i\frac{\partial}{\partial t},-i\frac{\partial}{\partial x}\right)u(t,x) = 0,$$

that is, solutions of the form

$$u(t,x) = r^{S(x)}u_1(\omega,x) + u_0(t,x). \tag{40}$$

The direct computations show that the coefficient $u_1(\omega,x)$ of the latter expansion satisfies the equation

$$\left[A_0^2(0,\omega,x,D_\omega) + A_0^1(0,\omega,x,D_\omega)S(x) + A_0^0(0,\omega,x)(S(x))^2\right]u_1(\omega,x) = 0, \tag{41}$$

which is (for x fixed) none more than a spectral family with parameter $S = S(x)$. This fact shows that the spectrum of this family can depend on x and this is the motivation of consideration of conormal asymptotics with power S dependent on x.

Remark 1 Here, we must attract the reader's attention to one important point of the theory in question. The fact is that, in essence, we had obtained a *family (in x) of spectral families of differential operators*

$$\mathcal{A}(z) = A_0^2(0,\omega,x,D_\omega) + zA_0^1(0,\omega,x,D_\omega) + z^2A_0^0(0,\omega,x)$$

with the spectral parameter z. Since the operator $\widehat{a}$ is elliptic, this family is meromorphically invertible. In general, the spectrum of this family depends on x:

$$\operatorname{Spec}\mathcal{A}(z) = \{z = S(x)\},$$

where $S(x)$ is a *multivalued* function in x. The set of x such that different branches $\{S_1(x), S_2(x), \ldots\}$ of this function coincide with one another is none more than a set of *focal points* for solutions to the corresponding homogeneous equation. Since in this paper we do not consider branching asymptotics, we require that *focal points are absent.* This means, in particular, that:

1. The branches $S_k(x)$ of the multivalued function $S(x)$ are regular functions in x on the whole manifold X.

2. The dimension of the kernel of the family $\mathcal{A}(z)$ is constant along X (we remark that, due to the fact that $\mathcal{A}(z)$ is invertible for some z one has

$$\dim\mathrm{Ker}\mathcal{A}(z) = \dim\mathrm{Coker}\mathcal{A}(z).$$

So, we arrive at the following problem for the operator $\widehat{a}$ in spaces with asymptotics:

$$\begin{cases} \widehat{a}u\,(t,x) = f\,(t,x) + r^{S(x)-2}c\,(x)\,\varphi^*\,(\omega,x)\,, \\ i^*u_0\,(t,x) + P_\varphi u_1\,(\omega,x) = g\,(x)\,, \end{cases}$$

where $u\,(t,x)$ is supposed to be of the form (40), $f\,(t,x)$ has similar asymptotic expansion

$$f\,(t,x) = r^{S(x)-2}f_1\,(\omega,x) + f_0\,(t,x)\,,$$

with $S\,(x)$ replaced by $S\,(x)-2$, and the functions $\varphi\,(\omega,x)$ and $\varphi^*\,(\omega,x)$ determine kernel and cokernel of equation (41), respectively[12].

The operator equation corresponding to such a problem is

$$\begin{pmatrix} \widehat{a} & \mathcal{P}^* & 0 \\ 0 & \widehat{a}_\omega & -P^*_{\varphi^*} \\ i^* & P_\varphi & 0 \end{pmatrix} \begin{pmatrix} u_0\,(t,x) \\ u_1\,(\omega,x) \\ c\,(x) \end{pmatrix} = \begin{pmatrix} f_0\,(t,x) \\ f_1\,(\omega,x) \\ g\,(x) \end{pmatrix},$$

where $\widehat{a}_\omega$ is the operator

$$\widehat{a}_\omega = A_0^2\,(0,\omega,x,D_\omega) + A_0^1\,(0,\omega,x,D_\omega)\,S\,(x) + A_0^0\,(0,\omega,x)\,(S\,(x))^2\,,$$

involved into equation (41), and $\mathcal{P}^*$ is a sum of compositions of adjoint projection operators $\mathcal{P}^*_\psi$ for ψ equal to one of the functions

$$r^{S(x)}\ln^j r,\ j=0,1,2 \text{ and } r^{S(x)-1}\ln^j r,\ j=0,1$$

with differential operators A_0^2, A_0^1, A_0^0 in y of order not more than two with coefficients depending on (ω,x).

1.4 Summary of results. Program of investigation in the general case.

In this subsection we shortly resume the results of the considerations of the examples above and try to list main questions to be investigated below in the framework of the general theory as well as the main objects to be introduced and investigated.

[12]To be short, we suppose that the kernel and the cokernel of the operator involved into (41) are both one-dimensional.

1. Geometrical situation. By (M, X) we denote, as above, a smooth manifold M and its smooth submanifold of codimension ν. Let (x, t) be coordinates on M near X such that the equation of X is $t = 0$. We denote by U_X a tube neighborhood of X in M and suppose that some bundle structure over X is chosen and fixed on U_X. For simplicity, we suppose that

$$U_X \simeq \left\{X \times [0,1] \times S^{\nu-1}\right\} / \left\{X \times \{0\} \times S^{\nu-1}\right\}$$

and denote by (r, ω) the standard polar coordinates on the ball

$$\left\{[0,1] \times S^{\nu-1}\right\} / \left\{\{0\} \times S^{\nu-1}\right\}$$

in the t-space. Clearly, the above assumption is fulfilled iff the conormal bundle of X in M is trivial.

2. Function classes. Here we shall consider function spaces with one-term asymptotics. The generalization of the theory to the general case is quite a simple task.

The following function spaces will be used:

1. For *solutions* of our future equation we use spaces with asymptotics. For one-term asymptotic expansions such a space is described near X as follows[13]:

   $$u \in H^s_{T_1}(M, X) \;\Leftrightarrow\; u = r^{S(x)} u_1(\omega, x) + u_0(t, x)$$

   (T_1 is an asymptotic type defined by $T_1 = (S(x), 0)$; for the definition see the beginning of Section 1) with

   $$u_1(\omega, x) \in H^s\left(X \times S^{\nu-1}\right), \; u_0(t, x) \in H^s(M),$$

   where $S(x) + \nu/2 \le s < S(x) + \nu/2 + 1$. This means that

   $$r^{S(x)} u_1(\omega, x) \notin H^s(M), \text{ but } r^{S(x)-m+1} u_1(\omega, x) \in H^{s-m}(M),$$

   where m is the order of the operator involved in the considered problem[14].

[13]Essentially, the function $u_1(\omega, x)$ (as well as the function $u_1(\omega, x)$ below) are a function on the subbundle of unit balls of the normal bundle $N_M(X)$. This subbundle is naturally interpreted as the set of directions of approach to X.

[14]More exactly, one should use the functions $r^{S(x)}\chi(r)$, where $\chi(r)$ is a cut-off function, that is, $\chi(r) = 1$ near the origin and $\chi(r) = 0$ for $r > 1/2$. For simplicity, we omit this function in the sequel.

2. For *right-hand parts* of our equation we use the similar spaces, e. g.:

$$f \in H^{s-m}_{T_2}(M,X) \;\Leftrightarrow\; f = r^{S(x)-m} f_1(\omega,x) + f_0(t,x),$$

$T_2 = (S(x) - m, 0)$, with

$$f_1(\omega,x) \in H^{s-m}\left(X \times S^{\nu-1}\right), \; f_0(t,x) \in H^{s-m}(M).$$

3. For right-hand parts of "boundary conditions" and for cokernel functions $c(x)$ we use the Sobolev spaces $H^{\sigma}(X)$ with corresponding values of σ.

3. Operators. To formulate the problem in question and to introduce the corresponding operator algebra, we need the following operators:

1. "Projectors"

$$\mathcal{P}_{\psi} : H^{\sigma}(M) \to H^{\sigma}\left(X \times S^{\nu-1}\right),$$
$$\mathcal{P}_{\psi}[u(r,\omega,x)] = \int_0^{\infty} u(r,\omega,x)\,\psi(r)\,r^2\,dr, \tag{42}$$

defined for any function $\psi(r)$ of the type $\psi(r) = r^{S_1(x)} \ln^j r$ with some smooth $S_1(x)$, j being a nonnegative integer for $\sigma > -\min_X S_1(x) - \nu/2$.

2. The corresponding "coprojectors", that is, conjugate operators for $\mathcal{P}_{\psi}$:

$$\mathcal{P}^*_{\psi} : H^{-\sigma}\left(X \times S^{\nu-1}\right) \to H^{-\sigma}(M),$$
$$\mathcal{P}^*_{\psi}[u(\omega,x)] = \psi(r)\,u(\omega,x). \tag{43}$$

3. "Projectors"

$$P_{\varphi} : H^{\sigma}\left(X \times S^{\nu-1}\right) \to H^{\sigma}(X),$$
$$P_{\varphi}[u(\omega,x)] = \int_{S^{\nu-1}} u(\omega,x)\,\varphi(\omega,x)\,ds_{\omega}, \tag{44}$$

defined for any smooth function $\varphi(\omega,x) \in C^{\infty}(X \times S^{\nu-1})$ for any values of s, and their conjugates

$$P^*_{\varphi} : H^{-\sigma}(X) \to H^{-\sigma}\left(X \times S^{\nu-1}\right),$$
$$P^*_{\varphi}[v(x)] = v(x)\,\varphi(\omega,x). \tag{45}$$

4. Usual boundary and coboundary operators (see [2], [21])

$$i^* : H^s(M) \to H^{s-\nu/2}(M),$$
$$i_* : H^{-s+\nu/2}(M) \to H^{-s}(M),$$

defined (and continuous) for any $s > \nu/2$.

