Max-Planck-Institut für Mathematik
Bonn

On Schur multiple zeta functions: A combinatoric generalization of multiple zeta functions

by

Maki Nakasuji
Ouamporn Phuksuwan
Yoshinori Yamasaki

Max-Planck-Institut für Mathematik
Preprint Series 2017 (25)

On Schur multiple zeta functions: A combinatoric generalization of multiple zeta functions

Maki Nakasuji
Ouamporn Phuksuwan
Yoshinori Yamasaki

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Information and Communication Science
Faculty of Science
Sophia University
Tokyo, Japan

Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Bangkok, Thailand

Graduate School of Science and Engineering
Ehime University
Ehime, Japan

MPIM 17-25

On Schur multiple zeta functions: A combinatoric generalization of multiple zeta functions

Maki Nakasuji[*], Ouamporn Phuksuwan and Yoshinori Yamasaki[†]

April 27, 2017

Abstract

We introduce Schur multiple zeta functions which interpolate both the multiple zeta and multiple zeta-star functions of the Euler-Zagier type combinatorially. We first study their basic properties including a region of absolute convergence and the case where all variables are the same. Next, under an assumption on variables, some determinant formulas coming from theory of Schur functions such as the Jacobi-Trudi, Giambelli and dual Cauchy formula are established with the help of Macdonald's ninth variation of Schur functions. Finally, we investigate the quasi-symmetric functions corresponding to the Schur multiple zeta functions. We obtain the similar results as above for them and, moreover, describe the images of them by the antipode of the Hopf algebra of quasi-symmetric functions explicitly.

2010 Mathematics Subject Classification: 11M41, 05E05.
Key words and phrases: *Multiple zeta functions, Schur functions, Jacobi-Trudi formula, quasi-symmetric functions*

Contents

1 Introduction **2**

2 Schur multiple zeta functions **5**
2.1 Combinatorial settings . . . 5
2.2 Definition of Schur multiple zeta functions . . . 5
2.3 A special case . . . 8

3 Jacobi-Trudi formulas **10**
3.1 A proof of the Jacobi-Trudi formula of H-type . . . 11
3.1.1 Rim decomposition of partition . . . 11
3.1.2 Patterns on the $\mathbb{Z}^2$ lattice . . . 12
3.1.3 Weight of patterns . . . 12
3.1.4 Proof . . . 13
3.2 A proof of the Jacobi-Trudi formula of E-type . . . 16

[*]Supported by Sophia Lecturing-Research Grants 2015, Grant-in-Aid for Young Scientists (B) 15K17519 and JSPS Joint Research Project with CNRS.
[†]Supported by Grant-in-Aid for Scientific Research (C) 15K04785.

4 Schur multiple zeta functions as variations of Schur functions 19
4.1 Schur multiple zeta functions of skew type 19
4.2 Macdonald's ninth variation of Schur functions 20
4.3 Jacobi-Trudi formulas 21
4.4 Giambelli formula 22
4.5 Dual Cauchy formula 23

5 Schur type quasi-symmetric functions 26
5.1 Quasi-symmetric functions 26
5.2 Relation between quasi-symmetric functions and multiple zeta values 26
5.3 Schur type quasi-symmetric functions 27

1 Introduction

The multiple zeta function and the multiple zeta-star function (MZF and MZSF for short) of the Euler-Zagier type are respectively defined by the series

$$\zeta(\boldsymbol{s}) = \sum_{m_1<\cdots<m_n} \frac{1}{m_1^{s_1}\cdots m_n^{s_n}}, \quad \zeta^\star(\boldsymbol{s}) = \sum_{m_1\le\cdots\le m_n} \frac{1}{m_1^{s_1}\cdots m_n^{s_n}},$$

where $\boldsymbol{s}=(s_1,\ldots,s_n)\in\mathbb{C}^n$. These series converge absolutely for $\Re(s_1),\ldots,\Re(s_{n-1})\ge 1$ and $\Re(s_n)>1$ (see, e.g., [Mat] for more precise description about the region of absolute convergence). One easily sees that a MZSF can be expressed as a linear combination of MZFs, and vice versa. For instance,

$$\begin{aligned}
\zeta^\star(s_1,s_2) &= \zeta(s_1,s_2)+\zeta(s_1+s_2),\\
\zeta(s_1,s_2) &= \zeta^\star(s_1,s_2)-\zeta^\star(s_1+s_2),\\
\zeta^\star(s_1,s_2,s_3) &= \zeta(s_1,s_2,s_3)+\zeta(s_1+s_2,s_3)+\zeta(s_1,s_2+s_3)+\zeta(s_1+s_2+s_3),\\
\zeta(s_1,s_2,s_3) &= \zeta^\star(s_1,s_2,s_3)-\zeta^\star(s_1+s_2,s_3)-\zeta^\star(s_1,s_2+s_3)+\zeta^\star(s_1+s_2+s_3),
\end{aligned}$$

where $\zeta(s)=\zeta^\star(s)$ is the Riemann zeta function. More generally, we have

$$\zeta^\star(\boldsymbol{s}) = \sum_{\boldsymbol{t}\preceq\boldsymbol{s}} \zeta(\boldsymbol{t}), \quad \zeta(\boldsymbol{s}) = \sum_{\boldsymbol{t}\preceq\boldsymbol{s}} (-1)^{n-\ell(\boldsymbol{t})}\zeta^\star(\boldsymbol{t}), \tag{1.1}$$

where, for $\boldsymbol{t}=(t_1,\ldots,t_m)\in\mathbb{C}^m$, $\ell(\boldsymbol{t})=m$ and $\boldsymbol{t}\preceq\boldsymbol{s}$ means that $\boldsymbol{t}$ is obtained from $\boldsymbol{s}$ by combining some of its adjacent parts. The special value of $\zeta(s_1,\ldots,s_n)$ and $\zeta^\star(s_1,\ldots,s_n)$ at positive integers were first introduced by Euler [E] for $n=2$, and by Hoffman [H1] and Zagier [Za] for general n, independently. Many different types of relations among such values have been studied in references such as [Zl, Mu, IKOO, OZ].

The purpose of the present paper is to introduce a generalization of both MZF and MZSF, which we call a *Schur multiple zeta function*, from the viewpoint of n-ple zeta functions. Indeed, it is defined similarly to the tableau expression of the Schur function as follows. For a partition λ of a positive integer n, let $T(\lambda,X)$ be the set of all Young tableaux of shape λ over a set X and, in particular, $\mathrm{SSYT}(\lambda)\subset T(\lambda,\mathbb{N})$ the set of all semi-standard Young tableaux of shape λ (see Section 2 for precise definitions). Recall that $M=(m_{ij})\in T(\lambda,\mathbb{N})$ is called semi-standard if $m_{i1} \le m_{i2} \le \cdots$ for all i and $m_{1j} < m_{2j} < \cdots$ for all j. For $\boldsymbol{s} = (s_{ij}) \in T(\lambda, \mathbb{C})$, the Schur multiple zeta function (SMZF for short) associated with λ is defined by the series

$$\zeta_\lambda(\boldsymbol{s}) = \sum_{M \in \mathrm{SSYT}(\lambda)} \frac{1}{M^{\boldsymbol{s}}},$$

where $M^{\boldsymbol{s}} = \prod_{(i,j) \in D(\lambda)} m_{ij}^{s_{ij}}$ for $M = (m_{ij}) \in \mathrm{SSYT}(\lambda)$ with $D(\lambda)$ being the Young diagram of λ. It is shown in Lemma 2.1 that the above series converges absolutely whenever $\boldsymbol{s} \in W_\lambda$ where

$$W_\lambda = \left\{ \boldsymbol{s} = (s_{ij}) \in T(\lambda, \mathbb{C}) \;\middle|\; \begin{array}{l} \Re(s_{ij}) \ge 1 \text{ for all } (i,j) \in D(\lambda) \setminus C(\lambda) \\ \Re(s_{ij}) > 1 \text{ for all } (i,j) \in C(\lambda) \end{array} \right\}$$

with $C(\lambda)$ being the set of all corners of λ. If (1^n) and (n) are denoted by the one column and one row partitions of n, then it is clear that $\zeta_{(1^n)}(\boldsymbol{s})$ ($\boldsymbol{s} \in T((1^n), \mathbb{C})$) and $\zeta_{(n)}(\boldsymbol{s})$ ($\boldsymbol{s} \in T((n), \mathbb{C})$) are nothing but MZF and MZSF, respectively. This shows that SMZFs actually interpolate both MZFs and MZSFs combinatorially. Remark that such interpolation multiple zeta functions were first mentioned in [Y] from the study of the multiple Dirichlet L-values.

In this paper, we study fundamental properties of SMZFs and establish some relations among them, which can be regard as analogues of those for Schur functions. Indeed, we obtain the following Jacobi-Trudi formulas for SMZFs, which is one of our main result of the paper. To describe the result, we need the set

$$W_\lambda^{\mathrm{diag}} = W_\lambda \cap T^{\mathrm{diag}}(\lambda, \mathbb{C}),$$

where, for a set X, $T^{\mathrm{diag}}(\lambda, X) = \{T = (t_{ij}) \in T(\lambda, X) \,|\, t_{ij} = t_{kl} \text{ if } j - i = l - k\}$. For a tableau $\boldsymbol{s} = (s_{ij}) \in W_\lambda^{\mathrm{diag}}$, we always write $a_k = s_{i,i+k}$ for $k \in \mathbb{Z}$ (and for any $i \in \mathbb{N}$). For example, when $\lambda = (4,3,3,2)$, $\boldsymbol{s} = (s_{ij}) \in W_{(4,3,3,2)}^{\mathrm{diag}}$ implies that $\boldsymbol{s}$ is of the form of

$$\boldsymbol{s} = \begin{array}{|c|c|c|c|} \hline s_{11} & s_{12} & s_{13} & s_{14} \\ \hline s_{21} & s_{22} & s_{23} \\ \cline{1-3} s_{31} & s_{32} & s_{33} \\ \cline{1-3} s_{41} & s_{42} \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|c|} \hline a_0 & a_1 & a_2 & a_3 \\ \hline a_{-1} & a_0 & a_1 \\ \cline{1-3} a_{-2} & a_{-1} & a_0 \\ \cline{1-3} a_{-3} & a_{-2} \\ \cline{1-2} \end{array}.$$

Theorem 1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition and $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ the conjugate of λ. Assume that $\boldsymbol{s} = (s_{ij}) \in W_\lambda^{\mathrm{diag}}$.*

(1) *Assume further that $\Re(s_{i,\lambda_i}) > 1$ for all $1 \le i \le r$. Then, we have*

$$\zeta_\lambda(\boldsymbol{s}) = \det \left[\zeta^\star(a_{-j+1}, a_{-j+2}, \ldots, a_{-j+(\lambda_i - i + j)}) \right]_{1 \le i,j \le r}. \tag{1.2}$$

Here, we understand that $\zeta^\star(\cdots) = 1$ if $\lambda_i - i + j = 0$ and 0 if $\lambda_i - i + j < 0$.

(2) *Assume further that $\Re(s_{\lambda'_i,i}) > 1$ for all $1 \le i \le s$. Then, we have*

$$\zeta_\lambda(\boldsymbol{s}) = \det \left[\zeta(a_{j-1}, a_{j-2}, \ldots, a_{j-(\lambda'_i - i + j)}) \right]_{1 \le i,j \le s}. \tag{1.3}$$

Here, we understand that $\zeta(\cdots) = 1$ if $\lambda'_i - i + j = 0$ and 0 if $\lambda'_i - i + j < 0$.

As is the case of Schur functions, we call (1.2) and (1.3) of H-type and E-type, respectively. From these formulas, as corollaries, one can obtain many algebraic relations given by determinants among MZFs and MZSFs. For example, considering the case $\lambda = (1^n)$ and $\lambda = (n)$, we have the following identities.

Corollary 1.2. *For $s_1, \ldots, s_n \in \mathbb{C}$ with $\Re(s_1), \ldots, \Re(s_n) > 1$, we have*

$$\zeta(s_1, \ldots, s_n) = \begin{vmatrix} \zeta^\star(s_1) & \zeta^\star(s_2, s_1) & & \cdots & \cdots & \zeta^\star(s_n, \ldots, s_2, s_1) \\ 1 & \zeta^\star(s_2) & & \cdots & \cdots & \zeta^\star(s_n, \ldots, s_2) \\ & 1 & \ddots & & & \vdots \\ & & \ddots & 1 & \zeta^\star(s_{n-1}) & \zeta^\star(s_n, s_{n-1}) \\ \text{\huge 0} & & & & 1 & \zeta^\star(s_n) \end{vmatrix},$$

$$\zeta^\star(s_1, \ldots, s_n) = \begin{vmatrix} \zeta(s_1) & \zeta(s_2, s_1) & & \cdots & \cdots & \zeta(s_n, \ldots, s_2, s_1) \\ 1 & \zeta(s_2) & & \cdots & \cdots & \zeta(s_n, \ldots, s_2) \\ & 1 & \ddots & & & \vdots \\ & & \ddots & 1 & \zeta(s_{n-1}) & \zeta(s_n, s_{n-1}) \\ \text{\huge 0} & & & & 1 & \zeta(s_n) \end{vmatrix}.$$

Moreover, just combining (1.2) and (1.3), we obtain a family of relations among MZFs and MZSFs. For example, considering the cases $\lambda = (2, 2, 1)$ and its conjugate $\lambda' = (3, 2)$, we have

$$\zeta_\lambda\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & a \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right) = \begin{vmatrix} \zeta^\star(a,b) & \zeta^\star(c,a,b) & \zeta^\star(d,c,a,b) \\ \zeta^\star(a) & \zeta^\star(c,a) & \zeta^\star(d,c,a) \\ 0 & 1 & \zeta^\star(d) \end{vmatrix} = \begin{vmatrix} \zeta(a,c,d) & \zeta(b,a,c,d) \\ \zeta(a) & \zeta(b,a) \end{vmatrix},$$

$$\zeta_{\lambda'}\left(\begin{array}{|c|c|c|}\hline a & c & d \\ \hline b & a & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\right) = \begin{vmatrix} \zeta^\star(a,c,d) & \zeta^\star(b,a,c,d) \\ \zeta^\star(a) & \zeta^\star(b,a) \end{vmatrix} = \begin{vmatrix} \zeta(a,b) & \zeta(c,a,b) & \zeta(d,c,a,b) \\ \zeta(a) & \zeta(c,a) & \zeta(d,c,a) \\ 0 & 1 & \zeta(d) \end{vmatrix},$$

where $a, b, c, d \in \mathbb{C}$ with $\Re(a), \Re(b), \Re(d) > 1$ and $\Re(c) \geq 1$. As you can see in the above examples and Corollary 1.2, these kind of relations hold even if we replace ζ with $\zeta^\star$ and vice versa.

It is also worth mentioning that both (1.2) and (1.3) give meromorphic continuations of $\zeta_\lambda(\boldsymbol{s})$ to $T^{\mathrm{diag}}(\lambda, \mathbb{C})$ $(= \mathbb{C}^{s+r-1}$ where $s = \lambda_1$ and $r = \lambda'_1)$ as a function of a_k for $1 - r \leq k \leq 1 + s$ because both MZFs and MZSFs admit meromorphic continuations to the whole complex space (see, e.g., [AET]).

The assumption on variables on the same diagonal lines is crucial. Actually, in Section 4, we find out that our SMZF, which can be easily generalized to the skew type, with the assumption is realized as (the limit of) a specialization of Macdonald's ninth variation of Schur function studied by Nakagawa, Noumi, Shirakawa and Yamada [NNSY]. Based on this fact, we present some results such as the Jacobi-Trudi formula of skew type, the Giambelli formula and the dual Cauchy formula for SMZFs. Notice that if we work for such formulas without the assumption, then we encounter extra terms (see Remark 3.10), which will be clarified in our future study.

Furthermore, in Section 5, we study SMZFs in a more general framework, that is, in the Hopf algebra QSym of quasi-symmetric functions studied by Gessel [G]. For a skew Young diagram ν, we define a special type of quasi-symmetric function $S_\nu(\boldsymbol{\alpha})$, which we call a *Schur type quasi-symmetric function*, similarly to SMZFs. (Note that there is a different type of quasi-symmetric functions, called *quasi-symmetric Schur functions* defined by Haglund, Mason, Luoto and Willigenburg [HLMW], as a basis of QSym, which arise from the combinatorics of Macdonald polynomials and refine Schur functions in a natural way.) Then, we also prove the Jacobi-Trudi formulas of both H-type and E-type for such quasi-symmetric functions under the same assumption as above. Notice that the former corresponds to (1.2) with entries in the essential quasi-symmetric functions and the latter to (1.3) with in the monomial quasi-symmetric functions. Remark that when ν is the one column and one row partitions, the corresponding formulas can be also respectively obtained by calculating the images of the essential and monomial quasi-symmetric functions by the antipode S of QSym in two different ways, as shown by Hoffman ([H2, Theorem 3.1]). More generally, for any skew Young diagram ν, we calculate the image of $S_\nu(\boldsymbol{\alpha})$ by S and see that it is essentially equal to the Schur type quasi-symmetric function again associated with $\nu^\#$, the anti-diagonal transpose of ν.

2 Schur multiple zeta functions

2.1 Combinatorial settings

We first set up some notions of partitions. A partition $\lambda=(\lambda_1,\dots,\lambda_r)$ of a positive integer n is a non-decreasing sequence of positive integers such that $|\lambda|=\sum_{i=1}^r\lambda_i=n$. We call $|\lambda|$ and $\ell(\lambda)=r$ the weight and length of λ, respectively. If $|\lambda|=n$, then we write $\lambda\vdash n$. We sometimes express $\lambda\vdash n$ as $\lambda=(1^{m_1(\lambda)}2^{m_2(\lambda)}\cdots n^{m_n(\lambda)})$ where $m_i(\lambda)$ is the multiplicity of i in λ. We identify $\lambda\vdash n$ with its Young diagram $D(\lambda)=\{(i,j)\in\mathbb{Z}^2\,|\,1\le i\le r,\ 1\le j\le\lambda_i\}$, depicted as a collection of n square boxes with λ_i boxes in the ith row. We say that $(i,j)\in D(\lambda)$ is a corner of λ if $(i+1,j)\notin D(\lambda)$ and $(i,j+1)\notin D(\lambda)$ and denote by $C(\lambda)\subset D(\lambda)$ the set of all corners of λ. For example, $C((4,3,3,2))=\{(1,4),(3,3),(4,2)\}$. A conjugate $\lambda'=(\lambda'_1,\dots,\lambda'_s)$ of λ is defined by $\lambda'_i=\#\{j\,|\,\lambda_j\ge i\}$. Namely, λ' is the partition whose Young diagram is the transpose of that of λ. For example, $(4,3,3,2)'=(4,4,3,1)$.

Let X be a set. For a partition λ, a Young tableau $T=(t_{ij})$ of shape λ over X is a filling of $D(\lambda)$ obtained by putting $t_{ij}\in X$ into (i,j) box of $D(\lambda)$. Similarly to the above, the conjugate tableau of T is defined by $T'=(t_{ji})$ whose shape is λ'. We denote by $T(\lambda,X)$ the set of all Young tableaux of shape λ over X, which is sometimes identified with $X^{|\lambda|}$. Moreover, we put

$$T^{\mathrm{diag}}(\lambda,X)=\{\,(t_{ij})\in T(\lambda,X)\,|\,t_{ij}=t_{kl}\text{ if }j-i=l-k\,\},$$

which is identified with $X^{\lambda_1+\ell(\lambda)-1}$. By a semi-standard Young tableau, we mean a Young tableau over the set of positive integers $\mathbb{N}$ such that the entries in each row are weakly increasing from left to right and those in each column are strictly increasing from top to bottom. We denote by $\mathrm{SSYT}(\lambda)$ the set of all semi-standard Young tableaux of shape λ.

