Corestriction Principle for non-abelian cohomology of reductive group schemes over Dedekind rings of integers of local and global fields

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Abstract

We prove some new results on Corestriction principle for non-abelian cohomology of group schemes over the rings of integers of local and global fields. Some connections with Grothendieck - Serre's conjecture are indicated, and applications to the study of class groups of algebraic groups over global fields are given.

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Introduction. In [T1] - [T4] we have proved some results on Corestriction principle for connecting maps of non-abelian Galois cohomology of reductive groups over local and global fields. In [T3] there was defined also a concept of Weak Corestriction principle for non-abelian Galois cohomology of such groups over arbitrary fields of charateristic 0.

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It is apparent and natural to consider similar notions for *groups of arithmetical types*, i.e., consider group schemes over arithmetical rings. Such a treatment over rings is necessary for various arithmetic considerations. For example, in [X], there has been proved the validity of Corestriction principle, under some restrictions, for spinor norms over the ring of $\mathfrak{p}$-adic integers, which has application in arithmetic theory of quadratic forms over global fields.

We consider in this paper the concept of Corestriction principle (resp. Weak Corestriction principle) in a setting, more general than that of Galois cohomology. The definitions are similar so we only briefly recall it below and refer the readers to [T1] - [T4] for more details. For the sake of arithmetical applications, we restrict ourselves only in the case of Dedekind rings (or their localizations or completions with respect to discrete valuations) and their quotient fields. We call such rings in this paper by *arithmetical rings.* Thus, for an arithmetical ring A and a flat A-group scheme (i.e. S-group scheme with $S = Spec(A)$), we denote as usual $\mathrm{H}^i_r(A, G) := \mathrm{H}^i_r(Spec(A), G)$, where r stands either for Zariski, étale, or flat (i.e., fppf) topology, whenever it makes sense. We assume once for all that, for $r =$ét and for all smooth *commutative* A-group schemes involved, there is a notion of corestriction homomorphism, that is, for any smooth commutative A-group scheme T and each extension A'/A belonging to certain category $\mathcal{C}_A$ of faithfully flat, étale extensions of finite type over A there is a functorial homomorphism

$$Cores_{A'/A,T} : \mathrm{H}^i_{et}(A', T_{A'}) \to \mathrm{H}^i_{et}(A, T),$$

and the same holds for localizations of A at finite sets of primes. Here we denote $T_{A'} = T \times_A A'$ the A'-group scheme obtained by base change from A to A'. One should notice that in general one may not expect such homomorphism to exist, and there is a general theory of trace handling this question by P. Deligne in [SGA 4], Exp. 17 (cf. also Gille [Gi]). However, thanks to [SGA 3], Exp. XXIV, Prop. 8.4, we may consider corestriction maps for étale cohomology groups (or some other cohomology groups for topology, other than étale, but still on the same small étale site), and the category $\mathcal{C}_A$ can be taken as that of all étale, finite extensions, which are integral closures of A in finite separable extensions k'/k. Once this is granted, one may then consider the concept of (Weak) Corestriction principle for images or kernels of connecting maps in a long exact sequence of cohomology.

Assume that we have a map which is functorial in A', $A' \in \mathcal{C}_A$:

$$\alpha_{A'} : \mathrm{H}^p_{et}(A', G_{A'}) \to \mathrm{H}^q_{et}(A', T_{A'}),$$

i. e., a map of functors $\alpha = (\alpha_{A'}) : (A' \mapsto \mathrm{H}^p_{et}(A', G_{A'})) \to (A' \mapsto \mathrm{H}^q_{et}(A', T_{A'}))$ where A' runs over all $\mathcal{C}_A$, T is a smooth commutative algebraic A-group scheme. It is natural to ask whether or not the following inclusion holds

$$Cores_{A'/A,T}(\mathrm{Im}\ (\alpha_{A'})) \subset \mathrm{Im}\ (\alpha_A).$$

If it is true, then we say that the Corestriction Principle holds for the image of the map $\alpha_A : \mathrm{H}^p_{et}(A, G) \to \mathrm{H}^q_{et}(A, T)$ with respect to extension A'/A. If it is the case for all $A' \in \mathcal{C}_A$, then we say that the Corestriction Principle holds for the image of the map $\alpha_A : \mathrm{H}^p_{et}(A, G) \to \mathrm{H}^q_{et}(A, T)$ for $\mathcal{C}_A$. We say that Weak Corestriction principle holds for the image of α_A with respect to the extension A'/A, if

$$Cores_{A'/A,T}(\mathrm{Im}\ (\alpha_{A'})) \subset \langle \mathrm{Im}\ (\alpha_A) \rangle,$$

where $\langle Im(\alpha_A) \rangle$ denotes the subgroup generated by Im (α_A) in the corresponding cohomology group. We may also consider similar notions for kernel of α_A, when G is commutative and T may be not.

In this paper we prove the following analogs of the results already proved in the case of local and global fields.

Theorem I. (Local Corestriction Principle) *Let A be a ring of integers of a non-archimedean local field k, A' the integral closure of A in a separable finite extension k' of k, belonging to $\mathcal{C}_{\mathcal{A}}$. Let G, T be reductive A-group schemes with T an A-torus, and let $\alpha_A : \mathrm{H}^p_{et}(A, G) \to \mathrm{H}^q_{et}(A, T)$, (resp. $\alpha_A : \mathrm{H}^1_{et}(A, T) \to \mathrm{H}^1_{et}(A, G)$) be a connecting map induced from an exact sequence of cohomologies of reductive A-group schemes involving G and T (resp. induced from A-morphism $T \to G$). Then Corestriction Principle holds for the image (resp. kernel) of α_A with respect to the extension A'/A.*

Theorem II. (Global Corestriction Principle) *Let A be a Dedelind ring with quotient field a global field k, V the set of all primes of A, $\alpha_A : \mathrm{H}^p_{et}(A, G) \to \mathrm{H}^q_{et}(A, T)$ (resp. $\alpha_A : \mathrm{H}^1_{et}(A, T) \to \mathrm{H}^1_{et}(A, G)$) a connecting map induced from an exact sequence of cohomologies of reductive A-group schemes involving G and T (resp. induced from an A-morphism $T \to G$), with T an A-torus.*

Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, there is a finite set $S \subset V$ such that the Corestriction Principle holds for the image (resp. kernel) of α_{A_S} : $\mathrm{H}^p_{et}(A_S, G_{A_S}) \to \mathrm{H}^q_{et}(A_S, T_{A_S})$ (resp. α_{A_S} : $\mathrm{H}^1_{et}(A_S, T_{A_S}) \to \mathrm{H}^1_{et}(A_S, G_{A_S})$) with respect to the extension A'_S/A_S, where A_S denotes the localization of A at S.

(By convention, in the case of global function field k, we call the ring of k-regular functions of a smooth projective k-curve also by the ring of integers of k.) We may also state above theorem a bit differently as follows.

