The Petersson Scalar Product

in the Cohomology of $\Gamma_0(p_0)$

by

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1. Introduction.

Let p_0 be a prime, $\Gamma_0(p_0)$, as usual, the congruence subgroup of $\Gamma = PSL_2(\mathbb{Z})$.

$$\Gamma_0(p_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \bmod p_0 \right\}.$$

Let $S_k(\Gamma_0(p_0), \eta)$ be the space of the cusp forms of weight k and nebentypus η. For $f \in S_k(\Gamma_0(p_0), \eta)$, $A \in \Gamma$ the period polynomial is defined by

$$\rho_f(A) = \int_0^{i\infty} (f|_k A)(z)(xz + y)^{k-2} dz.$$

Here the integral has to be taken along the line $z = it$, $t \geq 0$. For any function f, defined on the upper half plane H, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and any integer k we use

$$(f|_k A)(z) = (det(A))^{\frac{k}{2}} f(\frac{az + b}{cz + d})(cz + d)^{-k}.$$

Since f is a cusp form $f|_k A(z)$ is exponentially decreasing for $z \to 0, i\infty$, the above integral is absolutely convergent. The period polynomial $\rho_f(A)$ depends only on the left coset of A in $\Gamma_0(p_0) \setminus \Gamma$. Some behaviours of the period polynomials have been studied in [An] and [Sk] . The aim of this paper is to study the connection between the period polynomial of f and its Petersson scalar product. We will generalize a result in [KZ] p243 or [Ha] p280. The main result of this paper is the following:

Theorem: *Denote* $A_i := \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix}$ *for* $i = 0, 1, ..., p_0 + 1$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Then the Petersson scalar product of two cusp forms* $f, g \in S_{n+2}(\Gamma_0(p_0), \eta)$ *can be represented as the sum of the period polynomials of* f *and* g:

$$(f, g) = -\frac{1}{6(2i)^{n+1}} \sum_{i=0}^{p_0} \Big((T\rho_f(A_{i+1}), \rho_g(A_i))_M - (\rho_f(A_i), T\rho_g(A_{i+1}))_M \Big),$$

where $(\cdot,\cdot)_M$ is a pairing which will be defined in §4. In order to prove the main theorem we first describe the Shapiro isomorphism and the Eichler-Shimura isomorphism explicitly. In §4 we define and calculate the Petersson scalar product of two cusp forms in the cohomology of $\Gamma_0(p_0)$.

2. The Eichler-Shimura Isomorphism.

We consider the following $\Gamma_0(p_0)$-module

$$M_n = \left\{ \sum_{v=0}^{n} a_v x^v y^{n-v} \,\middle|\, a_v \in \mathbb{Q} \right\},$$

where $n > 0$. The group Γ acts on M_n via

$$r x^v y^{n-v} = (ax+cy)^v (bx+dy)^{n-v}, \quad r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let $\eta : (\mathbb{Z}/p_0)^* \to \mathbb{C}^*$ be a Dirichlet character and $\mathbb{Q}[\eta]$ the ring generated by $\mathbb{Q}$ and the values of η. Set $M_{n,\eta} = M_n \otimes \mathbb{Q}[\eta]$. We define an operation of $\Gamma_0(p_0)$ on $M_{n,\eta}$ via

$$r.x^v y^{n-v} = \eta(d)(ax+cy)^v (bx+dy)^{n-v}, \quad r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the Eichler-Shimura isomorphism says that the following sequence

$$0 \to S_{n+2}(\Gamma_0(p_0),\eta) \oplus \overline{S_{n+2}(\Gamma_0(p_0),\eta)} \overset{\phi\oplus\bar{\phi}}{\to} H^1(\Gamma_0(p_0), M_{n,\eta}\otimes\mathbb{C}) \to$$

$$\to \bigoplus_{s \text{ a cusp}} H^1(\Gamma_0(p_0)_s, M_{n,\eta}\otimes\mathbb{C}) \to 0$$

is exact, where s runs over cusps with respect to $\Gamma_0(p_0)$ and $\Gamma_0(p_0)_s := \{ r \in \Gamma_0(p_0) \mid rs = s \} =< T_s >$ is a cyclic infinite group. We describe now the map ϕ:

$$\phi : S_{n+2}(\Gamma_0(p_0),\eta) \to H^1(\Gamma_0(p_0), M_{n,\eta}\otimes\mathbb{C})$$

$$\phi_f(r) = \int_0^{r0} f(z)(xz+y)^n dz.$$

Denote H the upper half plane, $\overline{H} = H \cup \mathbb{Q} \cup \{\infty\}$, we show now:

