Simple Singularities
in Positive Characteristic

by

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MPI/88-19

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INTRODUCTION

Simple or ADE-singularities have attracted much attention during the last two decades, mostly because very different classification principles led finally to the same list of singularities. The list of normal forms consists of well known complex polynomials in $n+1$ variables, named A_k, D_k, E_6, E_7 and E_8, with isolated singularity at the origin. If $n = 2$, according to Artin [Ar], these are just the rational double points and if $n = 1$, they can also be characterized by their resolution, cf. [BPV]. All higher dimensions are obtained from the curve case by "suspension", namely by adding a certain number of squares in additional variables. Besides the characterization through resolution, there is another striking characterization due to Arnold [Arn] using deformation theory: The simple singularities in all dimensions are exactly the hypersurface singularities of <u>finite deformation type</u>, i.e. they are characterized by the fact that each one can be deformed only into finitely many other non isomorphic singularities. More recently, Knörrer [Kn] and Buchweitz-Greuel-Schreyer [BGS] proved, that the simple singularities and no other hypersurfaces have the property, that there exist only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules over their local ring, or, in other words, they are of finite Cohen-Macaulay type.

The above results hold actually also for power series over an algebraically closed field of characteristic 0. In this paper we extend both characterizations to algebraically closed fields of positive characteristic. Of course, the list of normal forms, which we continue to call simple or ADE-singularities, is now in general larger, depending on the characteristic of the field. It turns out that our list coincides in dimension 1 with the simple singularities in the sense of Barth-Peters-van de Ven [BPV], normal forms of which in characteristic > 0 had been obtained by Kiyek and Steinke [KS]. In dimension two we obtain just the rational double points of Artin [Ar]. Higher dimensional normal forms are obtained again by a certain suspension.

Although the main results are completely analogous to the characteristic zero case, there are some striking differences. While in characteristic zero the classification of simple singularities with respect to right equivalence and contact equivalence coincides, we have to use contact equivalence (e.g. E_8 is not of finite deformation type with respect to right equivalence in characteristic 5). Moreover, although the deformation pattern among the simple singularities is in general the same as in the classical case, we have some unexpected exceptional deformations. For instance $E_8^0 \longrightarrow A_8$ $(n = 1, \text{char} = 3)$ $E_6^0 \longrightarrow A_6^2$ $(n = 1, \text{char} = 2)$

and $E_8^j \longrightarrow D_8^j$, $i = 0,1,2$ ($n = 2$, char $= 2$) . This phenomenon was discovered independently by Knop [Kno] using a new description of simple singularities by simple groups. In the last paragraph we state the complete deformation relations (adjacency diagram) between the simple curve singularities and some adjacencies in the surface case.

The most difficult part of the proof is to show, that the simple singularities of our list are the only ones which have finite deformation type. In particular in dimension 2 and characteristic 2 this required an extensive partial classification of singularities. These calculations are entirely due to the second author and we refer to [Kr] for details. Following tradition we do not include them here, but following a suggestion of C.T.C. Wall we note down a list of basic subcases which can be used to determine singularities which are not in normal form.

During this work the second author was supported by a scholarship of the French-German exchange program PROCOPE. The final version was written down during a stay of the first author at the Max-Planck-Institut für Mathematik in Bonn. We like to express our gratitude to both institutions for support and for providing comfortable working conditions.

1. RESULTS

1.1 Let K be an algebraically closed field of arbitrary characteristic and $K[[\underline{x}]] = K[[x_0,\ldots,x_n]]$ the formal power series ring. Two power series $f,g \in K[[\underline{x}]]$ are called contact equivalent or isomorphic if the local K-algebras $K[[\underline{x}]]/(f)$ and $K[[\underline{x}]]/(g)$ are isomorphic (notation $f \sim g$). In the following lists, "dimension" refers to $n = \dim K[[\underline{x}]]/(f)$.

1.2 DEFINITION: A formal power series f is called simple or an ADE-singularity if it is contact equivalent to one of the following normal forms:

(I) char(K) $\neq$ 2

I.1 Dimension 1

name		normal form for $f \in K[[x,y]]$	
A_k		$x^2 + y^{k+1}$	$k \geq 1$
D_k		$x^2y + y^{k-1}$	$k \geq 4$
E_6	E_6^0	$x^3 + y^4$	
	E_6^1	$x^3 + y^4 + x^2y^2$	additionally in char = 3
E_7	E_7^0	$x^3 + xy^3$	
	E_7^1	$x^3 + xy^3 + x^2y^2$	additionally in char = 3
E_8	E_8^0	$x^3 + y^5$	
	E_8^1	$x^3 + y^5 + x^2y^3$	add. in char = 3
	E_8^2	$x^3 + y^5 + x^2y^2$	
	E_8^1	$x^3 + y^5 + xy^4$	additionally in char = 5

I.2 Dimension ≥ 2

$f \in K[[x_0,\ldots,x_n]]$

$$\boxed{f(x_0,\ldots,x_n) = g(x_0,x_1) + x_2^2 + \ldots + x_n^2}$$

where $g \in K[[x_0,x_1]]$ is one of the list I.1. The name of f is that of g.