2.2 Definition of Schur multiple zeta functions

For $\boldsymbol{s}=(s_{ij})\in T(\lambda,\mathbb{C})$, define

$$\zeta_\lambda(\boldsymbol{s})=\sum_{M\in\mathrm{SSYT}(\lambda)}\frac{1}{M^{\boldsymbol{s}}},\tag{2.1}$$

where $M^{\boldsymbol{s}}=\prod_{(i,j)\in D(\lambda)}m_{ij}^{s_{ij}}$ for $M=(m_{ij})\in\mathrm{SSYT}(\lambda)$. We also define $\zeta_\lambda=1$ for the empty partition $\lambda=\emptyset$. We call $\zeta_\lambda(\boldsymbol{s})$ a *Schur multiple zeta function* (SMZF for short) associated with

λ and sometimes write it shortly as $\boldsymbol{s}$ if there is no confusion. Clearly, this is an extension of both MZFs and MZSFs. Actually, one sees that

$$\zeta(s_1,\ldots,s_n)=\zeta_{(1^n)}\left(\begin{array}{|c|}\hline s_1\\\hline \vdots\\\hline s_n\\\hline\end{array}\right)=\begin{array}{|c|}\hline s_1\\\hline \vdots\\\hline s_n\\\hline\end{array},\quad \zeta^{\star}(s_1,\ldots,s_n)=\zeta_{(n)}\left(\begin{array}{|c|c|c|}\hline s_1&\cdots&s_n\\\hline\end{array}\right)=\begin{array}{|c|c|c|}\hline s_1&\cdots&s_n\\\hline\end{array}.$$

We first discuss a region where the series (2.1) is absolutely convergent.

Lemma 2.1. *Let*

$$W_\lambda=\left\{\boldsymbol{s}=(s_{ij})\in T(\lambda,\mathbb{C})\;\middle|\;\begin{array}{l}\Re(s_{ij})\geq 1\text{ for all }(i,j)\in D(\lambda)\setminus C(\lambda)\\ \Re(s_{ij})>1\text{ for all }(i,j)\in C(\lambda)\end{array}\right\}.$$

Then, the series (2.1) *converges absolutely if* $\boldsymbol{s}\in W_\lambda$.

Proof. Write $C(\lambda)=\{(i_1,j_1),\ldots,(i_k,j_k)\}$. Then, it can be written as $\lambda=(j_k^{i'_k}j_{k-1}^{i'_{k-1}}\cdots j_1^{i'_1})$ where $i'_l=i_l-i_{l-1}$ with $i_0=0$. Since $\Re(s_{ij})\geq 1$ for $(i,j)\in D(\lambda)\setminus C(\lambda)$, we have

$$\begin{aligned}\sum_{M\in\mathrm{SSYT}(\lambda)}\left|\frac{1}{M^{\boldsymbol{s}}}\right|&\leq\prod_{l=1}^{k}\sum_{(m_{ij})\in\mathrm{SSYT}(j_l^{i'_l})}\prod_{i=1}^{i_l}\prod_{j=1}^{j_l}\frac{1}{m_{ij}^{\Re(s_{ij})}}\\&\leq\prod_{l=1}^{k}\sum_{N_l=1}^{\infty}\frac{C_{i'_l,j_l}(N_l)}{N_l^{\Re(s_{i_l,j_l})}},\end{aligned}$$

where $C_{a,b}(N)$ is a finite sum defined by

$$C_{a,b}(N)=\sum_{\substack{(m_{ij})\in\mathrm{SSYT}(b^a)\\ m_{a,b}=N}}\prod_{\substack{i=1\\(i,j)\neq(a,b)}}^{a}\prod_{j=1}^{b}\frac{1}{m_{ij}}.$$

It is well known that, for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$, which is not dependent on N, such that $\sum_{m=1}^{N}\frac{1}{m}<C_\varepsilon N^{\varepsilon}$. Hence

$$|C_{a,b}(N)|\leq\prod_{\substack{i=1\\(i,j)\neq(a,b)}}^{a}\prod_{j=1}^{b}\sum_{m_{ij}=1}^{N}\frac{1}{m_{ij}}<C_\varepsilon^{ab-1}N^{\varepsilon(ab-1)}$$

and therefore

$$\begin{aligned}\sum_{M\in\mathrm{SSYT}(\lambda)}\left|\frac{1}{M^{\boldsymbol{s}}}\right|&\leq\prod_{l=1}^{k}\sum_{N_l=1}^{\infty}\frac{C_\varepsilon^{i'_lj_l-1}N_l^{\varepsilon(i'_lj_l-1)}}{N_l^{\Re(s_{i_l,j_l})}}\\&=\prod_{l=1}^{k}C_\varepsilon^{i'_lj_l-1}\zeta\left(\Re(s_{i_l,j_l})-\varepsilon(i'_lj_l-1)\right).\end{aligned}$$

This ends the proof because $\Re(s_{i_l,j_l})>1$ for $1\leq l\leq k$ and ε can be taken sufficiently small. $\square$

Remark 2.2. The condition $\boldsymbol{s}\in W_\lambda$ is a sufficient condition that the series (2.1) converges absolutely. It seems to be interesting to determine the region of absolute convergence of (2.1) with full description. See e.g., [Mat] for the cases of $\lambda=(1^n)$ and (n), that is, the cases of MZFs and MZSFs.

It should be noted that a SMZF can be also written as a linear combination of MZFs or MZSFs. In fact, for $\lambda \vdash n$, let $\mathcal{F}(\lambda)$ be the set of all bijections $f : D(\lambda) \to \{1, 2, \ldots, n\}$ satisfying the following two conditions:

(i) for all i, $f((i,j)) < f((i,j'))$ if and only if $j < j'$,

(ii) for all j, $f((i,j)) < f((i',j))$ if and only if $i < i'$.

Moreover, for $T = (t_{ij}) \in T(\lambda, X)$, put

$$V(T) = \left\{ \left(t_{f^{-1}(1)}, t_{f^{-1}(2)}, \ldots, t_{f^{-1}(n)}\right) \in X^n \;\middle|\; f \in \mathcal{F}(\lambda) \right\}.$$

Furthermore, when X has an addition $+$, we write $\boldsymbol{w} \preceq T$ for $\boldsymbol{w} = (w_1, w_2, \ldots, w_m) \in X^m$ if there exists $(v_1, v_2, \ldots, v_n) \in V(T)$ satisfying the following: for all $1 \le k \le m$, there exist $1 \le h_k \le m$ and $l_k \ge 0$ such that

(i) $w_k = v_{h_k} + v_{h_k+1} + \cdots + v_{h_k+l_k}$,

(ii) there are no i and i' such that $i \neq i'$ and $t_{ij}, t_{i'j} \in \{v_{h_k}, v_{h_k+1}, \ldots, v_{h_k+l_k}\}$ for some j,

(iii) $\bigsqcup_{k=1}^m \{h_k, h_k+1, \ldots, h_k+l_k\} = \{1, 2, \ldots, n\}$.

Then, by the definition, we have

$$\zeta_\lambda(\boldsymbol{s}) = \sum_{\boldsymbol{t} \preceq \boldsymbol{s}} \zeta(\boldsymbol{t}). \tag{2.2}$$

This clearly includes the first equation in (1.1) as the case $\lambda = (n)$. Moreover, by an inclusion-exclusion argument, one can also obtain its "dual" expression

$$\zeta_\lambda(\boldsymbol{s}) = \sum_{\boldsymbol{t} \preceq \boldsymbol{s}'} (-1)^{n-\ell(\boldsymbol{t})} \zeta^\star(\boldsymbol{t}), \tag{2.3}$$

which does the second one in (1.1) as the case $\lambda = (1^n)$.

Example 2.3. (1) For $\boldsymbol{s} = (s_{ij}) \in T((3,1), \mathbb{C})$, we have

$$V(\boldsymbol{s}) = \{(s_{11}, s_{12}, s_{13}, s_{21}), (s_{11}, s_{12}, s_{21}, s_{13}), (s_{11}, s_{21}, s_{12}, s_{13})\}.$$

One sees that $\boldsymbol{t} \preceq \boldsymbol{s}$ if and only if $\boldsymbol{t}$ is one of the followings:

$$\begin{aligned}
&(s_{11}, s_{12}, s_{13}, s_{21}), (s_{11}+s_{12}, s_{13}, s_{21}), (s_{11}, s_{12}+s_{13}, s_{21}), (s_{11}, s_{12}, s_{13}+s_{21}),\\
&(s_{11}+s_{12}+s_{13}, s_{21}), (s_{11}+s_{12}, s_{13}+s_{21}), (s_{11}, s_{12}+s_{13}+s_{21}), (s_{11}, s_{12}, s_{21}, s_{13}),\\
&(s_{11}+s_{12}, s_{21}, s_{13}), (s_{11}, s_{12}+s_{21}, s_{13}), (s_{11}, s_{21}, s_{12}, s_{13}), (s_{11}, s_{21}, s_{12}+s_{13}).
\end{aligned}$$

This shows that when $\boldsymbol{s} \in W_{(3,1)}$

$$\begin{aligned}
\begin{array}{|c|c|c|}\hline s_{11} & s_{12} & s_{13} \\ \hline s_{21} \\ \cline{1-1}\end{array}
&= \zeta(s_{11}, s_{12}, s_{13}, s_{21}) + \zeta(s_{11}+s_{12}, s_{13}, s_{21}) + \zeta(s_{11}, s_{12}+s_{13}, s_{21})\\
&\quad + \zeta(s_{11}, s_{12}, s_{13}+s_{21}) + \zeta(s_{11}+s_{12}+s_{13}, s_{21}) + \zeta(s_{11}+s_{12}, s_{13}+s_{21})\\
&\quad + \zeta(s_{11}, s_{12}+s_{13}+s_{21}) + \zeta(s_{11}, s_{12}, s_{21}, s_{13}) + \zeta(s_{11}+s_{12}, s_{21}, s_{13})\\
&\quad + \zeta(s_{11}, s_{12}+s_{21}, s_{13}) + \zeta(s_{11}, s_{21}, s_{12}, s_{13}) + \zeta(s_{11}, s_{21}, s_{12}+s_{13})\\
&= \zeta^\star(s_{11}, s_{21}, s_{12}, s_{13}) - \zeta^\star(s_{11}+s_{21}, s_{12}, s_{13}) - \zeta^\star(s_{11}, s_{21}+s_{12}, s_{13}),\\
&\quad + \zeta^\star(s_{11}, s_{12}, s_{21}, s_{13}) - \zeta^\star(s_{11}, s_{12}, s_{21}+s_{13}) + \zeta^\star(s_{11}, s_{12}, s_{13}, s_{21}).
\end{aligned}$$

Notice that the second equality follows from the discussion in (2).

(2) For $\boldsymbol{s}=(s_{ij})\in T((2,1,1),\mathbb{C})$, we have

$$V(\boldsymbol{s})=\{(s_{11},s_{12},s_{21},s_{31}),(s_{11},s_{21},s_{12},s_{31}),(s_{11},s_{21},s_{31},s_{12})\}.$$

One sees that $\boldsymbol{t}\preceq\boldsymbol{s}$ if and only if $\boldsymbol{t}$ is one of the followings:

$$\begin{gathered}(s_{11},s_{12},s_{21},s_{31}),(s_{11}+s_{12},s_{21},s_{31}),(s_{11},s_{12}+s_{21},s_{31}),\\(s_{11},s_{21},s_{12},s_{31}),(s_{11},s_{21},s_{12}+s_{31}),(s_{11},s_{21},s_{31},s_{12}).\end{gathered}$$

This shows that when $\boldsymbol{s}\in W_{(2,1,1)}$

$$\begin{aligned}\begin{array}{|c|c|}\hline s_{11}&s_{12}\\\hline s_{21}&\multicolumn{1}{c}{}\\\cline{1-1} s_{22}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}&=\zeta(s_{11},s_{12},s_{21},s_{31})+\zeta(s_{11}+s_{12},s_{21},s_{31})+\zeta(s_{11},s_{12}+s_{21},s_{31}),\\&\quad+\zeta(s_{11},s_{21},s_{12},s_{31})+\zeta(s_{11},s_{21},s_{12}+s_{31})+\zeta(s_{11},s_{21},s_{31},s_{12})\\&=\zeta^\star(s_{11},s_{21},s_{31},s_{12})-\zeta^\star(s_{11}+s_{21},s_{31},s_{12})-\zeta^\star(s_{11},s_{21}+s_{31},s_{12})\\&\quad-\zeta^\star(s_{11},s_{21},s_{31}+s_{12})+\zeta^\star(s_{11}+s_{21}+s_{31},s_{12})+\zeta^\star(s_{11}+s_{21},s_{31}+s_{12})\\&\quad+\zeta^\star(s_{11},s_{21}+s_{31}+s_{12})+\zeta^\star(s_{11},s_{21},s_{12},s_{31})-\zeta^\star(s_{11}+s_{21},s_{12},s_{31})\\&\quad-\zeta^\star(s_{11},s_{21}+s_{12},s_{31})+\zeta^\star(s_{11},s_{12},s_{21},s_{31})-\zeta^\star(s_{11},s_{12},s_{21}+s_{31}).\end{aligned}$$

Notice that the second equality follows from the discussion in (1).

Remark 2.4. By the definitions, it is clear that if $\boldsymbol{t}=(t_1,t_2,\ldots,t_m)\preceq\boldsymbol{s}\in T(\lambda,\mathbb{C})$, then t_m is expressed as a sum of s_{ij} where at least one of (i,j) is in $C(\lambda)$. This together with the expression (2.2) or (2.3) also leads Lemma 2.1.

2.3 A special case

We now consider a special case of variables; $\boldsymbol{s}=\{s\}^\lambda$ $(s\in\mathbb{C})$ where $\{s\}^\lambda=(s_{ij})\in T(\lambda,\mathbb{C})$ is the tableau given by $s_{ij}=s$ for all $(i,j)\in D(\lambda)$. In this case, one sees that our SMZF is realized as a specialization of the Schur function. Actually, for variables $\boldsymbol{x}=(x_1,x_2,\ldots)$, let

$$s_\lambda=s_\lambda(\boldsymbol{x})=\sum_{(m_{ij})\in\mathrm{SSYT}(\lambda)}\prod_{(i,j)\in D(\lambda)}x_{m_{ij}}$$

be the Schur function associated with λ. Then, for $s\in\mathbb{C}$ with $\Re(s)>1$, we have

$$\zeta_\lambda(\{s\}^\lambda)=e^{(s)}s_\lambda=s_\lambda(1^{-s},2^{-s},\ldots),$$

where $e^{(s)}$ is the function sending x_i to $\frac{1}{i^s}$. This means that $\zeta_\lambda(\{s\}^\lambda)$ can be written as a polynomial in $\zeta(s),\zeta(2s),\ldots$.

Proposition 2.5. *Let $\lambda\vdash n$. Then, for $s\in\mathbb{C}$ with $\Re(s)>1$, we have*

$$\zeta_\lambda(\{s\}^\lambda)=\sum_{\mu\vdash n}\frac{\chi^\lambda(\mu)}{z_\mu}\prod_{i=1}^{\ell(\mu)}\zeta(\mu_i s).\tag{2.4}$$

Here, $z_\mu=\prod_{i\geq1}i^{m_i(\mu)}m_i(\mu)!$ and $\chi^\lambda(\mu)\in\mathbb{Z}$ is the value of the character χ^λ attached to the irreducible representation of the symmetric group S_n of degree n corresponding to λ on the conjugacy class of S_n of the cycle type $\mu\vdash n$.

Proof. For a partition μ, let $p_\mu = p_\mu(\boldsymbol{x})$ be the power-sum symmetric function defined by $p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}$ where $p_r = p_r(\boldsymbol{x}) = \sum_{i=1}^{\infty} x_i^r$. We know that the Schur function can be written as a linear combination of power-sum symmetric functions (see [Mac]) as

$$s_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu.$$

Hence, one obtains the desired expression by noticing $e^{(s)} p_r = p_r(1^{-s}, 2^{-s}, \ldots) = \zeta(rs)$. $\square$

Remark 2.6. For variables $\boldsymbol{x} = (x_1, x_2, \ldots)$, let $e_n = e_n(\boldsymbol{x})$ and $h_n = h_n(\boldsymbol{x})$ be the elementary and complete symmetric functions of degree n, which are respectively defined by

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \quad h_n = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}.$$

Then, noticing $s_{(1^n)} = e_n$ and $s_{(n)} = h_n$ with $\chi^{(1^n)}(\mu) = |\mu| - \ell(\mu)$ and $\chi^{(n)}(\mu) = 1$, we have

$$\begin{aligned}
\zeta(s, \ldots, s) &= e^{(s)} e_n = e_n(1^{-s}, 2^{-s}, \ldots) = \sum_{\mu \vdash n} \frac{(-1)^{n-\ell(\mu)}}{z_\mu} \prod_{i=1}^{\ell(\mu)} \zeta(\mu_i s), \\
\zeta^\star(s, \ldots, s) &= e^{(s)} h_n = h_n(1^{-s}, 2^{-s}, \ldots) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \prod_{i=1}^{\ell(\mu)} \zeta(\mu_i s).
\end{aligned}$$

It is shown in e.g., [H1, Za, Mu] that $\zeta(2k, \ldots, 2k), \zeta^\star(2k, \ldots, 2k) \in \mathbb{Q}\pi^{2kn}$. These can be generalized to the Schur multiple zeta "values" as follows.

Corollary 2.7. *It holds that* $\zeta_\lambda(\{2k\}^\lambda) \in \mathbb{Q}\pi^{2k|\lambda|}$ *for* $k \in \mathbb{N}$.