Lemma: *For any* $r \in \Gamma$ *and* $t_0, t_1 \in \overline{H}$

$$\int_{rt_0}^{rt_1} f(z)(xz+y)^n dz = r \int_{t_0}^{t_1} (f|_{n+2}r)(z)(xz+y)^n dz. \tag{2.1}$$

Proof: Let $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Put $z = ru = \frac{au+b}{cu+d}$, then $dz = (cu+d)^{-2}du$. The integral can be rewritten as

$$\begin{aligned}
\int_{rt_0}^{rt_1} f(z)(xz+y)^n dz &= \int_{t_0}^{t_1} f(ru)(x\frac{au+b}{cu+d} + y)^n (cu+d)^{-2} du \\
&= \int_{t_0}^{t_1} f(ru)(cu+d)^{-n-2}((ax+cy)u + (bx+dy))^n du \\
&= \int_{t_0}^{t_1} (f|_{n+2}r)(z)((ax+cy)z + (bx+dy))^n dz \\
&= r\int_{t_0}^{t_1} (f|_{n+2}r)(z)(xz+y)^n dz.
\end{aligned}$$

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So for $r, s \in \Gamma_0(p_0)$ and $f \in S_{n+2}(\Gamma_0(p_0), \eta)$ we have

$$\begin{aligned}
\phi_f(rs) &= \int_0^{rs0} f(z)(xz+y)^n dz \\
&= \int_0^{r0} f(z)(xz+y)^n dz + \int_{r0}^{rs0} f(z)(xz+y)^n dz \\
&= \phi_f(r) + r\int_0^{s0} (f|_{n+2}r)(z)(xz+y)^n dz \\
&= \phi_f(r) + \eta(r)r\int_0^{s0} f(z)(xz+y)^n dz \qquad (f|_{n+2}r = \eta(r)f) \\
&= \phi_f(r) + r.\phi_f(s).
\end{aligned}$$

i.e. ϕ_f is a cocycle in $Z^1(\Gamma_0(p_0), M_{n,\eta} \otimes \mathbb{C})$.

3. The Shapiro Lemma.

We denote by $W_{n,\eta}$ the co-induced module of $M_{n,\eta}$ on Γ.

$$W_{n,\eta} = Coind_{\Gamma_0(p_0)}^{\Gamma} M_{n,\eta} = \{ f : \Gamma \to M_{n,\eta} \mid f(r_0 r) = r_0.f(r),\ r_0 \in \Gamma_0(p_0) \}$$

The operation of Γ on $W_{n,\eta}$ is defined by

$$(a.f)(r) := f(ra), \qquad a, r \in \Gamma, \quad f \in W_{n,\eta}$$

By the Shapiro lemma there is a canonical isomorphism

$$H^1(\Gamma_0(p_0), M_{n,\eta}) \cong H^1(\Gamma, W_{n,\eta}).$$

We describe now this isomorphism. Let $p : W_{n,\eta} \rightarrow M_{n,\eta}$ be a map which sends a function f to $f(1)$. Then the Shapiro isomorphism is the composition

$$S : H^1(\Gamma, W_{n,\eta}) \overset{\text{res}}{\rightarrow} H^1(\Gamma_0(p_0), W_{n,\eta}) \overset{p^*}{\rightarrow} H^1(\Gamma_0(p_0), M_{n,\eta}).$$

On the other hand the map $i : M_{n,\eta} \rightarrow W_{n,\eta}$ defined by

$$i(m)(r) = \begin{cases} r.m, & \text{if } r \in \Gamma_0(p_0) \\ 0, & \text{otherwise} \end{cases}$$

is a homomorphism. The inverse of the Shapiro isomorphisms is then the composition

$$S^{-1} : H^1(\Gamma_0(p_0), M_{n,\eta}) \overset{i_*}{\rightarrow} H^1(\Gamma_0(p_0), W_{n,\eta}) \overset{\text{cores}}{\rightarrow} H^1(\Gamma, W_{n,\eta})$$