Theorem II'. *Let A be the ring of integers of a global field k, $\alpha_k : \mathrm{H}^p(k, G) \to \mathrm{H}^q(k, T)$ (resp. $\alpha_k : \mathrm{H}^1(k, T) \to \mathrm{H}^1(k, G)$) a connecting map induced from an exact sequence of cohomologies of an exact sequence of smooth connected reductive k-groups (resp. induced from a k-morphism $T \to G$), with T a k-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, there is a finite set $S \subset V$ such that above exact sequence is obtained from an exact sequence of reductive A_S-group schemes $1 \to G' \to G \to T \to 1$ (resp. an A_S-morphism) by taking the fibers at generic point, and the Corestriction Principle holds for the image (resp. kernel) of α_{A_S} : $\mathrm{H}^p_{et}(A_S, G_{A_S}) \to \mathrm{H}^q_{et}(A_S, T_{A_S})$ (resp. α_{A_S} : $\mathrm{H}^1_{et}(A_S, T_{A_S}) \to \mathrm{H}^1_{et}(A_S, G_{A_S})$) with respect to the extension A'_S/A_S.*

As application of the results presented above, we derive the following norm principle for S-class groups of algebraic groups. We consider the class set of a given flat affine group scheme G of finite type over Dedekind ring A with smooth generic fiber G_k over the global quotient field k of A. Let $X = Spec(A)$, $\eta \in X$ the generic point of X, S a finite subset of $X_0 := X \setminus \{\eta\}$. The ring $A(S)$ of S-adèles is defined as

$$\mathbf{A}(S) := \prod_{v \in X_0 \setminus S} A_v \times \prod_{v \in S} k_v,$$

where k_v (resp. A_v) is the completion of k (resp. A) in the v-adic topology. We denote by $\mathbf{A} = ind.lim_S\ \mathbf{A}(S)$ the adèle ring of k (with respect to A !). Recall that (see e. g.[Bo], [Ha1], [Ni1], [PlR]) the S-class set, of G with respect to a finite set S of primes of A (denoted by $Cl_A(S, G)$), and the class set of G (denoted by $Cl_A(G)$), is the set of double classes

$$Cl_A(S, G) = G(\mathbf{A}(S)) \setminus G(\mathbf{A})/G(k),$$

and

$$Cl_A(G) = G(\mathbf{A}(\emptyset)) \setminus G(\mathbf{A})/G(k),$$

respectively. (Here $G(k)$ is embedded diagonally into $G(\mathbf{A})$. Another, more familiar notation for $Cl_A(G)$ using the set of infinite primes is given in Section 4.) It may happen that $Cl(S, G)$ (resp. $Cl_A(G)$ has a natural group structure (i.e. inherited from that of $G(\mathbf{A})$). In this case it is denoted by $\mathcal{G}Cl_A(S, G)$ (resp. $\mathcal{G}Cl_A(G)$).

Theorem III. (Norm principle for S-class groups of algebraic groups.)
1) *With notation as in Theorem II, assume further that for a finite set S of primes of k, containing the set ∞ of archimedean primes, and for the derived subgroup $G' = [G, G]$ of G, the topological group $\prod_{v \in S} G'(k_v)$ is non-compact. Then for any $A' \in \mathcal{C}_A$, the class set $Cl_{A'}(S, G)$ has a natural structure of finite abelian group, and we have a norm homomorphism, functorial in A'*

$$N_{A'/A} : \mathcal{G}Cl_{A'}(S, G) \to \mathcal{G}Cl_A(S, G).$$

2) *Let notation be as in Theorem II'. Then after localizing at a suitable finite set S of primes, for any $A' \in \mathcal{C}_A$, and for any finite set of primes T, containing S, the class set $Cl_{A'_S}(T, G)$ has a natural structure of finite abelian group, and we have a norm homomorphism, functorial in A'*

$$N_{A'/A} : \mathcal{G}Cl_{A'_S}(T, G) \to \mathcal{G}Cl_{A_S}(T, G).$$

A short presentation of the results obtained here was announced before (see [T8]).

1 Some preliminary results

1.1. Induced tori. We need the following analogs of some results proved in [Bo1], [Bo2], [Ko], [T3], [T4]. First, we recall the important notion of induced tori (see [Ha1], pp. 171 - 172). For a integrally closed noetherian domain A with quotient field k, we recall that (cf. [SGA 3], Exp. X, Théorème 5.16) for an A-torus T there is a finite étale extension A'/A, with quotient field k' such that $T_{A'}$ is A'-isomorphic to $\mathbf{G}_m^r$ for some r. We may assume that k'/k is a finite Galois extension, and that A'/A is also a Galois extension with the same Galois group $\Gamma := Gal(A'/A) = Gal(k'/k)$. Denote by $X_{A'}(T) := Hom_{A'}(T_{A'}, \mathbf{G}_m)$ the character group, which is a Γ-module and it determines the A-group scheme T up to a unique A-isomorphism ([SGA

3], Exp. X, Théorème 1.1). T is called A-induced if there are a subgroup $\Gamma_0 \subset \Gamma$ and a Γ-submodule $X_0 \subset X_{A'}(T)$ such that Γ_0 acts trivially on X_0 and

$$X_{A'}(T) = \bigoplus_{\sigma \in \Gamma/\Gamma_0} \sigma(X_0).$$

Then there is a uniquely defined subring $A_1 \subset A'$ such that $\Gamma_1 = Gal(A'/A_1)$, A_1/A is unramified, and one can checks that $T = R_{A_1/A}(\mathbf{G}_m^r)$, the restriction of scalars from A_1 to A (see [Ha], p. 172).

1.2. z-extensions. Now, as in the case of fields, for a ring A as above, and an exact sequence $1 \to Z \to H \to G \to 1$ of reductive A-group schemes, with Z an A-torus, we say that H is a *z-extension* of G if Z is an induced A-torus and the derived subgroup of H is simply connected (cf. [SGA 3], Exp. XXII, Sec. 4.3.3, for the corresponding notions). Now, if $x \in \mathrm{H}^1(A', G)$, we say that a z-extension $H \to G$ (over A) is *x-lifting* if $x \in \mathrm{Im}\,(\mathrm{H}^1(A', H_{A'}) \to \mathrm{H}^1(A', G_{A'}))$. By [Ha1], Lemma 1.4.1, and by the same arguments used by Harder [Ha1] in the proof of Satz 1.2.1, Borovoi [Bo1], [Bo2] and Kottwitz [Ko] (cf. also [T3], Lemma 2.1), we have the following assertion, and since its proof is basically similar so we omit it.

Lemma 1. *a*) ([Ha1], Lemma 1.4.1) *With notation as above we have*

$$\mathrm{H}^1_{flat}(A, T) = \mathrm{H}^1_{flat}(A_1, \mathbf{G}_m^r).$$

b) *With notations as above, for any given reductive A-group scheme G, there exist z-extensions of G.*
c) *Given an exact sequence $1 \to G_0 \to G_1 \to G_2 \to 1$ of reductive A-group schemes, there exists a z-extension of this sequence, i.e., an exact sequence $1 \to H_0 \to H_1 \to H_2 \to 1$ of reductive A-group schemes and a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \to & H_0 & \to & H_1 & \to & H_2 & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & G_0 & \to & G_1 & \to & G_2 & \to & 1 \end{array}$$

of reductive A-group schemes such that each A-group scheme H_i is a z-extension of G_i, $i = 0, 1, 2$.
d) Let A' belong to $\mathcal{C}_{\mathcal{A}}$, G a reductive A-group scheme. Then for any element $x \in \mathrm{H}^1_{et}(A', G)$ there exists a x-lifting z-extension H of G.
e) Let A' be as above and let $\pi : G_1 \to G_2$ be a morphism of reductive A-group schemes. Then for any given $x \in \mathrm{H}^1_{et}(A', G_{1A'})$ there exists a z-extension $\pi' : H_1 \to H_2$ of $\pi : G_1 \to G_2$, such that H_1 is x-lifting z-extension of G_1. ■

Notice that *b*) above is an extension of Ono's "cross diagram" lemma [O] (cf. also [Ha1]).

1.3. Deligne hypercohomology and abelianized cohomology. In [De], Sec. 2.4, Deligne has associated to each pair $f : G_1 \to G_2$ of algebraic groups defined over a field k, where f is a k-morphism, a category $[G_1 \to G_2]$ of G_2-trivialized G_1-torsors, and certain hypercohomology sets denoted by $\mathbf{H}^i(G_1 \to G_2)$, which fits into an exact sequence involving $G_1(k), G_2(k)$ and their first Galois cohomologies. In many important cases, the above category appears to be a strictly commutative Picard category (loc.cit). In [De], p. 276, there was also an indication that the construction given there can be done for sheafs of groups over any topos. Thus in [De], there was defined the hypercohomology sets $\mathbf{H}^i_r(G_1 \to G_2)$ for $i = -1, 0$, where r stands for étale or flat topology. (To be consistent, we use the notations of [Bo3] and [Br], while in [De], the degree of the hypercohomology sets corresponding to $G_1 \to G_2$ is shifted.) In particular, the existence of a norm map (i.e., the validity of Corestriction principle) for hypercohomology in degree 0 in the case of local and global fields was first proved by Deligne [De], Prop. 2.4.8.