Proof. This is a direct consequence of the expression (2.4) together with the fact $\zeta(2k) \in \mathbb{Q}\pi^{2k}$ obtained by Euler (and hence the rational part can be explicitly written in terms of the Bernoulli numbers). $\square$

Example 2.8. When $n = 3$, we have

$$\begin{aligned}
\begin{array}{|c|c|c|}\hline s & s & s \\ \hline\end{array} &= \frac{1}{6}\zeta(s)^3 + \frac{1}{2}\zeta(2s)\zeta(s) + \frac{1}{3}\zeta(3s) = \zeta^\star(s, s, s), \\
\begin{array}{|c|c|}\hline s & s \\ \hline s & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} &= \frac{2}{6}\zeta(s)^3 + \frac{0}{2}\zeta(2s)\zeta(s) + \frac{-1}{3}\zeta(3s), \\
\begin{array}{|c|}\hline s \\ \hline s \\ \hline s \\ \hline\end{array} &= \frac{1}{6}\zeta(s)^3 + \frac{-1}{2}\zeta(2s)\zeta(s) + \frac{1}{3}\zeta(3s) = \zeta(s, s, s).
\end{aligned}$$

Special values of $\zeta_\lambda(\{2k\}^\lambda)$ for $\lambda \vdash 3$ with small k are given as follows:

$\zeta_\lambda(\{2k\}^\lambda)$	$k=1$	$k=2$	$k=3$	$k=4$
$\begin{array}{\|c\|c\|c\|}\hline 2k & 2k & 2k \\ \hline\end{array}$	$\frac{31\pi^6}{15120}$	$\frac{4009\pi^{12}}{3405402000}$	$\frac{223199\pi^{18}}{194896477400625}$	$\frac{2278383389\pi^{24}}{1938427890852062610000}$
$\begin{array}{\|c\|c\|}\hline 2k & 2k \\ \hline 2k & \multicolumn{1}{\|c}{} \\ \cline{1-1}\end{array}$	$\frac{\pi^6}{840}$	$\frac{493\pi^{12}}{5108103000}$	$\frac{86\pi^{18}}{4331032831125}$	$\frac{116120483\pi^{24}}{2423034863565078262500 0}$
$\begin{array}{\|c\|}\hline 2k \\ \hline 2k \\ \hline 2k \\ \hline\end{array}$	$\frac{\pi^6}{5040}$	$\frac{\pi^{12}}{681080400}$	$\frac{2\pi^{18}}{64965492466875}$	$\frac{38081\pi^{24}}{4846069727130156525000 0}$

Example 2.9. When $n = 4$, we have

$$\begin{ytableau} s & s & s & s \end{ytableau} = \frac{1}{24}\zeta(s)^4 + \frac{1}{4}\zeta(2s)\zeta(s)^2 + \frac{1}{8}\zeta(2s)^2 + \frac{1}{3}\zeta(3s)\zeta(s) + \frac{1}{4}\zeta(4s) = \zeta^\star(s,s,s,s),$$

$$\begin{ytableau} s & s & s \\ s \end{ytableau} = \frac{3}{24}\zeta(s)^4 + \frac{1}{4}\zeta(2s)\zeta(s)^2 + \frac{-1}{8}\zeta(2s)^2 + \frac{0}{3}\zeta(3s)\zeta(s) + \frac{1}{4}\zeta(4s),$$

$$\begin{ytableau} s & s \\ s & s \end{ytableau} = \frac{2}{24}\zeta(s)^4 + \frac{0}{4}\zeta(2s)\zeta(s)^2 + \frac{2}{8}\zeta(2s)^2 + \frac{-1}{3}\zeta(3s)\zeta(s) + \frac{0}{4}\zeta(4s),$$

$$\begin{ytableau} s & s \\ s \\ s \end{ytableau} = \frac{3}{24}\zeta(s)^4 + \frac{-1}{4}\zeta(2s)\zeta(s)^2 + \frac{-1}{8}\zeta(2s)^2 + \frac{0}{3}\zeta(3s)\zeta(s) + \frac{1}{4}\zeta(4s),$$

$$\begin{ytableau} s \\ s \\ s \\ s \end{ytableau} = \frac{1}{24}\zeta(s)^4 + \frac{-1}{4}\zeta(2s)\zeta(s)^2 + \frac{1}{8}\zeta(2s)^2 + \frac{1}{3}\zeta(3s)\zeta(s) + \frac{-1}{4}\zeta(4s) = \zeta(s,s,s,s).$$

Special values of $\zeta_\lambda(\{2k\}^\lambda)$ for $\lambda \vdash 4$ with small k are given as follows:

$\zeta_\lambda(\{2k\}^\lambda)$	$k=1$	$k=2$	$k=3$	$k=4$
$\begin{ytableau} 2k & 2k & 2k & 2k \end{ytableau}$	$\frac{127\pi^8}{604800}$	$\frac{13739\pi^{16}}{1136785104000}$	$\frac{1202645051\pi^{24}}{1009597859818782609375}$	$\frac{3467913415992313\pi^{32}}{27995618815818008860855350000000}$
$\begin{ytableau} 2k & 2k & 2k \\ 2k \end{ytableau}$	$\frac{67\pi^8}{362880}$	$\frac{58489\pi^{16}}{8931882960000}$	$\frac{3670606169\pi^{24}}{6057587158912695656250}$	$\frac{49743652304257\pi^{32}}{799874823309085967453010000000}$
$\begin{ytableau} 2k & 2k \\ 2k & 2k \end{ytableau}$	$\frac{11\pi^8}{302400}$	$\frac{113\pi^{16}}{1838917080000}$	$\frac{14074\pi^{24}}{43895559122555765625}$	$\frac{30650383\pi^{32}}{155704220332691929148250000000}$
$\begin{ytableau} 2k & 2k \\ 2k \\ 2k \end{ytableau}$	$\frac{11\pi^8}{362880}$	$\frac{29\pi^{16}}{1786376592000}$	$\frac{98642\pi^{24}}{3028793579456347828125}$	$\frac{332561213\pi^{32}}{3999374116545429837265050000000}$
$\begin{ytableau} 2k \\ 2k \\ 2k \\ 2k \end{ytableau}$	$\frac{\pi^8}{362880}$	$\frac{\pi^{16}}{12504636144000}$	$\frac{4\pi^{24}}{432684797065192546875}$	$\frac{13067\pi^{32}}{9331872938606002953618450000000}$

3 Jacobi-Trudi formulas

The aim of this section is to give a proof of Theorem 1.1. To do that, we need some basic concepts in combinatorial method. Namely, we try to understand SMZF as a sum of weights of patterns on the $\mathbb{Z}^2$ lattice, similarly to Schur functions (more precisely, see, e.g., [LP, HG, Ste, Zi]). Now, we work on not SMZF itself but a truncated sum of SMZF, which may correspond to the Schur polynomial in theory of Schur functions. For $N \in \mathbb{N}$, let $\mathrm{SSYT}_N(\lambda)$ be the set of all $(m_{ij}) \in \mathrm{SSYT}(\lambda)$ such that $m_{ij} \leq N$ for all i,j. Define

$$\zeta_\lambda^N(\boldsymbol{s}) = \sum_{M \in \mathrm{SSYT}_N(\lambda)} \frac{1}{M^{\boldsymbol{s}}}.$$

In particular, put

$$\zeta^N(s_1,\ldots,s_n)=\zeta^N_{(1^n)}\left(\begin{array}{|c|}\hline s_1\\\hline \vdots\\\hline s_n\\\hline\end{array}\right),\quad \zeta^{N\star}(s_1,\ldots,s_n)=\zeta^N_{(n)}\left(\begin{array}{|c|c|c|}\hline s_1&\cdots&s_n\\\hline\end{array}\right).$$

Notice that $\lim\limits_{N\to\infty}\zeta^N_\lambda(\boldsymbol{s})=\zeta_\lambda(\boldsymbol{s})$ when $\boldsymbol{s}\in W_\lambda$. Similarly to (1.1), we have the expressions

$$\zeta^{N\star}(\boldsymbol{s})=\sum_{\boldsymbol{t}\preceq\boldsymbol{s}}\zeta^N(\boldsymbol{t}),\quad \zeta^N(\boldsymbol{s})=\sum_{\boldsymbol{t}\preceq\boldsymbol{s}}(-1)^{n-\ell(\boldsymbol{t})}\zeta^{N\star}(\boldsymbol{t}). \tag{3.1}$$

3.1 A proof of the Jacobi-Trudi formula of H-type

3.1.1 Rim decomposition of partition

A skew Young diagram θ is a diagram obtained as a set difference of two Young diagrams of partitions λ and μ satisfying $\mu\subset\lambda$, that is $\mu_i\le\lambda_i$ for all i. In this case, we write $\theta=\lambda/\mu$. It is called a *ribbon* if it is connected and contains no 2×2 block of boxes. Let λ be a partition. The maximal outer ribbon of λ is called the *rim of* λ. We can peel the diagram λ off into successive rims $\theta_t,\theta_{t-1},\ldots,\theta_1$ beginning from the outside of λ. We call $\Theta=(\theta_1,\ldots,\theta_t)$ a *rim decomposition of* λ. In other words, we consider a sequence of Young diagrams $\emptyset=\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(t)}=\lambda$ such that $\lambda^{(i-1)}\subset\lambda^{(i)}$ and $\lambda^{(i)}/\lambda^{(i-1)}$ is the ribbon θ_i for all $1\le i\le t$.

Example 3.1. The following $\Theta=(\theta_1,\theta_2,\theta_3,\theta_4)$ is a rim decomposition of $\lambda=(4,3,3,2)$;

$$\Theta=\begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&3&3&\\\cline{1-3} 2&3&4&\\\cline{1-3} 3&3&&\\\cline{1-2}\end{array},$$

which means that $\theta_1=\begin{array}{|c|c|}\hline &\\\hline\end{array}$, $\theta_2=\begin{array}{|c|}\hline \\\hline \\\hline\end{array}$, $\theta_3=\begin{array}{cccc}\cline{3-4} & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{}\\\cline{2-4} & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \\\cline{2-3} & \multicolumn{1}{|c|}{} & & \\\cline{1-2} \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & & \\\cline{1-2}\end{array}$ and $\theta_4=\begin{array}{|c|}\hline \\\hline\end{array}$.

Write $\lambda=(\lambda_1,\ldots,\lambda_r)$. We call a rim decomposition $\Theta=(\theta_1,\ldots,\theta_r)$ of λ an H-*rim decomposition* if each θ_i starts from $(i,1)$ for all $1\le i\le r$. Here, we permit $\theta_i=\emptyset$. We denote Rim^λ_H by the set of all H-rim decompositions of λ.

Example 3.2. The following $\Theta=(\theta_1,\theta_2,\theta_3,\theta_4)$ is an H-rim decomposition of $\lambda=(4,3,3,2)$;

$$\Theta=\begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 3&3&3&\\\cline{1-3} 3&4&4&\\\cline{1-3} 4&4&&\\\cline{1-2}\end{array},$$

which means that $\theta_1=\begin{array}{|c|c|}\hline &\\\hline\end{array}$, $\theta_2=\emptyset$, $\theta_3=\begin{array}{cccc}\cline{3-4} & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{}\\\cline{1-4} \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \\\cline{1-3} \multicolumn{1}{|c|}{} & & & \\\cline{1-1}\end{array}$ and $\theta_4=\begin{array}{ccc}\cline{2-3} & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{}\\\cline{1-3} \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \\\cline{1-2}\end{array}$. Note that the rim decomposition appeared in Example 3.1 is not an H-rim decomposition.

Remark 3.3. The H-rim decompositions are also appeared in [ELW], where they are called the *flat special rim-hooks*. They are used to compute the coefficients of the linear expansion of a given symmetric function via Schur functions.

3.1.2 Patterns on the $\mathbb{Z}^2$ lattice

Fix $N \in \mathbb{N}$. For a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, let a_i and b_i be lattice points in $\mathbb{Z}^2$ respectively given by $a_i = (r+1-i, 1)$ and $b_i = (r+1-i+\lambda_i, N)$ for $1 \le i \le r$. Put $A = (a_1, \dots, a_r)$ and $B = (b_1, \dots, b_r)$. An *H-pattern* corresponding to λ is a tuple $L = (l_1, \dots, l_r)$ of directed paths on $\mathbb{Z}^2$, whose directions are allowed only to go one to the right or one up, such that l_i starts from a_i and ends to $b_{\sigma(i)}$ for some $\sigma \in S_r$. We call such $\sigma \in S_r$ the *type* of L and denote it by $\sigma = \mathrm{type}(L)$. Note that the type of an H-pattern does not depend on N. The number of horizontal edges appearing in the path l_i is called the *horizontal distance* of l_i and is denoted by $\mathrm{hd}(l_i)$. When $\mathrm{type}(L) = \sigma$, we simply write $L : A \to B^\sigma$ where $B^\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(r)})$ and $l_i : a_i \to b_{\sigma(i)}$. It is easy to see that $\mathrm{hd}(l_i) = \lambda_{\sigma(i)} - \sigma(i) + i$ and $\sum_{i=1}^r \mathrm{hd}(l_i) = |\lambda|$.

Let $\mathcal{H}_\lambda^N$ be the set of all H-patterns corresponding to λ and $S_H^\lambda = \{\mathrm{type}(L) \in S_r \mid L \in \mathcal{H}_\lambda^N\}$. The following is a key lemma of our study, which is easily verified.

Lemma 3.4. *For $\Theta = (\theta_1, \dots, \theta_r) \in \mathrm{Rim}_H^\lambda$, there exists $L = (l_1, \dots, l_r) \in \mathcal{H}_\lambda^N$ such that $\mathrm{hd}(l_i) = |\theta_i|$ for all $1 \le i \le r$. Moreover, the map $\tau_H : \mathrm{Rim}_H^\lambda \to S_H^\lambda$ given by $\tau_H(\Theta) = \mathrm{type}(L)$ is a bijection.*

Example 3.5. Let $\lambda = (4, 3, 3, 2)$. Then, we have $\tau_H(\Theta) = (1243) \in S_4$ where Θ is the H-rim decomposition of λ appeared in Example 3.2.

3.1.3 Weight of patterns

Fix $\boldsymbol{s} = (s_{ij}) \in T(\lambda, \mathbb{C})$. We next assign a weight to $L = (l_1, \dots, l_r) \in \mathcal{H}_\lambda^N$ via the H-rim decomposition of λ as follows. Take $\Theta = (\theta_1, \dots, \theta_r) \in \mathrm{Rim}_H^\lambda$ such that $\tau_H(\Theta) = \mathrm{type}(L)$. Then, when the kth horizontal edge of l_i is on the jth row, we weight it with $\frac{1}{j^{s_{pq}}}$ where $(p, q) \in D(\lambda)$ is the kth component of θ_i. Now, the weight $w_{\boldsymbol{s}}^N(l_i)$ of the path l_i is defined to be the product of weights of all horizontal edges along l_i. Here, we understand that $w_{\boldsymbol{s}}^N(l_i) = 1$ if $\theta_i = \emptyset$. Moreover, we define the weight $w_{\boldsymbol{s}}^N(L)$ of $L \in \mathcal{H}_\lambda^N$ by

$$w_{\boldsymbol{s}}^N(L) = \prod_{i=1}^r w_{\boldsymbol{s}}^N(l_i).$$

Example 3.6. Let $\lambda = (4, 3, 3, 2)$. Consider the following $L = (l_1, l_2, l_3, l_4) \in \mathcal{H}_{(4,3,3,2)}^4$;

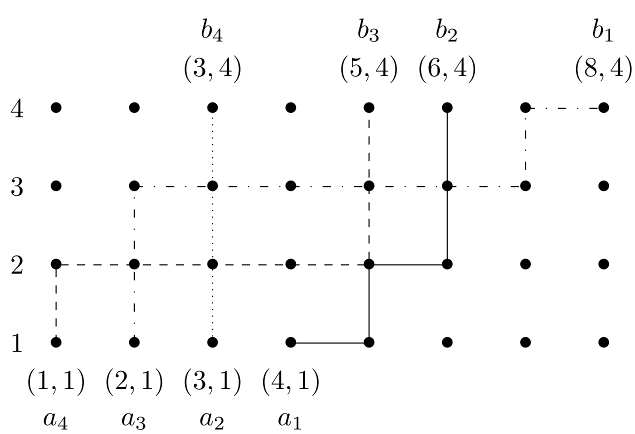

Figure 1: $L = (l_1, l_2, l_3, l_4) \in \mathcal{H}_{(4,3,3,2)}^4$

Since type$(L) = (1243)$, the corresponding H-rim decomposition of λ is nothing but the one appeared in Example 3.2.

Let $\boldsymbol{s} = \begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline e & f & g \\ \cline{1-3} h & i & j \\ \cline{1-3} k & l \\ \cline{1-2} \end{array} \in T((4,3,3,2),\mathbb{C})$. Then, the weight of l_i are given by

$$w_{\boldsymbol{s}}^4(l_1) = \frac{1}{1^a 2^b}, \quad w_{\boldsymbol{s}}^4(l_2) = 1, \quad w_{\boldsymbol{s}}^4(l_3) = \frac{1}{3^h 3^e 3^f 3^g 3^c 4^d}, \quad w_{\boldsymbol{s}}^4(l_4) = \frac{1}{2^k 2^l 2^i 2^j}.$$

In particular, when $\boldsymbol{s} = \begin{array}{|c|c|c|c|} \hline a_0 & a_1 & a_2 & a_3 \\ \hline a_{-1} & a_0 & a_1 \\ \cline{1-3} a_{-2} & a_{-1} & a_0 \\ \cline{1-3} a_{-3} & a_{-2} \\ \cline{1-2} \end{array} \in T^{\mathrm{diag}}((4,3,3,2),\mathbb{C})$, these are equal to

$$w_{\boldsymbol{s}}^4(l_1) = \frac{1}{1^{a_0} 2^{a_1}}, \quad w_{\boldsymbol{s}}^4(l_2) = 1, \quad w_{\boldsymbol{s}}^4(l_3) = \frac{1}{3^{a_{-2}} 3^{a_{-1}} 3^{a_0} 3^{a_1} 3^{a_2} 4^{a_3}}, \quad w_{\boldsymbol{s}}^4(l_4) = \frac{1}{2^{a_{-3}} 2^{a_{-2}} 2^{a_{-1}} 2^{a_0}}.$$

Notice that, in this case, from the definition of the weight, the tuple of indexes of the exponent of the denominator of $w_{\boldsymbol{s}}^4(l_i)$ along l_i should be equal to $(a_{1-i}, a_{1-i+1}, a_{1-i+2}, \ldots)$ for all i.

3.1.4 Proof

A proof of (1.2) is given by calculating the sum

$$X_\lambda^N(\boldsymbol{s}) = \sum_{L\in\mathcal{H}_\lambda^N} \varepsilon_{\mathrm{type}(L)} w_{\boldsymbol{s}}^N(L) = \sum_{\sigma\in S_H^\lambda} \varepsilon_\sigma \sum_{L:A\to B^\sigma} w_{\boldsymbol{s}}^N(L),$$

where ε_σ is the signature of $\sigma \in S_r$. First, the inner sum can be calculated as follows.

Lemma 3.7. *For $\sigma \in S_H^\lambda$, let $\Theta^\sigma = (\theta_1^\sigma, \ldots, \theta_r^\sigma) \in \mathrm{Rim}_H^\lambda$ be the H-rim decomposition such that $\tau_H(\Theta^\sigma) = \sigma$. Then, we have*

$$\sum_{L:A\to B^\sigma} w_{\boldsymbol{s}}^N(L) = \prod_{i=1}^r \zeta^{N\star}(\theta_i^\sigma(\boldsymbol{s})).$$

Here, for $\Theta = (\theta_1, \ldots, \theta_r) \in \mathrm{Rim}_H^\lambda$, $\theta_i(\boldsymbol{s}) \in \mathbb{C}^{|\theta_i|}$ is the tuple obtained by reading contents of the shape restriction of $\boldsymbol{s}$ to θ_i from the bottom left to the top right.