(cf. [AS] §1). In order to determine the isomorphism S^{-1} we consider the structur of the Γ-module $W_{n,\eta}$. Let

$$a_i = \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix}, \quad i = 0, 1, ..., p_0 - 1, \quad a_{p_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\{a_i\}$ is then a set of representatives of Γ with respect to $\Gamma_0(p_0)$:

$$\Gamma = \bigcup_{i=0}^{p_0} \Gamma_0(p_0) a_i.$$

An element $f \in W_{n,\eta}$ is determined by the values $f(a_0), f(a_1), ..., f(a_{p_0})$ by using the condition $f(r_0 r) = r_0 f(r)$. The dimension of $W_{n,\eta}$ is

$$(p_0 + 1) \cdot dim(M_{n,\eta}) = (p_0 + 1)(n + 1).$$

In other words, $W_{n,\eta}$ is generated by the elements $(w_0, w_1, ..., w_{p_0})$ with $w_i \in M_{n,\eta}$. For any $r \in \Gamma$ there exist always $r_i \in \Gamma_0(p_0)$ with $a_i r = r_i a_j$ for some j. Let $\omega \in H^1(\Gamma_0(p_0), M_{n,\eta}), s \in \Gamma$,

$$(S^{-1}\omega)(r)(s) = cores(i_*\omega)(r)(s) = \sum a_i^{-1}(i_*\omega)(r_i)(s)$$

$$= \sum (i_*\omega)(r_i)(sa_i^{-1}) = sa_i^{-1}.\omega(r_i) \quad \text{for } sa_i^{-1} \in \Gamma_0(p_0).$$

In particular, $(S^{-1}\omega)(r)(a_i) = \omega(r_i)$, i. e.,

$$(S^{-1}\omega)(r) = (\omega(r_0), ..., \omega(r_{p_0})) \in W_{n,\eta}.$$

Combining this with the Eichler-Shimura isomorphism we obtain a map Φ:

$$\Phi : S_{n+2}(\Gamma_0(p_0), \eta) \xrightarrow{\phi} H^1(\Gamma_0(p_0), M_{n,\eta} \otimes \mathbb{C}) \xrightarrow{S^{-1}} H^1(\Gamma, W_{n,\eta} \otimes \mathbb{C})$$

$$\Phi_f(r) = (\phi_f(r_0), ..., \phi_f(r_{p_0})) \in W_{n,\eta} \otimes \mathbb{C} \tag{3.1}$$

for $r \in \Gamma$ and $a_i r = r_i a_j$.

Since the group Γ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, and $T = SQ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the cohomology class Φ_f is determined by the value $\Phi_f(S)$, $\Phi_f(T)$. A simple computation shows that

$$\begin{cases} a_0 S = a_{p_0} \\ a_i S = S_i a_j \quad i \cdot j \equiv -1 \bmod p_0, \quad S_i = \begin{pmatrix} -j & -1 \\ 1+ij & i \end{pmatrix} \in \Gamma_0(p_0) \ , \\ a_{p_0} S = a_0 \end{cases}$$

and

$$\begin{cases} a_i T = a_{i+1}, \quad i = 0, 1, ..., p_0 - 2 \\ a_{p_0-1} T = U a_0 \\ a_{p_0} T = T a_{p_0} \end{cases} \qquad \begin{cases} a_0 Q = T a_{p_0} \\ a_1 Q = T^{-1} a_0 \\ a_i Q = S_i a_{j+1}, \quad i = 2, 3, ..., p_0 - 1 \\ a_{p_0} Q = a_1 \end{cases}$$

where $U := \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix}$. We calculate $\Phi_f(T), \Phi_f(S)$. Because

$$\phi_f(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = 0, \quad \phi_f(U) = 0,$$

we have $\Phi_f(T) = (0, ..., 0, \phi_f(T))$. Denote $\rho_f := \int_0^\infty f(z)(xz+y)^n dz$, then

$$\begin{aligned}
\phi_f(T) &= \int_0^{T0} f(z)(xz+y)^n dz \\
&= \int_0^\infty f(z)(xz+y)^n dz + \int_\infty^{T0} f(z)(xz+y)^n dz \\
&= \int_0^\infty f(z)(xz+y)^n dz + \int_{T\infty}^{T0} f(z)(xz+y)^n dz \\
&= \int_0^\infty f(z)(xz+y)^n dz + T. \int_\infty^{0} f(z)(xz+y)^n dz \\
&= (1-T). \int_0^\infty f(z)(xz+y)^n dz \\
&= (1-T).\rho_f.
\end{aligned}$$