4. The problem under consideration. We investigate the following Sobolev problem:

$$\begin{cases} \widehat{a}u = f + r^{S(x)-m} P^*_{\varphi^*} c(x) \text{ outside } X, \\ i^* \widehat{B} u_0 + P_\varphi u_1 = g, \end{cases} \tag{46}$$

with respect to the unknowns $u(x,t)$ and $c(x)$, where

$$u = r^{S(x)} u_1(\omega, x) + u_0(t, x).$$

Here

$$\widehat{a} = a\left(t, x, -i\frac{\partial}{\partial t} - i\frac{\partial}{\partial x}\right)$$

is an elliptic differential operator on M of order m, $\widehat{B}$ is some differential operator, and φ, φ^* are some smooth functions. The operator $\widehat{a}$ is considered as an operator in spaces

$$\widehat{a} : H^s_{T_1}(M, X) \to H^{s-m}_{T_2}(M, X),$$

and T_1, T_2 are the above asymptotic types.

5. Operator algebra. The solution to the problem considered will be carried out in the framework of an appropriate operator algebra. The elements of this algebra corresponding to problem (46) is

$$\begin{pmatrix} \widehat{a} & \mathcal{P}^* & 0 \\ 0 & \widehat{a}_\omega & P^*_{\varphi^*} \\ i^*\widehat{B} & P_\varphi & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ c \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ g \end{pmatrix} \tag{47}$$

(the definition of operator $\mathcal{P}^*$ and the operator family $\widehat{a}_\omega$ is quite similar to that considered in the end of the previous section[15] on the example of differential operator of second order in $\mathbf{R}^4$). In particular, the operator $\mathcal{P}^*$ is a finite sum of "coprojectors" $\mathcal{P}^*_\psi$ with different functions $\psi(r)$). The form of the obtained operator gives rise to the guess that one should investigate more general operators of the form

$$\begin{pmatrix} \widehat{a} & \mathcal{P}^* & \widehat{C} i_* \\ \mathcal{P} & \widehat{a}_\omega & P^*_{\varphi^*} \\ i^*\widehat{B} & P_\varphi & \widehat{a}_x \end{pmatrix}. \tag{48}$$

[15] The explicit form of these operators will be presented while constructing the general theory.

6. Problems to be solved. Here is the list of problems to be solved for the investigation elliptic differential equations in spaces with asymptotics:

1. One should describe the *operator algebra* which is closed with respect to conjugation and contains the resolving operators for Sobolev problems of the type (46). In doing so, one should take into account that the operator in this algebra must be of more general structure than (48). Actually, in the product of operators of the form (48) there arise additional terms, such as terms of the form $i_*\widehat{a}i^*$ or $\mathcal{P}^*\widehat{a}\mathcal{P}$ (where $\widehat{a}$ is some pseudodifferential operator on $X \times S^{\nu-1}$) in the upper left corner of the matrix, terms of the form $P^*_{\varphi^*}\widehat{a}P_\varphi$ in the middle of the matrix (here $\widehat{a}$ is some pseudodifferential operator on X), and so on. Clearly, such an operators must be described in rather general terms. One of the possible descriptions can be performed similar to the papers [22], [21] by the authors since the operators $\mathcal{P}_\psi$ and P_φ introduced here are strongly related with operators π^* and π_* introduced in the above cited paper. Actually, if we denote by π the projection
$$\pi : X \times S^{\nu-1} \to X,$$
then one has
$$P_\varphi = \varphi \circ \pi^*.$$
Here by φ we have denoted the operator of multiplication by the function $\varphi(\omega, x)$. The connection between the operator $\mathcal{P}$ and the corresponding projection is a little bit more complicated since the corresponding functions $\psi(r)$ may have a singularity at the origin. However, the above connection allows one to describe the elements of an operator algebra, for instance, with the help of Fourier integral operators similar to the results of the papers [22], [21]. Clearly, one should investigate the action of these operators on the above introduced functional spaces.

2. One should prove the corresponding *pseudodifferentiality theorems.* For example, the fact that the operators
$$P_\varphi\widehat{a}P^*_{\varphi^*} \text{ and } \mathcal{P}_\psi\widehat{a}\mathcal{P}^*_\psi$$
are pseudodifferential operators on the manifolds X and $X \times S^{\nu-1}$, respectively, for any pseudodifferential operator $\widehat{a}$ must be proved.

3. One should formulate the conditions of *ellipticity* of the operators of the constructed algebra and prove the corresponding finiteness theorems. These conditions are written down in the form of ellipticity of some pseudodifferential operators.

4. Finally, one should derive the formulas for the index of an element of the operator algebra provided that this element is an elliptic one.

The answers to all these questions the reader will find in the next section.

To conclude this section we remark that one can use *weighted Sobolev spaces* for the investigation of the Sobolev problems in spaces with asymptotics as well; the results will be quite similar to that obtained here for the classical Sobolev spaces. We have chosen the considerations in the classical Sobolev spaces since these spaces are the most appropriate ones in consideration of the Sobolev problems (see [1], [2], [21]).

Besides, the same technique can be applied to the investigation of differential equations on *manifolds with singularities* in spaces with asymptotics.

2 Spaces with asymptotics

2.1 Main definitions

In this section, we present the exact definitions of spaces with asymptotics to be used in the sequel for constructing the Sobolev problems theory.

So, let M be a smooth manifold of dimension n and X be its smooth submanifold of codimension ν. As above, we use the coordinates (r, ω, x) in a tubular neighborhood U_X of X. For simplicity, we assume that U_X is diffeomorphic to the direct product $X \times \mathcal{D}^\nu$, where $\mathcal{D}^\nu$ is a ν-dimensional unit disk.

Definition 1 The tuple

$$T = \{ S_k, m_k | \ k = 1, 2, \dots N\}$$

is named an *asymptotic type* for the given index s of the Sobolev space. Here:

1. $\{S_k\}$ is a sequence of complex numbers in the complex plane $\mathbf{C}_s$, depending on the parameter x contained as a whole in the half-plane $\operatorname{Re} S \le s - \nu/2$ and invariant with respect to the shift by $+1$ in the plane $\mathbf{C}_s$. We suppose that $S_k(x) \ne S_j(x)$ for $k \ne j$. These numbers will be called *degrees*.

2. $\{m_k\}$ is a tuple of positive integers called *multiplicities* of the corresponding values of S_k.

The requirement $\operatorname{Re} S \le s - \nu/2$ expresses the fact that all the terms of asymptotic expansion written in the explicit form do not belong to the space $H^s(M)$ used for the remainder.

Definition 2 *The Sobolev space*[16] $H_T^s(M,X)$ *of order* s *with asymptotic type* T is a space of functions $u(t,x)$, subject to the following conditions:

1. $u(t,x) \in H_{\mathrm{loc}}^s(M \setminus X)$ [17], that is, the function $u(t,x)$ belongs to the Sobolev space of order s *everywhere outside* X.

2. The function $u(t,x)$ admits the following representation:

$$u(t,x) = \chi(r) \sum_{k=1}^{N} r^{S_k(x)} \sum_{j=0}^{m_k-1} \frac{\ln^j r}{j!} u_j^k(\omega,x) + u_0(t,x), \tag{49}$$

   where

$$u_j^k(\omega,x) \in H^s(X \times S^{\nu-1}),$$
$$u_0(t,x) \in H^s(M).$$

Here $\chi(r)$ is a cut-off function equal to 1 near the origin which vanishes identically for sufficiently large r. This function must be chosen in such a way that the local coordinates (r,ω,x) are defined on its support $\operatorname{supp}\chi(r)$.

We remark that the space $H_T^s(M,X)$ can be treated as a direct sum[18] of Sobolev spaces $H^s(X \times S^{\nu-1})$ for coefficients $u_j^k(\omega,x)$, $k = 1,\dots,N$, $j = 0,\dots,m_k-1$ and the space $H^s(M)$ for regular component $u_0(t,x)$ of the corresponding element. Due to this treatment, the space $H_T^s(M,X)$ has a natural structure of a Banach space which will be used in the sequel while considering the continuity of operators acting in spaces of the above described type.

2.2 Boundedness theorems

Let us examine the action of differential operators in spaces $H_T^s(M,X)$. For simplicity we shall consider the case when the set of degrees S_k involved into the asymptotic type considered is a lattice originated from $S(x)$ with the step 1, that is,

$$S_k(x) = S(x) + k,\ k = 0,\dots,N-1$$

[16] As it was shown above, it can be necessary to consider Sobolev spaces having different smoothness along X and in transversal variables. We restrict ourselves here by the consideration of "homogeneous" Sobolev spaces since the needed generalization is not a hard task though leads to significant complication of formulas.

[17] The space $H_{\mathrm{loc}}^s(M\backslash X)$, consists of functions $u(t,x)$ such that for any C^∞-function ψ with support in $M\backslash X$ the product ψu belongs to $H^s(M)$.

[18] The space H_T^s can fail to be a direct sum of spaces H^s only in the case when one of numbers S is a nonnegative integer. For example, the function $r\cos\varphi$ in the plane $\mathbf{R}^2$ is infinitely smooth and, hence, the expansion $u = r\cos\varphi + u_0$ is ambiguous. In such exceptional cases we shall use the direct sum avoiding the consideration of a space of the type H_T^s itself.

(we recall that the set $\{S_k\}$ must be invariant under the shift by 1 in the s-plane, so that the considered set of degrees is a minimal one). We shall use the following affirmations on the action of projectors and coprojectors in spaces with asymptotics.

Proposition 1 *Let $H^s_T(M,X)$ be a Sobolev space with the asymptotic type T and α be a real number subject to the inequality $\alpha > s - \nu/2$. Then for any function $\psi(r) = \left|(d/dr)^j \psi(r)\right| \le Cr^\alpha$ for any $j \ge 0$ the operators*

$$\mathcal{P}_\psi \,:\, H^s_T(M,X) \to H^s\left(X \times S^{\nu-1}\right)$$

and

$$\mathcal{P}^*_\psi \,:\, H^{-s}\left(X \times S^{\nu-1}\right) \to H^{-s}_T(M,X)$$

are continuous.

Proposition 2 *Let $\varphi(\omega,x) \in H^s(X \times S)^{\nu-1}$ be an arbitrary smooth function on $X \times S^{\nu-1}$. Then the operators*

$$P_\varphi \,:\, H^s\left(X \times S^{\nu-1}\right) \to H^s(X)$$

and

$$P^*_\varphi \,:\, \to H^{-s}(X)\, H^{-s}\left(X \times S^{\nu-1}\right)$$

are continuous.