Later on, Borovoi in [Bo3] and Breen in [Br] gave a detailed exposition and extension of such hypercohomology, and in [Bo3] (resp. [Br]), there was defined also the set $\mathbf{H}^1(G_1 \to G_2)$ (resp. $\mathbf{H}^1_r(G_1 \to G_2)$, where the setting in [Br] works over any topos T_r). In the particular case when the base scheme is the spectrum of a field of characteristic 0, the theory coincides with the one given by Borovoi [Bo3]). As in [Bo3], by using [Br], we may also define the abelianization map $ab_G : \mathrm{H}^i_r(A, G) \to \mathbf{H}^i(\tilde{G} \to G)$, for a reductive A-group scheme G, where $\tilde{G}$ is the simply connected semisimple A-group scheme, which is the universal covering of $G' = [G, G]$, the semisimple part of G, and $i = 0, 1$. In fact, it has been proved that if $\tilde{Z}$ (resp. Z) is the center of $\tilde{G}$ (resp. of G), then there are an equivalence of categories $[\tilde{Z} \to Z] \simeq [\tilde{G} \to G]$, and quasi-isomorphisms of complexes

$$(\tilde{Z} \to Z) \simeq (\tilde{T} \to T) \simeq (\tilde{G} \to G),$$

where $\tilde{T}$ (resp. T) is a maximal A-torus of $\tilde{G}$ (resp. G), with $f^{-1}(T) = \tilde{T}$. One defines $\mathrm{H}^i_{ab,r}(A, G) := \mathbf{H}^i_r(\tilde{G} \to G)$ and call it the *abelianized cohomology* of degree i of G (in the corresponding topos; here r stands for "ét" or "flat", if one of group schemes involved is not smooth).

1.4. Equivalent conditions for Corestriction principle. Let G be a reductive A-group scheme. Denote by G' the derived subgroup scheme of G, $\tilde{G}$ the simply connected covering of G', $Ad(G)$ the adjoint group scheme of G (see [SGA 3], Exp. XXII, 4.3.3), $\tilde{F} :=$ Ker $(\tilde{G} \to Ad(G))$, $F :=$ Ker $(\tilde{G} \to G')$ and let $\tilde{Z}$, Z be as above. Since $\tilde{Z}$ and Z are commutative, the resulting cohomology sets $\mathbf{H}^i_r(\tilde{Z} \to Z)$ have natural structure of abelian groups. In the case of spectrum of a field of characteristic 0, it is known that there exists functorial corestriction homomorphisms for $\mathrm{H}^i_{ab,et}(G)$ (which follows from [Pe], cf. [T2]). It can be also extended to the case of positive characteristic, if we assume that the center $\tilde{Z}$ of $\tilde{G}$ is smooth. However, in the general case (étale or flat case) it is not clear whether such functorial homomorphisms always exist. Thus we make the following assumption.

(Hyp_A) *For $A' \in \mathcal{C}_A$, for any G as above such that $\tilde{Z}$ is smooth, there exist functorial corestriction homomorphisms $Cores_{A'/A} : \mathrm{H}^i_{ab,et}(G_{A'}) \to \mathrm{H}^i_{ab,et}(G), i = 0, 1$.*

Let $\alpha : \mathrm{H}^p_{et}(A, G) \to \mathrm{H}^q_{et}(A, T)$ be a connecting map of cohomologies and assume that an extension A'/A, $A' \in \mathcal{C}_A$, is fixed. Under the assumption of (Hyp_A), we consider the following statements.

a) *The (Weak) Corestriction principle holds, with respect to the extension A'/A, for the image of any connecting map $\alpha : \mathrm{H}^p_{et}(A, G) \to \mathrm{H}^q_{et}(A, T)$ for reductive A-group schemes G, T, with T an A-torus, $0 \leq p \leq 1, p \leq q \leq p+1$.*

b) *For any reductive A-group scheme G, such that $\tilde{Z}$ is smooth, the (Weak) Corestriction principle holds, with respect to the extension A'/A, for the im-*

ages of functorial map $ab_G : \mathrm{H}^p_{et}(A,G) \to \mathrm{H}^p_{ab,et}(A,G), 0 \le p \le 1.$

c) The (Weak) Corestriction principle holds for the image of any connecting (coboundary) map $\mathrm{H}^p_{et}(A,G) \to \mathrm{H}^{p+1}_{et}(A,T), 0 \le p \le 1$, *with respect to the extension* A'/A, *where*

$$1 \to T \to G_1 \to G \to 1$$

is any exact sequence of reductive A*-group schemes, and* T *is a smooth central* A*-subgroup scheme.*

d) The same statement as in c), but G_1, G *are semisimple.*

e) The (Weak) Corestriction principle holds for the image of any connecting (coboundary) map $\mathrm{H}^p_{et}(A, Ad(G)) \to \mathrm{H}^{p+1}_{et}(A,F)$, *with respect to the extension* A'/A, *where*

$$1 \to F \to G \to Ad(G) \to 1$$

is any exact sequence of semisimple A*-group schemes, and* F *is a smooth (central) subgroup.*

f) The (Weak) Corestriction principle holds for the image of any connecting (coboundary) map $\mathrm{H}^p_{et}(A, Ad(G)) \to \mathrm{H}^{p+1}_{et}(A,F)$, *with respect to the extension* A'/A, *where*

$$1 \to F \to G \to Ad(G) \to 1$$

is any exact sequence of semisimple A*-group schemes, and* F *is a smooth (central) subgroup and* G *is simply connected.*

Notice that we always have obvious implications $c) \Rightarrow d) \Rightarrow e) \Rightarrow f)$. For the statements above, denote by $x(p)$ (resp. $x(p,q)$) the corresponding statement evaluated at p (resp. at p,q). For example $a(0,1)$ means the statement $a)$ with $p = 0, q = 1$. One of main results of [T3] is Theorem 2.10, which says that in the case of $A = Spec(k)$, the spectrum of a field, the above statements are all equivalent. All proofs given there are functorial and can be formally extended to our case provided that the Hilbert-90 Theorem holds, namely $\mathrm{H}^1_{et}(A,T) = 0$ for any induced torus T over A. This holds, for example, if

A is a local ring by [Gr1], p. 190-15. Thus we have the following theorem, and since its proof is almost identical with that of Theorem 2.10 in [T3], so is omitted (however, see [T7]). (Here a statement x) holds if it holds for any possible values of p, q.)

Theorem 2. *a) Assuming* (Hyp_A), *there are the following equivalence relations*

$$a) \Leftrightarrow b), \quad c) \Leftrightarrow d), \quad e) \Leftrightarrow f).$$

b) The following relations between above statements for certain values of p,q hold. For low dimension we have

$$a(0,1) \Leftarrow a(0,0) \Leftrightarrow b(0) \Leftrightarrow c(0) \Leftrightarrow d(0) \Leftrightarrow e(0) \Leftrightarrow f(0).$$

For higher dimension we have

$$\begin{array}{c} a(1,2) \Leftarrow a(1,1) \Leftrightarrow b(1) \\ c(1) \Leftrightarrow d(1) \\ \Downarrow \\ e(1) \Leftrightarrow f(1) \end{array}$$

and if A is a ring such that $\mathrm{H}^1_{et}(A', T) = 0$ *for any induced* A'*-torus* T, $A' \in \mathcal{C}_A$, *then the following implications*

$$\begin{array}{c} a(1,2) \Leftarrow a(1,1) \Leftrightarrow b(1) \\ \Downarrow \\ c(1) \Leftrightarrow d(1) \\ \Downarrow \\ e(1) \Leftrightarrow f(1) \end{array}$$

hold true.
c) In general, without assuming (Hyp_A), *by ignoring* $b(i)$, *all above implications without* $b(i)$ *involving, hold true.* ∎

1.5. Remark. Notice that in the case of spectrum of a field, all condition related with smoothness can be omitted, and we can consider flat cohomology instead ([T4]).