Proof. Let $L = (l_1, \ldots, l_r) \in \mathcal{H}_\lambda^N$ be an H-pattern of type σ. Then l_i is a path from a_i to $b_{\sigma(i)}$ with $\mathrm{hd}(l_i) = \lambda_{\sigma(i)} - \sigma(i) + i = |\theta_i^\sigma|$. For simplicity, write $k_i = \lambda_{\sigma(i)} - \sigma(i) + i$ and $\theta_i^\sigma(\boldsymbol{s}) = (s_{i,1}, \ldots, s_{i,k_i})$. Suppose that l_i has n_j steps on the jth row for $1 \le j \le N$. Then, from the definition of the weight, we have

$$w_{\boldsymbol{s}}^N(l_i) = \underbrace{\frac{1}{1^{s_{i,1}}} \cdots \frac{1}{1^{s_{i,n_1}}}}_{n_1 \text{ terms}} \underbrace{\frac{1}{2^{s_{i,n_1+1}}} \cdots \frac{1}{2^{s_{i,n_1+n_2}}}}_{n_2 \text{ terms}} \cdots \underbrace{\frac{1}{N^{s_{i,n_1+\cdots+n_{N-1}+1}}} \cdots \frac{1}{N^{s_{i,n_1+\cdots+n_N}}}}_{n_N \text{ terms}}$$

with $n_1+\cdots+n_N=k_i$. This shows that

$$\sum_{L:A\to B^\sigma} w_{\boldsymbol{s}}^N(L)=\prod_{i=1}^r \sum_{l_i:a_i\to b_{\sigma(i)}} w_{\boldsymbol{s}}^N(l_i)$$
$$=\prod_{i=1}^r \sum_{1\le m_1\le\cdots\le m_{k_i}\le N} \frac{1}{m_1^{s_{i,1}}\cdots m_{k_i}^{s_{i,k_i}}}$$
$$=\prod_{i=1}^r \zeta^{N\star}(s_{i,1},\dots,s_{i,k_i}).$$

□

From Lemma 3.7, we have

$$X_\lambda^N(\boldsymbol{s})=\sum_{\sigma\in S_H^\lambda}\varepsilon_\sigma\prod_{i=1}^r\zeta^{N\star}\left(\theta_i^\sigma(\boldsymbol{s})\right). \tag{3.2}$$

Let $\mathcal{H}_{\lambda,0}^N$ be the set of all $L=(l_1,\dots,l_r)\in\mathcal{H}_\lambda^N$ such that any distinct pair of l_i and l_j has no intersection. Define

$$X_{\lambda,0}^N(\boldsymbol{s})=\sum_{L\in\mathcal{H}_{\lambda,0}^N}\varepsilon_{\mathrm{type}(L)}w_{\boldsymbol{s}}^N(L),\quad X_{\lambda,1}^N(\boldsymbol{s})=\sum_{L\in\mathcal{H}_\lambda^N\setminus\mathcal{H}_{\lambda,0}^N}\varepsilon_{\mathrm{type}(L)}w_{\boldsymbol{s}}^N(L).$$

Clearly we have $X_\lambda^N(\boldsymbol{s})=X_{\lambda,0}^N(\boldsymbol{s})+X_{\lambda,1}^N(\boldsymbol{s})$. Moreover, since $\mathrm{type}(L)=\mathrm{id}$ for all $L\in\mathcal{H}_{\lambda,0}^N$ where id is the identity element of S_r and id corresponds to the trivial H-rim decomposition $(\theta_1,\dots,\theta_r)=((\lambda_1),\dots,(\lambda_r))$, employing the well-known identification between non-intersecting lattice paths and semi-standard Young tableaux, we have

$$X_{\lambda,0}^N(\boldsymbol{s})=\sum_{L\in\mathcal{H}_{\lambda,0}^N}w_{\boldsymbol{s}}^N(L)=\zeta_\lambda^N(\boldsymbol{s}).$$

Therefore, from (3.2), we reach the expression

$$\zeta_\lambda^N(\boldsymbol{s})=\sum_{\sigma\in S^\lambda}\varepsilon_\sigma\prod_{i=1}^r\zeta^{N\star}\left(\theta_i^\sigma(\boldsymbol{s})\right)-X_{\lambda,1}^N(\boldsymbol{s}). \tag{3.3}$$

Now, one can obtain (1.2) by taking the limit $N\to\infty$ of (3.3) under suitable assumptions on $\boldsymbol{s}$ described in Theorem 1.1 together with the following proposition.

Proposition 3.8. *Assume that $\boldsymbol{s}=(s_{ij})\in T^{\mathrm{diag}}(\lambda,\mathbb{C})$. Write $a_k=s_{i,i+k}$ for $k\in\mathbb{Z}$.*

(1) *We have*

$$X_\lambda^N(\boldsymbol{s})=\det\left[\zeta^{N\star}(a_{-j+1},a_{-j+2},\dots,a_{-j+(\lambda_i-i+j)})\right]_{1\le i,j\le r}. \tag{3.4}$$

(2) *It holds that*

$$X_{\lambda,1}^N(\boldsymbol{s})=0. \tag{3.5}$$

Proof. We first notice that, if $\boldsymbol{s} = (s_{ij}) \in T^{\mathrm{diag}}(\lambda, \mathbb{C})$, then we have

$$\theta_i^\sigma(\boldsymbol{s}) = (a_{1-i}, a_{1-i+1}, \ldots, a_{1-i+(\lambda_{\sigma(i)}-\sigma(i)+i)-1})$$

for all $1 \le i \le r$. Therefore, understanding that $\zeta^N_{(k)} = 0$ for $k < 0$, from (3.2), we have

$$\begin{aligned} X^N_\lambda(\boldsymbol{s}) &= \sum_{\sigma \in S^\lambda} \varepsilon_\sigma \prod_{i=1}^r \zeta^{N\star}(\theta_i^\sigma(\boldsymbol{s})) \\ &= \sum_{\sigma \in S_r} \varepsilon_\sigma \prod_{i=1}^r \zeta^{N\star}(a_{1-i}, a_{1-i+1}, \ldots, a_{1-i+(\lambda_{\sigma(i)}-\sigma(i)+i)-1}) \\ &= \det\left[\zeta^{N\star}(a_{1-i}, a_{1-i+1}, \ldots, a_{1-i+(\lambda_j-j+i)-1})\right]_{1\le i,j\le r} \\ &= \det\left[\zeta^{N\star}(a_{-i+1}, a_{-j+2}, \ldots, a_{-j+(\lambda_i-i+j)})\right]_{1\le i,j\le r}. \end{aligned}$$

Hence, we obtain (3.4).

We next show the second assertion. To do that, we employ the well-known involution $L \mapsto \overline{L}$ on $\mathcal{H}^N_\lambda \setminus \mathcal{H}^N_{\lambda,0}$ defined as follows. For $L = (l_1, \ldots, l_r) \in \mathcal{H}^N_\lambda \setminus \mathcal{H}^N_{\lambda,0}$ of type σ, consider the first (rightmost) intersection point appearing in L, at which two paths say l_i and l_j cross. Let $\overline{L}$ be an H-pattern that contains every paths in L except for l_i and l_j and two more paths $\overline{l_i}$ and $\overline{l_j}$. Here, $\overline{l_i}$ (resp. $\overline{l_j}$) follows l_i (resp. l_j) until it meets the first intersection point and after that follows l_j (resp. l_i) to the end. Notice that, if $\boldsymbol{s} = (s_{ij}) \in T^{\mathrm{diag}}(\lambda, \mathbb{C})$, then we have $w^N_{\boldsymbol{s}}(\overline{L}) = w^N_{\boldsymbol{s}}(L)$ since there is no change of horizontal edges between L and $\overline{L}$. Moreover, we have $\mathrm{type}(\overline{L}) = -\mathrm{type}(L)$ because the end points of L and $\overline{L}$ are just switched. These imply that

$$\begin{aligned} X^N_{\lambda,1}(\boldsymbol{s}) &= \sum_{L \in \mathcal{H}^N_\lambda \setminus \mathcal{H}^N_{\lambda,0}} \varepsilon_{\mathrm{type}(\overline{L})} w^N_{\boldsymbol{s}}(\overline{L}) \\ &= -\sum_{L \in \mathcal{H}^N_\lambda \setminus \mathcal{H}^N_{\lambda,0}} \varepsilon_{\mathrm{type}(L)} w^N_{\boldsymbol{s}}(L) \\ &= -X^N_{\lambda,1}(\boldsymbol{s}) \end{aligned}$$

and therefore lead (3.5). □

Remark 3.9. When $\boldsymbol{s} \in T^{\mathrm{diag}}(\lambda, \mathbb{C})$, (3.3) can be also written in terms of the H-rim decomposition as follows;

$$\zeta^N_\lambda(\boldsymbol{s}) = \sum_{\Theta=(\theta_1,\theta_2,\ldots,\theta_r)\in \mathrm{Rim}^\lambda_H} \varepsilon_H(\Theta) \zeta^{N\star}(\theta_1(\boldsymbol{s}))\, \zeta^{N\star}(\theta_2(\boldsymbol{s})) \cdots \zeta^{N\star}(\theta_r(\boldsymbol{s})), \tag{3.6}$$

where $\varepsilon_H(\Theta) = \varepsilon_{\tau_H(\Theta)}$. Note that $\varepsilon(\Theta) = (-1)^{n-\#\{i\,|\,\theta_i\neq\emptyset\}}$ when $\lambda = (1^n)$.

Remark 3.10. In some cases, $X^N_\lambda(\boldsymbol{s})$ actually has a determinant expression without the assumption on variables;

$$X^N_{(2,2)}\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}\right) = \begin{vmatrix} \zeta^{N\star}(a,b) & \zeta^{N\star}(c,d,b) \\ \zeta^{N\star}(a) & \zeta^{N\star}(c,d) \end{vmatrix},$$

$$X^N_{(2,2,1)}\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline e & \\ \cline{1-1}\end{array}\right) = \begin{vmatrix} \zeta^{N\star}(a,b) & \zeta^{N\star}(c,d,b) & \zeta^{N\star}(e,c,d,b) \\ \zeta^{N\star}(a) & \zeta^{N\star}(c,d) & \zeta^{N\star}(e,c,d) \\ 0 & 1 & \zeta^{N\star}(e) \end{vmatrix}.$$

However, in general, $X_\lambda^N(\boldsymbol{s})$ can not be written as a determinant. For example, we have

$$
\begin{aligned}
X^N_{(2,2,2)}\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline e & f \\ \hline \end{array}\right) &= \zeta^{N\star}(a,b)\zeta^{N\star}(c,d)\zeta^{N\star}(e,f) - \zeta^{N\star}(a,b)\zeta^{N\star}(c)\zeta^{N\star}(e,f,d) \\
&\quad - \zeta^{N\star}(c,a)\zeta^{N\star}(e,f,d,b) - \zeta^{N\star}(a)\zeta^{N\star}(c,d,b)\zeta^{N\star}(e,f) \\
&\quad + \zeta^{N\star}(c,a,b)\zeta^{N\star}(e,f,d) + \zeta^{N\star}(a)\zeta^{N\star}(c)\zeta^{N\star}(e,f,d,b)
\end{aligned}
$$

and see that the righthand side does not seem to be expressed as a determinant (but is close to the determinant).

Similarly, $X^N_{\lambda,1}(\boldsymbol{s})$ does not vanish in general. For example,

$$
\begin{aligned}
X^2_{(2,2),1}\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}\right) &= \Bigg(\frac{1}{1^a1^b1^c1^d} + \frac{1}{1^a1^b1^c2^d} + \frac{1}{1^a2^b1^c1^d} + \frac{1}{1^a2^b1^c2^d} \\
&\qquad \frac{1}{1^a2^b2^c2^d} + \frac{1}{2^a2^b1^c1^d} + \frac{1}{2^a2^b1^c2^d} + \frac{1}{2^a2^b2^c2^d}\Bigg) \\
&\quad - \Bigg(\frac{1}{1^a1^b1^c1^d} + \frac{1}{1^a2^b1^c1^d} + \frac{1}{1^a2^b1^c2^d} + \frac{1}{1^a2^b2^c2^d} \\
&\qquad + \frac{1}{2^a1^b1^c1^d} + \frac{1}{2^a2^b1^c1^d} + \frac{1}{2^a2^b1^c2^d} + \frac{1}{2^a2^b2^c2^d}\Bigg) \\
&= \frac{1}{1^a1^b1^c2^d} - \frac{1}{2^a1^b1^c1^d},
\end{aligned}
$$

which actually vanishes when $a = d$.

3.2 A proof of the Jacobi-Trudi formula of E-type

To prove (1.3), we need to consider another type of patterns on the $\mathbb{Z}^2$ lattice. Because the discussion are essentially the same as the previous subsection, we omit all proofs of the results in this subsection.

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition and $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ the conjugate of λ. A rim decomposition $\Theta = (\theta_1, \dots, \theta_s)$ of λ is called an *E-rim decomposition* if each θ_i starts from $(1, i)$ for all $1 \le i \le s$. Here, we again permit $\theta_i = \emptyset$. We denote by Rim_E^λ the set of all E-rim decompositions of λ.

Example 3.11. The following $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ is an E-rim decomposition of $\lambda = (4, 3, 3, 2)$;

$$
\Theta = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array},
$$

which means that $\theta_1 = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}$, $\theta_2 = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$, $\theta_3 = \begin{array}{ccc} \cline{3-3} & & \multicolumn{1}{|c|}{} \\ \cline{3-3} & & \multicolumn{1}{|c|}{} \\ \cline{2-3} & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \\ \cline{1-2} \end{array}$ and $\theta_4 = \begin{array}{|c|}\hline \\ \hline \end{array}$.

Fix $N \in \mathbb{N}$. Let c_i and d_i be lattice points in $\mathbb{Z}^2$ respectively given by $c_i = (s + 1 - i, 1)$ and $d_i = (s + 1 - i + \lambda'_i, N + 1)$ for $1 \le i \le s$. Put $C = (c_1, \dots, c_s)$ and $D = (d_1, \dots, d_s)$.

An *E-pattern* corresponding to λ is a tuple $L = (l_1, \ldots, l_s)$ of directed paths on $\mathbb{Z}^2$, whose directions are allowed only to go one to the northeast or one up, such that l_i starts from c_i and ends to $d_{\sigma(i)}$ for some $\sigma \in S_s$. We also call such $\sigma \in S_s$ the *type* of L and denote it by $\sigma = \mathrm{type}(L)$. The number of northeast edges appearing in the path l_i is called the *northeast distance* of l_i and is denoted by $\mathrm{ned}(l_i)$. When $\mathrm{type}(L) = \sigma$, we simply write $L : C \to D^\sigma$ where $D^\sigma = (d_{\sigma(1)}, \ldots, d_{\sigma(s)})$ and $l_i : c_i \to d_{\sigma(i)}$. It is easy to see that $\mathrm{ned}(l_i) = \lambda'_{\sigma(i)} - \sigma(i) + i$ and $\sum_{i=1}^{s} \mathrm{ned}(l_i) = |\lambda|$.

Let $\mathcal{E}_\lambda^N$ be the set of all E-patterns corresponding to λ and $S_E^\lambda = \{\mathrm{type}(L) \in S_s \mid L \in \mathcal{E}_\lambda^N\}$.

Lemma 3.12. *For $\Theta = (\theta_1, \ldots, \theta_s) \in \mathrm{Rim}_E^\lambda$, there exists $L = (l_1, \ldots, l_s) \in \mathcal{E}_\lambda^N$ such that $\mathrm{ned}(l_i) = |\theta_i|$ for all $1 \le i \le s$. Moreover, the map $\tau_E : \mathrm{Rim}_E^\lambda \to S_E^\lambda$ given by $\tau_E(\Theta) = \mathrm{type}(L)$ is a bijection.*

Fix $\boldsymbol{s} = (s_{ij}) \in T(\lambda, \mathbb{C})$. A weight on $L = (l_1, \ldots, l_s) \in \mathcal{E}_\lambda^N$ is similarly defined via the E-rim decomposition of λ as follows. Take $\Theta = (\theta_1, \ldots, \theta_s) \in \mathrm{Rim}_E^\lambda$ such that $\tau_E(\Theta) = \mathrm{type}(L)$. Then, when the kth northeast edge of l_i lies from the jth row to $(j+1)$th row, we weight it with $\frac{1}{j^{s_{pq}}}$ where $(p, q) \in D(\lambda)$ is the kth component of θ_i. Now, the weight $w_{\boldsymbol{s}}^N(l_i)$ of the path l_i is defined to be the product of weights of all northeast edges along l_i. Here, we understand that $w_{\boldsymbol{s}}^N(l_i) = 1$ if $\theta_i = \emptyset$. Moreover, we define the weight $w_{\boldsymbol{s}}^N(L)$ of $L \in \mathcal{E}_\lambda^N$ by

$$w_{\boldsymbol{s}}^N(L) = \prod_{i=1}^{s} w_{\boldsymbol{s}}^N(l_i).$$

Example 3.13. Let $\lambda = (4, 3, 3, 2)$. Consider the E-rim decomposition $\Theta \in \mathrm{Rim}_E^\lambda$ of λ appeared in Example 3.11. It is easy to see that $\tau_E(\Theta) = (123) \in S_4$ via the following $L = (l_1, l_2, l_3, l_4) \in \mathcal{E}_{(4,3,3,2)}^6$;

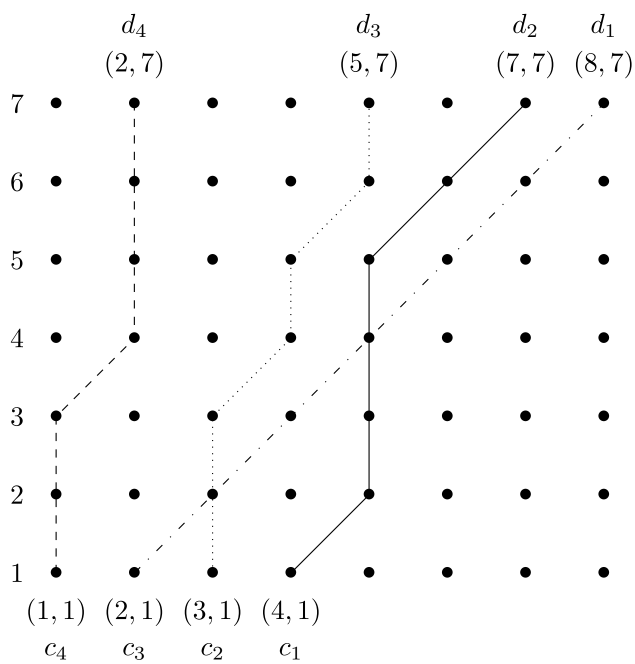

Figure 2: $L = (l_1, l_2, l_3, l_4) \in \mathcal{E}_{(4,3,3,2)}^6$

Let $\boldsymbol{s} = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & g & \\ \cline{1-3} h & i & j & \\ \cline{1-3} k & l & & \\ \cline{1-2}\end{array} \in T((4,3,3,2),\mathbb{C})$. Then, the weight of l_i are given by

$$w_{\boldsymbol{s}}^4(l_1) = \frac{1}{1^a 5^e 6^h}, \quad w_{\boldsymbol{s}}^4(l_2) = \frac{1}{3^b 5^f}, \quad w_{\boldsymbol{s}}^4(l_3) = \frac{1}{1^c 2^g 3^j 4^i 5^l 6^k}, \quad w_{\boldsymbol{s}}^4(l_4) = \frac{1}{3^d}.$$

In particular, when $\boldsymbol{s} = \begin{array}{|c|c|c|c|}\hline a_0 & a_1 & a_2 & a_3 \\ \hline a_{-1} & a_0 & a_1 & \\ \cline{1-3} a_{-2} & a_{-1} & a_0 & \\ \cline{1-3} a_{-3} & a_{-2} & & \\ \cline{1-2}\end{array} \in T^{\mathrm{diag}}((4,3,3,2),\mathbb{C})$, these are equal to

$$w_{\boldsymbol{s}}^4(l_1) = \frac{1}{1^{a_0} 5^{a_{-1}} 6^{a_{-2}}}, \quad w_{\boldsymbol{s}}^4(l_2) = \frac{1}{3^{a_1} 5^{a_0}}, \quad w_{\boldsymbol{s}}^4(l_3) = \frac{1}{1^{a_2} 2^{a_1} 3^{a_0} 4^{a_{-1}} 5^{a_{-2}} 6^{a_{-3}}}, \quad w_{\boldsymbol{s}}^4(l_4) = \frac{1}{3^{a_3}}.$$

Notice that, in this case, from the definition of the weight, the tuple of indexes of the exponent of the denominator of $w_{\boldsymbol{s}}^4(l_i)$ along l_i should be equal to $(a_{-1+i}, a_{-1+i-1}, a_{-1+i-2}, \dots)$ for all i.