(II) char(K) = 2

II.1 Dimension 1

name		normal form for $f \in K[[x,y]]$	
A_{2m-1}		$x^2 + xy^m$	$m \geq 1$
A_{2m}	A_{2m}^0	$x^2 + y^{2m+1}$	$m \geq 1$
	A_{2m}^r	$x^2 + y^{2m+1} + xy^{2m-r}$	$m \geq 1,\ 1 \leq r \leq m-1$
D_{2m}		$x^2y + xy^m$	$m \geq 2$
D_{2m+1}	D_{2m+1}^0	$x^2y + y^{2m}$	$m \geq 2$
	D_{2m+1}^r	$x^2y + y^{2m} + xy^{2m-r}$	$m \geq 2,\ 1 \leq r \leq m-1$
E_6	E_6^0	$x^3 + y^4$	
	E_6^1	$x^3 + y^4 + xy^3$	
E_7		$x^3 + xy^3$	
E_8		$x^3 + y^5$	

II.2 Dimension 2

name		normal form for $f \in K[[x,y,z]]$	
A_k		$z^{k+1} + xy$	
D_{2m}	D^0_{2m}	$z^2 + x^2y + xy^m$	$m \geq 2$
	D^r_{2m}	$z^2 + x^2y + xy^m + xy^{m-r}z$	$m \geq 2,\ 1 \leq r \leq m-1$
D_{2m+1}	D^0_{2m+1}	$z^2 + x^2y + y^mz$	$m \geq 2$
	D^r_{2m+1}	$z^2 + x^2y + y^mz + xy^{m-r}z$	$m \geq 2,\ 1 \leq r \leq m-1$
E_6	E^0_6	$z^2 + x^3 + y^2z$	
	E^1_6	$z^2 + x^3 + y^2z + xyz$	
E_7	E^0_7	$z^2 + x^3 + xy^3$	
	E^1_7	$z^2 + x^3 + xy^3 + x^2yz$	
	E^2_7	$z^2 + x^3 + xy^3 + y^3z$	
	E^3_7	$z^2 + x^3 + xy^3 + xyz$	
E_8	E^0_8	$z^2 + x^3 + y^5$	
	E^1_8	$z^2 + x^3 + y^5 + xy^3z$	
	E^2_8	$z^2 + x^3 + y^5 + xy^2z$	
	E^3_8	$z^2 + x^3 + y^5 + y^3z$	
	E^4_8	$z^2 + x^3 + y^5 + xyz$	

II.3 Dimension ≥ 3

$f \in K[[x_0,\dots,x_n]]$

$f(x_0,\dots,x_n) = g(x_0,x_1) + x_2x_3 + \dots + x_{2k}x_{2k+1}$	$n = 2k+1$
$f(x_0,\dots,x_n) = g(x_0,x_1,x_2) + x_3x_4 + \dots + x_{2k-1}x_{2k}$	$n = 2k$

where $g \in K[[x_0,x_1]]$ resp. $K[[x_0,x_1,x_2]]$ is one of the list II.1 resp. II.2. The name of f is that of g .

1.3. Remarks: (1) The normal forms in dimension 2 are exactly the normal forms of rational double points which were classified by Artin [Ar]. Moreover, Lipman showed in [Li] that a twodimensional double point is rational if and only if it is absolutely isolated, i.e. can be resolved by a finite sequence of blowing up points. This criterion will be used for the proof that ADE-singularities are of finite deformation type.

(2) The normal forms in dimension 1 are exactly the normal forms of functions f which (a) are reduced, (b) have multiplicity 2 or 3 and (c) the reduced total transform of f after one blowing up has also property (b). This was proved by Kiyek-Steinke in [KS]. Note that our notation differs slightly from that of Kiyek and Steinke. Ours harmonizes with Artins and fits more natural into the deformation pattern of these singularities. The upper index 0 denotes the classical normal form, which is the most special with respect to deformations.

(3) A power series f of the form $f(x_0,\dots,x_n) = x_0x_1 + g(x_2,\dots,x_n)$ is called a double suspension of g . Since $x_0x_1 \sim x_0^2 + x_1^2$ if $\mathrm{char}(K) \neq 2$, we see that each normal form in dimension ≥ 3 is obtained from a simple curve or surface singularity by a certain numer of double suspensions. Note that the simple singularities are all in m^2 where m denotes the maximal ideal of $K[[\underline{x}]]$ and that they have isolated singularities.

The main result of this paper is the following

1.4. *Theorem. Let* $f \in K[[x_0,\dots,x_n]]$ *be in* m^2. *The following statements are equivalent*:

(i) f *is simple*,

(ii) f *is of finite deformation type*,

(iii) f *is of finite Cohen-Macaulay type*.

For a precise definition of finite deformation type over an arbitrary algebraically closed field see 2.1.

1.5. *Remarks*: (1) The result is of course well known in characteristic 0, cf. [Arn], [Au], [BGS], [Es], [GK], [Kn]. The equivalence of (i) and (iii) in dimension 1 and 2 (and in any dimension if $\mathrm{char}(K) \neq 2$) and the implication (i) $\Rightarrow$ (iii) in any dimension and positive characteristic is also known, cf. [Au], [KS], [BGS], [So].

We prove the equivalence of (i) and (ii) and the implication (iii) $\Rightarrow$ (i) in paragraph 3.