2 Local case

2.1. Serre - Grothendieck conjecture. Let S be an integral, regular, Noetherian scheme with function field K, G a reductive group scheme over S, and let E be a G-torsor over S, i.e., a principal homogeneous space of G over S locally trivial for the étale topology of S. We say that E is *rationally trivial* if it has a section over K.

First we recall the following conjecture due to Serre and Grothendieck, in the most general form given by Grothendieck. J.-P. Serre and A. Grothendieck in C. Chevalley's Seminar in 1958 ([SCh], Exp. I and Exp. V) and A. Grothendieck in a Bourbaki Seminar [Gr2] in 1966 formulated the following conjecture.

Conjecture. ([Gr2], Remarque 1.11.) *Let S be a locally noetherian regular scheme, G a semisimple group scheme over S. Then any G-torsor over S which is trivial at maximal points is also locally trivial.*

In the case of arbitrary reductive group schemes, the following is a more general formulation of this conjecture (cf. [Ni1], [CTO]):

(*) *If S is as above and G is a reductive S-group scheme, then every rationally trivial G-torsor is locally trivial for the Zariski topology of S.*

In other form the conjecture says (cf. [Ni1], [CTO])

(**) *The following sequence of (pointed) cohomology sets*

$$1 \to \mathrm{H}^1_{Zar}(S, G) \to \mathrm{H}^1_{et}(S, G) \to \mathrm{H}^1(K, G_K)$$

is exact.

Equivalently, it says that

(***) *If S, G are as above, η is the generic point of S and $A = \mathcal{O}_x$ is any local ring at $x \in S \setminus \{\eta\}$, then the natural map of cohomology*

$$\mathrm{H}^1_{et}(A, G) \to \mathrm{H}^1(K, G_K)$$

has trivial kernel.

Partial results obtained are due to Harder [Ha1], Tits (unpublished, but see [Ni1], Theorem 4.1,) Nisnevich [Ni1], [Ni2], Theorem 4.2, Colliot-Thélène and Sansuc [CTS] and Colliot-Thélène and Ojanguren [CTO]. Some very general formalism has been treated recently in [Mo], Chapter I. We mainly need only the following

Theorem 3. *a*) (Tits, cf. [Ni1], Theorems 4.1.) *If A is a complete discrete valuation ring with quotient field K, and G is a semisimple A-group scheme, then the above Conjecture* (***) holds.
b) ([Ni1], Théorème 4.2) *If S is a regular one-dimensional noetherian scheme and G is a semisimple S-group scheme, then the above conjectures hold.*
c) ([Ni1], Théorème 4.5) *If S = Spec R, R is a regular local henselian ring and G is S-semisimple group scheme, then above conjectures hold.*

2.2. We have the following

Proposition 4. *a*) *Let A be the ring of integers of a non-archimedean local field k, $1 \to G' \to G \to T \to 1$ an exact sequence of reductive A-group schemes, with T an A-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, the Corestriction Principle holds for the image of $\mathrm{H}^i_{et}(A,G) \to \mathrm{H}^i_{et}(A,T)$, with respect to the extension A'/A, where i = 0,1.*
b) *Let A, k be as above and let $1 \to T \to G_1 \to G \to 1$ be an exact sequence of reductive A-group schemes, with T an A-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, the Corestriction Principle holds for the image of $\mathrm{H}^i_{et}(A,G) \to \mathrm{H}^{i+1}_{et}(A,T)$, with respect to the extension A'/A, where i = 0,1.*

First we need the following

Lemma 5. *Let A be the ring of integers of a local non-archimedean field k, G a semisimple quasi-split A-group. Then G has a maximal A-torus T, the generic fiber of which is an anisotropic k-torus T_k.*

Proof. It is clear that we may assume G to be simply connected. By the structure theorem on decomposition of G into factors, which correspond to irreducible factors of its Dynkin scheme (see [SGA 3], Exp. XXIV, Sec. 5.3, Prop. 5.5, Prop. 5.9), we may assume, without any loss of generality, that the Dynkin diagram of G is irreducible, i.e., G_k is (absolutely) almost simple. First we consider the case G is A-split. By the inspection of the proof of Lemma 6.15 of [PlR], one sees that there exists a maximal k-torus T of G_k which is k-anisotropic and also split over a finite unramified extension of k. Such a torus, as is well-known, can be lifted to a A-torus, which is a maximal A-torus of G.

Next assume that G_k is quasi-split, but not split. We fix a Borel A-subgroup B containing a maximal A-torus T of G, which always exists according to [Ha1], Satz 3.1. Thus the generic fiber G_k is a quasi-split k-group, which is not split, and T_k is a maximal k-torus containing a maximal k-split subtorus S_k of G_k. Hence G_k is either of type ${}^2A$, ${}^2D$, ${}^2E_6$, or quasi-split trialitarian types ${}^3D_4$, ${}^6D_4$. By choosing a suitable matrix realization, one can verify the assertion directly for types ${}^2A$ and ${}^2D$. It remains to check quasi-split type ${}^2E_6$ and trialitarian types ${}^3D_4, {}^6D_4$. Notice that for any $s \in S = SpecA$ with residue field $k(s)$, the fiber over s is either a split or quasi-split $k(s)$-group. Let denote by $Tits(G)_s$ the Tits index of $G_s = G \otimes k(s)$ (i.e., Dynkin diagram with the action of the Galois group $Gal(k(s)_s/k(s))$ on the vertices of the Dynkin diagram (see [Ti])), $R(G)_s$ the root system of G_s with respect to $T_s := T \times k(s)$, with a basis Δ_s, which corresponds to the set of vertices of $Tits(G)_s$. Denote by $\tilde{\alpha}_s$ the highest root in $R(G)_s$ (cf. [Bou], Table II). First assume that G_k is of type ${}^2E_6$. Then one checks as in [T], [T3], that the root subgroup H_s corresponding to $\tilde{\alpha}_s$ is defined over $k(s)$ and also $k(s)$-split, and is of type A_1. These groups give rise to a split semisimple A-group scheme H of G, each fiber of which is of split type A_1. By a direct inspection, one sees that there is a torus, denoted by T_H, which is a maximal A-torus of H and satisfies the requirement of the lemma (i.e., the generic fiber is an anisotropic maximal k-torus of G_k). The centralizer of the A-subgroup scheme H gives rise to a reductive A-subgroup scheme, each fiber of which has type ${}^2A_5$. Now we can finish the proof by reducing to the type ${}^2A$ already considered.

The proof in the trialitarian case is similar. ∎

2.2.1. Remark. One should notice that, as another argument of the proof, we may also use a result of [SGA 3], Exp. XXIV, Corol. 1.12, which allows us to lift isomorphisms of group schemes on residues fields to local henselian rings.

2.3. *Proof of Proposition 4.* Some of ideas of the proof goes back to [De],

[MS], [T1] - [T4]. It is clear that we may assume G' to be the derived subgroup of G (see [SGA 3], Exp. XXII, Théorème 6.2.1).
a) *Case* $i = 0$. First we assume that the derived subgroup $G' = [G, G]$ is simply connected. We consider the following commutative diagram with exact rows

$$\begin{array}{ccccc} G(A') & \xrightarrow{\pi'} & T(A') & \to & \mathrm{H}^1_{et}(A', G'_{A'}) \\ & & \downarrow & & \\ G(A) & \xrightarrow{\pi} & T(A) & \to & \mathrm{H}^1_{et}(A, G') \end{array}$$

By a theorem of Bruhat - Tits - Kneser ([BrT], [Kn1], [Kn2]), $\mathrm{H}^1(k, G'_k)$ (resp. $\mathrm{H}^1(k', G'_{k'})$) is trivial, since G' is simply connected. By a result of Tits above (a special case of Serre - Grothendieck conjecture), the natural map $\mathrm{H}^1_{et}(A, G') \to \mathrm{H}^1(k, G'_k)$ (resp. $\mathrm{H}^1_{et}(A', G') \to \mathrm{H}^1(k', G'_{k'})$) has trivial kernel. Therefore π, π' are surjective maps, and the assertion is trivial.