We similarly give a proof of (1.3) by calculating the sum

$$Y_\lambda^N(\boldsymbol{s}) = \sum_{L \in \mathcal{E}_\lambda^N} \varepsilon_{\mathrm{type}(L)} w_{\boldsymbol{s}}^N(L) = \sum_{\sigma \in S_E^\lambda} \varepsilon_\sigma \sum_{L: C \to D^\sigma} w_{\boldsymbol{s}}^N(L),$$

Lemma 3.14. *For $\sigma \in S_E^\lambda$, let $\Theta^\sigma = (\theta_1^\sigma, \dots, \theta_s^\sigma) \in \mathrm{Rim}_E^\lambda$ be the E-rim decomposition such that $\tau_E(\Theta^\sigma) = \sigma$. Then, we have*

$$\sum_{L: C \to D^\sigma} w_{\boldsymbol{s}}^N(L) = \prod_{i=1}^s \zeta^N\left(\theta_i^\sigma(\boldsymbol{s})\right).$$

Here, for $\Theta = (\theta_1, \dots, \theta_s) \in \mathrm{Rim}_E^\lambda$, $\theta_i(\boldsymbol{s}) \in \mathbb{C}^{|\theta_i|}$ is the tuple obtained by reading contents of the shape restriction of $\boldsymbol{s}$ to θ_i from the top right to the bottom left.

From Lemma 3.14, we have

$$Y_\lambda^N(\boldsymbol{s}) = \sum_{\sigma \in S_E^\lambda} \varepsilon_\sigma \prod_{i=1}^s \zeta^N\left(\theta_i^\sigma(\boldsymbol{s})\right). \tag{3.7}$$

Define $\mathcal{E}_{\lambda,0}^N$ similarly to $\mathcal{H}_{\lambda,0}^N$ and also $Y_{\lambda,0}^N(\boldsymbol{s})$ and $Y_{\lambda,1}^N(\boldsymbol{s})$. It holds that

$$Y_{\lambda,0}^N(\boldsymbol{s}) = \sum_{L \in \mathcal{E}_{\lambda,0}^N} w_{\boldsymbol{s}}^N(L) = \zeta_\lambda^N(\boldsymbol{s}).$$

Hence, from (3.7), we reach the expression

$$\zeta_\lambda^N(\boldsymbol{s}) = \sum_{\sigma \in S_E^\lambda} \varepsilon_\sigma \prod_{i=1}^s \zeta^N\left(\theta_i^\sigma(\boldsymbol{s})\right) - Y_{\lambda,1}^N(\boldsymbol{s}). \tag{3.8}$$

Now, (1.3) is obtained by taking the limit $N \to \infty$ of (3.8) under suitable assumptions on $\boldsymbol{s}$ described in Theorem 1.1 together with the following proposition.

Proposition 3.15. *Assume that* $\boldsymbol{s} = (s_{ij}) \in T^{\mathrm{diag}}(\lambda, \mathbb{C})$. *Write* $a_k = s_{i,i+k}$ *for* $k \in \mathbb{Z}$.

(1) *We have*

$$Y_\lambda^N(\boldsymbol{s}) = \det\left[\zeta^N(a_{j-1}, a_{j-2}, \ldots, a_{j-(\lambda_i'-i+j)})\right]_{1\le i,j\le s}.$$

(2) *It holds that*

$$Y_{\lambda,1}^N(\boldsymbol{s}) = 0.$$

Remark 3.16. When $\boldsymbol{s} \in T^{\mathrm{diag}}(\lambda, \mathbb{C})$, (3.8) can be also written in terms of the E-rim decomposition as follows;

$$\zeta_\lambda^N(\boldsymbol{s}) = \sum_{\Theta=(\theta_1,\theta_2,\ldots,\theta_s)\in \mathrm{Rim}_E^\lambda} \varepsilon_E(\Theta)\zeta^N\left(\theta_1(\boldsymbol{s})\right)\zeta^N\left(\theta_2(\boldsymbol{s})\right)\cdots\zeta^N\left(\theta_s(\boldsymbol{s})\right), \tag{3.9}$$

where $\varepsilon_E(\Theta) = \varepsilon_{\tau_E(\Theta)}$. Note that $\varepsilon_E(\Theta) = (-1)^{n-\#\{i\,|\,\theta_i\neq\emptyset\}}$ when $\lambda = (n)$.

4 Schur multiple zeta functions as variations of Schur functions

4.1 Schur multiple zeta functions of skew type

Our SMZFs are naturally extended to those of skew type as follows. Let λ and μ be partitions satisfying $\mu \subset \lambda$. We use the same notations $D(\lambda/\mu)$, $T(\lambda/\mu, X), T^{\mathrm{diag}}(\lambda/\mu, X)$ for a set X, $\mathrm{SSYT}(\lambda/\mu)$ and $\mathrm{SSYT}_N(\lambda/\mu)$ for a positive integer $N \in \mathbb{N}$ as the previous sections.

Let $\boldsymbol{s} = (s_{ij}) \in T(\lambda/\mu, \mathbb{C})$. We define a skew SMZF associated with λ/μ by

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \sum_{M\in \mathrm{SSYT}(\lambda/\mu)} \frac{1}{M^{\boldsymbol{s}}} \tag{4.1}$$

and its truncated sum

$$\zeta_{\lambda/\mu}^N(\boldsymbol{s}) = \sum_{M\in \mathrm{SSYT}_N(\lambda/\mu)} \frac{1}{M^{\boldsymbol{s}}},$$

where $M^{\boldsymbol{s}} = \prod_{(i,j)\in D(\lambda/\mu)} m_{ij}^{s_{ij}}$ for $M = (m_{ij}) \in \mathrm{SSYT}(\lambda/\mu)$. As we have seen in Lemma 2.1, the series (4.1) converges absolutely if $\boldsymbol{s} \in W_{\lambda/\mu}$ where $W_{\lambda/\mu}$ is also similarly defined as W_λ (note that $C(\lambda/\mu) \subset C(\lambda)$). We have again

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \sum_{\boldsymbol{t}\preceq\boldsymbol{s}} \zeta(\boldsymbol{t}), \quad \zeta_{\lambda/\mu}(\boldsymbol{s}) = \sum_{\boldsymbol{t}\preceq\boldsymbol{s}'} (-1)^{|\lambda/\mu|-\ell(\boldsymbol{t})}\zeta^\star(\boldsymbol{t}), \tag{4.2}$$

where $\preceq$ is naturally generalized to the skew types.

Example 4.1. (1) For $\boldsymbol{s} = (s_{ij}) \in W_{(2,2,2)/(1,1)}$, we have

$$\begin{aligned}
\begin{array}{cc} & \boxed{s_{12}} \\ & \boxed{s_{22}} \\ \boxed{s_{31}} & \boxed{s_{32}} \end{array}
&= \zeta(s_{31}, s_{12}, s_{22}, s_{32}) + \zeta(s_{31}+s_{12}, s_{22}, s_{32}) + \zeta(s_{12}, s_{31}+s_{22}, s_{32}), \\
&\quad + \zeta(s_{12}, s_{31}, s_{22}, s_{32}) + \zeta(s_{12}, s_{22}, s_{31}+s_{32}) + \zeta(s_{12}, s_{22}, s_{31}, s_{32}) \\
&= \zeta^\star(s_{31}, s_{12}, s_{22}, s_{32}) - \zeta^\star(s_{31}+s_{12}, s_{22}, s_{32}) - \zeta^\star(s_{31}, s_{12}+s_{22}, s_{32}) \\
&\quad - \zeta^\star(s_{31}, s_{12}, s_{22}+s_{32}) + \zeta^\star(s_{31}+s_{12}+s_{22}, s_{32}) + \zeta^\star(s_{31}+s_{12}, s_{22}+s_{32}) \\
&\quad + \zeta^\star(s_{31}, s_{12}+s_{22}+s_{32}) + \zeta^\star(s_{12}, s_{31}, s_{22}, s_{32}) - \zeta^\star(s_{12}, s_{31}, s_{22}+s_{32}) \\
&\quad + \zeta^\star(s_{12}, s_{22}, s_{31}, s_{32}) - \zeta^\star(s_{12}+s_{22}, s_{31}, s_{32}) - \zeta^\star(s_{12}, s_{22}+s_{31}, s_{32}).
\end{aligned}$$

(2) For $\boldsymbol{s} = (s_{ij}) \in W_{(3,3)/(2)}$, we have

$$\begin{array}{|c|c|c|}\cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & s_{13} \\ \hline s_{21} & s_{22} & s_{23} \\ \hline \end{array} = \zeta(s_{13}, s_{21}, s_{22}, s_{23}) + \zeta(s_{13} + s_{21}, s_{22}, s_{23}) + \zeta(s_{13}, s_{21} + s_{22}, s_{23})$$
$$+ \zeta(s_{13}, s_{21}, s_{22} + s_{23}) + \zeta(s_{13} + s_{21} + s_{22}, s_{23}) + \zeta(s_{13} + s_{21}, s_{22} + s_{23})$$
$$+ \zeta(s_{13}, s_{21} + s_{22} + s_{23}) + \zeta(s_{21}, s_{13}, s_{22}, s_{23}) + \zeta(s_{21}, s_{13} + s_{22}, s_{23})$$
$$+ \zeta(s_{21}, s_{13}, s_{22} + s_{23}) + \zeta(s_{21}, s_{22}, s_{13}, s_{23}) + \zeta(s_{21} + s_{22}, s_{13}, s_{23})$$
$$= \zeta^\star(s_{13}, s_{21}, s_{22}, s_{23}) - \zeta^\star(s_{13} + s_{21}, s_{22}, s_{23}) - \zeta^\star(s_{21}, s_{13} + s_{22}, s_{23}),$$
$$+ \zeta^\star(s_{21}, s_{13}, s_{22}, s_{23}) - \zeta^\star(s_{21}, s_{22}, s_{23} + s_{13}) + \zeta^\star(s_{21}, s_{22}, s_{13}, s_{23}).$$

As the same discussion performed in Section 2.3, one sees that $\zeta_{\lambda/\mu}(\{s\}^{\lambda/\mu}) = e^{(s)} s_{\lambda/\mu} = s_{\lambda/\mu}(1^{-s}, 2^{-s}, \ldots)$ for $s \in \mathbb{C}$ with $\Re(s) > 1$ where $s_{\lambda/\mu}$ is the skew Schur function associate with λ/μ (see [Mac]). In particular, since $s_{\lambda/\mu}$ is a symmetric function and hence can be expressed as a linear combination of the power-sum symmetric functions, we have $\zeta_{\lambda/\mu}(\{2k\}^{\lambda/\mu}) \in \mathbb{Q}\pi^{2k(|\lambda|-|\mu|)}$ for $k \in \mathbb{N}$. Notice that it is shown in [Sta] that $\zeta_{\lambda/\mu}(\{2k\}^{\lambda/\mu})$ for a special choice of λ/μ with $k = 1, 2, 3$ is involved with $f^{\lambda/\mu}$; the number of standard Young tableaux of shape λ/μ.

4.2 Macdonald's ninth variation of Schur functions

Let $W^{\mathrm{diag}}_{\lambda/\mu} = W_{\lambda/\mu} \cap T^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$. We now show that, when $\boldsymbol{s} \in W^{\mathrm{diag}}_{\lambda/\mu}$, the skew SMZF $\zeta_{\lambda/\mu}(\boldsymbol{s})$ is realized as (the limit of) a specialization of the ninth variation of skew Schur functions studied by Nakagawa, Noumi, Shirakawa and Yamada [NNSY]. As in the previous discussion, we write $a_k = s_{i,i+k}$ for $k \in \mathbb{Z}$ (and for any $i \in \mathbb{N}$) for $\boldsymbol{s} = (s_{ij}) \in W^{\mathrm{diag}}_{\lambda/\mu}$.

Let r and s be positive integers. Put $\eta = r + s$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ be partitions satisfying $\mu \subset \lambda \subset (s^r)$ (we here allow $\lambda_i = 0$ or $\mu_i = 0$) and $J = \{j_1, j_2, \ldots, j_r\}$ with $j_a = \lambda_{r+i-a} + a$ and $I = \{i_1, i_2, \ldots, i_r\}$ with $i_b = \mu_{r+i-b} + b$ the corresponding Maya diagrams, respectively. Notice that I and J are subsets of $\{1, 2, \ldots, \eta\}$ satisfying $j_1 < j_2 < \cdots < j_r$ and $i_1 < i_2 < \cdots < i_r$. Then, Macdonald's ninth variation of skew Schur function $S^{(r)}_{\lambda/\mu}(X)$ associated with a general matrix $X = [x_{ij}]_{1 \le i,j \le \eta}$ of size η is defined by

$$S^{(r)}_{\lambda/\mu}(X) = \xi^I_J(X_+).$$

Here, we have used the Gauss decomposition $X = X_- X_0 X_+$ of X where X_-, X_0 and X_+ are lower unitriangular, diagonal and upper unitriangular matrices, respectively, which are determined uniquely as matrices with entries in the field of rational functions in the variables x_{ij} for $1 \le i, j \le \eta$. Moreover, $\xi^I_J(X_+)$ is the minor determinant of X_+ corresponding to I and J. Put

$$e^{(r)}_n(X) = S^{(r)}_{(1^n)}(X) = \xi^{1,\ldots,r}_{1,\ldots,r-1,r+n}(X_+),$$
$$h^{(r)}_n(X) = S^{(r)}_{(n)}(X) = \xi^{1,\ldots,r}_{1,\ldots,\widehat{r-n+1},r+1}(X_+),$$

which are variations of the elementary and complete symmetric polynomials, respectively. Here $\widehat{r-n+1}$ means that we ignore $r-n+1$. For convenience, we put $e^{(r)}_0(X) = h^{(r)}_0(X) = 1$ and $e^{(r)}_n(X) = h^{(r)}_n(X) = 0$ for $n < 0$.

For $N \in \mathbb{N}$, let $U = U^{(N)}$ be an upper unitriangular matrix of size η defined by $U = U_1 U_2 \cdots U_N$ where

$$U_k = \left(I_\eta + u^{(1)}_k E_{12}\right)\left(I_\eta + u^{(2)}_k E_{23}\right) \cdots \left(I_\eta + u^{(\eta-1)}_k E_{\eta-1,\eta}\right).$$

Here, $u_k^{(i)}$ are variables for $1 \le k \le N$ and $1 \le i \le \eta - 1$ and I_η and E_{ij} are the identity and unit matrix of size η, respectively. The following is crucial in this section.

Lemma 4.2. *Let $\boldsymbol{s} = (s_{ij}) \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{C})$. Write $a_k = s_{i,i+k}$ for $k \in \mathbb{Z}$. If $u_k^{(i)} = k^{-a_{i-r}}$, then we have*

$$\zeta^N_{\lambda/\mu}(\boldsymbol{s}) = S^{(r)}_{\lambda/\mu}(U). \tag{4.3}$$

Proof. It is shown in [NNSY] that $S^{(r)}_{\lambda/\mu}(U)$ has a tableau representation

$$S^{(r)}_{\lambda/\mu}(U) = \sum_{(m_{ij}) \in \mathrm{SSYT}_N(\lambda/\mu)} \prod_{(i,j) \in D(\lambda/\mu)} u^{(r-i+j)}_{m_{ij}}. \tag{4.4}$$

Hence the claim immediately follows because $u^{(r-i+j)}_{m_{ij}} = m_{ij}^{-a_{j-i}} = m_{ij}^{-s_{ij}}$ if $u_k^{(i)} = k^{-a_{i-r}}$. □

As corollaries of the results in [NNSY], we obtain the following formulas for skew SMZFs.

4.3 Jacobi-Trudi formulas

It is shown in [NNSY] that $S^{(r)}_{\lambda/\mu}(X)$ satisfies the Jacobi-Trudi formulas

$$S^{(r)}_{\lambda/\mu}(X) = \det\left[h^{(\mu_j+r-j+1)}_{\lambda_i-\mu_j-i+j}(X)\right]_{1\le i,j\le r}, \tag{4.5}$$

$$S^{(r)}_{\lambda/\mu}(X) = \det\left[e^{(r-1-\mu'_j+j)}_{\lambda'_i-\mu'_j-i+j}(X)\right]_{1\le i,j\le s}, \tag{4.6}$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ and $\mu' = (\mu'_1, \ldots, \mu'_s)$ are the conjugates of λ and μ, respectively (we again allow $\lambda'_i = 0$ or $\mu'_i = 0$).

Theorem 4.3. *Retain the above notations. Assume that $\boldsymbol{s} = (s_{ij}) \in W^{\mathrm{diag}}_{\lambda/\mu}$.*

(1) *Assume further that $\Re(s_{i,\lambda_i}) > 1$ for all $1 \le i \le r$. Then, we have*

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \det\left[\zeta^\star(a_{\mu_j-j+1}, a_{\mu_j-j+2}, \ldots, a_{\mu_j-j+(\lambda_i-\mu_j-i+j)})\right]_{1\le i,j\le r}. \tag{4.7}$$

Here, we understand that $\zeta^\star(\cdots) = 1$ if $\lambda_i - \mu_j - i + j = 0$ and 0 if $\lambda_i - \mu_j - i + j < 0$.

(2) *Assume further that $\Re(s_{\lambda'_i,i}) > 1$ for all $1 \le i \le s$. Then, we have*

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \det\left[\zeta(a_{-\mu'_j+j-1}, a_{-\mu'_j+j-2}, \ldots, a_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)})\right]_{1\le i,j\le s}. \tag{4.8}$$

Here, we understand that $\zeta(\cdots) = 1$ if $\lambda'_i - \mu'_j - i + j = 0$ and 0 if $\lambda'_i - \mu'_j - i + j < 0$.