It follows that

$$\Phi_f(T) = (1-T).(0, ..., 0, \rho_f) \tag{3.2}$$

In particular, it implies that Φ_f is a class in $H_p^1(\Gamma, W_{n,\eta})$, the cuspidal cohomology groups of $\Gamma_0(p_0)$. Similarly, $\Phi_f(S)$ can be written in the form

$$\Phi_f(S) = (0, \phi_f(S_1), ..., \phi_f(S_{p_0-1}), 0). \tag{3.3}$$

Furthermore, the integral $\phi_f(S_i)$ can be represented by the period polynomial of f for $0 < i < p_0$:

$$\begin{aligned}
\phi_f(S_i) &= \int_0^{S_i 0} f(z)(xz+y)^n dz = \int_0^{S_i a_j \infty} f(z)(xz+y)^n dz \qquad (a_j\infty = 0) \\
&= \int_{a_i\infty}^{a_i S\infty} f(z)(xz+y)^n dz = a_i \int_\infty^{S\infty} (f|_{n+2}a_i)(z)(xz+y)^n dz \\
&= a_i \int_\infty^0 (f|_{n+2}a_i)(z)(xz+y)^n dz.
\end{aligned}$$

The period polynomial of f for an element $A \in \Gamma$ is defined by

$$\rho_f(A) := \int_0^\infty (f|_{n+2}A)(z)(xz+y)^n dz$$

(cf. [An]). For $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we see that $\rho_f(I) = \rho_f$. Therefore we obtain a relation between the period polynomial and the cohomology class

$$\phi_f(S_i) = -a_i \rho_f(a_i) \quad \text{for } 0 < i < p_0 \tag{3.4}$$

4. Petersson scalar product in the cohomology groups.

There is a well-known pairing on $M_{n,\eta}$:

$$(\cdot,\cdot)_M : M_{n,\eta} \otimes M_{n,\eta} \rightarrow \mathbb{Q}[\eta]$$

$$(v,w)_M := \sum_{i=0}^{n} (-1)^i \binom{n}{i}^{-1} v_i \bar{w}_{n-i}$$

for $v = \sum v_i x^i y^{n-i}, w = \sum w_i x^i y^{n-i} \in M_{n,\eta}$. The form $(\cdot,\cdot)_M$ has the following properties:

1. $(\cdot,\cdot)_M$ is non-degenerate.

2. $(\cdot,\cdot)_M$ is hermitian, i.e. $\overline{(v,w)_M} = (v,w)_M$.

3. $(rv, rw)_M = (v,w)_M$ for $r \in \Gamma$, $v, w \in M_n$.

   $(r_0.v, r_0.w)_M = (v,w)_M$ for $r_0 \in \Gamma_0(p_0)$, $v, w \in M_{n,\eta}$.

From the diagram

$$\begin{array}{ccccc} M_{n,\eta} & \otimes & M_{n,\eta} & \stackrel{(\cdot,\cdot)_M}{\rightarrow} & \mathbb{Q}[\eta] \\ \downarrow p & & \uparrow i & & \\ W_{n,\eta} & \otimes & W_{n,\eta} & \rightarrow & \mathbb{Q}[\eta] \end{array}$$

where the map p, i are given in §3:

$$p(f) := f(1), \qquad i(m)(x) := \begin{cases} x.m, & x \in \Gamma_0(p_0) \\ 0, & \text{otherwise} \end{cases},$$

we can define a reduced pairing $(\cdot,\cdot)_W$ on $W_{n,\eta}$ by setting:

$$(v,w)_W := \sum_{i=0}^{p_0} (v(a_i), w(a_i))_M, \quad \forall v, w \in W_{n,\eta}$$

which has the properties:

1. $(\cdot,\cdot)_W$ is non-degenerate.

2. $(\cdot,\cdot)_W$ is hermitian.

3. $(r.v, r.w)_W = (v,w)_W$ for $v, w \in W_{n,\eta}$ and $r \in \Gamma$.

4. $(w, i(m))_W = (p(w), m)_M$ for $w \in W_{n,\eta}$, $m \in M_{n,\eta}$;

1. and 2. follow easily from the properties of $(\cdot,\cdot)_M$.