2. DETERMINANCY

2.1. As before let K denote an algebraically closed field of arbitrary characteristic, $K[[\underline{x}]]$ the formal power series ring in $n+1$ indeterminates $x_0,\dots,x_n$ and m its maximal ideal. $f \in K[[\underline{x}]]$ is called a hypersurface singularity if $f \in m^2$ and $f \neq 0$. Occasionally we call also the local ring $K[[\underline{x}]]/(f)$ or its formal spectrum $(X,0) = \mathrm{Spf}(K[[\underline{x}]]/(f))$ a (hypersurface) singularity. The singularity f is called isolated if there exists a $k > 0$ such that $m^k \subset j(f)$, where $j(f) = (f,\frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_m})$ denotes the Jacobian ideal of f. f is an isolated singularity if and only if its Tjurina number $\tau = \dim_K K[[\underline{x}]]/j(f)$ is finite. Note that in characteristic $p > 0$ the Milnor number $\dim_K K[[\underline{x}]]/(\frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_n})$ is not an invariant of the contact class of f (e.g. $x^p + y^{p+1}$ and $(1+x)(x^p+y^{p+1})$).

2.2. If $f \in K[[\underline{x}]]$ happens to be a polynomial, it defines the affine K-variety $X = V(f) = \mathrm{Spec}(K[\underline{x}]/(f))$ where $K[\underline{x}]$ denotes the polynomial ring. By a singularity or a singular point of X we mean a closed point $x \in X$ and its complete local ring $\hat{\mathcal{O}}_{X,x}$. Since any isolated singularity $f \in K[[\underline{x}]]$ is contact equivalent to a polynomial $g \in K[\underline{x}]$ (cf. 2.6), it can be realized as the singular point 0 of the affine variety $X = V(g)$.

2.3 Let $f \in K[[\underline{x}]]$ be an isolated singularity. By (2.6) we may assume that f is a polynomial. We can choose polynomials $g_1,\dots,g_\tau \in K[\underline{x}]$, $g_1 = 1$, $g_i(0) = 0$ for $i > 1$, which represent a K-basis of $K[[\underline{x}]]/j(f)$. Let $F \in K[\underline{x},t_2,\dots,t_\tau]$ be defined by

$$F = f + \sum_{i=2}^{\tau} t_i g_i ,$$

$f_t(x) = F(x,t)$ for $t = (t_2,\dots,t_\tau) \in K^{\tau-1}$.
Let X_t be the affine variety $V(f_t)$ and note that $X_0 = V(f)$.
It is well known that

$$\mathbb{A}^{n+1} \times_K \mathbb{A}^{\tau-1} \longrightarrow \mathbb{A} \times_K \mathbb{A}^{\tau-1}$$

$$(x,t) \longmapsto (F(x,t),t)$$

is an algebraic representative of the miniversal (or semiuniversal) deformation of f , i.e. of the singular point $0 \in X_0$.

2.4. An isolated singularity $f \in K[[\underline{x}]]$ is said to be of finite deformation type, if for an algebraic representative of the miniversal deformation of f as above the following holds: there exist Zariski-open neighbourhoods $U \subset \mathbb{A}^{n+1}$ of 0 and $W \subset \mathbb{A}^{\tau-1}$ of 0 such that the set of isomorphism classes of singularities of $X_t \cap U$, t running through all closed points of W , is finite.
If f is a polynomial with a non isolated singularity at 0 then we do not have a finite dimensional miniversal deformation space for f . Nevertheless we can say that f is of finite deformation type if there exists a finite list of singularities such that, as above, f deforms only to singularities of this list for every algebraic deformation of f . Our classification in paragraph 3 however shows that no isolated hypersurface singularities are never of finite deformation type.

2.5. LEMMA: $f \in K[[x_0,\ldots,x_n]]$ is not of finite deformation type if either $n \geq 2$ and $\mathrm{mult}(f) \geq 3$ or $n \geq 1$ and $\mathrm{mult}(f) \geq 4$.

Proof: This is just a dimension count. Let $\ell = \mathrm{mult}(f)$, i.e. $f \in m^{\ell}$ but $f \notin m^{\ell+1}$. The contact group $\{(u,\varphi)\,|\,u$ a unit of $K[x]$, φ a formal coordinate transformation of $(K^{n+1},0)\}$ induces an operation of $K^* \times GL_{n+1}(K)$ on $m^{\ell}/m^{\ell+1}$. It can be checked easily, that the dimension of the orbit of $f^{(\ell)}$ under this group is smaller than the dimension of $m^{\ell}/m^{\ell+1}$ under the hypotheses of the lemma. Therefore infinitely many orbits occur and $f^{(\ell)}$ and hence f cannot be of finite deformation type. (Here and in the following $f^{(\ell)}$ denotes the ℓ-jet of f).

2.6. $f \in K[[\underline{x}]]$ is called k-determined (with respect to contact equivalence) if it is contact equivalent to $f^{(k)}$, the k-jet of f , i.e. the power series expansion of f up to and including order k . The minimum k is the index of determinancy.

LEMMA: If for $f \in K[[x]]$, $m^k \subset j(f)$, then f is 2k-determined. In particular, an isolated singularity is 2τ-determined.