Proof. From (4.4), we have

$$e^{(r)}_n(U) = \sum_{m_1<m_2<\cdots<m_n\le N} u^{(r)}_{m_1} u^{(r-1)}_{m_2} \cdots u^{(r-n+1)}_{m_n},$$

$$h^{(r)}_n(U) = \sum_{m_1\le m_2\le\cdots\le m_n\le N} u^{(r)}_{m_1} u^{(r+1)}_{m_2} \cdots u^{(r+n-1)}_{m_n}.$$

Now, write $r' = \mu_j + r - j + 1$ and $k' = \lambda_i - \mu_j - i + j$ for simplicity. Then, we have $u_{m_i}^{(r'+i-1)} = m_i^{-a_{\mu_j-j+i}}$ if $u_k^{(i)} = k^{-a_{i-r}}$ and hence

$$\begin{aligned}
h_{\lambda_i-\mu_j-i+j}^{(\mu_j+r-j+1)}(U) &= \sum_{m_1 \leq m_2 \leq \cdots \leq m_{k'} \leq N} u_{m_1}^{(r')} u_{m_2}^{(r'+1)} \cdots u_{m_{k'}}^{(r'+k'-1)} \\
&= \sum_{m_1 \leq m_2 \leq \cdots \leq m_{k'} \leq N} m_1^{-a_{\mu_j-j+1}} m_2^{-a_{\mu_j-j+2}} \cdots m_{k'}^{-a_{\mu_j-j+k'}} \\
&= \zeta^{N\star}(a_{\mu_j-j+1}, a_{\mu_j-j+2}, \ldots, a_{\mu_j-j+(\lambda_i-\mu_j-i+j)}).
\end{aligned}$$

This shows that (4.7) follows from (4.5) by letting $N \to \infty$. Similarly, (4.6) is obtained from (4.8) via the expression

$$e_{\lambda_i'-\mu_j'-i+j}^{(r-1-\mu_j'+j)}(U) = \zeta^N(a_{-\mu_j'+j-1}, a_{-\mu_j'+j-2}, \ldots, a_{-\mu_j'+j-(\lambda_i'-\mu_j'-i+j)}).$$

$\square$

Example 4.4. When $\lambda/\mu = (4,3,2)/(2,1)$, we have

$$\begin{ytableau} \none & \none & a_2 & a_3 \\ \none & a_0 & a_1 \\ a_{-2} & a_{-1} \end{ytableau} = \begin{vmatrix} \zeta^\star(a_2,a_3) & \zeta^\star(a_0,a_1,a_2,a_3) & \zeta^\star(a_{-2},a_{-1},a_0,a_1,a_2,a_3) \\ 1 & \zeta^\star(a_0,a_1) & \zeta^\star(a_{-2},a_{-1},a_0,a_1) \\ 0 & 1 & \zeta^\star(a_{-2},a_{-1}) \end{vmatrix},$$

$$\begin{ytableau} \none & \none & a_2 & a_3 \\ \none & a_0 & a_1 \\ a_{-2} & a_{-1} \end{ytableau} = \begin{vmatrix} \zeta(a_{-2}) & \zeta(a_0,a_{-1},a_{-2}) & \zeta(a_2,a_1,a_0,a_{-1},a_{-2}) & \zeta(a_3,a_2,a_1,a_0,a_{-1},a_{-2}) \\ 1 & \zeta(a_0,a_{-1}) & \zeta(a_2,a_1,a_0,a_{-1}) & \zeta(a_3,a_2,a_1,a_0,a_{-1}) \\ 0 & 1 & \zeta(a_2,a_1) & \zeta(a_3,a_2,a_1) \\ 0 & 0 & 1 & \zeta(a_3) \end{vmatrix}.$$

4.4 Giambelli formula

For a partition λ, we define two sequences of indices $p_1, \ldots, p_t$ and $q_1, \ldots, q_t$ by $p_i = \lambda_i - i + 1$ and $q_i = \lambda_i' - i$ for $1 \leq i \leq t$ where t is the number of diagonal entries of λ. Notice that $p_1 > p_2 > \cdots > p_t > 0$ and $q_1 > q_2 > \cdots > q_t \geq 0$ and $\lambda = (p_1 - 1, \ldots, p_t - 1 \,|\, q_1, \ldots, q_t)$ is the Frobenius notation of λ. It is shown in [NNSY] that $S_\lambda^{(r)}(X)$ satisfies the Giambelli formula

$$S_\lambda^{(r)}(X) = \det\left[S_{(p_i,1^{q_j})}^{(r)}(X)\right]_{1 \leq i,j \leq t}. \tag{4.9}$$

Theorem 4.5. *Retain the above notations. Assume that $\boldsymbol{s} = (s_{ij}) \in W_\lambda^{\mathrm{diag}}$. Moreover, assume further that $\Re(s_{i,\lambda_i}) = \Re(a_{p_i-1}) > 1$ and $\Re(s_{\lambda_i',i}) = \Re(a_{-q_i}) > 1$ for $1 \leq i \leq t$. Then, we have*

$$\zeta_\lambda(\boldsymbol{s}) = \det\left[\zeta_{(p_i,1^{q_j})}(\boldsymbol{s}_{i,j})\right]_{1 \leq i,j \leq t},$$

where $\boldsymbol{s}_{i,j} = \begin{ytableau} a_0 & a_1 & a_2 & \cdots & a_{p_i-1} \\ a_{-1} \\ \vdots \\ a_{-q_j} \end{ytableau} \in W_{(p_i,1^{q_j})}.$

Proof. Putting $u_k^{(i)} = k^{-a_{i-r}}$, from (4.3) and (4.9), we have

$$\zeta_\lambda^N(\boldsymbol{s}) = S_\lambda^{(r)}(U) = \det\left[S^{(r)}_{(p_i,1^{q_j})}(U)\right]_{1\le i,j\le t} = \det\left[\zeta^N_{(p_i,1^{q_j})}(\boldsymbol{s}_{i,j})\right]_{1\le i,j\le t}.$$

This leads the desired equation by letting $N \to \infty$. □

Example 4.6. When $\lambda = (4,3,3,2) = (3,1,0\,|\,3,2,0)$, we have

$$\begin{array}{|c|c|c|c|}\hline a_0 & a_1 & a_2 & a_3 \\ \hline a_{-1} & a_0 & a_1 & \multicolumn{1}{c}{} \\ \cline{1-3} a_{-2} & a_{-1} & a_0 & \multicolumn{1}{c}{} \\ \cline{1-3} a_{-3} & a_{-2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} = \left|\begin{array}{ccc}
\begin{array}{|c|c|c|c|}\hline a_0 & a_1 & a_2 & a_3 \\ \hline \multicolumn{1}{|c|}{a_{-1}} \\ \cline{1-1} \multicolumn{1}{|c|}{a_{-2}} \\ \cline{1-1} \multicolumn{1}{|c|}{a_{-3}} \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|c|}\hline a_0 & a_1 & a_2 & a_3 \\ \hline \multicolumn{1}{|c|}{a_{-1}} \\ \cline{1-1} \multicolumn{1}{|c|}{a_{-2}} \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|c|}\hline a_0 & a_1 & a_2 & a_3 \\ \hline\end{array} \\[2em]
\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline \multicolumn{1}{|c|}{a_{-1}} \\ \cline{1-1} \multicolumn{1}{|c|}{a_{-2}} \\ \cline{1-1} \multicolumn{1}{|c|}{a_{-3}} \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline \multicolumn{1}{|c|}{a_{-1}} \\ \cline{1-1} \multicolumn{1}{|c|}{a_{-2}} \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline a_0 & a_1 \\ \hline\end{array} \\[2em]
\begin{array}{|c|}\hline a_0 \\ \hline a_{-1} \\ \hline a_{-2} \\ \hline a_{-3} \\ \hline\end{array} &
\begin{array}{|c|}\hline a_0 \\ \hline a_{-1} \\ \hline a_{-2} \\ \hline\end{array} &
\begin{array}{|c|}\hline a_0 \\ \hline\end{array}
\end{array}\right|.$$

4.5 Dual Cauchy formula

It is shown in [N1] (see also [N2]) that the dual Cauchy formula

$$\sum_{\lambda\subset(s^r)} (-1)^{|\lambda|} S_\lambda^{(r)}(X) S_{\lambda^*}^{(s)}(Y) = \Psi^{(r,s)}(X,Y) \tag{4.10}$$

holds for $X = [x_{ij}]_{1\le i,j\le\eta}$ and $Y = [y_{ij}]_{1\le i,j\le\eta}$. Here, for a partition $\lambda = (\lambda_1,\ldots,\lambda_r) \subset (s^r)$, $\lambda^* = (s-\lambda'_s,\ldots,s-\lambda'_1)$ and $\Psi^{(r,s)}(X,Y)$ is the dual Cauchy kernel defined by

$$\Psi^{(r,s)}(X,Y) = \frac{\xi^{1,\ldots,r+s}_{1,\ldots,r+s}(Z)}{\xi^{1,\ldots,r}_{1,\ldots,r}(X)\,\xi^{1,\ldots,s}_{1,\ldots,s}(Y)}, \quad Z = \left[\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1\eta} \\ \vdots & \vdots & & \vdots \\ x_{r1} & x_{r2} & \cdots & x_{r\eta} \\ \hline y_{11} & y_{12} & \cdots & y_{1\eta} \\ \vdots & \vdots & & \vdots \\ y_{s1} & y_{s2} & \cdots & y_{s\eta} \end{array}\right].$$

Remark that when both X and Y are unitriangular, we have $\Psi^{(r,s)}(X,Y) = \det(Z)$.

We now show an analogue of (4.10) for SMZFs. To do that, we first simplify the formula (4.10) in the case where $X = U$ and $Y = V$. Here, for $M \in \mathbb{N}$, $V = V^{(M)}$ is an upper unitriangular matrix of size η similarly defined as U, that is, $V = V_1V_2\cdots V_M$ where

$$V_k = \left(I_\eta + v_k^{(1)}E_{12}\right)\left(I_\eta + v_k^{(2)}E_{23}\right)\cdots\left(I_\eta + v_k^{(\eta-1)}E_{\eta-1,\eta}\right)$$

with $v_k^{(i)}$ being variables for $1 \le k \le M$ and $1 \le i \le \eta - 1$.

Write $U = [u_{ij}]_{1\le i,j\le \eta}$ and $V = [v_{ij}]_{1\le i,j\le \eta}$. We first show that

$$u_{ij} = \begin{cases} h^{(i)}_{j-i}(U) & i \le j, \\ 0 & i > j, \end{cases} \qquad v_{ij} = \begin{cases} h^{(i)}_{j-i}(V) & i \le j, \\ 0 & i > j. \end{cases} \tag{4.11}$$

Since these are clearly equivalent, let us show only the former. Because U is an upper unitiangular matrix, we have $u_{ij} = 0$ unless $i \le j$. When $i \le j$, we have

$$u_{ij} = \sum_{l_1,\dots,l_{N-1}=1}^{\eta} (U_1)_{i,l_1}(U_2)_{l_1,l_2}\cdots(U_N)_{l_{N-1},j}.$$

Here, for a matrix A, we denote by $(A)_{i,j}$ the (i,j) entry of A. Since

$$(U_k)_{a,b} = \begin{cases} \prod_{h=0}^{b-a-1} u_k^{(a+h)} & a \le b, \\ 0 & a > b, \end{cases}$$

we have

$$u_{ij} = \sum_{i\le l_1\le\cdots\le l_{N-1}\le j} \left(\prod_{h_1=0}^{l_1-i-1} u_1^{(i+h_1)}\right)\left(\prod_{h_2=0}^{l_2-l_1-1} u_2^{(l_1+h_2)}\right)\cdots\left(\prod_{h_N=0}^{j-l_{N-1}-1} u_N^{(l_{N-1}+h_N)}\right).$$

Furthermore, writing $j = i + p$, we have

$$\begin{aligned} u_{i,i+p} &= \sum_{i\le l_1\le\cdots\le l_{N-1}\le i+p} \left(\prod_{h_1=0}^{l_1-i-1} u_1^{(i+h_1)}\right)\left(\prod_{h_2=0}^{l_2-l_1-1} u_2^{(l_1+h_2)}\right)\cdots\left(\prod_{h_N=0}^{i+p-l_{N-1}-1} u_N^{(l_{N-1}+h_N)}\right) \\ &= \sum_{1\le m_1\le\cdots\le m_p\le N} u_{m_1}^{(i)} u_{m_2}^{(i+1)}\cdots u_{m_p}^{(i+p-1)} \\ &= h_p^{(i)}(U), \end{aligned}$$

whence we obtain the claim.

When $X = U$ and $Y = V$, from (4.11), (4.10) can be written as follows.

Corollary 4.7. *It holds that*

$$\sum_{\lambda\subset(s^r)} (-1)^{|\lambda|} S^{(r)}_{\lambda}(U) S^{(s)}_{\lambda^*}(V) = \det \left[\begin{array}{ccccccc} 1 & h_1^{(1)}(U) & h_2^{(1)}(U) & \cdots & h_r^{(1)}(U) & \cdots & h_{\eta-1}^{(1)}(U) \\ 0 & 1 & h_1^{(2)}(U) & \cdots & h_{r-1}^{(2)}(U) & \cdots & h_{\eta-2}^{(2)}(U) \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & h_1^{(r)}(U) & \cdots & h_{\eta-r}^{(r)}(U) \\ \hline 1 & h_1^{(1)}(V) & h_2^{(1)}(V) & \cdots & h_s^{(1)}(V) & \cdots & h_{\eta-1}^{(1)}(V) \\ 0 & 1 & h_1^{(2)}(V) & \cdots & h_{s-1}^{(2)}(V) & \cdots & h_{\eta-2}^{(2)}(V) \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & h_1^{(s)}(V) & \cdots & h_{\eta-s}^{(s)}(V) \end{array}\right]. \tag{4.12}$$

Theorem 4.8. *Assume that* $\boldsymbol{s}=(s_{ij})\in W^{\mathrm{diag}}_{(s^r)}$ *and* $\boldsymbol{t}=(t_{ij})\in W^{\mathrm{diag}}_{(r^s)}$ *with* $a_k=s_{i,i+k}$ *and* $b_k=t_{i,i+k}$ *for* $k\in\mathbb{Z}$*. Moreover, assume that* $\Re(s_{rj})>1$ *for all* $1\le j\le s$ *and* $\Re(t_{sj})>1$ *for all* $1\le j\le r$*. Then, we have*

(4.13)

$$
\sum_{\lambda\subset(s^r)}(-1)^{|\lambda|}\zeta_{\lambda}\left(\boldsymbol{s}|_{\lambda}\right)\zeta_{\lambda^*}\left(\boldsymbol{t}|_{\lambda^*}\right)
$$

$$
=\det\left[\begin{array}{cccccccc}
1 & \zeta^\star(a_{1-r}) & \zeta^\star(a_{1-r},a_{2-r}) & \cdots & \zeta^\star(a_{1-r},\dots,a_0) & \cdots & \zeta^\star(a_{1-r},\dots,a_{\eta-1-r})\\
0 & 1 & \zeta^\star(a_{2-r}) & \cdots & \zeta^\star(a_{2-r},\dots,a_0) & \cdots & \zeta^\star(a_{2-r},\dots,a_{\eta-1-r})\\
\vdots & \ddots & \ddots & \ddots & \vdots & & \vdots\\
0 & \cdots & 0 & 1 & \zeta^\star(a_0) & \cdots & \zeta^\star(a_0,\dots,a_{\eta-1-r})\\
\hline
1 & \zeta^\star(b_{1-s}) & \zeta^\star(b_{1-s},b_{2-s}) & \cdots & \zeta^\star(b_{1-s},\dots,b_0) & \cdots & \zeta^\star(b_{1-s},\dots,a_{\eta-1-s})\\
0 & 1 & \zeta^\star(b_{2-s}) & \cdots & \zeta^\star(b_{2-s},\dots,a_0) & \cdots & \zeta^\star(b_{2-s},\dots,a_{\eta-1-s})\\
\vdots & \ddots & \ddots & \ddots & \vdots & & \vdots\\
0 & \cdots & 0 & 1 & \zeta^\star(b_0) & \cdots & \zeta^\star(b_0,\dots,b_{\eta-1-s})
\end{array}\right].
$$

Here, $\boldsymbol{s}|_{\lambda}\in W^{\mathrm{diag}}_{\lambda}$ *and* $\boldsymbol{t}|_{\lambda^*}\in W^{\mathrm{diag}}_{\lambda^*}$ *are the shape restriction of* $\boldsymbol{s}$ *and* $\boldsymbol{t}$ *to* λ *and* λ^**, respectively.*

Proof. Putting $u^{(i)}_k=k^{-a_{i-r}}$ and $v^{(i)}_k=k^{-b_{i-r}}$, we have

$$
\begin{aligned}
h^{(i)}_k(U)&=\sum_{m_1\le\cdots\le m_k\le N}u^{(i)}_{m_1}u^{(i+1)}_{m_2}\cdots u^{(i+k-1)}_{m_k}\\
&=\sum_{m_1\le\cdots\le m_k\le N}m_1^{-a_{i-r}}m_2^{-a_{i+1-r}}\cdots m_k^{-a_{i+k-1-r}}\\
&=\zeta^{N\star}(a_{i-r},a_{i+1-r},\dots,a_{i+k-1-r})
\end{aligned}
$$

and similarly

$$
h^{(i)}_k(V)=\zeta^{M\star}(b_{i-s},b_{i+1-s},\dots,b_{i+k-1-s}).
$$

Therefore, (4.12) immediately yields (4.13) by letting $N,M\to\infty$. □

Example 4.9. When $r=2$ and $s=3$, we have

$$
\begin{aligned}
(\text{LHS of }(4.13)) = & -\begin{ytableau} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \end{ytableau}\cdot 1-\begin{ytableau} a_0 & a_1 & a_2 \\ a_{-1} & a_0 \end{ytableau}\cdot\begin{ytableau} b_0 \end{ytableau}+\begin{ytableau} a_0 & a_1 & a_2 \\ a_{-1} \end{ytableau}\cdot\begin{ytableau} b_0 \\ b_{-1} \end{ytableau}\\
& -\begin{ytableau} a_0 & a_1 & a_2 \end{ytableau}\cdot\begin{ytableau} b_0 \\ b_{-1} \\ b_{-2} \end{ytableau}+\begin{ytableau} a_0 & a_1 \\ a_{-1} & a_0 \end{ytableau}\cdot\begin{ytableau} b_0 & b_1 \end{ytableau}-\begin{ytableau} a_0 & a_1 \\ a_{-1} \end{ytableau}\cdot\begin{ytableau} b_0 & b_1 \\ b_{-1} \end{ytableau}\\
& +\begin{ytableau} a_0 & a_1 \end{ytableau}\cdot\begin{ytableau} b_0 & b_1 \\ b_{-1} \\ b_{-2} \end{ytableau}+\begin{ytableau} a_0 \\ a_{-1} \end{ytableau}\cdot\begin{ytableau} b_0 & b_1 \\ b_{-1} & b_0 \end{ytableau}-\begin{ytableau} a_0 \end{ytableau}\cdot\begin{ytableau} b_0 & b_1 \\ b_{-1} & b_0 \\ b_{-2} \end{ytableau}+1\cdot\begin{ytableau} b_0 & b_1 \\ b_{-1} & b_0 \\ b_{-2} & b_{-1} \end{ytableau}.
\end{aligned}
$$

On the other hand, we have

$$
(\text{RHS of } (4.13)) = \det \left[\begin{array}{ccccc} 1 & \zeta^\star(a_{-1}) & \zeta^\star(a_{-1},a_0) & \zeta^\star(a_{-1},a_0,a_1) & \zeta^\star(a_{-1},a_0,a_1,a_2) \\ 0 & 1 & \zeta^\star(a_0) & \zeta^\star(a_0,a_1) & \zeta^\star(a_0,a_1,a_2) \\ \hline 1 & \zeta^\star(b_{-2}) & \zeta^\star(b_{-2},b_{-1}) & \zeta^\star(b_{-2},b_{-1},b_0) & \zeta^\star(b_{-2},b_{-1},b_0,b_1) \\ 0 & 1 & \zeta^\star(b_{-1}) & \zeta^\star(b_{-1},b_0) & \zeta^\star(b_{-1},b_0,b_1) \\ 0 & 0 & 1 & \zeta^\star(b_0) & \zeta^\star(b_0,b_1) \end{array} \right].
$$

5 Schur type quasi-symmetric functions

We finally investigate SMZFs from the view point of the quasi-symmetric functions introduced by Gessel [G].