Next we consider the general case. We take any z-extension

$$1 \to Z \to H \to G \to 1$$

of G. By Grothendieck's Hilbert-90 Theorem ([Gr1]) and Lemma 1, we have

$$\mathrm{H}^1_{et}(A, Z) = 0, \mathrm{H}^1_{et}(A', Z_{A'}) = 0,$$

so the homomorphisms $\pi' : H(A') \to G(A')$ and $\pi : H(A) \to G(A)$ are surjective. Consider the exact sequence

$$1 \to \tilde{G} \to H \to S \to 1,$$

where $\tilde{G} = [H, H]$, $S := H/\tilde{G}$. From above we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & \tilde{G} & \to & H & \to & S & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & G' & \to & G & \to & T & \to & 1 \end{array}$$

From this we derive the following commutative diagrams

$$\begin{array}{ccc} H(A') & \xrightarrow{\alpha'} & S(A') \\ \downarrow \pi' & & \downarrow \beta' \\ G(A') & \xrightarrow{\gamma'} & T(A') \end{array}$$

$$\begin{array}{ccc} H(A) & \xrightarrow{\alpha} & S(A) \\ \downarrow \pi & & \downarrow \beta \\ G(A) & \xrightarrow{\gamma} & T(A) \end{array}$$

and also

$$\begin{array}{ccc} S(A') & \xrightarrow{\beta'} & T(A') \\ p \downarrow & & \downarrow q \\ S(A) & \xrightarrow{\beta} & T(A) \end{array}$$

where p, q are corestriction homomorphisms. Let $g' \in G(A'), t' = \gamma'(g') \in T(A'), t = Cores(t) \in T(A)$. From above, there exists $h' \in H(A')$ such that $\pi'(h') = g'$. Let $s' = \alpha'(h') \in S(A'), s = Cores(s') \in S(A)$. By previous part, there is $h \in H(A)$ such that $\alpha(h) = s$. Therefore for $g = \pi(h)$, we have

$$
\begin{aligned}
\gamma(g) &= \gamma(\pi(h)) \\
&= \beta(\gamma(h)) \\
&= \beta(s) \\
&= \beta(Cores(s')) \\
&= Cores(\beta'(s')) \\
&= Cores(\beta'(\alpha'(h'))) \\
&= Cores(\gamma'(\pi'(h'))) \\
&= Cores(\gamma'(g')) \\
&= Cores(t') = t,
\end{aligned}
$$

i.e., $t \in Im(\gamma)$ as required.

Case $i = 1$. Let F' be the center of G', $Ad(G) := G'/F'$ the adjoint group of G, and one can define as in the classical case the A-group scheme G^q which is the quasi-split inner form of G, i.e., the cohomology class from $\mathrm{H}^1_{et}(A, Aut(G^q))$ corresponding to G belongs to the image of the canonical map $\mathrm{H}^1_{et}(A, G^q) \to \mathrm{H}^1_{et}(A, Aut(G^q))$. Then one sees that we have $G/F' = Ad(G) \times S$ (direct product), where S is a A-subtorus of G/F'. Then the same argument as in [T3] (see Theorem 2 above) shows that it suffices to show the Corestriction principle to hold for the image of the map

$$\Delta : \mathrm{H}^1_{flat}(A, Ad(G)) \to \mathrm{H}^2_{flat}(A, \tilde{F}),$$

where $\tilde{F}$ is the center of the simply connected covering $\tilde{G}$ of $Ad(G)$, which is a flat finite A-group scheme of multiplicative type. If $char.k = 0$, we can use étale cohomology, so there is a corestriction homomorphism $\mathrm{H}^2_{et}(A', \tilde{F}) \to \mathrm{H}^2_{et}(A, \tilde{F})$. In fact, by a result of Colliot-Thélène and Sansuc [CTS], for any torus T over A we have an injective homomorphism $\mathrm{H}^2_{et}(A, T) \to \mathrm{H}^2(k, T_k)$. If T_1 is a maximal A-torus of the quasi-split inner form G'^q of G' as in Lemma 5, and $Ad(T_1)$ is the corresponding maximal A-torus of $Ad(G)$, then $T_{1,k}$ is k-anisotropic, so $\mathrm{H}^2(k, T_{1,k}) = 0$, so one checks that the coboundary map

$\Delta : \mathrm{H}^1_{flat}(A, Ad(T_1)) \to \mathrm{H}^2_{flat}(A, \tilde{F})$ is surjective. Therefore the same is true for

$$\Delta^q : \mathrm{H}^1_{flat}(A, Ad(G^q)) \to \mathrm{H}^2_{flat}(A, \tilde{F}).$$

The method of proof of Lemma 2.5 of [T4] also shows that then the same is true for G, i.e. the map

$$\Delta : \mathrm{H}^1_{flat}(A, Ad(G)) \to \mathrm{H}^2_{flat}(A, \tilde{F})$$

is surjective. The case of positive characteristic is trivial, due to the fact that $\mathrm{H}^r_{flat}(A, F) = 0$, for $r > 2$ (see [Mi1], Chap. III, Sec.7).
b) Follows from part *a*) and Theorem 2. The proposition is therefore completely proved. ∎

2.4. As the proofs above suggest, we have the following analogs of some results of Kneser (cf. [Kn1], [Kn2], [PlR]). For the sake of convenience, we state also the global analogs here, which will be used in next section.

Proposition 6. *a*) *Let A be the ring of integers of a local non-archimedean field k, G a semisimple A-group, $\tilde{G}$ the simply connected A-group scheme which is covering G, F the kernel of canonical morphism $\tilde{G} \to G$. Then the coboundary map $\Delta : \mathrm{H}^1_{flat}(A, G) \to \mathrm{H}^2_{flat}(A, F)$ is bijective.*
b) ([Do1], [Do2, Ch. VIII, Corol. 2.5]) *Assume that A is a Dedekind ring with quotient field a global field k, G a semisimple A-group, $\tilde{G}$ the simply connected A-group scheme which is covering G, F the kernel of canonical morphism $\tilde{G} \to G$. Then the coboundary map $\Delta : \mathrm{H}^1_{flat}(A, G) \to \mathrm{H}^2_{flat}(A, F)$ is surjective.*
c) *With notation as in b), assume further that A is the ring of integers of a global function field k. Then Δ is bijective.*

Proof. *a*) The surjectivity follows from above. To prove the bijectivity, notice that by Theorem 3 (Tits theorem), we have Ker $(\mathrm{H}^1_{et}(A, \tilde{G}) \to \mathrm{H}^1(k, \tilde{G})) = 0$, while $\mathrm{H}^1(k, \tilde{G}) = 0$ by Kneser - Bruhat - Tits theorem ([Kn1], [Kn2, [BrT]). Now the bijectivity follows by using twisting with the cocycles (see [Gir], Chap. 4, Prposition 4.3.4).
c) By *b*), Δ is surjective. To prove the injectivity of Δ we make use of Theorem 3, b). By using twisting, we need only show that Ker $(\Delta) = 0$. By Prasad - Margulis theorem (cf. e.g. [Ma], [Pr]), we know that over global function field k any simply connected semisimple group $\tilde{G}$ has strong approximation

over k. Then by Theorem 3 we have the following exact sequence

$$1 \to \mathrm{H}^1_{Zar}(A, \tilde{G}) \to \mathrm{H}^1_{et}(A, \tilde{G}) \to \mathrm{H}^1(k, \tilde{G}_k)$$

and a result of Harder [Ha1], Korollar 2.3.1, shows that $\mathrm{H}^1_{Zar}(A, \tilde{G}) = 0$. Thus the commutative diagram together with the triviality of $\mathrm{H}^1(k, \tilde{G}_k)$ (by a theorem of Harder [Ha2])

$$\begin{array}{ccccc}
\mathrm{H}^1_{Zar}(A, \tilde{G}) = 0 & & & & \\
\downarrow & & & & \\
\mathrm{H}^1_{flat}(A, \tilde{G}) & \xrightarrow{p} & \mathrm{H}^1_{flat}(A, G) & \xrightarrow{\Delta} & \mathrm{H}^2_{flat}(A, F) \\
q \downarrow & & \downarrow r & & \downarrow s \\
\mathrm{H}^1(k, \tilde{G}_k) & \xrightarrow{p_k} & \mathrm{H}^1(k, G_k) & \xrightarrow{\Delta_k} & \mathrm{H}^2(k, F_k)
\end{array}$$

tells us that $\mathrm{H}^1_{et}(A, \tilde{G}) = 0$, and the assertion follows. ∎

2.5. Remarks. 1) Kneser [Kn2] first proved above results in the case of local and global field of characteristic 0. In [Do1], [Do2], *a*) has been proved for the case of spectrum of a local field and this result and *b*) have been proved by using the method of bands (gerbes) (see [Gir]).
2) It may happen that Δ as in *b*) is not bijective (in the number field case).