Proof: Let $g \in K[[\underline{x}]]$ such that $g-f \in m^{2k+1}$. We have to show that there exist a unit $u \in K[[x]]$ and a coordinate transformation $\varphi : K^n \longrightarrow K^n$ such that $g-u\cdot f(\varphi) = 0$. For this purpose we construct inductively units $u^p(x) \in K[[x]]$ and (n+1) tupels of power series of sufficiently high order $a^1(x),\ldots,a^p(x)$ such that

$$g(x)-u^p(x)f(x+a^1(x)+\ldots+a^p(x)) \in M^{2k+p+1} .$$

Then u^p tends to u and $x+a^1(x)+\ldots+a^p(x)$ to φ if p goes to infinity.

The details, which are similar to those of the proof given in [BL] for right equivalence, are left to the reader.

2.7. The above bound for determinancy is in general much to high. In characteristic 0 better bounds are known, but they fail usually in positive characteristic.

The simple singularities in characteristic $\neq 2$ have index of determinancy d where d is the maximum degree of monomials occuring in the classical normal form. This is no longer true if $\mathrm{char}(K) = 2$. Then we have the following indices of determinancy for surface singularities:

A_k	D^r_{2m}, $r \geq 0$	D^r_{2m+1}, $r \geq 0$	E^r_6	$E^{0,1}_7$	$E^{2,3}_7$	E^r_8
$k+1$	$\max(2r,m+1)$	$\max(2r+1,m+1)$	3	5	4	5

2.8. Since the Tjurina number τ is upper semicontinuous with respect to deformations, lemma 2.6 implies the following: Given an arbitrary set of singularities which is closed under deformations and which, for each τ, has only finitely many members of Tjurina number $\leq \tau$, then each member is of finite deformation type. This remark will be used to show that the simple singularities are of finite deformation type.

3. PROOFS

3.1. Curve singularities (Proof of theorem 1.4, (i) $\Leftrightarrow$ (ii))

We use the characterization of simple singularities in remark 1.3(2).

" $\Rightarrow$ " : Let $f(x,y)$ be a simple curve singularity and consider an algebraic representative of the miniversal deformation $f_t(x,y)$ as in 2.3. We blow up the origin in the (x,y)-plane and check that for each t the reduced total transform of f_t has only singular points of multiplicity 2 or 3. Hence the singularities of f_t must be simple again. By 2.8 f is of finite deformation type.

" $\Leftarrow$ " : By Lemma 2.5 we have to show that non simple curve singularities of multiplicity 2 or 3 are of infinite deformation type. Let $f(x,y)$ be arbitrary with $\text{mult}(f) = 2,3$.

$\text{mult}(f) = 2 \Rightarrow f \sim A_k$ for some $k \geq 1$ or $f \sim x^2$ (type A_∞).

$\text{mult}(f) = 3$ and f has ≥ 2 different tangents $\Rightarrow f \sim D_k$ for some k or $f \sim x^2y$ (D_∞) .

$\text{mult}(f) = 3$ and f has one triple tangent: f can be written as

$$f(x,y) = x^3 + a(y)x^2y^2 + \bar{b}(y)xy^3 + \bar{c}(y)y^4$$

and we have according to [KS]:

$$\begin{aligned}
&\text{mult}(\bar{c}) = 0 && \Leftrightarrow f \sim E_6\\
&\text{mult}(\bar{b}) = 0,\ \text{mult}(\bar{c}) \geq 1 && \Leftrightarrow f \sim E_7\\
&\text{mult}(\bar{b}) \geq 1,\ \text{mult}(\bar{c}) = 1 && \Leftrightarrow f \sim E_8 .
\end{aligned}$$

In the remaining case we can write

$$f(x,y) = x^3 + a(y)x^2y^2 + b(y)xy^4 + c(y)y^6 .$$

We replace $a(y)$, $b(y)$, $c(y)$ respectively by

$$\bar{a}(y,t) = a(y) - a(0) + \sigma_1(t-\alpha)$$
$$\bar{b}(y,t) = b(y) - b(0) + \sigma_2(t-\alpha)$$
$$\bar{c}(y,t) = b(y) - c(0) + \sigma_3(t-\alpha)$$

where the σ_i denote the elementary symmetric functions in three variables. Moreover $(t-\alpha) = (t_1-\alpha_1, t_2-\alpha_2, t_3-\alpha_3)$ where α_i are the zeros of $x^3 + a(0)x^2 + b(0)x + c(0)$. In this way we obtain a deformation f_t of f . On the second reduced total transform of f_t we have four points on an exceptional $\mathbb{P}^1$, the cross ratio of which varies with t . The result follows.

3.2. Singularities of dimension > 1, char(K) $\neq$ 2

The equivalence (i) $\Longleftrightarrow$ (ii) of theorem 1.4 follows immediately from lemma 2.5, the curve case and, since the higher dimensional singularities are simple suspensions of curves, from the following two facts:

If $f \in K[[x_0,\dots,x_n]]$ is a simple suspension of $g \in K[[x_1,\dots,x_n]]$ (i.e. $f = x_0^2 + g$) then each deformation of f is a simple suspension of a deformation of g .

If $f \in K[[x_0,\dots,x_n]]$ is of multiplicity 2 then f is a simple suspension of some $g \in K[[x_1,\dots,x_n]]$ (splitting lemma in characteristic $\neq 2$).

3.3. Classification of double points $f \in K[[x,y,z]]$, $\mathrm{char}(K) = 2$

We only give a rough pattern for what is needed in the next section. For complete details see [Kr]. The forms A_∞ und $D_\infty(n)$ are defined in the next section 3.4.