Proof of 3.: For $r \in \Gamma$ there are $r_i \in \Gamma_0(p_0)$ with $a_i r = r_i a_j$ for some j. So

$$\begin{aligned}(r.v, r.w) &= \sum((r.v)(a_i), (r.w)(a_i))_M = \sum(v(a_i r), w(a_i r))_M \\ &= \sum(r_i v(a_j), r_i w(a_j))_M = \sum(v(a_j), w(a_j))_M = (v,w)_W.\end{aligned}$$

Proof of 4.:

$$\begin{aligned}(w, i(m))_W &= \sum_{i=0}^{p_0}(w(a_i), i(m)(a_i))_M = \sum_{i=0,\, a_i\in\Gamma_0(p_0)}^{p_0} (w(a_i), a_i.m)_M \\ &= (w(1), m)_M = (p(w), m)_M.\end{aligned}$$

The compositions of cup product and the above pairing give us two scalar product on the cohomologies:

$$<\cdot,\cdot>_M: H^1_c(\Gamma_0(p_0), M_{n,\eta}) \otimes H^1(\Gamma_0(p_0), M_{n,\eta}) \stackrel{\cup}{\to} H^2_c(\Gamma_0(p_0), M_{n,\eta}\otimes M_{n,\eta}) \stackrel{(\cdot,\cdot)_M}{\to}$$

$$\stackrel{(\cdot,\cdot)_M}{\to} H^2_c(\Gamma_0(p_0), \mathbb{Q}[\eta]\ ) \stackrel{\epsilon^*}{\to} \mathbb{Q}[\eta]$$

$$<\cdot,\cdot>_W: H^1_c(\Gamma, W_{n,\eta}) \otimes H^1(\Gamma, W_{n,\eta}) \stackrel{\cup}{\to} H^2_c(\Gamma, W_{n,\eta}\otimes W_{n,\eta}) \stackrel{(\cdot,\cdot)_W}{\to}$$

$$\stackrel{(\cdot,\cdot)_W}{\to} H^2_c(\Gamma, \mathbb{Q}[\eta]\ ) \stackrel{\epsilon^*}{\to} \mathbb{Q}[\eta]\ .$$

where $H^*_c(*,*)$ denotes the cohomology group with compact support. A simple consequence of the property 4. is then:

$$< S(\phi), S(\psi) >_M = < \phi, \psi >_W, \quad \forall \phi \in H^1_c(\Gamma, W_{n,\eta}), \psi \in H^1(\Gamma, W_{n,\eta}).$$

Now we determine the scalar product $< \cdot, \cdot >_W$ explicitly. For two cocycle $\phi \in Z^1_c(\Gamma, W_{n,\eta})$ and $\psi \in Z^1(\Gamma, W_{n,\eta})$ the cup product $\phi \cup \psi$ is given by

$$(\phi \cup \psi)(a, b) = \phi(a) \otimes a\psi(b) \in W_{n,\eta} \otimes W_{n,\eta}, \quad \forall a, b \in \Gamma.$$

The isomorphism ϵ^* is already calculated in [Hab] p278:

$$\epsilon^*(u) = \frac{1}{2}u(S, S) + \frac{1}{3}\Big(u(Q, Q) + u(Q, Q^2)\Big) - u(S, Q), \quad \forall u \in H^2_c(\Gamma, \mathbb{Q}[\eta]\ ).$$

Because $(1 + S)\phi(S) = 0$ and $(1 + Q + Q^2)\psi(Q) = 0$, the scalar product $< \cdot, \cdot >_W$ can be written as

$$\begin{aligned}
< \phi, \psi >_W =& \frac{1}{2}(\phi(S), S\psi(S))_W + \frac{1}{3}\Big((\phi(Q), Q\psi(Q))_W + (\phi(Q), Q\psi(Q^2))_W\Big) \\
& - (\phi(S), S\psi(Q))_W \\
=& -\frac{1}{2}(\phi(S), \psi(S))_W + \frac{1}{3}\Big((\phi(Q), Q\psi(Q))_W + (\phi(Q), (Q + Q^2)\psi(Q))_W\Big) \\
& + (\phi(S), \psi(Q))_W \\
=& (\phi(S), \psi(Q))_W - \frac{1}{2}(\phi(S), \psi(S))_W + \frac{1}{3}(\phi(Q), (Q - 1)\psi(Q))_W.
\end{aligned}$$