2.6. The proof of Theorem I is completed after we prove the following

Proposition 7. *Let A be the ring of integers of a non-archimedean local field k, $T \to G$ a morphism of reductive A-group schemes, with T a A-torus. Then for any finite separable unramified extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, the Corestriction principle holds for the kernel of $\mathrm{H}^1_{et}(A, T) \to \mathrm{H}^1_{et}(A, G)$, with respect to the extension A'/A.*

Proof. It is clear that we may assume that T is a maximal A-torus of G. Let

G' be the derived subgroup scheme of G and $S := G/G'$, an A-torus. We have the following commutative diagram

$$\begin{array}{ccccc} & & \mathrm{H}^1_{et}(A,T) & \stackrel{\gamma}{\rightarrow} & \mathrm{H}^1_{et}(A,S) \\ & & \downarrow \alpha & & \downarrow = \\ \mathrm{H}^1_{et}(A,G') & \stackrel{\delta}{\rightarrow} & \mathrm{H}^1_{et}(A,G) & \stackrel{\beta}{\rightarrow} & \mathrm{H}^1_{et}(A,S) \end{array}$$

First we assume that G' is simply connected. Then by Tits theorem quoted above, $\mathrm{H}^1_{et}(A,G')$ is injected into $\mathrm{H}^1(k,G'_k)$, where the latter is trivial (Bruhat -Tits - Kneser Theorem, [BrT], [Kn1, Kn2]). Thus we have Ker (α) = Ker $(\beta\circ\alpha)$ = Ker (γ), for which the Corestriction principle trivially holds.

In the general case, let $x' \in$ Ker $(\mathrm{H}^1_{et}(A',T) \rightarrow \mathrm{H}^1_{et}(A',G))$. By Lemma 1 there exists a z-extension $\tilde{T} \rightarrow H$ of the pair $T \rightarrow G$, which is x'-lifting. Again by considering the commutative diagrams

$$\begin{array}{ccc} \mathrm{H}^1_{et}(A',\tilde{T}_{A'})) & \stackrel{\alpha'}{\rightarrow} & \mathrm{H}^1_{et}(A',H_{A'}) \\ \downarrow \pi' & & \downarrow \beta' \\ \mathrm{H}^1_{et}(A',T_{A'}) & \stackrel{\gamma'}{\rightarrow} & \mathrm{H}^1_{et}(A',G_{A'}) \end{array}$$

$$\begin{array}{ccc} \mathrm{H}^1_{et}(A,\tilde{T}) & \stackrel{\alpha}{\rightarrow} & \mathrm{H}^1_{et}(A,H) \\ \downarrow \pi & & \downarrow \beta \\ \mathrm{H}^1_{et}(A,T) & \stackrel{\gamma}{\rightarrow} & \mathrm{H}^1_{et}(A,G) \end{array}$$

and also

$$\begin{array}{ccc} \mathrm{H}^1_{et}(A', \tilde{T}_{A'}) & \stackrel{\beta'}{\rightarrow} & \mathrm{H}^1_{et}(A', T_{A'}) \\ p \downarrow & & \downarrow q \\ \mathrm{H}^1_{et}(A, \tilde{T}) & \stackrel{\beta}{\rightarrow} & \mathrm{H}^1_{et}(A, T) \end{array}$$

we can finish by using similar arguments in the case treated above. ∎

Theorem I now follows from Proposition 4 and Proposition 7.

2.7. Example. Let $\mathcal{O}_v$ be the ring of integers of a local non-archimedean field k, G, T reductive $\mathcal{O}_v$-group schemes, where T is a torus, and let $\pi : G \to T$ be a $\mathcal{O}_v$-morphism of group schemes. For any finite separable unramified extension k'/k with the ring of integers $\mathcal{O}_w$ there is a natural norm homomorphism

$$N := N_{\mathcal{O}_w/\mathcal{O}_v} : T(\mathcal{O}_w) \to T(\mathcal{O}_v),$$

and in the following diagram

$$\begin{array}{ccc} G(\mathcal{O}_w) & \stackrel{\beta'}{\rightarrow} & T(\mathcal{O}_w) \\ & & \downarrow N \\ G(\mathcal{O}_v) & \stackrel{\beta}{\rightarrow} & T(\mathcal{O}_v) \end{array}$$

we have

$$N(\beta'(G(\mathcal{O}_w))) \subset \beta(G(\mathcal{O}_v)).$$

3 Global case

3.1. Some notations. Now let A be the ring of integers of a global field k (see the convention in Introduction) or more generally, a Dedekind ring with quotient field a global field k, and let A' be the integral closure of A in a finite separable extension k' of k. Let V_k be the set of all non-equivalent rank 1 valuations of k, S a finite non-empty subset of V_k containing all archimedean valuations. For $v \in V_k$, denote by k_v (resp. $\mathcal{O}_v$) the completion of k at v (resp. the ring of integers of k_v). Denote by A_S the ring of S-integers of k, containing A, i.e.,

$$A_S = \cap_{v \notin S} \mathcal{O}_v.$$

If k'/k is a finite extension of k, S as above, then we denote by S' the extensions to k' of valuations belonging to S, thus $S' \subset V_{k'}$, and by A' the integral closure of A in k' (hence A'_S is also the integral closure of A_S in k') and we assume $A \in \mathcal{C}_A$, thus $A'_S \in \mathcal{C}_{A_S}$.

3.2. We have the following

Proposition 8. a) *Let A be a Dedekind ring with quotient field a global field k, $1 \to G' \to G \to T \to 1$ an exact sequence of reductive A-group schemes, with T an A-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, there is a finite set $S \subset V$ such that the Corestriction principle holds for the image of $\mathrm{H}^i_{et}(A_S, G_{A_S}) \to \mathrm{H}^i_{et}(A_S, T_{A_S})$, with respect to the extension A'_S/A_S where i = 0,1.*
b) *Let A be as above, and let $1 \to G'_k \to G_k \to T_k \to 1$ an exact sequence of connected reductive k-groups, with T a k-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, there is a finite set $S \subset V$ such that above exact sequence is obtained from an exact sequence of reductive A_S-group schemes $1 \to G' \to G \to T \to 1$ by taking the fibers at generic point, and the Corestriction principle holds for the image of $\mathrm{H}^i_{et}(A_S, G_{A_S}) \to \mathrm{H}^i_{et}(A_S, T_{A_S})$ with respect to the extension A'_S/A_S where i = 0,1.*

Proof. It is clear that *b*) $\Leftrightarrow$ *a*) for S sufficiently large, so it suffices to prove *a*). It is clear also, that we may assume G' to be the derived subgroup of G (see [SGA 3], Exp. XXI).