The proof of these affirmations goes by the direct estimates of norms of operators given by explicit formulas (42) and (45).

The following affirmation takes place.

Proposition 3 *Let*

$$\widehat{a} = a\left(t, x, -i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}\right)$$

be a differential operator of order m in (t,x) with smooth coefficients. Suppose that $m_{k+1} \ge m_k + 1$ for all $k = 0, \dots, N-1$. Then the operator $\widehat{a}$ determines a continuous mapping of Banach spaces

$$\widehat{a} \,:\, H^s_T(M,X) \to H^{s-m}_{T-m}(M,X), \tag{50}$$

where $T - m$ is an asymptotic type determined by degrees $S_k - m$ with multiplicities m_k.

The requirement $m_{k+1} \geq m_k + 1$ is needed due to the fact that the application of the operator $r\partial/\partial x$ enlarges the multiplicity by 1 (see below).

Proof. Since the Banach structure of $H^s_T(M,X)$ is induced by its representation in the form of the direct sum

$$H^s_T(M,X) = \bigoplus_{k=1}^{N} \bigoplus_{j=0}^{m_k-1} H^s(X \times S^{\nu-1}) \oplus H^s(M), \tag{51}$$

determined with the help of decomposition (49), and the same affirmation takes place for the space $H^{s-m}_{T-m}(M,X)$, for the proof of the theorem one has to:

1. Write down the operator (50) as the matrix in accordance to representation (51) of $H^s_T(M,X)$ and similar representation of $H^{s-m}_{T-m}(M,X)$.
2. Verify that all elements of the obtained matrix are continuous operators in the corresponding Sobolev spaces.

Let us proceed with the first step of this process.

Let

$$a\left(t,x,-i\frac{\partial}{\partial t},-i\frac{\partial}{\partial x}\right) = r^{-m} a_1\left(r,\omega,x,r\frac{\partial}{\partial r},D_\omega,-ir\frac{\partial}{\partial x}\right)$$

be the expression of the operator $\widehat{a}$ in (r,ω,x). One has

$$\widehat{a} = r^{-m} \sum_{j=0}^{m} \left(\sum_{|\alpha|\leq m-j} a_{\alpha j}\left(r,\omega,x,D_\omega\right) \left(-ir\frac{\partial}{\partial x}\right)^\alpha \right) \left(r\frac{\partial}{\partial r}\right)^j,$$

where $a_{\alpha j}\left(r,\omega,x,D_\omega\right)$ are differential operators of order $m-j-|\alpha|$ on the sphere $S^{\nu-1}$. Let us apply the operator $\widehat{a}$ to the function $u\left(t,x\right) \in H^s_T\left(M,X\right)$, written down in the form (49):

$$\widehat{a}u = r^{-m} \sum_{j=0}^{m} \left(\sum_{|\alpha|\leq m-j} a_{\alpha j}\left(r,\omega,x,D_\omega\right) \left(-ir\frac{\partial}{\partial x}\right)^\alpha \right) \left(r\frac{\partial}{\partial r}\right)^j$$

$$\times \chi\left(r\right) \sum_{k=1}^{N} r^{S_k(x)} \sum_{j=0}^{m_k-1} \frac{\ln^j r}{j!} u^k_j\left(\omega,x\right) + \widehat{a}u_0\left(t,x\right).$$

Let us compute the first term on the right in the latter formula. To do this, we remark that

$$r\frac{\partial}{\partial r}\left(r^{S_k(x)}\frac{\ln^j r}{j!}\right) = S_k\left(x\right) r^{S_k(x)}\frac{\ln^j r}{j!} + r^{S_k(x)}\frac{\ln^{j-1} r}{(j-1)!}$$

for $j \geq 1$,

$$r\partial/\partial r\left(r^{S_k(x)}\right) = S_k(x)\, r^{S_k(x)},$$

as well as

$$-ir\frac{\partial}{\partial x^l}\left(r^{S_k(x)}\frac{\ln^j r}{j!}\right) = -i\frac{\partial S_k(x)}{\partial x^l}r^{S_k(x)+1}\frac{\ln^{j+1} r}{j!}.$$

Using the last two formulas and expanding the coefficients of the operator $\widehat{a}$ in powers of r, one can show that the function $\widehat{a}u$ is representable as a finite linear combination of functions (we recall that $S_k(x) = S(x) + k$)

$$\varphi_{kjl}(r) = r^{S(x)+k-m}\chi^{(l)}(r)\frac{\ln^j r}{j!},\ k = 0,1,\ldots,\ j = 0,\ldots,m_k-1,\ l = 0,\ldots,m, \quad (52)$$

of the form

$$\widehat{a}u = \sum_{k,j,l}\left(\sum_{k',l'} a_{kjlk'l'}(r,\omega,x,D_\omega,D_x)\, u_{l'}^{k'}(\omega,x)\right)\varphi_{kjl}(r), \quad (53)$$

where $D_x = -i\partial/\partial x$ (the properties of differential operators $a_{kjlk'l'}(r,\omega,x,D_\omega,D_x)$ will be refined below).

The sum on the right in (53) can be split into the following three subsums:

a) The subsum involving terms with $l = 0$, such that $\operatorname{Re} S(x) + k \leq s - \nu/2$.
b) The subsum involving terms with $l = 0$, such that $\operatorname{Re} S(x) + k > s - \nu/2$.
c) The subsum involving terms with $l > 0$.

The terms contained in the second and third sums, as well as the corresponding functions $\varphi_{kjl}(r)$, we shall call *inessential.*

It is easy to see that these terms are elements from the space $H^{s-m}(M)$, such that the norms of these elements in the space $H^{s-m}(M)$ can be evaluated via $H^s(M)$-norms of the corresponding coefficients $u_{l'}^{k'}(\omega,x)$, since orders of the operators

$$a_{kjlk'l'}(r,\omega,x,D_\omega,D_x),$$

clearly do not exceed m.

Later on, the following is valid for the terms from the first subsum:

- the coefficients of the operators $a_{kj0k'l'}(r,\omega,x,D_\omega,D_x)$, involved in this subsum do not depend on r (since the expansion of coefficients of the initial operator can be done up to a sufficiently high order so that the remainders of these expansions will be included into one of the last two subsums);

- the matrix containing the operators $a_{kj0k'l'}(r,\omega,x,D_\omega,D_x)$, is a triangle one with respect to a lexicographic ordering of pairs (k,j) such that $(k,j)<(k',j')$ for $k>k'$ or $k=k'$, $j<j'$ (it is clear that for $(k,j)<(k',j')$ the function $\varphi_{kj0}(r)$, determined by (52) decreases faster than the function $\varphi_{k'j'0}(r)$);

- the operators involved to the diagonal blocks of the above mentioned matrix (that is, corresponding to the indices $k=k'$), do not contain differentiation in x;

- each diagonal block with the number k is, in turn, a triangle matrix with one and the same operator

$$\widehat{a}^k_\omega = a\,(0,\omega,x,S\,(x)+k,D_\omega,0) \tag{54}$$

  on the diagonal.

So, we have derived the matrix representation of the operator $\widehat{a}$ in terms of expansions (51); to be short, we shall use the block form for matrices, including coefficients corresponding one and the same value of k in one and the same block. Denoting by U^k the vector

$$U^k = \left(u^k_0,\ldots u^k_{m_k-1}\right)^t,$$

we have

$$\widehat{a}\begin{pmatrix} u_0 \\ U^0 \\ U^1 \\ \vdots \\ U^{N-1}\end{pmatrix} = \begin{pmatrix} \widehat{a} & \mathcal{P}^*_0 & \mathcal{P}^*_1 & \ldots & \mathcal{P}^*_{N-1} \\ 0 & \mathcal{A}^0_\omega & \star & \ldots & \star \\ 0 & 0 & \mathcal{A}^1_\omega & \ldots & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mathcal{A}^{N-1}_\omega \end{pmatrix}\begin{pmatrix} u_0 \\ U^0 \\ U^1 \\ \vdots \\ U^{N-1}\end{pmatrix}. \tag{55}$$

In the latter formula:

- $\mathcal{P}^*_j$ are sums of compositions of coprojectors $\mathcal{P}^*_{\varphi(r)}$ with $\varphi(r)=\varphi_{kjl}(r)$ of the form (45) and differential operators of order not more than m over all inessential functions $\varphi_{kjl}(r)$;

- $\star$ denote matrices of differential operators of order not more than m on the manifold $X\times S^{\nu-1}$;

- by $\mathcal{A}^k_\omega$ we denote triangle matrices of differential operators of order not more than m on $S^{\nu-1}$ with coefficients smooth in x:

$$\mathcal{A}^k_\omega = \begin{pmatrix} \widehat{a}^k_\omega & \star & \cdots & \star \\ 0 & \widehat{a}^k_\omega & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \widehat{a}^k_\omega \end{pmatrix},$$

with operator (54) on the diagonal.

Let us check the continuity of elements involved into matrix (55) in the corresponding Sobolev spaces. It is clear that all the operators $\widehat{a}^k_\omega$, as well as all the operators marked with stars are continuous from $H^s(X \times S^{\nu-1})$ to $H^{s-m}(X \times S^{\nu-1})$, since all these operators are differential operators of order not more than m. It is clear also that the operator

$$\widehat{a} \; : \; H^s(M) \;\to\; H^{s-m}(M)$$

is continuous. The only thing rest is to check that all the operators $\mathcal{P}^*_j$ are continuous in spaces

$$\mathcal{P}^*_j : \; H^s(X \times S^{\nu-1}) \to H^{s-m}(M).$$

This affirmation is directly follows from the definition of the set of inessential functions φ_{kjl}.