For $\phi, \psi \in H^1_c(\Gamma, W_{n,\eta})$ one has $\phi(S) = \psi(Q), \psi(S) = \psi(Q)$ and it follows that

$$\begin{aligned}
< \phi, \psi >_W &= (\phi(S), \psi(S))_W - \frac{1}{2}(\phi(S), \psi(S))_W + \frac{1}{3}(\phi(S), (Q - 1)\psi(S))_W \\
&= \frac{1}{6}\Big((\phi(S), \psi(S))_W + 2(\phi(S), Q\psi(S))_W\Big) \\
&= \frac{1}{6}\Big((\phi(S), \psi(S))_W + (Q^{-1}\phi(S), \psi(S))_W + (\phi(S), Q\psi(S))_W\Big) \\
&= \frac{1}{6}\Big((\phi(Q), \psi(S))_W + (Q^2\phi(Q), \psi(S))_W + (\phi(S), Q\psi(S))_W\Big) \\
&= \frac{1}{6}\Big(-(Q\phi(S), \psi(S))_W + (\phi(S), Q\psi(S))_W\Big) \\
&= \frac{1}{6}\Big((T\phi(S), \psi(S))_W - (\phi(S), T\psi(S))_W\Big).
\end{aligned}$$

We consider the following exact sequence:

$$\cdots \to H^1_c(\Gamma, W_{n,\eta}) \xrightarrow{j^*} H^1(\Gamma, W_{n,\eta}) \to H^1(\Gamma_\infty, W_{n,\eta}) \to \cdots$$

The image $j^*(H^1_c(\Gamma, W_{n,\eta}))$ is the cuspidal cohomology $H^1_p(\Gamma, W_{n,\eta})$. For a class $\phi \in H^1_p(\Gamma, W_{n,\eta})$ one has $\phi(T) = (1-T)w$ for some $w \in W_{n,\eta}$. We set $\phi'(S) = \phi(S) - (1-S)w$ and $\phi'(Q) = \phi'(S)$, the ϕ' is then a class in $H^1_c(\Gamma, W_{n,\eta})$, i.e., $\phi'(T) = 0$, and the image $j^*(\phi') = \phi$. We define then a bilinear form $< \cdot, \cdot >$ on $H^1_p(\Gamma, W_{n,\eta})$ by

$$< \phi, \psi > := < \phi', \psi' >_W, \quad \forall \phi, \psi \in H^1_p(\Gamma, W_{n,\eta}).$$

On the other hand, one can define a pairing on $S_{n+2}(\Gamma_0(p_0), \eta)$. For a vector $\binom{u}{v} \in \mathbb{C}^2$ the $n+1$-dimensional column vector $\binom{u}{v}^n$ is defined by

$$\begin{pmatrix} u \\ v \end{pmatrix}^n = {}^t(u^n, u^{n-1}v, \ldots, uv^{n-1}, v^n)$$

The matrix Θ with respect to the pairing $(\cdot, \cdot)_M$ has the property

$${}^t\begin{pmatrix} u \\ v \end{pmatrix}^n \Theta \begin{pmatrix} x \\ y \end{pmatrix}^n = (det \begin{pmatrix} u & x \\ v & y \end{pmatrix})^n. \tag{4.1}$$

For any $f \in S_{n+2}(\Gamma_0(p_0), \eta)$ we denote $\delta(f) = f(z)\binom{z}{1}^n dz$. δ is thus a homomorphism from $S_{n+2}(\Gamma_0(p_0), \eta)$ to the de-Rham cohomology group $H^1_p(\Gamma_0(p_0) \setminus H, \widetilde{M}_{n,\eta})$. The integral

$$A(f,g) = \int_{\Gamma_0(p_0)\setminus H} {}^t\delta(f) \wedge \Theta\overline{\delta(g)}$$

defines a pairing on $H^1_p(\Gamma_0(p_0) \setminus H, \widetilde{M}_{n,\eta}) \cong H^1_p(\Gamma_0(p_0), M_{n,\eta}) \cong H^1_p(\Gamma, W_{n,\eta})$. As shown in [Hi] p279 we have $A(f,g) = < \Phi_f, \Phi_g >$. By a simple computation we see that ${}^t\delta(f) \wedge \Theta\overline{\delta(g)} = -(2i)^{n+1} f\bar{g}y^n dx \wedge dy$ which yields

$$A(f,g) = \int_{\Gamma_0(p_0)\setminus H} {}^t\delta(f) \wedge \Theta\overline{\delta(g)} = -(2i)^{n+1} \int_{\Gamma_0(p_0)\setminus H} f\bar{g}y^n dx \wedge dy = -(2i)^{n+1}(f,g),$$

where (f,g) is the well-known Petersson scalar product. It follows that

$$(f,g) = -\frac{1}{(2i)^{n+1}} < \Phi_f, \Phi_g > \tag{4.2}$$

(cf. [Hida] p279 or [KZ] p244).