Case i=0. First we assume that G has simply connected semisimple part (i.e., the derived subgroup scheme). We have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^0_{et}(A', G_{A'}) & \xrightarrow{\pi'} & \mathrm{H}^0_{et}(A', T_{A'}) & \xrightarrow{\gamma} & \mathrm{H}^1_{et}(A', G'_{A'}) \\
 & & \downarrow & & \\
\mathrm{H}^0_{et}(A, G) & \xrightarrow{\pi} & \mathrm{H}^0_{et}(A, T) & \xrightarrow{\alpha} & \mathrm{H}^1_{et}(A, G')
\end{array}$$

and also the following

$$\begin{tikzcd}
 & & \mathrm{H}^1_{et}(A, G') \arrow[r, "\psi"] & \mathrm{H}^1(k, G'_k) \\
\mathrm{H}^1_{et}(A', G'_{A'}) \arrow[r, "\phi"] & \mathrm{H}^1(k', G'_{k'}) & & \\
 & & \mathrm{H}^0_{et}(A, T) \arrow[r, "\psi'"] \arrow[uu, "\alpha"] & \mathrm{H}^0(k, T_k) \arrow[uu, "\beta"] \\
\mathrm{H}^0_{et}(A', T_{A'}) \arrow[r, "\phi'"] \arrow[uu, "\gamma"] \arrow[urr] & \mathrm{H}^0(k', T_{k'}) \arrow[uu, "\delta"] \arrow[urr] & &
\end{tikzcd}$$

where the south-east arrows are corestriction homomorphisms. We have also similar diagrams where A is replaced by A_S, the localization of A at a finite set of valuations S

$$
\begin{array}{ccccc}
 & & \mathrm{H}^1_{et}(A_S, G'_{A_S}) \xrightarrow{\psi} \mathrm{H}^1(k, G'_k) & & \\
\mathrm{H}^1_{et}(A'_S, G'_{A'_S}) \xrightarrow{\phi} \mathrm{H}^1(k', G'_{k'}) & & \uparrow \alpha \qquad\qquad \uparrow \beta & & \\
\uparrow \gamma \qquad\qquad \uparrow \delta & & \mathrm{H}^0_{et}(A_S, T_{A_S}) \xrightarrow{\psi'} \mathrm{H}^0(k, T_k) & & \\
\nearrow \qquad\qquad \nearrow & & & & \\
\mathrm{H}^0_{et}(A'_S, T_{A'_S}) \xrightarrow{\phi'} \mathrm{H}^0(k', T_{k'}) & & & &
\end{array}
$$

We may also assume that, by passing from A to A_S for suitable S, G' has strong approximation theorem with respect to S, i.e., $G'(A_S)$ is dense in the product $G'_S := \prod_{v \in S} G'(k_v)$ with respect to the diagonal embedding, or equivalently, $G'_S G'(k)$ is dense in $G'(\mathbf{A})$. This is possible due to fundamental results of Kneser, Platonov, Margulis and Prasad (see [PlR], [Ma], [Pr] and related references there). In fact, G' has strong approximation with respect to S if and only if the topological group $\prod_{v \in S} G'(k_v)$ is non-compact. Also, it is well-known that the set S_0 of all valuations v of k, where $G'(k_v)$ is compact, is finite, so we just take S such that $S \setminus S_0 \neq \emptyset$. Then the same proof of [Ha1], Korollar 2.3.2, shows that $\mathrm{H}^1_{Zar}(A'_S, G'_{A'_S}) = 0$. Therefore by Nisnevich results (Theorem 3), the maps ϕ, ψ have trivial kernels. Let $x' \in \mathrm{Im}\ (\pi')$ Then $x' \in \mathrm{Ker}\ (\gamma) = \mathrm{Ker}\ (\phi \circ \gamma) = \mathrm{Ker}\ (\delta \circ \phi')$. By [T1], [T2], the Corestriction principle holds for Ker (δ), therefore for $x = Cores(x')$ we have $\psi'(x) \in \mathrm{Ker}\ (\beta)$. Hence

$$
\begin{aligned}
\psi(\alpha((Cores(x')))) &= \beta(\psi'(Cores(x'))) \\
&= \beta(Cores(\phi'(x'))) \\
&= 0,
\end{aligned}
$$

i.e., $x \in \mathrm{Ker}\ (\alpha)$, since ψ has trivial kernel.

In the general case, by Lemma 1, we may take any z-extension (H, S) of the pair (G, T), i.e., we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \to & Z & \to & S & \to & T & \to & 1 \\
 & & \downarrow= & & \downarrow & & \downarrow & & \\
1 & \to & Z & \to & H & \to & G & \to & 1
\end{array}$$

and further the proof is similar to that of Proposition 4, Case a).

Case $i = 1$. The proof in this case is similar to the local case, by using Theorem 2 and Propositions 6, 7. ∎

3.3. Next we consider the case of exact sequence $1 \to T_1 \to G_1 \to G \to 1$.

Proposition 9. a) *Let A a Dedekind ring with quotient field a global field k, $1 \to T \to G_1 \to G \to 1$ an exact sequence of reductive A-group schemes, with T a central A-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, there is a finite set $S \subset V$ such that the Corestriction principle holds for the image of $\mathrm{H}^i_{et}(A_S, G_{A_S}) \to \mathrm{H}^{i+1}_{et}(A_S, T_{A_S})$, with respect to the extension A'_S/A_S, where i = 0,1.*
b) *Let A be the ring of integers of a global field k, $1 \to T_k \to G_{1,k} \to G_k \to 1$ an exact sequence of connected reductive k-groups, with T_k a k-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, there is a finite set $S \subset V$ such that above exact sequence is obtained from an exact sequence of reductive A_S-group schemes $1 \to T \to G_1 \to G \to 1$ by taking the fibers at generic point, and the Corestriction principle holds for the image of $\mathrm{H}^i_{et}(A_S, G_{A_S}) \to \mathrm{H}^{i+1}_{et}(A_S, T_{A_S})$, with respect to the extension A'_S/A_S where i = 0,1.*

Proof. It is clear that $b) \Leftrightarrow a)$, and to prove $b)$ it is clear that we may and shall assume that conditions of $a)$ hold. This time we need to take S sufficiently large such that $\tilde{G}$ has strong approximation over A_S and such that A'_S is a factorial ring, thus $\mathrm{H}^1_{et}(A'_S, Z_{A'_S}) = 0$ for any induced A'_S-torus Z (cf. [Be]). Then the proposition follows by using Theorem 2 and Proposition 8. ∎

3.4. Finally, we treat the case of kernel of a connecting map induced from a morphism $T \to G$, where T is an A-torus and G a reductive A-group scheme.

Proposition 10. a) *Let A a Dedekind ring with quotient field a global field k, $\alpha : T \to G$ a morphism of reductive A-group schemes, with T an A-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, there is a finite set $S \subset V$ such that the Corestriction principle holds for the kernel of α_{A_S} : $\mathrm{H}^1_{et}(A_S, T_{A_S}) \to \mathrm{H}^1_{et}(A_S, G_{A_S})$ with respect to the extension A'_S/A_S.*
b) *Let A a Dedekind ring with quotient field a global field k, $\alpha : T_k \to G_k$ a morphism of connected reductive k-groups, with T_k a k-torus. Then for any finite separable extension k'/k with the ring of integers A' belong to $\mathcal{C}_A$, there is a finite set $S \subset V$ such that α is induced from a morphism (denoted also by the same symbol) of reductive A_S-group schemes $\alpha : T \to G$, and the Corestriction principle holds for the kernel of α_{A_S} : $\mathrm{H}^1_{et}(A_S, T_{A_S}) \to \mathrm{H}^1_{et}(A_S, G_{A_S})$, with respect to the extension A'_S/A_S, where $A'_S := A_S \otimes_{A_S} A', A' \in \mathcal{C}_A$.*

Proof. By enlarging A it is clear that *b*) $\Rightarrow$ *a*), thus it suffices to prove *b*). We apply the method of proof of Proposition 7. First we may take S such that A'_S is a principal ideal domain, thus $\mathrm{H}^1_{et}(A'_S, Z_{A'_S}) = 0$ for any induced A'_S-torus Z (cf. [Be]). Next we may assume that T is a maximal A-torus of G.