Remark 2 If an asymptotic type T contains several lattices originated from $S(x)$, noncomparable with one another modulo integers then the operator $\widehat{a}$ can be represented in the form

$$\widehat{a} = \begin{pmatrix} \widehat{a} & \mathcal{P}^*_1 & \mathcal{P}^*_2 & \ldots & \mathcal{P}^*_L \\ 0 & A_1 & 0 & \ldots & 0 \\ 0 & 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A_L \end{pmatrix}, \qquad (56)$$

where L is the number of lattices, and A_j are blocks of the form

$$\begin{pmatrix} \widehat{\mathcal{A}}^0_\omega & \star & \cdots & \star \\ 0 & \widehat{\mathcal{A}}^1_\omega & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \widehat{\mathcal{A}}^{N-1}_\omega \end{pmatrix},$$

corresponding to these lattices.

Remark 3 While proving Proposition 3 we have derived *the matrix form* of the operator $\widehat{a}$ given by formulas (55) and (56).

3 An algebra of matrix operators

3.1 Sobolev problems and corresponding operators

In this section, we investigate the correspondence between Sobolev problems in their general statements and the corresponding matrix operators. As above, for simplicity, we restrict ourselves by the case when the asymptotic type considered contains only one lattice of the form

$$S_k(x) = S(x) + k,\ k = 0, 1, \ldots, N-1.$$

The changes to be done for the consideration of the general case are quite evident.

Let $H^s_T(M, X)$ be a Sobolev space with the asymptotic type T and let $\widehat{a}$ be an operator of the above described type. Then, due to Proposition 1, this operator determines a continuous mapping

$$\widehat{a} \ :\ H^s_T(M, X) \ \to\ H^{s-m}_{T-m}(M, X).$$

Consider the following problem:

$$\left\{\begin{array}{l} \widehat{a}u \equiv f + \sum\limits_{k=0}^{N-1} r^{S(x)-m+k} \sum\limits_{j=0}^{m_k-1} \frac{\ln^j r}{j!} \widehat{C}_{kj} P^*_{\varphi^*_{kj}} \left[c^k_j(x)\right], \\ \sum\limits_{k,j} P_{\varphi_{kj}} \left[\widehat{B}^l_{kj} u^k_j(\omega, x)\right] + i^* \left[\widehat{B}^l_0 u_0(t, x)\right] = g^l,\ l = 0, \ldots, L, \end{array}\right. \tag{57}$$

where the first equation is valid everywhere on M except for the submanifold X. Here $u^k_j(\omega, x)$ and $u_0(t, x)$ are functions involved into expansion (49), $P^*_{\varphi^*_{kj}}$ and $P_{\varphi_{kj}}$ are operators (44) and (45) corresponding to some smooth functions $\varphi^*_{kj}(\omega)$ and $\varphi_{kj}(\omega)$, $\widehat{B}_{kj}$, $\widehat{B}^l_0$, and $\widehat{C}_{kj}$ are pseudodifferential operators on manifolds $X \times S^{\nu-1}$ and M, respectively, and L is some integer. It is supposed that the functions u and f belong to $H^s_T(M, X)$ and $H^{s-m}_{T-m}(M, X)$, correspondingly.

Let us write down the operator equation corresponding to problem (57):

$$\begin{pmatrix} \widehat{a} & \mathcal{P}^* & 0 \\ 0 & \widehat{\mathcal{A}}_\omega & P^*_{\varphi^*} \\ i^*\widehat{B} & P_\varphi & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ U \\ c \end{pmatrix} = \begin{pmatrix} f_0 \\ F \\ g \end{pmatrix}.$$

The latter equation uses the following notation.

1. By U, we denote a vector with components $\left\{u^k_j(\omega, x)\right\}$ lexicographically ordered. Similar, c is a vector with components $\left\{c^k_j(x)\right\}$.

2. By F, we denote a vector containing the asymptotics components of the right-hand part f of problem (57).

3. By $\mathcal{P}^*$, we denote a string containing the operators $\mathcal{P}_0^*, \ldots, \mathcal{P}_{N-1}^*$, involved into the description (55) of the operator $\widehat{a}$.

4. $\widehat{\mathcal{A}}_\omega$ is a triangle matrix with differential operators of order not more than m as its elements. The diagonal elements of this matrix are formed from the operators $\widehat{a}_\omega^k$, defined by (54).

5. $P_{\varphi^*}^*$ is a diagonal matrix with operators $P_{\varphi_{kj}^*}^*$ involved into the right-hand part of (57) as its elements. Similar, the matrix P_φ is built from the operators $P_{\varphi_{kj}}\widehat{B}_{kj}^l$, involved into the left-hand part of the boundary condition of problem (57).

6. Finally, $\widehat{B}$ is a matrix with operators $\widehat{B}_0^l$ involved into the left-hand part of the boundary conditions of (57) as its elements.

To simplify the notation, we shall carry out the theory for spaces with one-term asymptotics. The changes needed for the consideration of the general case are clear enough though lead to significant complication of formulas.

3.2 Geometric situation and the corresponding mappings of functional spaces

So, let us consider the space $H_T^s(M, X)$ with the asymptotic type T determined by a single point[19] $s = S(x)$ for any $x \in X$ with multiplicity 0 and let $H_{T-m}^{s-m}(M, X)$ be a Sobolev space with asymptotics corresponding to the asymptotic type $T - m$, determined by a single point $s = S(x) - m$ with multiplicity 0. Let $\widehat{a}$ be, as above, a differential operator on M with infinitely smooth coefficients.

Consider the corresponding matrix operator

$$\begin{pmatrix} \widehat{a} & \mathcal{P}^* & 0 \\ 0 & \widehat{\mathcal{A}}_\omega & P_{\varphi^*}^* \\ i^*\widehat{B} & P_\varphi & 0 \end{pmatrix}, \tag{58}$$

where, in the case considered, all operators involved in the latter matrix are scalar ones. Our aim is to widen the set of operators of the form (58) up to an operator algebra with involution. To solve this problem, let us consider the geometrical

[19] In the case of one-term asymptotic expansions the above mentioned lattices reduce up to one point each.

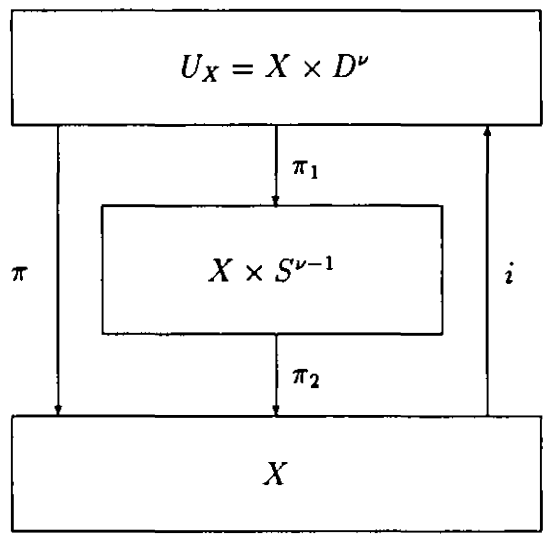

Diagram 1.

situation in more detail. Since all differences between the considered case and the case of elliptic operators on smooth manifolds without boundary are concentrated in a neighborhood of X, we shall carry out all the considerations in a tubular neighborhood U_X of X.

As it was alredy mentioned, we suppose that $U_X = X \times D^\nu$, where D^ν is a unit disk in the ν-dimensional Cartesian space. The geometrical mappings connected with U_X are drawn on Diagram 1.

The operators involved into this diagram determine the operators π^*, π_1^*, π_2^*, and i^*

$$\begin{aligned}
\pi^* &: H^s(X) \to H^s(X \times D^\nu), \ \pi^*[u(x)] = u(x) \otimes 1(t);\\
\pi_1^* &: H^s(X \times S^{\nu-1}) \to H^s(X \times D^\nu), \ \pi_1^*[u(\omega, x)] = u(\omega, x) \otimes 1(r);\\
\pi_2^* &: H^s(X) \to H^s(X \times S^{\nu-1}), \ \pi_2^*[u(x)] = u(x) \otimes 1(\omega);\\
i^* &: H^s(X \times D^\nu) \to H^{s-\nu/2}(X), \ i^*[u(t,x)] = u(0,x),
\end{aligned}$$

the latter mapping being defined for $s > \nu/2$.

In turn, these operators determine the adjoint operators in the Sobolev space scale with respect to the following parings:

$$\langle u, v \rangle = \int_X u(x)\, v(x)\, dx$$

in a space of functions on X (we suppose that a nondegenerate positive measure is fixed on X and that the coordinates x are chosen in such a way that the density of this measure equals 1),

$$\langle u, v\rangle = \int\limits_{X\times S^{\nu-1}} u(\omega,x)\, v(\omega,x)\, ds_\omega dx,$$

on the manifold $X \times S^{\nu-1}$ (here by ds_ω we denote the standard volume element on the unit $(\nu-1)$-dimensional sphere), and

$$\langle u, v\rangle = \int\limits_M u(t,x)\, v(t,x)\, dt dx,$$

on the manifold M (more exactly, on the tubular neighborhood $X \times D^{\nu-1}$ of the manifold X; by dt we denote ν-dimensional volume element in $\mathbf{R}^\nu$). These adjoint operators are realized in the Sobolev spaces in the following way:

$$\pi_* : H^{-s}(X \times D^\nu) \to H^{-s}(X), \qquad \pi_*[u(t,x)] = \int\limits_{D^\nu} u(t,x)\, dt;$$

$$\pi_{1*} : H^{-s}(X \times D^\nu) \to H^{-s}(X \times S^{\nu-1}), \quad \pi_{1*}[u(r,\omega,x)] = \int\limits_0^1 u(t,x)\, r^{\nu-1} dr;$$

$$\pi_{2*} : H^{-s}(X \times S^{\nu-1}) \to H^{-s}(X), \qquad \pi_{2*}[u(\omega,x)] = \int\limits_{S^{\nu-1}} u(\omega,x)\, ds_\omega;$$

$$i_* : H^{-s+\nu/2}(X) \to H^{-s}(X \times D^\nu), \qquad i_*[u(x)] = u(t,x) \otimes \delta(t);$$

and the latter mapping is defined for $s > \nu/2$. All these mappings are drawn on Diagram 2.

On this diagram, we do not show the exact values of indices of the Sobolev spaces in question and denote all functional spaces by the letter $\mathcal{F}$.