5.1 Quasi-symmetric functions

Let $\boldsymbol{t} = (t_1, t_2, \ldots)$ be variables and $\mathfrak{P}$ a subalgebra of $\mathbb{Z}[\![t_1, t_2, \ldots]\!]$ consisting of all formal power series with integer coefficients of bounded degree. We call $p = p(\boldsymbol{t}) \in \mathfrak{P}$ a *quasi-symmetric function* if the coefficient of $t_{k_1}^{\alpha_1} t_{k_2}^{\alpha_2} \cdots t_{k_n}^{\alpha_n}$ of p is the same as that of $t_{h_1}^{\alpha_1} t_{h_2}^{\alpha_2} \cdots t_{h_n}^{\alpha_l}$ of p whenever $k_1 < k_2 < \cdots < k_n$ and $h_1 < h_2 < \cdots < h_n$. The algebra of all quasi-symmetric functions is denoted by Qsym. For a composition $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of a positive integer, define the *monomial quasi-symmetric function* $M_{\boldsymbol{\alpha}}$ and the *essential quasi-symmetric function* $E_{\boldsymbol{\alpha}}$ respectively by

$$
M_{\boldsymbol{\alpha}} = \sum_{m_1 < m_2 < \cdots < m_n} t_{m_1}^{\alpha_1} t_{m_2}^{\alpha_2} \cdots t_{m_n}^{\alpha_n}, \quad E_{\boldsymbol{\alpha}} = \sum_{m_1 \le m_2 \le \cdots \le m_n} t_{m_1}^{\alpha_1} t_{m_2}^{\alpha_2} \cdots t_{m_n}^{\alpha_n}.
$$

We know that these respectively form integral basis of Qsym. Notice that

$$
E_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta} \preceq \boldsymbol{\alpha}} M_{\boldsymbol{\beta}}, \quad M_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta} \preceq \boldsymbol{\alpha}} (-1)^{n-\ell(\boldsymbol{\beta})} E_{\boldsymbol{\beta}}. \tag{5.1}
$$

5.2 Relation between quasi-symmetric functions and multiple zeta values

A relation between the multiple zeta values and the quasi-symmetric functions is studied by Hoffman [H2] (remark that the notations of MZF and MZSF in [H2] are different from ours; they are $\zeta(s_n, s_{n-1}, \ldots, s_1)$ and $\zeta^\star(s_n, s_{n-1}, \ldots, s_1)$, respectively, in our notations). Let $\mathfrak{H} = \mathbb{Z}\langle x, y\rangle$ be the noncommutative polynomial algebra over $\mathbb{Z}$. We can define a commutative and associative multiplication $*$, called a $*$-product, on $\mathfrak{H}$. We call $(\mathfrak{H}, *)$ the (integral) harmonic algebra. Let $\mathfrak{H}^1 = \mathbb{Z}1 + y\mathfrak{H}$, which is a subalgebra of $\mathfrak{H}$. Notice that every $w \in \mathfrak{H}^1$ can be written as an integral linear combination of $z_{\alpha_1} z_{\alpha_2} \cdots z_{\alpha_n}$ where $z_\alpha = yx^{\alpha-1}$ for $\alpha \in \mathbb{N}$. For each $N \in \mathbb{N}$, define the homomorphism $\phi_N : \mathfrak{H}^1 \to \mathbb{Z}[t_1, t_2, \ldots, t_N]$ by $\phi_N(1) = 1$ and

$$
\phi_N(z_{\alpha_1} z_{\alpha_2} \cdots z_{\alpha_n}) = \begin{cases} \displaystyle\sum_{m_1 < m_2 < \cdots < m_n \le N} t_{m_1}^{\alpha_1} t_{m_2}^{\alpha_2} \cdots t_{m_n}^{\alpha_n} & n \le N, \\ 0 & \text{otherwise}, \end{cases}
$$

and extend it additively to $\mathfrak{H}^1$. There is a unique homomorphism $\phi : \mathfrak{H}^1 \to \mathfrak{P}$ such that $\pi_N \phi = \phi_N$ where π_N is the natural projection from $\mathfrak{P}$ to $\mathbb{Z}[t_1, t_2, \ldots, t_N]$. We have $\phi(z_{\alpha_1} z_{\alpha_2} \cdots z_{\alpha_n}) = M_{(\alpha_1, \alpha_2, \ldots, \alpha_n)}$. Moreover, as is described in [H2], ϕ is an isomorphism between $\mathfrak{H}^1$ and Qsym.

Let e be the function sending t_i to $\frac{1}{i}$. Moreover, define $\rho_N : \mathfrak{H}^1 \to \mathbb{R}$ by $\rho_N = e\phi_N$. For a composition $\boldsymbol{\alpha}$, we have

$$\rho_N \phi^{-1}(M_{\boldsymbol{\alpha}}) = \zeta^N(\boldsymbol{\alpha}), \quad \rho_N \phi^{-1}(E_{\boldsymbol{\alpha}}) = \zeta^{N\star}(\boldsymbol{\alpha}).$$

Here, the second formula follows from the first equations of (3.1) and (5.1). Define the map $\rho : \mathfrak{H}^1 \to \mathbb{R}^{\mathbb{N}}$ by $\rho(w) = (\rho_N(w))_{N\in\mathbb{N}}$ for $w \in \mathfrak{H}^1$. Notice that if $w \in \mathfrak{H}^0 = \mathbb{Z}1 + y\mathfrak{H}x$, which is a subalgebra of $\mathfrak{H}^1$, then we may understand that $\rho(w) = \lim_{N\to\infty} \rho_N(w) \in \mathbb{R}$. In particular, for a composition $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_n \geq 2$, we have

$$\rho\phi^{-1}(M_{\boldsymbol{\alpha}}) = \zeta(\boldsymbol{\alpha}), \quad \rho\phi^{-1}(E_{\boldsymbol{\alpha}}) = \zeta^{\star}(\boldsymbol{\alpha}). \tag{5.2}$$

5.3 Schur type quasi-symmetric functions

Now, one easily reaches the definition of the following *Schur type quasi-symmetric functions* (of skew type). For partitions λ, μ satisfying $\mu \subset \lambda \subset (s^r)$ and $\boldsymbol{\alpha} = (\alpha_{ij}) \in T(\lambda/\mu, \mathbb{N})$, define

$$S_{\lambda/\mu}(\boldsymbol{\alpha}) = \sum_{(m_{ij})\in \mathrm{SSYT}(\lambda)} \prod_{(i,j)\in D(\lambda/\mu)} t_{m_{ij}}^{\alpha_{ij}},$$

which is actually in Qsym. Clearly we have

$$S_{(1^n)}\left(\begin{array}{|c|}\hline \alpha_1 \\ \hline \vdots \\ \hline \alpha_n \\ \hline \end{array}\right) = M_{(\alpha_1,\ldots,\alpha_n)}, \quad S_{(n)}\left(\begin{array}{|c|c|c|}\hline \alpha_1 & \cdots & \alpha_n \\ \hline \end{array}\right) = E_{(\alpha_1,\ldots,\alpha_n)}.$$

Hence $S_{\lambda/\mu}(\boldsymbol{\alpha})$ interpolates both the monomial and essential quasi-symmetric functions. Moreover, one sees that this is the quasi-symmetric function corresponding to the Schur multiple zeta value in the sense of (5.2).

Lemma 5.1. *Let*

$$I_{\lambda/\mu} = \left\{\boldsymbol{\alpha} = (\alpha_{ij}) \in T(\lambda/\mu, \mathbb{N}) \mid \alpha_{ij} \geq 2 \text{ for all } (i,j) \in C(\lambda/\mu)\right\}.$$

Then, for $\boldsymbol{\alpha} \in I_{\lambda/\mu}$, we have

$$\rho\phi^{-1}(S_{\lambda/\mu}(\boldsymbol{\alpha})) = \zeta_{\lambda/\mu}(\boldsymbol{\alpha}).$$

Proof. This follows from one of the following expressions

$$S_{\lambda/\mu}(\boldsymbol{\alpha}) = \sum_{\boldsymbol{u} \preceq \boldsymbol{\alpha}} M_{\boldsymbol{u}}, \quad S_{\lambda/\mu}(\boldsymbol{\alpha}) = \sum_{\boldsymbol{u} \preceq \boldsymbol{\alpha}'} (-1)^{|\lambda/\mu| - \ell(\boldsymbol{u})} E_{\boldsymbol{u}}, \tag{5.3}$$

similarly obtained as (4.2), together with (4.2) and (5.2). □

Remark 5.2. There is another important class of quasi-symmetric functions called the *fundamental* or *ribbon quasi-symmetric function* defined by $F_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}\succeq\boldsymbol{\alpha}} M_{\boldsymbol{\beta}}$ for a composition $\boldsymbol{\alpha}$. We remark that they are not in the class of Schur type quasi-symmetric functions.

We again concentrate on the case $\boldsymbol{\alpha} = (\alpha_{ij}) \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{N})$. Write $a_k = \alpha_{i,i+k}$ for $k \in \mathbb{Z}$ (and for any $i \in \mathbb{N}$). Then, from the tableau expression (4.4) of the ninth variation of the Schur function $S^{(r)}_{\lambda/\mu}(U)$, if we put $u^{(i)}_k = t_k^{a_{i-r}}$, then we have $u^{(r-i+j)}_{m_{ij}} = t^{\alpha_{ij}}_{m_{ij}}$ and hence

$$\begin{aligned} S^{(r)}_{\lambda/\mu}(U) &= \sum_{(m_{ij})\in \mathrm{SSYT}_N(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} u^{(r-i+j)}_{m_{ij}} \\ &= \sum_{(m_{ij})\in \mathrm{SSYT}_N(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} t^{\alpha_{ij}}_{m_{ij}} \\ &= \phi_N \phi^{-1}(S_{\lambda/\mu}(\boldsymbol{\alpha})). \end{aligned}$$

This shows that, when $\boldsymbol{\alpha} \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{N})$, the Schur type quasi-symmetric function $S_{\lambda/\mu}(\boldsymbol{\alpha})$ is also realized as (the limit of) a specialization of the ninth variation of the Schur functions, whence we can similarly obtain the Jacobi-Trudi, Giambelli and dual Cauchy formulas for such quasi-symmetric functions. Notice that the following formulas actually hold in the algebra of formal power series, which means that we do not need any further assumptions on variables such as appeared in the corresponding results in the previous section for SMZFs.

Theorem 5.3. *Assume that* $\boldsymbol{\alpha} = (\alpha_{ij}) \in T^{\mathrm{diag}}(\lambda/\mu, \mathbb{N})$ *and write* $a_k = \alpha_{i,i+k}$ *for* $k \in \mathbb{Z}$.

(1) *We have*

$$S_{\lambda/\mu}(\boldsymbol{\alpha}) = \det\left[E_{(a_{\mu_j-j+1},a_{\mu_j-j+2},\dots,a_{\mu_j-j+(\lambda_i-\mu_j-i+j)})}\right]_{1\le i,j\le r}. \tag{5.4}$$

Here, we understand that $E_{(\cdots)} = 1$ *if* $\lambda_i - \mu_j - i + j = 0$ *and* 0 *if* $\lambda_i - \mu_j - i + j < 0$.

(2) *We have*

$$S_{\lambda/\mu}(\boldsymbol{\alpha}) = \det\left[M_{(a_{-\mu'_j+j-1},a_{-\mu'_j+j-2},\dots,a_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)})}\right]_{1\le i,j\le s}. \tag{5.5}$$

Here, we understand that $M_{(\cdots)} = 1$ *if* $\lambda'_i - \mu'_j - i + j = 0$ *and* 0 *if* $\lambda'_i - \mu'_j - i + j < 0$.

Theorem 5.4. *Let* $\lambda = (p_1 - 1, \dots, p_t - 1 \,|\, q_1, \dots, q_t)$ *be a partition written in the Frobenius notation. Assume that* $\boldsymbol{\alpha} = (\alpha_{ij}) \in T^{\mathrm{diag}}(\lambda, \mathbb{N})$ *and write* $a_k = \alpha_{i,i+k}$ *for* $k \in \mathbb{Z}$. *Then, we have*

$$S_\lambda(\boldsymbol{\alpha}) = \det\left[S_{(p_i,1^{q_j})}(\boldsymbol{\alpha}_{i,j})\right]_{1\le i,j\le t},$$

where $$\boldsymbol{\alpha}_{i,j} = \begin{array}{|c|c|c|c|c|} \hline a_0 & a_1 & a_2 & \cdots & a_{p_i-1} \\ \hline a_{-1} & \multicolumn{4}{c}{} \\ \cline{1-1} \vdots & \multicolumn{4}{c}{} \\ \cline{1-1} a_{-q_j} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \in T((p_i, 1^{q_j}), \mathbb{N}).$$

Theorem 5.5. *Assume that* $\boldsymbol{\alpha} = (\alpha_{ij}) \in T^{\mathrm{diag}}((s^r), \mathbb{N})$ *and* $\boldsymbol{\beta} = (\beta_{ij}) \in T^{\mathrm{diag}}((r^s), \mathbb{N})$ *with*

$a_k = \alpha_{i,i+k}$ and $b_k = \beta_{i,i+k}$ for $k \in \mathbb{Z}$. Write $\eta = r + s$. Then, we have

$$(5.6)\qquad \sum_{\lambda \subset (s^r)} (-1)^{|\lambda|} S_\lambda(\boldsymbol{\alpha}|_\lambda)\, S_{\lambda^*}(\boldsymbol{\beta}|_{\lambda^*})$$

$$= \det \left[\begin{array}{ccccccc} 1 & E_{(a_{1-r})} & E_{(a_{1-r},a_{2-r})} & \cdots & E_{(a_{1-r},\dots,a_0)} & \cdots & E_{(a_{1-r},\dots,a_{\eta-1-r})} \\ 0 & 1 & E_{(a_{2-r})} & \cdots & E_{(a_{2-r},\dots,a_0)} & \cdots & E_{(a_{2-r},\dots,a_{\eta-1-r})} \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & E_{(a_0)} & \cdots & E_{(a_0,\dots,a_{\eta-1-r})} \\ \hline 1 & E_{(b_{1-s})} & E_{(b_{1-s},b_{2-s})} & \cdots & E_{(b_{1-s},\dots,b_0)} & \cdots & E_{(b_{1-s},\dots,b_{\eta-1-s})} \\ 0 & 1 & E_{(b_{2-s})} & \cdots & E_{(b_{2-s},\dots,b_0)} & \cdots & E_{(b_{2-s},\dots,b_{\eta-1-s})} \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & E_{(b_0)} & \cdots & E_{(b_0,\dots,b_{\eta-1-s})} \end{array}\right].$$

Here, $\boldsymbol{\alpha}|_\lambda \in T^{\mathrm{diag}}(\lambda, \mathbb{N})$ and $\boldsymbol{\beta}|_{\lambda^*} \in T^{\mathrm{diag}}(\lambda^*, \mathbb{N})$ are the shape restriction of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to λ and λ^*, respectively.

Remark 5.6. In [MR], a more general type of quasi-symmetric function is defined by a set of equality and inequality conditions. One can see that this includes both the Schur type quasi-symmetric functions and the fundamental quasi-symmetric functions as special cases and actually leads a generalized multiple zeta function via $\rho\phi^{-1}$. However, because it is too complicated in general, it seems to be difficult to expect that such generalized quasi-symmetric and multiple zeta functions satisfy the similar kind of determinant formulas as above.

We know that Qsym has a commutative Hopf algebra structure (see [H2, K, MM, Sw]). The antipode S, which is an automorphism of Qsym satisfying $S^2 = \mathrm{id}$, is explicitly given as follows.

Theorem 5.7 ([H2, Theorem 3.1])**.** *For a composition $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, we have*

(1) $S(M_{\boldsymbol{\alpha}}) = \displaystyle\sum_{\boldsymbol{\alpha}_1 \sqcup \boldsymbol{\alpha}_2 \sqcup \cdots \sqcup \boldsymbol{\alpha}_m = \boldsymbol{\alpha}} (-1)^m M_{\boldsymbol{\alpha}_1} M_{\boldsymbol{\alpha}_2} \cdots M_{\boldsymbol{\alpha}_m}.$

(2) $S(M_{\boldsymbol{\alpha}}) = (-1)^n E_{\overline{\boldsymbol{\alpha}}}.$

Here, $\boldsymbol{\alpha}_1 \sqcup \boldsymbol{\alpha}_2 \sqcup \cdots \sqcup \boldsymbol{\alpha}_m$ is just the juxtaposition of non-empty compositions $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_m$ and $\overline{\boldsymbol{\alpha}} = (\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$.

Combining these formulas, we reach the expressions

$$(5.7)\qquad M_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}_1 \sqcup \boldsymbol{\alpha}_2 \sqcup \cdots \sqcup \boldsymbol{\alpha}_m = \overline{\boldsymbol{\alpha}}} (-1)^{n-m} E_{\boldsymbol{\alpha}_1} E_{\boldsymbol{\alpha}_2} \cdots E_{\boldsymbol{\alpha}_m},$$

$$(5.8)\qquad E_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}_1 \sqcup \boldsymbol{\alpha}_2 \sqcup \cdots \sqcup \boldsymbol{\alpha}_m = \overline{\boldsymbol{\alpha}}} (-1)^{n-m} M_{\boldsymbol{\alpha}_1} M_{\boldsymbol{\alpha}_2} \cdots M_{\boldsymbol{\alpha}_m}.$$

One sees by induction on n that (5.7) and (5.8) are respectively equivalent to the formulas

$$M_{(\alpha_1,\dots,\alpha_n)} = \begin{vmatrix} E_{(\alpha_1)} & E_{(\alpha_2,\alpha_1)} & \cdots & \cdots & E_{(\alpha_n,\dots,\alpha_2,\alpha_1)} \\ 1 & E_{(\alpha_2)} & \cdots & \cdots & E_{(\alpha_n,\dots,\alpha_2)} \\ & 1 & \ddots & & \vdots \\ & & \ddots \quad 1 & E_{(\alpha_{n-1})} & E_{(\alpha_n,\alpha_{n-1})} \\ \mathbf{0} & & & 1 & E_{(\alpha_n)} \end{vmatrix},$$

$$E_{(\alpha_1,\dots,\alpha_n)} = \begin{vmatrix} M_{(\alpha_1)} & M_{(\alpha_2,\alpha_1)} & \cdots & \cdots & M_{(\alpha_n,\dots,\alpha_2,\alpha_1)} \\ 1 & M_{(\alpha_2)} & \cdots & \cdots & M_{(\alpha_n,\dots,\alpha_2)} \\ & 1 & \ddots & & \vdots \\ & & \ddots \quad 1 & M_{(\alpha_{n-1})} & M_{(\alpha_n,\alpha_{n-1})} \\ \mathbf{0} & & & 1 & M_{(\alpha_n)} \end{vmatrix},$$

which are obtained from the Jacobi-Trudi formulas (5.4) and (5.5), respectively.