According to the classification of quadratic forms we have three main cases (recall that $f^{(k)}$ denotes the k-jet) :

(1) $f^{(2)} \sim xy + z^2$ $\qquad (\Rightarrow f \sim A_1)$

(2) $f^{(2)} \sim xy$

(3) $f^{(2)} \sim z^2$.

Case (2): Using transformation of the form $x \longrightarrow x+\alpha(x,y,z)$ and $y \longrightarrow y+\beta(x,y,z)$ one obtains $f \sim xy + a_k z^k + O(k+1)$ for some $k \geq 3$.

(2.1) $a_k \neq 0$ for at least one $k \Rightarrow f \sim A_{k-1}$ (for k minimal)

(2.2) $a_k = 0$ for all $k \Rightarrow f \sim A_\infty$.

Case (3): Write f in the form

$$f(x,y,z) = z^2 + \psi(x,y) + \varphi(x,y)z \;,$$

$O(\psi) \geq 3$, $O(\psi) \geq 2$.

(3.1) $\psi^{(3)} \equiv 0$: all not simple (case B) 2.1, 2.2, 2.3 next section)

(3.2) $\psi^{(3)} \sim x^2y + xy^2 \Rightarrow f \sim D_4$

(3.3) $\psi^{(3)} \sim x^2y \Rightarrow$

$f \sim z^2 + x^2y + a\,xyz + by^2z + O(4)$

(3.3.1) $a, b \neq 0 \Rightarrow f \sim D_5^1$

(3.3.2) $a = 0,\ b \neq 0 \Rightarrow f \sim D_5^0$

(3.3.3) $a \neq 0,\ b = 0 \Rightarrow$ (for some $k \geq 4$)

$f \sim z^2 + x^2y + xyz + cy^{k-1}z + dxy^{k-1} + ey^k + O(k+1)$

(3.3.3.1) $e = 0 \Rightarrow f$ of type D_k for some $k < \infty$ or

$f \sim D_\infty(n),\ n \geq 1$

(3.3.3.2) $e \neq 0$, k odd $\Rightarrow f \sim D_{k+2}^r,\ r = (k-1)/2$

(3.3.3.3) $e \neq 0$, k even $\Rightarrow f \sim D_{k+2}^{k/2}$ or possibly $\sim D_\infty(1)$ if

$d \neq 0$

(3.3.4) $a = b = 0 : \Rightarrow$ (for some $k \geq 4$)

$f \sim z^2 + x^2y + \alpha xy^{k-2}z + \beta y^{k-1}z + \gamma xy^{k-1} + O(k+1)$

(3.3.4.1) $\gamma \neq 0 : f \sim D_{2k-2}^1\ (\alpha \neq 0),\ f \sim D_{2k-2}^0\ (\alpha = 0)$

(3.3.4.2) $\gamma = 0,\ \beta \neq 0 : f \sim D_{2k-1}^1\ (\alpha \neq 0),\ f \sim D_{2k-1}^0\ (\alpha = 0)$

(3.3.4.3) $\gamma = \beta = 0,\ \alpha \neq 0$: different D_k's or $D_\infty(n),\ n \geq 1$

(3.3.4.4) $\alpha = \beta = \gamma = 0$: Case (3.3.4) for bigger k . If

$\alpha_k = \beta_k = \gamma_k = 0$ for all $k \Rightarrow f \sim D_\infty(0)$

(3.4) $\psi^{(3)} = x^3 \Rightarrow$

$f \sim z^2 + x^3 + axyz + by^2z + O(4)$

(3.4.1) $a, b \neq 0 \Rightarrow f \sim E_6^1$

(3.4.2) $a = 0,\ b \neq 0 \Rightarrow f \sim E_6^0$

(3.4.3) $a \neq 0,\ b = 0 \Rightarrow$

$f \sim z^2 + x^3 + xyz + cxy^3 + O(5)$

(3.4.3.1) $c \neq 0 \Rightarrow f \sim E_7^3$

(3.4.3.2) $c = 0 \Rightarrow f \sim z^2 + x^3 + xyz + dy^5 + O(6)$; then either

$f \sim E_8^4$ for $d \neq 0$ or of type B)3.5 if $d = 0$

(3.4.4) $a = b = 0 \Rightarrow$

$f \sim z^2 + x^3 + \alpha xy^2z + \beta y^3z + \gamma xy^3 + O(5)$

(3.4.4.1) $\gamma \neq 0 \Rightarrow f \sim E_7^0$ or $E_7^1\ (\beta = 0)$ or $E_7^2\ (\beta \neq 0)$

(3.4.4.2) $\gamma = 0,\ \beta \neq 0 \Rightarrow f \sim E_8^3$ (if y^5 occurs) or of type B) 3.4

(3.4.4.3) $\gamma = \beta = 0,\ \alpha \neq 0 \Rightarrow f \sim E_8^2$ (if y^5 occurs) or of type B) 3.3

(3.4.4.4) $\alpha = \beta = \gamma = 0 \Rightarrow f \sim E_8^0,\ E_8^1$ or of type B) 3.1, 3.2.

3.4. Surface singularities in characteristic 2 (Proof of 1.4 (i) $\Longleftrightarrow$ (ii))

" $\Rightarrow$ " : Like in the curve case we blow up the miniversal deformation of each simple surface singularity. By explicit calculation one shows that they are absolutely isolated. Remark 1.3 (1) together with 2.8 shows that they are of finite deformation type.