While constructing an operator algebra one has to take into account that the compositions of matrix operators of the form (58) contains operators of *more general form* than that included into the initial operators of the form (58). Clearly, at any place in the matrix of an operator included in the algebra under construction, only compositions of the operators involved into the latter diagram with pseudodifferential operators acting on the required manifolds can appear. For example, operators in the upper left corner of the matrix must take functions on M into functions on

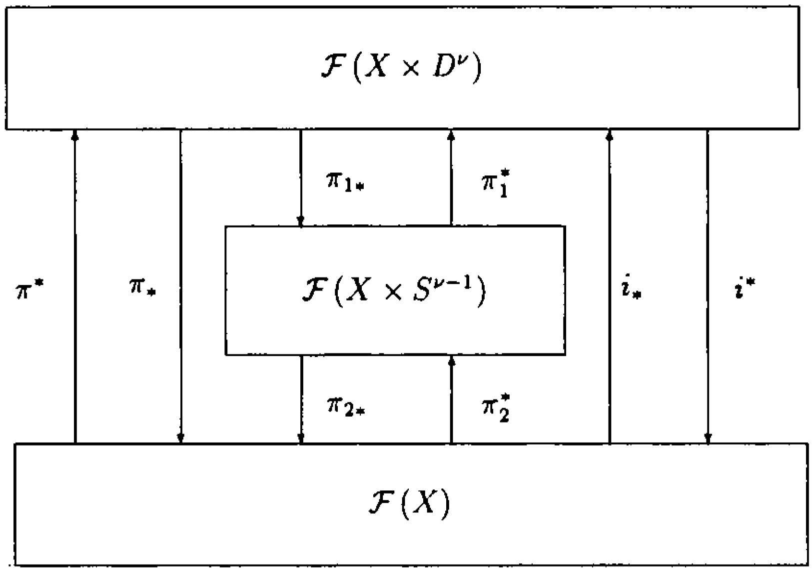

Diagram 2.

the same manifold. Operators standing in the second row of the first column must take functions on M into functions on $X \times S^{\nu-1}$, etc.

It is clear that the number of operators involved into such compositions increases infinitely and the description of elements of the constructed algebra becomes quite transcendental. Fortunately, some of the above mentioned compositions occur to be pseudodifferential operators. The similar situation takes place in the construction of the operator algebra corresponding to the situation of a single embedding (classical Sobolev problems, see [2], [21]), as well as in the consideration of some class of non-local problems corresponding to a pointed bundle (see [22]).

It occurs that the results of the last cited paper can be used in the construction of the operator algebra including matrices (58). In doing so, however, it is necessary to represent the operators involved into Diagram 2 in a somewhat different form. This will be done in the next subsection.

3.3 Graphic schemes and matrix operators

To describe the above mentioned representation we remark that the disk D^ν can be considered as a cylinder over the sphere $S^{\nu-1}$ (with shrinked lower boundary):

$$D^\nu = S^{\nu-1} \times [0,1] / S^{\nu-1} \times \{0\}.$$

So, functions on $U_X = X \times D^\nu$ can be treated as functions on $X \times S^{\nu-1} \times [0,1]$ and vice verse (we remark that the values of the functions considered on a set of zero measure is not essential for the definition of Sobolev spaces at least for nonnegative values of the Sobolev index). Functions from this space are characterized both by differential properties of the function in question on the open cylinder $X \times S^{\nu-1} \times (0,1]$ and by the behavior of this function as $r \to 0$.

Later on, if we represent the tubular neighborhood of X as a cylinder, the embedding of X in M is induced by

$$X \times S^{\nu-1} \times \{0\} \xrightarrow{i_1} X \times S^{\nu-1} \times [0,1].$$

The operator i^* can be thus represented as a composition

$$i^* = \frac{1}{V_{\nu-1}} \pi_{2*} \circ i_{1*},$$

where $V_{\nu-1}$ is a Riemannian volume of the $(\nu-1)$-dimensional unit sphere. Taking into account the relations

$$\pi^* = \pi_1^* \circ \pi_2^*,$$
$$\pi_* = \pi_{2*} \circ \pi_{1*},$$

which are the consequences of the naturality of the operation $*$, one sees that the tuple of operators involved into Diagram 3 (where $\dot{D}^\nu$ is the deleted disk) can be used instead of the tuple π^*, π_1^*, π_2^*, i^*, π_*, π_{1*}, π_{2*} and i_*.

Clearly, one has to control carefully the behavior of symbols of pseudodifferential operators near $r = 0$.

Now operators which can appear while computing compositions of an arbitrary number of matrices of the type (58) in any place of the resulting matrix can be described in the form of convenient graphical form allowing one to obtain easily the general form of the operator matrix invariant with respect to compositions and conjugations. Let us illustrate this on several examples.

The possible compositions of operators from the latter diagram and pseudodifferential operators which can appear in the upper left corner of the result are illustrated by the Diagram 4.

The horizontal lines on this diagram denote the function spaces given on the corresponding manifolds (from above to below: $X \times S^{\nu-1} \times [0,1]$, $X \times S^{\nu-1}$, X), each arrow denotes one of the operators drawn on Diagram 3, and the endpoints of these arrows correspond to pseudodifferential operators. So, the set 1 of arrows on Diagram 4 represents one of operators of the form

$$\widehat{A} i_{1*} \widehat{B} \pi_2^* \widehat{C} \pi_{2*} \widehat{D} i_1^* \widehat{E},$$

$$\widehat{A} i_{1*} \widehat{B} \pi_2^* \widehat{C} \pi_{2*} \widehat{D} \pi_{1*} \widehat{E},$$

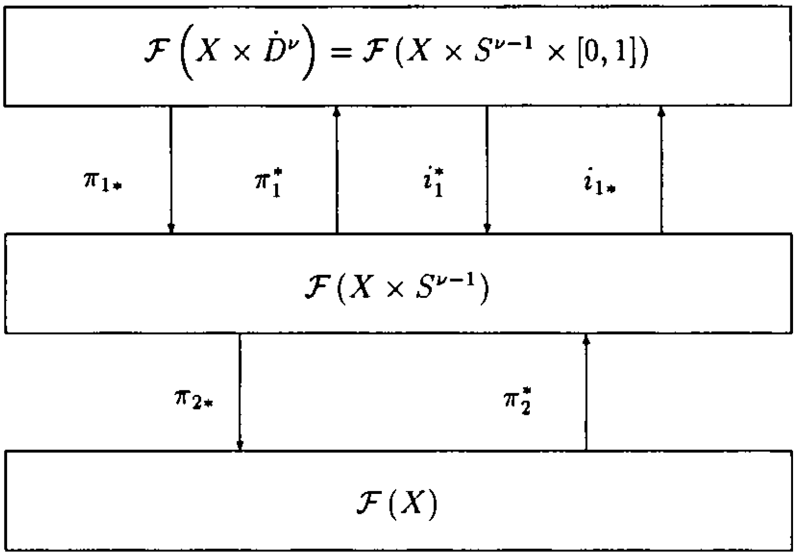

Diagram 3.

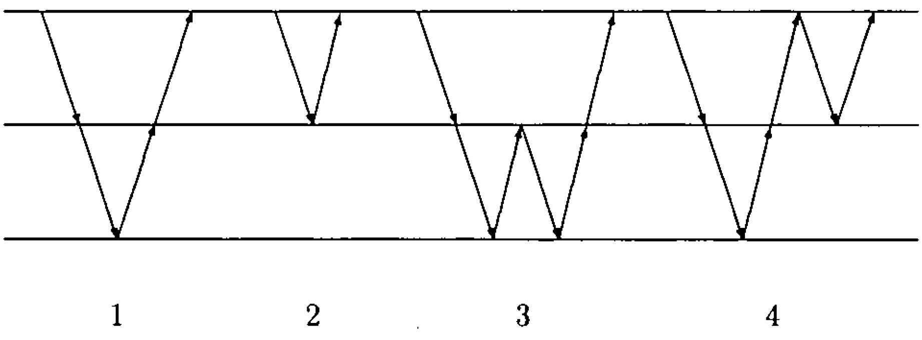

Diagram 4.

$$\widehat{A}\pi_1^*\widehat{B}\pi_2^*\widehat{C}\pi_{2*}\widehat{D}i_1^*\widehat{E},$$

$$\widehat{A}\pi_1^*\widehat{B}\pi_2^*\widehat{C}\pi_{2*}\widehat{D}\pi_{1*}\widehat{E},$$

where $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{D}$ and $\widehat{E}$ are some pseudodifferential operators on the corresponding manifolds. Such sets of arrows will be called *graphic schemes*, the arrows themselves will be called *edges* and their endpoints will be called *vertexes*. So, each edge of the graphic scheme represents one of the operators from Diagram 3, and each vertex represents a pseudodifferential operator on the corresponding manifold. We remark that one and the same graphic scheme includes several types of compositions of operators from Diagram 3 with pseudodifferential operators.

Later on, all pseudodifferential operators used for the constructing operators corresponding to graphic schemes are operators on the corresponding manifolds with smooth symbols. The only exception is that one can use the multiplication by a function $\psi(r)$ subject to the estimates

$$\left|\left(r\frac{d}{dr}\right)^j \psi(r)\right| \le C_j r^\alpha \tag{59}$$

with some fixed α in the composition with one of the operators π_{1*} or π_1^*.

Let us describe the action of operators corresponding to graphic schemes in the Sobolev space scale. Clearly, it suffices to consider the action of elementary operators of the form

$$\pi_{1*}\psi(r),\ \psi(r)\pi_1^*,\ i_1^*,\ i_{1*},\ \pi_{2*},\ \pi_2^*,$$

since the action of pseudodifferential operators in Sobolev spaces is well known. Moreover, the operators i^*, i_*, π^*, π_* in the Sobolev space scale are also examined (see, e. g. [22]). So, the following result must be proved:

Lemma 1 *Let $\psi(r)$ be a function vanishing outside some neighborhood of the origin subject to estimates (59). Then the operators $\pi_{1*}\psi(r)$ and $\psi(r)\pi_1^*$ are continuous in the following function spaces:*

$$\psi(r)\pi_1^* : H^s(X\times S^{\nu-1}) \to H^s(X\times \dot{D}^\nu),$$

$$\pi_{1*}\psi(r) : H^{-s}(X\times \dot{D}^\nu) \to H^{-s}(X\times S^{\nu-1}),$$

for $s < \nu/2+\alpha-2$.