We calculate the value $<\Phi_f, \Phi_g>$. In view of formulas (3.2),(3.3) we have

$$\Phi'_f(S) = \Phi_f(S) - (1-S)(0,...,0,\rho_f) = (\rho_f, \phi_f(S_1), ..., \phi_f(S_{p_0-1}), -\rho_f)$$

and

$$\begin{aligned} T\Phi'_f(S)(a_i) &= \Phi'_f(a_iT) = \Phi'_f(a_{i+1}) = \phi_f(S_{i+1}), \quad i = 0,1,...,p_0-2 \\ T\Phi'_f(S)(a_{p_0-1}) &= \Phi'_f(a_{p_0-1}T) = \Phi'_f(U\alpha_0) = U\rho_f \\ T\Phi'_f(S)(a_{p_0}) &= \Phi'_f(a_{p_0}T) = \Phi'_f(T\alpha_{p_0}) = -T\rho_f, \end{aligned}$$

i.e., $T\Phi'_f(S) = (\phi_f(S_1), ..., \phi_f(S_{p_0-1}), U\rho_f, -T\rho_f)$. We'll show

Theorem: *Denote* $A_i := \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix}$ *for* $i = 0,1,...,p_0+1$. *Then the Petersson scalar product of two cusp forms* $f,g \in S_{n+2}(\Gamma_0(p_0),\eta)$ *can be represented as the sum of the period polynomials of* f *and* g*:*

$$(f,g) = -\frac{1}{6(2i)^{n+1}} \sum_{i=0}^{p_0} \Big((T\rho_f(A_{i+1}), \rho_g(A_i))_M - (\rho_f(A_i), T\rho_g(A_{i+1}))_M \Big)$$

Proof: For $i = 0,...,p_0-1$ we have $A_i = a_i$, $A_{p_0} = UA_0$, $A_{p_0+1} = UA_1$. we get

$$<\Phi_f, \Phi_g> = \frac{1}{6} \sum_{i=0}^{p_0} \Big((T\Phi'_f(S)(a_i), (\Phi'_g(S))(a_i))_M - ((\Phi'_f(S)(a_i), (T\Phi'_g(S))(a_i))_M \Big)$$

For $i = 1,...,p_0-2$ we find that

$$\begin{aligned} &(T\Phi'_f(S)(a_i), \Phi'_g(S)(a_i))_M - (\Phi'_f(S)(a_i), T\Phi'_g(S)(a_i))_M \\ =&(\phi_f(S_{i+1}), \phi_g(S_i))_M - (\phi_f(S_i), \phi_g(S_{i+1}))_M \\ =&(-a_{i+1}\rho_f(a_{i+1}), -a_i\rho_g(a_i))_M - (-a_i\rho_f(a_i), -a_{i+1}\phi_g(a_{i+1}))_M \qquad \text{(from (3.4))} \\ =&(a_iT\rho_f(a_{i+1}), a_i\rho_g(a_i))_M - (a_i\rho_f(a_i), a_iT\phi_g(a_{i+1}))_M \qquad \text{(because } a_iT = a_{i+1}) \\ =&(T\rho_f(a_{i+1}), \rho_g(a_i))_M - (\rho_f(a_i), T\phi_g(a_{i+1}))_M \\ =&(T\rho_f(A_{i+1}), \rho_g(A_i))_M - (\rho_f(A_i), T\phi_g(A_{i+1}))_M. \end{aligned}$$

We consider the terms $i = 0, p_0-1, p_0$. It is easy to prove the following properties:

$$\begin{aligned} &\phi_f(S_1) = (1-T^{-1})\rho_f, \quad \phi_f(S_{p_0-1}) = U(1-T)\rho_f \\ &\rho_f(A_0) = -A_0^{-1}\rho_f, \quad \rho_f(A_{p_0-1}) = -TA_0^{-1}(1-T)\rho_f, \quad \rho_f(A_1) = -A_1^{-1}(1-T^{-1})\rho_f \\ &\rho_f(A_{p_0}) = \rho_f(A_0), \quad \rho_f(A_{p_0+1}) = \rho_f(A_1), \end{aligned}$$