Example 5.8. When $n = 3$, we have

$$\begin{aligned} M_{(\alpha_1,\alpha_2,\alpha_3)} &= E_{(\alpha_3,\alpha_2,\alpha_1)} - E_{(\alpha_3,\alpha_2)}E_{(\alpha_1)} - E_{(\alpha_3)}E_{(\alpha_2,\alpha_1)} + E_{(\alpha_3)}E_{(\alpha_2)}E_{(\alpha_1)} \\ &= \begin{vmatrix} E_{(\alpha_1)} & E_{(\alpha_2,\alpha_1)} & E_{(\alpha_3,\alpha_2,\alpha_1)} \\ 1 & E_{(\alpha_2)} & E_{(\alpha_3,\alpha_2)} \\ 0 & 1 & E_{(\alpha_3)} \end{vmatrix}, \\ E_{(\alpha_1,\alpha_2,\alpha_3)} &= M_{(\alpha_3,\alpha_2,\alpha_1)} - M_{(\alpha_3,\alpha_2)}M_{(\alpha_1)} - M_{(\alpha_3)}M_{(\alpha_2,\alpha_1)} + M_{(\alpha_3)}M_{(\alpha_2)}M_{(\alpha_1)} \\ &= \begin{vmatrix} M_{(\alpha_1)} & M_{(\alpha_2,\alpha_1)} & M_{(\alpha_3,\alpha_2,\alpha_1)} \\ 1 & M_{(\alpha_2)} & M_{(\alpha_3,\alpha_2)} \\ 0 & 1 & M_{(\alpha_3)} \end{vmatrix}. \end{aligned}$$

For a skew Young diagram ν, we denote by $\nu^{\#}$ the transpose of ν with respect to the anti-diagonal. Similarly, the anti-diagonal transpose of a skew Young tableaux $T \in T(\nu, X)$ is denoted by $T^{\#} \in T(\nu^{\#}, X)$. In the following discussion, we also encounter $(T^{\#})' \in T((\nu^{\#})', X)$, the conjugate of $T^{\#}$. For example,

$$\begin{array}{|c|c|c|}\hline \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \hline \alpha_{21} \\ \cline{1-1} \end{array}^{\#} = \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{\alpha_{13}} \\ \cline{2-2} & \multicolumn{1}{|c|}{\alpha_{12}} \\ \hline \multicolumn{1}{|c|}{\alpha_{21}} & \multicolumn{1}{c|}{\alpha_{11}} \\ \hline \end{array}, \quad \left(\begin{array}{|c|c|c|}\hline \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \hline \alpha_{21} \\ \cline{1-1} \end{array}^{\#} \right)' = \begin{array}{ccc} \cline{3-3} & & \multicolumn{1}{|c|}{\alpha_{21}} \\ \hline \multicolumn{1}{|c|}{\alpha_{13}} & \multicolumn{1}{c|}{\alpha_{12}} & \multicolumn{1}{c|}{\alpha_{11}} \\ \hline \end{array}$$

Namely, $(T^{\#})'$ is just the rotation of T by π around the center of ν. Now, the image of the Schur type quasi-symmetric functions by the antipode S is explicitly calculated as follows.

Theorem 5.9. *For a skew Young diagram ν, we have*

$$S(S_\nu(\boldsymbol{\alpha})) = (-1)^{|\nu|} S_{\nu^{\#}}(\boldsymbol{\alpha}^{\#}). \tag{5.9}$$

Moreover, when $\boldsymbol{\alpha} \in T^{\mathrm{diag}}(\nu, \mathbb{N})$, we have

$$S(S_\nu(\boldsymbol{\alpha})) = (-1)^{|\nu|} \sum_{\Theta=(\theta_1,\theta_2,\dots,\theta_r)\in \mathrm{Rim}_H^{\nu^{\#}}} \varepsilon_H(\Theta) E_{\theta_1(\boldsymbol{\alpha}^{\#})} E_{\theta_2(\boldsymbol{\alpha}^{\#})} \cdots E_{\theta_r(\boldsymbol{\alpha}^{\#})}, \tag{5.10}$$

$$S(S_\nu(\boldsymbol{\alpha})) = (-1)^{|\nu|} \sum_{\Theta=(\theta_1,\theta_2,\dots,\theta_s)\in \mathrm{Rim}_E^{\nu^{\#}}} \varepsilon_E(\Theta) M_{\theta_1(\boldsymbol{\alpha}^{\#})} M_{\theta_2(\boldsymbol{\alpha}^{\#})} \cdots M_{\theta_s(\boldsymbol{\alpha}^{\#})}. \tag{5.11}$$

Proof. From (5.3) and Theorem 5.7 (2), we have

$$
\begin{aligned}
S(S_\nu(\boldsymbol{\alpha})) &= \sum_{\boldsymbol{u} \preceq \boldsymbol{\alpha}} S(M_{\boldsymbol{u}}) \\
&= \sum_{\overline{\boldsymbol{u}} \preceq \boldsymbol{\alpha}} (-1)^{\ell(\boldsymbol{u})} E_{\boldsymbol{u}} \\
&= (-1)^{|\nu|} \sum_{\boldsymbol{u} \preceq (\boldsymbol{\alpha}^{\#})'} (-1)^{|\nu|-\ell(\boldsymbol{u})} E_{\boldsymbol{u}} \\
&= (-1)^{|\nu|} S_{\nu^{\#}}(\boldsymbol{\alpha}^{\#}).
\end{aligned}
$$

Notice that, in the third equality, we have used the fact that $\overline{\boldsymbol{u}} \preceq \boldsymbol{\alpha}$ if and only if $\boldsymbol{u} \preceq (\boldsymbol{\alpha}^{\#})'$, which can be verified directly. This shows (5.9). Now, the rest of assertions are immediately obtained from

$$
\begin{aligned}
S_\nu(\boldsymbol{\alpha}) &= \sum_{\Theta=(\theta_1,\theta_2,\ldots,\theta_r)\in \mathrm{Rim}^\nu_H} \varepsilon_H(\Theta) E_{\theta_1(\boldsymbol{\alpha})} E_{\theta_2(\boldsymbol{\alpha})} \cdots E_{\theta_r(\boldsymbol{\alpha})}, \\
S_\nu(\boldsymbol{\alpha}) &= \sum_{\Theta=(\theta_1,\theta_2,\ldots,\theta_s)\in \mathrm{Rim}^\nu_E} \varepsilon_E(\Theta) M_{\theta_1(\boldsymbol{\alpha})} M_{\theta_2(\boldsymbol{\alpha})} \cdots M_{\theta_s(\boldsymbol{\alpha})},
\end{aligned}
$$

which are similarly obtained as (3.6) and (3.9) (hence we need the assumption $\boldsymbol{\alpha} \in T^{\mathrm{diag}}(\nu, \mathbb{N})$) and lead the Jacobi-Trudi formulas (5.4) and (5.5) for the Schur type quasi-symmetric functions. This completes the proof. $\square$

Remark 5.10. The formula (5.11) with $\nu = (1^n)$ is nothing but the one in Theorem 5.7 (1).

Example 5.11. When $\nu = (3,1)$, we have from (5.9)

$$
\begin{aligned}
S\left(S_{(3,1)}\left(\begin{ytableau} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} \end{ytableau}\right)\right) &= S_{(2,2,2)/(1,1)}\left(\begin{ytableau} \none & \alpha_{13} \\ \none & \alpha_{12} \\ \alpha_{21} & \alpha_{11} \end{ytableau}\right) \\
&= E_{(\alpha_{21},\alpha_{13},\alpha_{12},\alpha_{11})} - E_{(\alpha_{21}+\alpha_{13},\alpha_{12},\alpha_{11})} - E_{(\alpha_{21},\alpha_{13}+\alpha_{12},\alpha_{11})} \\
&\quad - E_{(\alpha_{21},\alpha_{13},\alpha_{12}+\alpha_{11})} + E_{(\alpha_{21}+\alpha_{13}+\alpha_{12},\alpha_{11})} + E_{(\alpha_{21}+\alpha_{13},\alpha_{12}+\alpha_{11})} \\
&\quad + E_{(\alpha_{21},\alpha_{13}+\alpha_{12}+\alpha_{11})} + E_{(\alpha_{13},\alpha_{21},\alpha_{12},\alpha_{11})} - E_{(\alpha_{13},\alpha_{21},\alpha_{12}+\alpha_{11})} \\
&\quad + E_{(\alpha_{13},\alpha_{12},\alpha_{21},\alpha_{11})} - E_{(\alpha_{13}+\alpha_{12},\alpha_{21},\alpha_{11})} - E_{(\alpha_{13},\alpha_{12}+\alpha_{21},\alpha_{11})} \\
&= M_{(\alpha_{21},\alpha_{13},\alpha_{12},\alpha_{11})} + M_{(\alpha_{21}+\alpha_{13},\alpha_{12},\alpha_{11})} + M_{(\alpha_{13},\alpha_{21}+\alpha_{12},\alpha_{11})} \\
&\quad + M_{(\alpha_{13},\alpha_{21},\alpha_{12},\alpha_{11})} + M_{(\alpha_{13},\alpha_{12},\alpha_{21}+\alpha_{11})} + M_{(\alpha_{13},\alpha_{21}+\alpha_{12},\alpha_{11})}.
\end{aligned}
$$

Here, the second and third equations are similarly obtained as in Example 4.1. On the other hand, we have from (5.10)

$$
\begin{aligned}
S\left(S_{(3,1)}\left(\begin{ytableau} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} \end{ytableau}\right)\right) &= E_{(\alpha_{13})} E_{(\alpha_{12})} E_{(\alpha_{21},\alpha_{11})} - E_{(\alpha_{12},\alpha_{13})} E_{(\alpha_{21},\alpha_{11})} \\
&\quad - E_{(\alpha_{13})} E_{(\alpha_{21},\alpha_{11},\alpha_{12})} + E_{(\alpha_{21},\alpha_{11},\alpha_{12},\alpha_{13})}
\end{aligned}
$$

where each term corresponds to the H-rim decomposition $\begin{ytableau} \none & 1 \\ \none & 2 \\ 3 & 3 \end{ytableau}$, $\begin{ytableau} \none & 2 \\ \none & 2 \\ 3 & 3 \end{ytableau}$, $\begin{ytableau} \none & 1 \\ \none & 3 \\ 3 & 3 \end{ytableau}$ and $\begin{ytableau} \none & 3 \\ \none & 3 \\ 3 & 3 \end{ytableau}$ of $(3,1)^{\#} = (2,2,2)/(1,1)$, respectively, and from (5.11)

$$S\left(S_{(3,1)}\left(\begin{ytableau} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} \end{ytableau}\right)\right) = M_{(\alpha_{21})}M_{(\alpha_{13},\alpha_{12},\alpha_{11})} - M_{(\alpha_{13},\alpha_{12},\alpha_{11},\alpha_{21})},$$

where each term to the E-rim decomposition $\begin{ytableau} \none & 2 \\ \none & 2 \\ 1 & 2 \end{ytableau}$ and $\begin{ytableau} \none & 2 \\ \none & 2 \\ 2 & 2 \end{ytableau}$, respectively.

Remark 5.12. The equation (5.9) is essentially obtained by Malvenuto and Reutenauer [MR, Theorem 3.1] for their quasi-symmetric functions. Notice that $\nu^{\#}$ is called the conjugate of ν in their notion. If Jacobi-Trudi formulas are obtained for such quasi-symmetric functions, then one may also establish the similar kind of expressions like (5.10) and (5.11) for them.

Using Theorem 5.9, one automatically gets another relation from a given relation among quasi-symmetric functions by mapping it by the antipode S. For instance, from (5.6), we obtain the following equation.

Corollary 5.13. *Assume that* $\boldsymbol{\alpha} = (\alpha_{ij}) \in T^{\mathrm{diag}}((s^r),\mathbb{N})$ *and* $\boldsymbol{\beta} = (\beta_{ij}) \in T^{\mathrm{diag}}((r^s),\mathbb{N})$ *with* $a_k = \alpha_{i,i+k}$ *and* $b_k = \beta_{i,i+k}$ *for* $k \in \mathbb{Z}$. *Write* $\eta = r+s$. *Then, we have*

$$\sum_{\lambda\subset(r^s)} (-1)^{|\lambda|} S_{(r^s)/\lambda}\left((\boldsymbol{\alpha}|_{\lambda^*})^{\#}\right) S_{(s^r)/\lambda^*}\left((\boldsymbol{\beta}|_{\lambda})^{\#}\right)$$

$$= \det \left[\begin{array}{cccccc}
1 & -M_{(a_{1-r})} & M_{(a_{2-r},a_{1-r})} & \cdots & (-1)^r M_{(a_0,\dots,a_{1-r})} & \cdots & (-1)^{\eta-1} M_{(a_{\eta-1-r},\dots,a_{1-r})} \\
0 & 1 & -M_{(a_{2-r})} & \cdots & (-1)^{r-1} M_{(a_0,\dots,a_{2-r})} & \cdots & (-1)^{\eta-2} M_{(a_{\eta-1-r},\dots,a_{2-r})} \\
\vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & -M_{(a_0)} & \cdots & (-1)^{\eta-r} M_{(a_{\eta-1-r},\dots,a_0)} \\
\hline
1 & -M_{(b_{1-s})} & M_{(b_{2-s},b_{1-s})} & \cdots & (-1)^s M_{(b_0,\dots,b_{1-s})} & \cdots & (-1)^{\eta-1} M_{(b_{\eta-1-s},\dots,b_{1-s})} \\
0 & 1 & -M_{(b_{2-s})} & \cdots & (-1)^{s-1} M_{(b_0,\dots,b_{2-s})} & \cdots & (-1)^{\eta-2} M_{(b_{\eta-1-s},\dots,b_{2-s})} \\
\vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & -M_{(b_0)} & \cdots & (-1)^{\eta-s} M_{(b_{\eta-1-s},\dots,b_0)}
\end{array}\right].$$

Here, $\boldsymbol{\alpha}|_{\lambda^*} \in T^{\mathrm{diag}}(\lambda^*,\mathbb{N})$ *and* $\boldsymbol{\beta}|_{\lambda} \in T^{\mathrm{diag}}(\lambda,\mathbb{N})$ *are the shape restriction of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *to* λ^* *and* λ, *respectively.*

Remark that mapping this equation by $\rho\phi^{-1}$ under suitable convergence assumptions, one obtains the corresponding relation among the Schur multiple zeta values.

Acknowledgement

We would like to express our appreciation to all those who gave us valuable advice for this article: Prof. Masatoshi Noumi who provided expertise that greatly helped us to prove the results on Schur multiple zeta functions in Section 4, Prof. Masanobu Kaneko who gave guidance in quasi-symmetric functions and inspired us to establish the generalized result for such functions and Prof. Takeshi Ikeda who gave meaningful suggestion for our work. We would also like to thank Prof. Hiroshi Naruse, Prof. Takashi Nakamura, Prof. Soichi Okada and Prof. Yasuo Ohno for their useful comments in many aspects. Finally, the third author is very grateful to the Max Planck Institut für Mathematik in Bonn for the hospitality and support during his research stay at the Institute.

References

[AET] S. Akiyama, S. Egami and Y. Tanigawa, Analytic continuation of multiple zeta functions and their values at non-positive integers, *Acta Arith.*, **98** (2001), no. 2, 107–116.

[ELW] E. Eggea, N. Loehrb and G. Warringtonc, From quasisymmetric expansions to Schur expansions via a modified inverse Kostka matrix, *European J. Combin.*, **31** (2010), no. 8, 2014–2027.

[E] L. Euler, Meditationes circa singulare serierum genus, *Novi Comm. Acad. Sci. Petropol.*, **20** (1775) 140–186; Reprinted in: Opera Omnia, Ser. I, vol. 15, B.G. Teubner, Berlin, 1927, pp. 217–267.

[G] I. M. Gessel, Multipartite P-functions and inner products of skew Schur functions, combinatorics and algebra, *Contemp. Math.*, **34** (1984), 289–301.

[HLMW] J. Haglund, K. Luoto, S. Mason and S. van Willigenburg, Quasi symmetric Schur functions, *J. Combin. Theory Ser. A* **118** (2011), no. 2, 463–490.

[HG] A.M. Hamel and I.P. Goulden, Planar decompositions of tableaux and Schur function determinants, *European, J. Combin.*, **16** (1995), no. 5, 461–477.

[H1] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.*, **152** (1992), no. 2, 275–290.

[H2] M. E. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, *Kyushu J. Math.*, **69** (2015), no. 2, 345–366.

[IKOO] K. Ihara, J. Kajikawa, Y. Ohno and J. Okuda, Multiple zeta values vs. multiple zeta-star values, *J. Algebra*, **332** (2011), 187–208.

[K] C. Kassel, Quantum groups, Graduate Texts in Mathematics, **155**. Springer-Verlag, New York, 1995.

[LP] A. Lascoux and P. Pagacz, Ribbon Schur functions, *European, J. Combin.*, **9** (1988), no. 6, 561–574.

[Mac] I. G. Macdonald, Schur functions: theme and variations, Sḿinaire Lotharingien de Combinatoire (Saint-Nabor, 1992), pp. 5–39, Publ. Inst. Rech. Math. Av., 498, Univ. Louis Pasteur, Strasbourg, 1992.

[MR] C. Malvenuto and C. Reutenauer, Plethysm and conjugation of quasi-symmetric functions, Selected papers in honor of Adriano Garsia (Taormina, 1994), *Discrete Mathe.*, **193** (1998), no. 1-3, 225–233.

[Mat] K. Matsumoto, On the analytic continuation of various multiple-zeta functions, *Number Theory for the Millennium* (Urbana, 2000), Vol. II, M.A. Bennett et. al. (eds.), A. K. Peters, Natick, MA, 2002, pp. 417–440.

[MM] J. W. Milnor and J. C. Moore, On the structure of Hopf algebras, *Ann. of Math.*, **81** (1965), no.2, 211–264.

[Mu] S. Muneta, On some explicit evaluations of multiple zeta-star values, *J. Number Theory*, **128** (2008), no. 9, 2538–2548.

[NNSY] J. Nakagawa, M. Noumi, M. Shirakawa and Y. Yamada, Tableau representation for Macdonald's ninth variation of Schur functions, *(English summary) Physics and combinatorics* (Nagoya, 2000), pp. 180-195, World Sci. Publ., River Edge, NJ, 2001.

[N1] M. Noumi, Remarks on elliptic Schur functions, talk at the international workshop on "Analysis, Geometry and Group Representations for Homogeneous Spaces", November 22-26, 2010, Lorentz Center, Leiden, The Netherlands. (http://www.lorentzcenter.nl/lc/web/2010/423/presentations/Noumi.pdf)

[N2] M. Noumi, Painlevé Equations through Symmetry, *Translations of Mathematical Monographs*, 223 (2004).

[OZ] Y. Ohno and W.Zudilin, Zeta stars, *Commun. Number Theory Phys.*, **2** (2008), no.2, 325–347.

[Sta] R. Stanley, Two remarks on skew tableaux, *Electron. J. Combin.*, **18** (2011), no. 2, Paper 16, 8 pp.

[Ste] J. Stembridge, Nonintersecting paths, Pfaffians, and plane partitions, *Adv. Math.*, **83** (1990), no. 1, 96–131.

[Sw] M. E. Sweedler, Hopf algebras, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York 1969.

[Y] Y. Yamasaki, Evaluations of multiple Dirichlet L-values via symmetric functions, *J. Number Theory*, **129** (2009), no. 10, 2369–2386.

[Za] D. Zagier, Values of zeta functions and their applications. First European Congress of Mathematics, Vol. II (Paris, 1992), 497–512, *Progr. Math.*, **120**, Birkhauser, Basel, 1994.

[Zi] P. Zinn-Justin, Six-vertex, Loop and Tiling model : Integrability and Combinatorics, arXiv:0901.0665.

[Zl] S. A. Zlobin, Relations for multiple zeta values, *Mat. Zametki*, **84** (2008), no.6, 825–837; translation in *Math. Notes*, **84** (2008), no.5-6, 771–782.

Maki Nakasuji
Department of Information and Communication Science, Faculty of Science,
Sophia University, Tokyo, Japan
nakasuji@sophia.ac.jp

Ouamporn Phuksuwan
Department of Mathematics and Computer Science, Faculty of Science,
Chulalongkorn University, Bangkok, Thailand
ouamporn.p@chula.ac.th

Yoshinori Yamasaki
Graduate School of Science and Engineering,
Ehime University, Ehime, Japan
yamasaki@math.sci.ehime-u.ac.jp