Proof. Since the operators considered are adjoint to each other, it is sufficient to prove the continuity of the first one. Clearly, we have

$$\psi(r)\pi_1^*[u(\omega,x)] = \psi(r)\,u(\omega,x)$$

and, hence,

$$||\psi(r)\pi_1^*[u(\omega,x)]||_s \le ||\psi(r)||_s \cdot ||u(\omega,x)||_s = C\cdot ||u(\omega,x)||_s,$$

since for $s < \nu/2+\alpha-2$ the function $\psi(r)$ belongs to the space $H^s(D^\nu)$. This completes the proof.

Let us introduce now some terminology. Suppose that an operator corresponding to some graphic scheme is given. Then one can construct a chain of the Sobolev spaces corresponding to this operator. For example, such a chain corresponding to the operator

$$\widehat{A}\psi_1(r)\pi_1^*\widehat{B}\pi_{1*}\psi_2(r)\widehat{C} \tag{60}$$

determined by the graphic scheme 2 on Diagram 4 (where $\psi_1(r)$ and $\psi_2(r)$ are functions of the above described type subject to the inequality (59)), is

$$H^s(X\times \dot{D}^\nu) \xrightarrow{\widehat{C}} H^{s-m_C}(X\times \dot{D}^\nu) \xrightarrow{\pi_{1*}\psi_2(r)} H^{s-m_C}(X\times S^{\nu-1}) \xrightarrow{\widehat{B}}$$
$$\xrightarrow{\widehat{B}} H^{s-m_B-m_C}(X\times S^{\nu-1}) \xrightarrow{\psi_1(r)\pi_1^*} H^{s-m_B-m_C}(X\times \dot{D}^\nu) \xrightarrow{\widehat{A}}$$
$$\xrightarrow{\widehat{A}} H^{s-m_A-m_B-m_C}(X\times \dot{D}^\nu).$$

The number $m_A+m_B+m_C$ will be called *the order*[20] of the operator (60). The operator itself will be called *admissible* for the given index s of a Sobolev space if inequalities needed for the corresponding operator to be continuous are fulfilled on each step; for example, operator (60) is admissible for given s if

$$s-m_C < \frac{\nu}{2}+\alpha_2-2,\quad s-m_B-m_C < \frac{\nu}{2}+\alpha_1-2,$$

where α_1 and α_2 are numbers involved into estimate (59) for the functions ψ_1 and ψ_2, respectively. The following affirmations are direct consequences of Lemma 1 and the definitions above.

Proposition 4 *Let $\widehat{A}$ be an operator of order m corresponding some graphic scheme admissible for some index s of a Sobolev space. Then this operator is continuous in spaces*

$$\widehat{A}: H^s(M_i)\to H^{s-m}(M_j),$$

where M_i and M_j are manifolds from the list $(X\times\dot{D}^\nu,\ X\times S^{\nu-1},\ X)$, corresponding to the origin and the endpoint of the considered graphic scheme, respectively.

[20]More exactly, by Sobolev order, so that, for instance, the orders of the operators i_1^* and i_{1*} are both equal to $\nu/2$.

The following affirmation is also quite evident.

Proposition 5 *Let $\widehat{A}$ and $\widehat{B}$ be the two operators of orders m_A and m_B corresponding to the graphic schemes Σ_A and Σ_B such that the endpoint of the scheme Σ_A coincides with the origin of the scheme Σ_B. Suppose that the operators $\widehat{A}$ and $\widehat{B}$ are admissible for s and $s-m_A$, respectively, for some given s. Then the composition $\widehat{B}\circ\widehat{A}$ is a well-defined operator admissible for s. This operator corresponds to the concatenation $\Sigma_A\Sigma_B$ of graphic schemes Σ_A and Σ_B.*

Later on, if $\widehat{A}$ is an operator of order m corresponding to a graphic scheme Σ_A admissible for some s, then $\widehat{A}^*$ is the operator admissible for $-(s-m)$. This operator corresponds to the graphic scheme Σ_A^{-1} obtained from Σ_A by inversion of directions of all its arrows.

Proposition 6 *The composition of the two operators corresponding to the two consequent arrows of any graphic scheme such that first of them goes up and the other goes down by one step, is a pseudodifferential operator.*

Proof of this affirmation goes in one and the same way for all operators subject to conditions of Proposition 6. So, we present the proof for one of these operators, e. g.

$$\pi_{1*}\psi_1(r)\widehat{A}\psi_2(r)\pi_1^*, \tag{61}$$

where $\widehat{A}$ is some pseudodifferential operator on $X\times D^\nu$.

To carry out the proof (cf. [21]), we use the well-known Hörmander criterium [23] of pseudodifferentiality of an operator applying the latter operator to the function of the form $e^{\lambda S(x,\omega)}\varphi(x,\omega)$, where $S(x,\omega)$ and $\varphi(x,\omega)$ are smooth functions on $X\times S^{\nu-1}$. We have

$$\widehat{A}\psi_2(r)\pi_1^*[e^{\lambda S(x,\omega)}\varphi(x,\omega)] = \widehat{A}\{e^{\lambda S(x,\omega)}\varphi(x,\omega)\psi_2(r)\}.$$

Since $\widehat{A}$ is a pseudodifferential operator, the expansion

$$\widehat{A}\psi_2(r)\pi_1^*[e^{\lambda S(x,\omega)}\varphi(x,\omega)] = e^{\lambda S(x,\omega)}\sum_{j=0}^{\infty}\lambda^{m-j}F_j(r,x,\omega)$$

takes place, where $F(r,x,\omega)$ are some functions depending on the derivatives of the functions S and φ up to a certain order. We remark that the functions $F_j(r,\omega,x)$ have not more than power increase in r.

Applying the operator $\pi_{1*}\psi_1(r)$ to the latter formula, we obtain

$$\pi_{1*}\psi_1(r)\widehat{A}\psi_2(r)\pi_1^*[e^{\lambda S(x,\omega)}\varphi(x,\omega)] = e^{\lambda S(x,\omega)}\sum_{j=0}^{\infty}\lambda^{m-j}\int_0^1\psi_1(r)F_j(r,x,\omega)r^{\nu-1}\,dr,$$

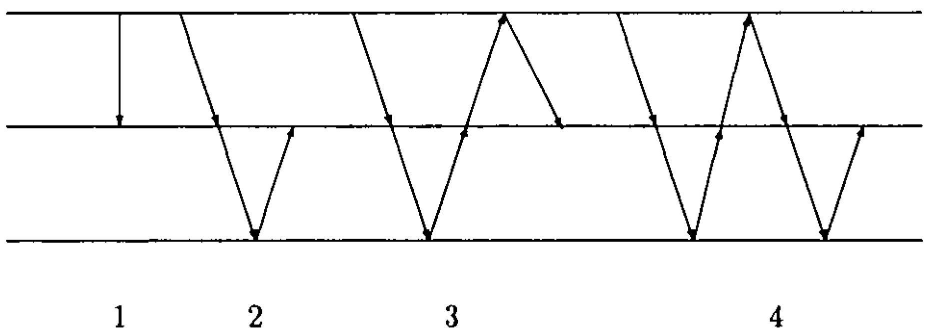

Diagram 5.

where the integration must be treated as an application of a distribution in r to the test function equal to unity on $[0, 1]$. The existence of the latter expansion proves the pseudodifferential character of operator (61).

The affirmation proved motivates the following terminology. The set of the two subsequent edges of a graphic scheme will be called *reducible* if the first of them goes upwards and the second goes downwards. A graphic scheme is called *simple* if it does not contain reducible subschemes. For example, the graphic schemes 1 and 2 on Diagram 4 are simple ones, and the schemes 3 and 4 are not. Moreover, 1 and 2 are the only simple schemes among all schemes with the same endpoints.

Let us present some more graphic schemes corresponding to the intersection of the first column and the second row of the matrix. These schemes are drawn on Diagram 5.

Here schemes 1 and 2 are simple, and all the rest are not. Moreover, 1 and 2 are the only simple schemes among all schemes with the same position in the matrix. Generally, *for each position in the matrix there exist only a finite number of simple schemes.*

Denote by A_{ij} a finite sum of operators corresponding to a graphic scheme with the origin on ith and the endpoint on jth level.

The above considerations lead us to the following statement.

Theorem 1 *The operator*

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

containing finite sums of operators corresponding to simple graphic schemes deter-

mine a continuous mapping

$$A: H^{s_1}(M) \oplus H^{s_2}(X \times S^{\nu-1}) \oplus H^{s_3}(X) \to H^{\sigma_1}(M) \oplus H^{\sigma_2}(X \times S^{\nu-1}) \oplus H^{\sigma_3}(X),$$

if all the operators involved into the jth column are admissible for the index s_j. The set $\mathcal{A}$ of matrix operators of this kind form an algebra with involution.

3.4 Ellipticity and finiteness theorems

In this subsection, we investigate the question of ellipticity for operators of the above introduced algebra. In other words, we are intended to derive the conditions under which the corresponding operators are almost invertible[21]. To make our presentation more transparent, we shall consider matrix operators corresponding to Sobolev problems in spaces with one-term asymptotics

$$\begin{aligned} u(x,t) &= r^{S(x)} u_1(x,\omega) + u_0(x,t), \\ f(x,t) &= r^{S(x)-m} f_1(x,\omega) + f_0(x,t), \end{aligned}$$

where m is an order of the corresponding elliptic differential operator $\widehat{a}$, and the functions u_j, f_j belong to the corresponding Sobolev spaces.