" $\Leftarrow$ " : By lemma 2.5 we have to consider only double points. The classification of 3.3 shows that we have two cases

A) f is equivalent to one of the normal forms of 1.2 II.2.

B) f is not of this form and then f belongs to one of the following classes:

1\. 1.1 $\quad f \sim xy \quad (A_\infty)$

1.2 $\quad f \sim z^2 + x^2y \quad (D_\infty(0))$

1.3 $\quad f \sim z^2 + x^2y + xy^nz \quad (D_\infty(n),\ n \geq 1)$

2\. $f^{(3)} \sim z^2 + z\varphi^{(2)}(x,y)$ for some $\varphi \in m^2$. Depending on $\varphi^{(2)}$ we have

2.1 $\quad f \sim z^2 + O(4)$

2.2 $\quad f \sim z^2 + x^2z + O(4)$

2.3 $\quad f \sim z^2 + xyz + O(4)$

3. 3.1 $f \sim z^2 + x^3 + O(6)$

3.2 $f \sim z^2 + x^3 + xy^3z + dy^4z + O(6)$ $(d \in K)$

3.3 $f \sim z^2 + x^3 + xy^2z + dxy^4 + O(6)$

3.4 $f \sim z^2 + x^3 + y^3z + d\,xy^4 + O(6)$

3.5 $f \sim z^2 + x^3 + xyz + O(6)$

Functions of class 1. deform obviously into infinitely many A_k or D_k singularities. For the remaining one can find easily deformations into class 3.5. Each f of class 3.5, not belonging to A), can be transformed into $z^2 + x^3 + xyz + ay^k + O(k+1)$ for some $k \geq 6$. Let $f_t = z^2 + x^3 + xyz + ty^6 + O(7)$. Explicit calculation shows that $f_t \sim f_s$ if and only if $t = s$. Hence the elements of class 3.5 and by transitivity all the others are not of finite deformation type.

3.5. Singularities of dimension > 2, char(K) = 2

Before proving the equivalence (i) $\Longleftrightarrow$ (ii) of 1.4 in this case, we need a kind of splitting lemma in characteristic 2. First of all let us recall the classification of quadratic forms over an algebraically closed field of characteristic 2.

LEMMA 1: Let $f \in K[x_0,\dots,x_n]$ be a quadratic form. After a suitable change of coordinates f is one of the following normal forms:

(1) x_0^2

(2) $x_0^2 + x_1x_2 + \dots + x_{2k-1}x_{2k}$, $2k \leq n$

(3) $x_0x_1 + x_2x_3 + \dots + x_{2k}x_{2k+1}$, $2k+1 \leq n$.

The proof is elementary.

<u>LEMMA 2</u>: $ax_0^k + bx_1x_2 + h(x_0,x_1,x_2,\dots,x_n) \in K[[x_0,\dots,x_n]]$ <u>with</u> $a,b \in K$, $b \neq 0$, $h \in m^3$ <u>is contact equivalent to</u> $ax_0^k + x_1x_2 + \tilde{h}(x_0,x_3,\dots,x_n)$ with $\tilde{h} \in m^3$.

<u>Proof</u>: let $h = x_1g_1 + x_2g_2 + g_3$, $g_1,g_2 \in m^2$ and g_3 not depending on x_1,x_2 . The change of coordinates $x_1 \longmapsto \frac{1}{b}(x_1+g_2)$, $x_2 \longmapsto (x_2+\frac{1}{b}g_1)$ suffices to increase the multiplicity of g_1,g_2 . We thus finish by induction.

<u>COROLLARY 3</u> (Splitting lemma in characteristic 2): <u>Let</u> $f \in K[[x_0,\dots,x_n]]$, $\mathrm{mult}(f) = 2$, <u>then either</u>

(a) $f \sim x_0^2 + x_1x_2 + \dots + x_{2\ell-1}x_{2\ell} + h(x_0,x_{2\ell+1},\dots,x_n)$
<u>with</u> $0 \le 2\ell \le n$, $h \in m^3$, <u>or</u>

(b) $f \sim x_0x_1 + \dots + x_{2\ell}x_{2\ell+1} + h(x_{2\ell+2},\dots,x_n)$
<u>with</u> $1 \le 2\ell+1 \le n$, $h \in m^3$.

Note that f is actually right equivalent to (a) or (b).

<u>LEMMA 4</u>: $x_0^2 + h(x_0,\dots,x_n) \in K[[x_0,\dots,x_n]]$, $h \in m^3$, $n \ge 3$ <u>is not of finite deformation type</u>.

<u>Proof</u>: $x_0^2 + m^3/m^4$ has dimension $\binom{n+3}{n}$ but it is easy to check that the image of $x_0^2 + h$ under the contact group in $x_0^2 + m^3/m^4$ has only dimension $(n+1)^2 + 1$. Therefore, if $n \ge 3$, there must be infinitely many orbits.

For the proof of 1.4(i) $\Longleftrightarrow$ (ii) we may assume that f is of type (a) or (b) of Corollary 3. In case (a), f is of finite deformation type iff $x_0^2 + h(x_0, x_{2\ell+1}, \ldots, x_n)$ is. By lemma 4 this can only happen if $n \le 2\ell+2$. But then we are in the surface case which was already treated. Similarly, using lemma 2.5, case (b) reduces to the curve case.