Applying the above identities, the pairings can be written as

$$
\begin{aligned}
&(T\Phi'_f(S)(a_0), \Phi'_g(S)(a_0))_M - (\Phi'_f(S)(a_0), T\Phi'_g(S)(a_0))_M \\
=&(\phi_f(S_1), \rho_g)_M - (\rho_g, \phi_g(S_1))_M \\
=&((1-T^{-1})\rho_f, \rho_g)_M - (\rho_f, (1-T^{-1})\rho_g)_M \\
=&(T\rho_f, \rho_g)_M - (\rho_f, T\rho_g)_M \\
&(T\Phi'_f(S)(a_{p_0-1}), \Phi'_g(S)(a_{p_0-1}))_M - (\Phi'_f(S)(a_{p_0-1}), T\Phi'_g(S)(a_{p_0-1}))_M \\
=&(U\rho_f, \phi_g(S_{p_0-1}))_M - (\phi_f(S_{p_0-1}), U\rho_g)_M \\
=&(U\rho_f, U(1-T)\rho_g)_M - (U(1-T)\rho_f, U\rho_g)_M \\
=&(\rho_f, (1-T)\rho_g)_M - ((1-T)\rho_f, \rho_g)_M \\
=&(T\rho_f, \rho_g)_M - (\rho_f, T\rho_g)_M \\
&(T\Phi'_f(S)(a_{p_0}), \Phi'_g(S)(a_{p_0}))_M - (\Phi'_f(S)(a_{p_0}), T\Phi'_g(S)(a_{p_0}))_M \\
=&(-T\rho_f, -\rho_g)_M - (-\rho_f, -T\rho_g)_M \\
=&(T\rho_f, \rho_g)_M - (\rho_f, T\rho_g)_M
\end{aligned}
$$

and

$$
\begin{aligned}
&(T\rho_f(A_1), \rho_g(A_0))_M - (\rho_f(A_0), T\rho_g(A_1))_M \\
=&(TA_1^{-1}(1-T^{-1})\rho_f, -A_0^{-1}\rho_g)_M - (-A_0^{-1}\rho_f, -TA_1^{-1}(1-T^{-1})\rho_g)_M \\
=&((1-T^{-1})\rho_f, \rho_g)_M - (\rho_f, (1-T^{-1})\rho_g)_M \\
=&((T\rho_f, \rho_g)_M - (\rho_f, T\rho_g)_M \\
&(T\rho_f(A_{p_0}), \rho_g(A_{p_0-1}))_M - (\rho_f(A_{p_0-1}), T\rho_g(A_{p_0}))_M \\
=&(T\rho_f(A_0), \rho_g(A_{p_0-1}))_M - (\rho_f(A_{p_0-1}), T\rho_g(A_0))_M \\
=&(-TA_0^{-1}\rho_f, -TA_0^{-1}(1-T)\rho_g)_M - (-TA_0^{-1}(1-T)\rho_f, -TA_0^{-1}\rho_g)_M \\
=&(\rho_f, (1-T)\rho_g)_M - ((1-T)\rho_f, \rho_g)_M \\
=&((T\rho_f, \rho_g)_M - (\rho_f, T\rho_g)_M \\
&(T\rho_f(A_{p_0+1}), \rho_g(A_{p_0}))_M - (\rho_f(A_{p_0}), T\rho_g(A_{p_0+1}))_M \\
=&(T\rho_f(A_1), \rho_g(A_0))_M - (\rho_f(A_0), T\rho_g(A_1))_M) \\
=&(-TA_1^{-1}(1-T^{-1})\rho_f, -A_0^{-1}\rho_g)_M - (-A_0^{-1}\rho_f, -TA_1^{-1}(1-T^{-1})\rho_g)_M \\
=&((1-T^{-1})\rho_f, \rho_g)_M - (\rho_f, (1-T^{-1})\rho_g)_M \\
=&((T\rho_f, \rho_g)_M - (\rho_f, T\rho_g)_M .
\end{aligned}
$$

This completes the proof of the theorem.

- qed -

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