This means that we consider an matrix operator corresponding to the following problem

$$\left\{ \begin{array}{l} \widehat{a}u \equiv f + r^{S(x)-m} \widehat{C} P_\varphi^* c, \\ i^* \widehat{B}_1 u_0 + P_\psi \widehat{B}_2 u_1 = g \end{array} \right. \tag{62}$$

of the type (57). Here:

- the operator $\widehat{a}$ is an elliptic pseudodifferential operator on M of order m:

  $$\widehat{a} = r^{-m} a\left(r, \omega, x, r\frac{\partial}{\partial r}, D_\omega, rD_x\right)$$

  near X;

- the operators $\widehat{C}$, $\widehat{B}_1$, and $\widehat{B}_2$ are some pseudodifferential operators on the manifolds M and $X \times S^{\nu-1}$;

- the operator P_ψ is defined by

  $$(P_\psi u_1)_j(x) = \int\limits_{S^{\nu-1}} u_1(x,\omega)\, \psi_j(x,\omega)\, ds_\omega, \; j = 1, \ldots, N$$

  with some (smooth, for simplicity) functions $\psi_j(x,\omega)$ on $X \times S^{\nu-1}$;

[21]That is, invertible modulo compact operators

- the operator P_φ^* is defined by

$$\left(P_\varphi^* c\right)(x,\omega)=\sum_{j=1}^{N}\varphi_j(x,\omega)\,c_j(x)$$

with some smooth functions $\varphi_j(x,\omega)$, $j=1,\dots,N$;

- c and g are vector-valued functions on the manifold M, $c=(c_1(x),\dots,c_N(x))$, $g=(g_1(x),\dots,g_N(x))$.

The corresponding operator equation has the form

$$\begin{pmatrix} \widehat{a} & \mathcal{P}^* & 0 \\ 0 & \widehat{a}_\omega & \widehat{C}P_\varphi^* \\ i^*\widehat{B}_1 & P_\psi\widehat{B}_2 & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ c \end{pmatrix}=\begin{pmatrix} f_0 \\ f_1 \\ g \end{pmatrix}, \tag{63}$$

where $\widehat{a}_\omega$ is defined similar to the formula (54) above:

$$\widehat{a}_\omega=a\left(0,\omega,x,S(x),D_\omega,0\right). \tag{64}$$

We remark that no connection between the operator $\widehat{a}$ and the projectors P_φ^* and P_ψ is supposed. Our aim is to derive the ellipticity conditions for operator (63) (and, hence, to problem (62)) in terms of the operators involved into (63).

Since, as we have seen on the examples above, the nonresonance case is more or less trivial, we shall consider the resonance case (which has also the physical interest). This means that the operator (64) is degenerate, or, in other words, that the number $S(x)$ is a spectral number for the family

$$\widehat{a}_\omega(z)=a\left(0,\omega,x,z,D_\omega,0\right)$$

for any fixed value of x. Again, we assume that the kernel and the cokernel of operator (64) smoothly depend on the variable x along the manifold X.

Due to the above assumptions, there exists a decompositions

$$H^{s_2}\left(X\times S^{\nu-1}\right)=L_1\oplus\operatorname{Ker}\widehat{a}_\omega, \tag{65}$$

where $\operatorname{Ker}\widehat{a}_\omega$ is isomorphic to the space of sections of some finite-dimensional bundle K over the manifold X, and

$$H^{\sigma_2}\left(X\times S^{\nu-1}\right)=\operatorname{Im}\widehat{a}_\omega\oplus L_2, \tag{66}$$

where L_2 (the cokernel of the operator $\widehat{a}_\omega$) is isomorphic to the space of sections of some other finite-dimensional bundle C over X.

Let us fix the mentioned bundles and isomorphisms. Then the decompositions (65) and (66) become

$$H^{s_2}\left(X\times S^{\nu-1}\right) = L_1\oplus H^{s_2}(X,K), \tag{67}$$
$$H^{\sigma_2}\left(X\times S^{\nu-1}\right) = \operatorname{Im}\widehat{\mathrm{a}}_\omega\oplus \mathrm{H}^{\sigma_2}(\mathrm{X},\mathrm{C}). \tag{68}$$

In accordance to decompositions (67) and (68), the action of the operator $\widehat{a}_\omega$ can be written down in the form

$$\widehat{a}_\omega u_1 = \begin{pmatrix} \widehat{a}^0_\omega & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1^0 \\ u_1^1 \end{pmatrix} = \begin{pmatrix} f_1^0 \\ f_1^1 \end{pmatrix},$$

where

$$\widehat{a}^0_\omega : L_1 \to \operatorname{Im}\widehat{\mathrm{a}}_\omega$$

is an isomorphism. Clearly, both L_1 and $\operatorname{Im}\widehat{\mathrm{a}}_\omega$ can be interpreted as sections of infinite-dimensional bundles over X and $\widehat{a}^0_\omega$ is a *family of isomorphisms* in fibers of these bundles.

So, the ellipticity of the initial operator induces splitting of the isomorphic part of the corresponding family such that the reminder is finite-dimensional.

Now let us try to rewrite equation (63) in terms of the decompositions (67), (68). To do this we denote by

$$\varphi^0 = \left(\varphi_1^0,\ldots,\varphi_k^0\right)$$

the orhtonormal (in the L_2-sense) basis in the kernel of the operator $\widehat{a}_\omega$, corresponding to the isomorphism in question, and by

$$\psi^0 = \left(\psi_1^0,\ldots,\psi_k^0\right)$$

the similar basis in the cokernel of this operator (we remark that the dimension of kernel and cokernel of the operator $\widehat{a}_\omega$ are equal and we denote it by k). Then

- the relation $P_{\varphi^0}P^*_{\varphi^0} = \mathbf{1}_k$ takes place;
- the operator $P^*_{\varphi^0}P_{\varphi^0}$ is a projector to the kernel of the operator $\widehat{a}_\omega$ since

$$\left(P^*_{\varphi^0}P_{\varphi^0}\right)\left(P^*_{\varphi^0}P_{\varphi^0}\right) = P^*_{\varphi^0}\left(P_{\varphi^0}P^*_{\varphi^0}\right)P_{\varphi^0} = P^*_{\varphi^0}\mathbf{1}_k P_{\varphi^0} = P^*_{\varphi^0}P_{\varphi^0};$$

Now the decomposition of the function $u_1(\omega,x)$ corresponding to decomposition (67) is

$$u_1 = u_1^0\oplus u_1^1,$$

where

1. $u_1^0 = \left(1 - P^*_{\varphi^0} P_{\varphi^0}\right) u_1$ is a projection of the function u_1 on the space L_1, the space L_1 is automatically defined here as the image of the projector $\left(1 - P^*_{\varphi^0} P_{\varphi^0}\right)$;

2. $u_1^1 = P_{\varphi^0} \left(P^*_{\varphi^0} P_{\varphi^0}\right) u_1 = P_{\varphi^0} u_1$ is a section of the bundle K identified with $\operatorname{Ker} \widehat{a}_\omega$ with the help of the operator P_{φ^0}.

Similar, the decomposition

$$f_1 = f_1^0 \oplus f_1^1$$

is defined by

1. $f_1^0 = \left(1 - P^*_{\psi^0} P_{\psi^0}\right) f_1$ is a projectionof the function f_1 on the space $\operatorname{Im} \widehat{a}_\omega$;

2. $f_1^1 = P_{\psi^0} f_1$ is a section of the (finite-dimensional) bundle C identified with the cokernel of the operator $\widehat{a}_\omega$ with the help of the projector P_{ψ^0}.

Now, equation (63) can be rewritten as

$$\begin{pmatrix} \widehat{a} & \mathcal{P}_0^* & \mathcal{P}_1^* & 0 \\ 0 & \widehat{a}_\omega^0 & 0 & \left(1 - P^*_{\psi^0} P_{\psi^0}\right) \widehat{C} P^*_\varphi \\ 0 & 0 & 0 & \widehat{C}_1 \\ i^* \widehat{B}_1 & P_{1\psi} \widehat{B}_{21} & \widehat{B}_{22} & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1^0 \\ u_1^1 \\ c \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1^0 \\ f_1^1 \\ g \end{pmatrix}. \qquad (69)$$

Here

- $\mathcal{P}_0^*$ is just a restriction of the operator $\mathcal{P}^*$ to the space L_1;
- $\mathcal{P}_1^* u_1^1 = \mathcal{P}^* P^*_{\varphi^0} u_1^1$ since the function $P^*_{\varphi^0} u_1^1$ is a function in (ω, x) corresponding to the section u_1^1 of the bundle K;
- $P_{1\psi} \widehat{B}_{21}$ is a restriction of the operator $P_\psi \widehat{B}_2$ to the space L_1;
- the operator $\widehat{C}_1$ is defined by

  $$\widehat{C}_1 = P_{\psi^0} \widehat{C} P^*_\varphi;$$

  this operator is a *pseudodifferential* one in sections of finite-dimensional bundles due to Proposition 6;

- the operator $\widehat{B}_{22}$ is a *pseudodifferential* operator in sections of finite-dimensional bundles defined by[22]

$$\widehat{B}_{22} = P_{\psi}\widehat{B}_2 P^*_{\varphi^0}.$$

Let us reduce the obtained system to a pseudodifferential system on X by "exluding unknowns" method omitting compact operators.

First, we derive the function u_0 from the first equation

$$\widehat{a}u_0 + \mathcal{P}_0^* u_1^0 + \mathcal{P}_1^* u_1^1 = f_0$$

of system (69). The result is

$$u_0 = \widehat{a}^{-1}\left(f_0 - \mathcal{P}_0^* u_1^0 - \mathcal{P}_1^* u_1^1\right), \tag{70}$$

where $\widehat{a}^{-1}$ is the almost inverse for the operator $\widehat{a}$.

Second, we derive the unknown u_1^0 from the second equation

$$\widehat{a}^0_\omega u_1^0 + \left(1 - P^*_{\psi^0}P_{\psi^0}\right)\widehat{C}P^*_\varphi c = f_1^0$$

of system (69). We obtain

$$u_1^0 = \left(\widehat{a}^0_\omega\right)^{-1}\left(f_1^0 - \left(1 - P^*_{\psi^0}P_{\psi^0}\right)\widehat{C}P^*_\varphi c\right). \tag{71}$$

Equations (70) and (71) separate the "infinite-dimensional" part of the solution expressing it via its "finite-dimensional" part. Clearly, this is possible only under the condition of ellipticity of the initial operator $\widehat{a}$.