3.6. Finite Cohen-Macaulay type in characteristic 2

It remains to show that a singularity of dimension > 2 which is of finite Cohen-Macaulay type (CM-type for short) has to be simple (char(K) = 2). The main theorem of Solberg [So], which is an extension of an earlier result of Knörrer to characteristic 2, says that $g(x_2, \ldots, x_n)$ is of finite CM-type if its double suspension $f = x_0x_1 + g$ is. Moreover, by [BGS], Prop. 3.1, $f \in K[[x_0, \ldots, x_n]]$, $n \ge 3$, has infinite CM-type if $\operatorname{mult}(f) \ge 3$. By corollary 3 we are therefore reduced to the case $f \sim x_0^2 + h(x_0, x_1, \ldots, x_n)$, $\operatorname{mult}(h) \ge 3$, $n \ge 3$. In order to show that f is of finite CM-type, it suffices to construct infinitely many ideals $I \subset K[[x_0, \ldots, x_n]]$ such that $f \in I^2$ (cf. Cor. 1.7 of [BGS]). Let

$$C := \{\lambda = (0 : \lambda_1 : \ldots : \lambda_n) \in \mathbb{P}^n(K) \mid h^{(3)}(\lambda) = 0\}$$

where $h^{(3)}$ is the homogeneous part of degree 3 of h. Since $n \ge 3$, $\dim C \ge 1$ and since K is algebraically closed, C contains infinitely many points λ. Let $I(\lambda) \subset K[[\underline{x}]]$ be the ideal generated by $\lambda_i x_j - \lambda_j x_i$, $i,j = 0, \ldots, n$ and $I_\lambda := I(\lambda) + m^2$. $I(\lambda)$ is the ideal of the point $\lambda \in C$ and therefore $I_\lambda \neq I_\mu$ for $\lambda \neq \mu$. We have to show $f \in I_\lambda^2$. Since $x_0^2 + h^{(3)}$ vanishes in λ, $f \in I(\lambda) + M^4$ and since $x_0 \in I(\lambda)$ it suffices to show $h^{(3)} \in I(\lambda)M^2$. But this holds because $h^{(3)}(\lambda) = 0$.

4. ADJACENCIES

4.1. The methods to prove theorem 1.4 allow a nearly complete description of all deformations between simple singularities in all characteristics. Except for surface singularities in characteristic 2 we calculated all possible relations.
The adjacencies in characteristic 0 are called the classical ones and are the following

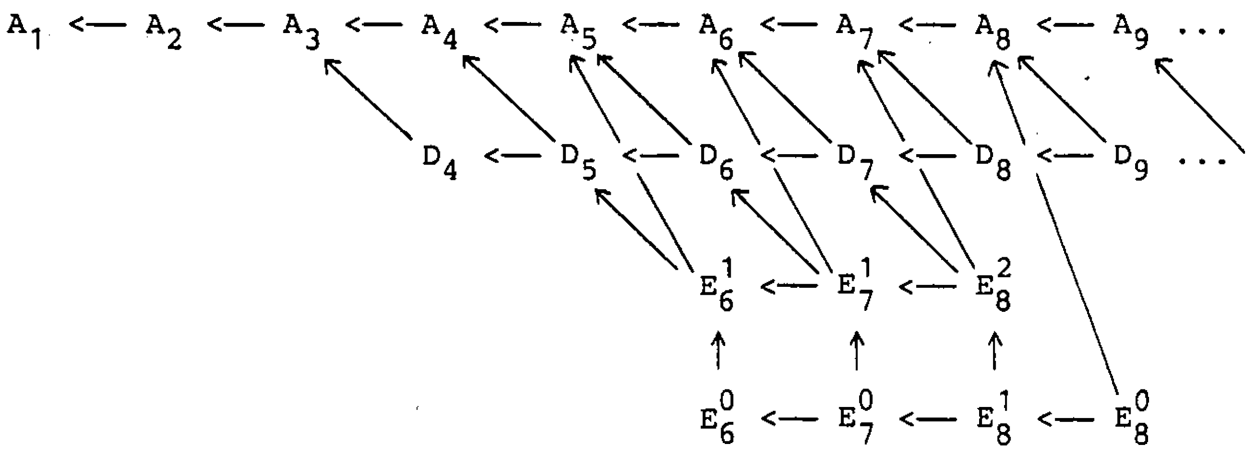
$$A_k \to A_{k-1};\ D_k \to D_{k-1}, A_{k-1};\ E_k \to E_{k-1}, D_{k-1}, A_{k-1}\ .$$

No other deformations up to transitivity occur (those ones which are obtained by transitivity will not be mentioned explicitly).
Adjacencies between singularities in even resp. odd dimension are the same as in dimension 2 resp. 1. This is clear since a double suspension does not alter adjacencies even in characteristic 2.
Our notation coincides with [Ar] for surfaces but differs slightly from [KS] for curves.

4.2. Dimension 1, $\mathrm{char}(K) \geq 3$
If $\mathrm{char}(K) \neq 2,3$ we have only the classical adjacencies
$\mathrm{char}(K) = 5$ this means that both E_8-singularities deform into A_7, D_7 and E_7 .

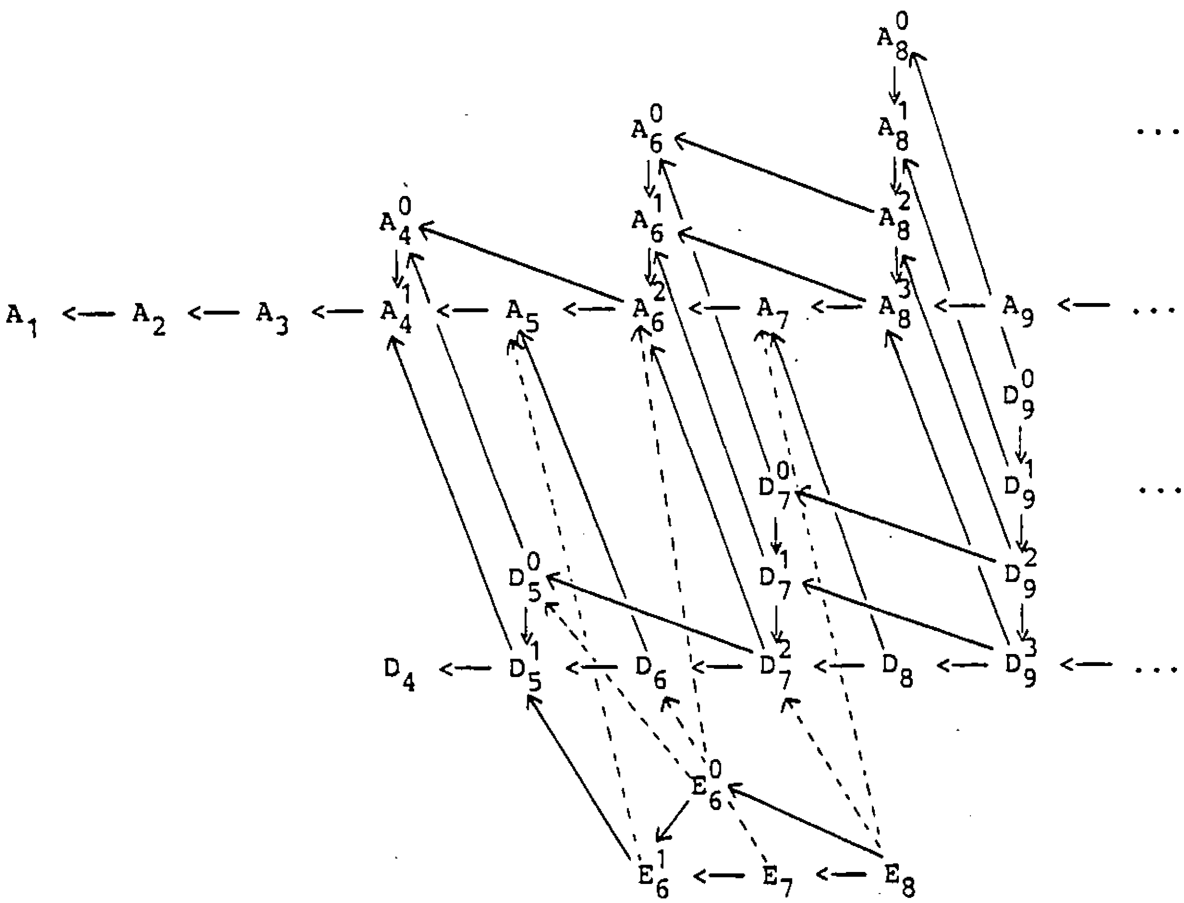
char(K) = 3

Unexpected is only $E_8^0 \longrightarrow A_8$ which can be realized by
$f_t(x,y) = x^3 + y^5 + t^5x^2 - 2t^4xy^2 - txy^3 + t^3y^4$.



4.3. Dimension 1, char(K) = 2

(dotted lines are only for lucidity)



Unexpected is $E_6^0 \longrightarrow A_6^2$ $(f_t(x,y) = x^3 + y^4 + t^4x^2)$ and $E_7 \not\longrightarrow A_6$ $(E_6^0$ plays the role of $E_7)$.

4.4. Dimension 2

Since all surfaces in $\mathrm{char}(K) \neq 2$ are simple suspensions of simple curve singularities, they have the corresponding deformation relations (described in 4.2).

In characteristic 2 one can verify the following:

$$A_k \longrightarrow A_{k-1};\ D_{2n+1}^r \longrightarrow D_{2n}^r,\ D_{2n}^r \longrightarrow D_{2n-1}^{r-1}\ ;$$

$$E_6^r \longrightarrow D_5^r \quad (r = 0,1)\ ,$$

$$E_7^r \longrightarrow E_6^{r-2},\ E_7^r \longrightarrow D_6^{r-1} \quad (r = 1,2,3)\ ,$$

$$E_8^r \longrightarrow E_7^{r-1},\ E_8^r \longrightarrow D_7^{r-2} \quad (r = 3,4) \quad \text{and the exception}$$

$$E_8^r \longrightarrow D_8^r \quad (r = 0,1,2) \quad \text{given by} \quad f_t(x,y,z) = f(x,y,z) + t^4x^2y\ ,$$

where f denotes the normal form of E_8^r .

Deformations $D \longrightarrow A$ and $E \longrightarrow A$ have not be considered and we do not claim completeness in the other cases. Actually, Knop [Kn] has shown $E_7^0 \longrightarrow A_7$ for which we did not find a realization.

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