Substituting relations (70) and (71) into the last two equations of system (69), we arrive at the following system of equations for the unknowns c and u_1^1:

$$\begin{pmatrix} 0 & \widehat{C}_1 \\ \Delta_1 & \Delta_2 \end{pmatrix}\begin{pmatrix} u_1^1 \\ c \end{pmatrix} = \begin{pmatrix} f_1^1 \\ \widetilde{g} \end{pmatrix}, \tag{72}$$

where the operators Δ_1 and Δ_2 are (matrix, in general) pseudodifferential (due to Proposition 6) operators given by

$$\begin{aligned} \Delta_1 &= \widehat{B}_{22} - i^*\widehat{B}_1\widehat{a}^{-1}\mathcal{P}_1^*, \\ \Delta_2 &= \left(i^*\widehat{B}_1\widehat{a}^{-1}\mathcal{P}_0^* - P_{1\psi}\widehat{B}_{21}\right)\left(\widehat{a}^0_\omega\right)^{-1}\left(1 - P^*_{\psi^0}P_{\psi^0}\right)\widehat{C}P^*_\varphi, \end{aligned}$$

[22] In some sense, the operators $\widehat{C}_1$ and $\widehat{B}_{22}$ *compare* the boundary and coboundary operators involved into the considered problem with projectors on kernel and cokernel of the corresponding operator family. These operators play an essential role in the investigation of ellipticity of the matrix operator in question.

and the function $\widetilde{g}$ equals to

$$\widetilde{g} = g - i^* \widehat{B}_1 \widehat{a}^{-1} f_0 + \left(i^* \widehat{B}_1 \widehat{a}^{-1} \mathcal{P}_0^* - P_{1\psi} \widehat{B}_{21} \right) \left(\widehat{a}_\omega^0 \right)^{-1} f_1^0.$$

So, the ellipticity condition for matrix operator involved into equation (69) is the requirement of ellipticity of the pseudodifferential operator in (72). Due to the particular form of this (matrix) operator this condition can be written down in the form:

a) The operator $\widehat{a}$ is elliptic.

b) The operator $\widehat{C}_1$ is elliptic.

c) The operator Δ_1 is elliptic.

Definition 3 The operator

$$\mathcal{A} = \begin{pmatrix} \widehat{a} & \mathcal{P}_0^* & \mathcal{P}_1^* & 0 \\ 0 & \widehat{a}_\omega^0 & 0 & \left(1 - P_{\psi^0}^* P_{\psi^0}\right) \widehat{C} P_\varphi^* \\ 0 & 0 & 0 & \widehat{C}_1 \\ i^* \widehat{B}_1 & P_{1\psi} \widehat{B}_{21} & \widehat{B}_{22} & 0 \end{pmatrix} \tag{73}$$

involved into (63) is called to be *elliptic* if the conditions a) — c) above are fulfilled.

The following affirmation is valid:

Theorem 2 *Let operator* (73) *be elliptic. Then it possesses the Fredholm property.*

Proof. The carried out excluding of unknowns method supplies us with the matrix operator (almost) inverse to operator (73). In particular, the finiteness theorem in the corresponding spaces[23] follows from this fact.

The form of the almost inverse for (73) is:

$$\begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} & \widehat{A}_{13} & \widehat{A}_{14} \\ \widehat{A}_{21} & \widehat{A}_{22} & \widehat{A}_{23} & \widehat{A}_{24} \\ \widehat{A}_{31} & \widehat{A}_{32} & \widehat{A}_{33} & \widehat{A}_{34} \\ \widehat{A}_{41} & \widehat{A}_{42} & \widehat{A}_{43} & \widehat{A}_{44} \end{pmatrix}, \tag{74}$$

where

[23]We are not presenting here the exact indices of the corresponding Sobolev spaces: these cumbersome expressions hardly can be applied for the concrete problems where they can easily computed directly.

$$
\begin{aligned}
\widehat{A}_{11} &= \widehat{a}^{-1}\left(1+\mathcal{P}_1^*\Delta_1^{-1}i^*\widehat{B}_1\widehat{a}^{-1}\right),\\
\widehat{A}_{12} &= \widehat{a}^{-1}\mathcal{P}_0^*\left(\widehat{a}_\omega^0\right)^{-1}-\widehat{a}^{-1}\mathcal{P}_1^*\Delta_1^{-1}\\
&\quad\times\left(i^*\widehat{B}_1\widehat{a}^{-1}\mathcal{P}_0^*-P_{1\psi}\widehat{B}_{21}\right)\left(\widehat{a}_\omega^0\right)^{-1},\\
\widehat{A}_{13} &= \widehat{a}^{-1}\mathcal{P}_0^*\left(\widehat{a}_\omega^0\right)^{-1}\left(1-P_{\psi_0}^*P_{\psi_0}\right)\widehat{C}P_\varphi^*\widehat{C}_1^{-1}\\
&\quad+\widehat{a}^{-1}\mathcal{P}_1^*\Delta_1^{-1}\Delta_2\widehat{C}_1^{-1},\\
\widehat{A}_{14} &= -\widehat{a}^{-1}\mathcal{P}_1^*\Delta_1^{-1},\\
\widehat{A}_{22} &= \left(\widehat{a}_\omega^0\right)^{-1},\\
\widehat{A}_{23} &= \left(\widehat{a}_\omega^0\right)^{-1}\left(1-P_{\psi_0}^*P_{\psi_0}\right)\widehat{C}P_\varphi^*\widehat{C}_1^{-1},\\
\widehat{A}_{31} &= \Delta_1^{-1}i^*\widehat{B}_1\widehat{a}^{-1},\\
\widehat{A}_{32} &= \Delta_1^{-1}\left(i^*\widehat{B}_1\widehat{a}^{-1}\mathcal{P}_0^*-P_{1\psi}\widehat{B}_{21}\right)\left(\widehat{a}_\omega^0\right)^{-1},\\
\widehat{A}_{33} &= \Delta_1^{-1}\Delta_2\widehat{C}_1^{-1},\\
\widehat{A}_{34} &= \Delta_1^{-1},\\
\widehat{A}_{43} &= \widehat{C}_1^{-1},
\end{aligned}
$$

and all the rest elements of the matrix vanish. The verification of the facts that (74) is an matrix operator of the type described in Subsection 3.3 and that (74) is almost inverse for (73) is left to the reader.

Similar considerations can be used for deriving the ellipticity conditions for general matrix operators of the type described in the Subsection 3.3. We shall not carry out these considerations here since they are rather complicated in form and we leave them to the reader. We remark only that in this case the ellipticity condition for the operator $\widehat{a}$ will be replaced by the ellipticity condition for an operator of the form

$$
\widehat{a}-\mathcal{P}^*\left(\widehat{a}_\omega^0\right)^{-1}\mathcal{P}
$$

with some $\mathcal{P}^*$ and $\mathcal{P}$ involved into the algebra described in [22].

3.5 Index of a matrix operator

In this subsection, we present the computation of the index of the elliptic operator from the above constructed algebra. Similar to the previous subsection, we restrict ourselves by consideration of operator (73) corresponding to a Sobolev problem in spaces with asymototics in the resonance case.

The computation of the index will be carried out with the help of a homotopy connectiong the initial operator to the diagonal one (the similar procedure was used in [2] for usual Sobolev problems).

So, let us consider a homotopy

$$\mathcal{A}(t) = \begin{pmatrix} \widehat{a} & t\mathcal{P}_0^* & t\mathcal{P}_1^* & 0 \\ 0 & \widehat{a}_\omega^0 & 0 & t\left(1 - P_{\psi^0}^* P_{\psi^0}\right)\widehat{C}P_\varphi^* \\ 0 & 0 & 0 & \widehat{C}_1 \\ ti^*\widehat{B}_1 & tP_{1\psi}\widehat{B}_{21} & \widehat{B}(t) & 0 \end{pmatrix}, \tag{75}$$

where the operator $\widehat{B}(t)$ is given by

$$\widehat{B}(t) = \widehat{B}_{22} - \left(1 - t^2\right) i^* \widehat{B}_1 \widehat{a}^{-1} \mathcal{P}_1^*.$$

Operator (75), clearly, coincides with (73) for $t = 1$, and becomes

$$\mathcal{A}(0) = \begin{pmatrix} \widehat{a} & 0 & 0 & 0 \\ 0 & \widehat{a}_\omega^0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{C}_1 \\ 0 & 0 & \Delta_1 & 0 \end{pmatrix} \tag{76}$$

at $t = 0$. As above, the ellipticity conditions for operator (75) are reduced to the ellipticity of the operator $\widehat{a}$ and the following pseudodifferential operator

$$\begin{pmatrix} 0 & \widehat{C}_1 \\ \Delta_1 & \Delta(t) \end{pmatrix}$$

acting on sections of finite-dimensional bundles over X. Here (this expression is not essential in the sequel)

$$\Delta(t) = t^2 \left(ti^* \widehat{B}_1 \widehat{a}^{-1} \mathcal{P}_0^* - P_{1\psi}\widehat{B}_{21}\right) \left(\widehat{a}_\omega^0\right)^{-1} \widehat{C}_1 P_{1\varphi}^*.$$

So, if the operator (73) is an elliptic one, then $\mathcal{A}(t)$ for each $t \in [0,1]$ is elliptic, as well. Therefore, the index of $\mathcal{A}(1)$ coincides with $\mathcal{A}(0)$, and we arrive at the following statement:

Theorem 3 *The index of matrix operator* (73) *equals to*

$$\operatorname{index} \mathcal{A} = \operatorname{index} \mathcal{A}(1) = \operatorname{index} \mathcal{A}(0) = \operatorname{index} \widehat{a} + \operatorname{index} \widehat{C}_1 + \operatorname{index} \Delta_1.$$

Actually, the latter formula follows from the fact that the operator $\hat{H}_\omega^0$ involved into (76) is an isomorphism.

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