Step 1. First we reduce to the case of semisimple groups. Denote by P the radical of G, i.e., the maximal central A-torus of G, and by G' the derived subgroup of G. We have $P \subset T$ and also finite surjective A-morphism $t : G' \times P \to G$, with kernel F a finite A-group scheme of multiplicative type (see [SGA 3], Exp. XXII, Sec. 6.2.3). Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & F & \to & T' \times P & \to & T & \to & 1 \\ & & \downarrow= & & \downarrow s & & \downarrow t & & \\ 1 & \to & F & \to & G' \times P & \to & G & \to & 1 \end{array}$$

Here T' is a maximal A-torus of G', embedded into T. We take the cohomology and consider the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^1_{flat}(A'_S, F_{A'_S}) & \xrightarrow{p'} & \mathrm{H}^1_{flat}(A'_S, T'_{A'_S} \times P_{A'_S}) & \xrightarrow{q'} & \mathrm{H}^1_{flat}(A'_S, T_{A'_S}) & \xrightarrow{\Delta'} & \mathrm{H}^2_{flat}(A'_S, F_{A'_S}) \\
\downarrow = & & \downarrow s & & \downarrow t & & \downarrow = \\
\mathrm{H}^1_{et}(A'_S, F_{A'_S}) & \xrightarrow{p} & \mathrm{H}^1_{flat}(A'_S, G'_{A'_S} \times P_{A'_S}) & \xrightarrow{q} & \mathrm{H}^1_{flat}(A'_S, G_{A'_S}) & \xrightarrow{\Delta} & \mathrm{H}^2_{flat}(A'_S, F_{A'_S})
\end{array}$$

Let $x \in \mathrm{Ker}\,(t : \mathrm{H}^1_{flat}(A'_S, T_{A'_S}) \to \mathrm{H}^1_{flat}(A'_S, G_{A'_S}))$. Then we have $0 = \Delta(t(x)) = \Delta'(x)$, therefore there exists $y \in \mathrm{H}^1_{flat}(A'_S, T'_{A'_S} \times P_{A'_S})$ such that $x = q'(y)$, thus $q(s(y)) = 0$, i.e., $s(y) = p(f) = s(p'(f))$, $f \in \mathrm{H}^1_{flat}(A'_S, F_{A'_S})$. In order to prove the Corestriction principle in this case, by twisting with the cocycle representing f (see [Gir], Chap. IV, Proposition 4.3.4), we may assume that $f = 0$. Therefore we are reduced to proving the assertion for the map

$$\mathrm{H}^1_{flat}(A'_S, T'_{A'_S} \times P_{A'_S}) \to \mathrm{H}^1_{flat}(A'_S, G'_{A'_S} \times P_{A'_S}).$$

It is now clear that we may reduce further to consider the case of the map $\mathrm{H}^1_{flat}(A'_S, T'_{A'_S}) \to \mathrm{H}^1_{flat}(A'_S, G'_{A'_S})$, i.e., to the semisimple case.

Step 2. Next we reduce to the case of simply connected semisimple groups. Namely we want to lift the problem to the simply connected covering $\tilde{G}$ of G'. We fix an element $x' \in \mathrm{Ker}\ \ (\mathrm{H}^1_{et}(A'_S, T'_{A'_S}) \to \mathrm{H}^1_{et}(A'_S, G'_{A'_S}))$. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & F_0 & \to & \tilde{T} & \to & T & \to & 1 \\
 & & \downarrow = & & \downarrow u & & \downarrow v & & \\
1 & \to & F_0 & \to & \tilde{G} & \to & G & \to & 1
\end{array}$$

Here $\tilde{T}$ is a maximal A-torus of $\tilde{G}$, covering T, and $F_0 = \ker(\tilde{G} \to G')$. By taking the cohomology and consider the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^1_{flat}(A'_S, F_{0,A'_S}) & \xrightarrow{p'} & \mathrm{H}^1_{flat}(A'_S, \tilde{T}_{A'_S}) & \xrightarrow{q'} & \mathrm{H}^1_{flat}(A'_S, T'_{A'_S}) & \xrightarrow{\Delta'} & \mathrm{H}^2_{flat}(A'_S, F_{0,A'_S}) \\
\downarrow = & & \downarrow u & & \downarrow v & & \downarrow = \\
\mathrm{H}^1_{flat}(A'_S, F_{0,A'_S}) & \xrightarrow{p} & \mathrm{H}^1_{flat}(A'_S, \tilde{G}_{A'_S}) & \xrightarrow{q} & \mathrm{H}^1_{flat}(A'_S, G'_{A'_S}) & \xrightarrow{\Delta} & \mathrm{H}^2_{flat}(A'_S, F_{0,A'_S})
\end{array}$$

The same arguments as above show that we are reduced to the pair $(\tilde{T}, \tilde{G})$. To finish the proof, we consider the following commutative diagram

$$\begin{array}{ccccccc}
1 & \to & \mathrm{H}^1_{Zar}(A'_S, \tilde{T}_{A'_S}) & \xrightarrow{p'} & \mathrm{H}^1_{et}(A'_S, \tilde{T}_{A'_S}) & \xrightarrow{q'} & \mathrm{H}^1(k', \tilde{T}_{A'_S}) \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
1 & \to & \mathrm{H}^1_{Zar}(A'_S, \tilde{G}_{A'_S}) & \xrightarrow{p} & \mathrm{H}^1_{et}(A'_S, \tilde{G}_{A'_S}) & \xrightarrow{q} & \mathrm{H}^1(k', \tilde{G}_{k'})
\end{array}$$

where all its rows are exact according to Nisnevich Theorem (see Theorem 3). As in the proof of Proposition 8, we may also take S sufficiently large, so that $\tilde{G}_k$ has strong approximation theorem with respect to S, i.e., $\tilde{G}(A_S)$ is dense in the product $\prod_{v \in S} \tilde{G}(k_v)$ with respect to the diagonal embedding. Then, as above we have $\mathrm{H}^1_{Zar}(A'_S, \tilde{G}_{A'_S}) = 0$. Therefore from above commutative diagram it follows that

$$\mathrm{Ker}\ (\beta) = \mathrm{Ker}\ (\beta \circ q) = \mathrm{Ker}\ (\gamma \circ q').$$

By [T1], the Corestriction principle holds for Ker (γ), it follows that the same holds true for Ker $(\gamma \circ q')$, and we are done. ∎

Theorem II now follows from above propositions.

4 Applications

We consider in this section some applications of results and methods described in previous sections.

4.1. We consider the class set of a given flat affine group scheme G of finite type over Dedekind ring A with smooth generic fiber G_k over the quotient field k of A. Let $X = Spec(A)$, $\eta \in X$ the generic point of X, S a finite subset of $X_0 := X \setminus \{\eta\}$. The ring $\mathbf{A}(S)$ of S-adèles is defined as

$$\mathbf{A}(S) := \prod_{v \in X_0 \setminus S} A_v \times \prod_{v \in S} k_v,$$

where k_v (resp. A_v) is the completion of k (resp. A) in the v-adic topology. We denote by $\mathbf{A} = ind.lim_S\, \mathbf{A}(S)$ the adèle ring of k (with respect to A !). We recall (see [Ha1], [Ni1], [Ni3], [Ni4]) that the local class set for a prime $v \in X_0$ (denoted by $Cl_v(G)$), the S-class set, of G with respect to a finite set S of primes of A (denoted by $Cl(S, G)$), and the class set of G (denoted by $Cl_A(G)$), is the set of double classes

$$Cl_v(G) := G(A_v) \setminus G(k_v)/G(k),$$

$$Cl_A(S, G) = G(A(S)) \setminus G(\mathbf{A})/G(k),$$

and

$$Cl_A(G) = G(\mathbf{A}(\emptyset)) \setminus G(\mathbf{A})/G(k),$$

respectively. Here $G(k)$ is embedded diagonally into $G(\mathbf{A})$. The double class $G(\mathbf{A}(\emptyset)).1.G(k)$ is called the principal class. In the classical case (and notation) of the algebraic groups G defined over a Dedekind ring A with quotient field a global field k, the class set is nothing else than the usual class set of the group G, i.e., if ∞ is the set of all infinite primes of A, $\mathbf{A}(\infty)$ the set of integral adèles of $\mathbf{A}$:

$$\mathbf{A}(\infty) := \prod_{v \notin \infty} A_v \times \prod_{v \in \infty} k_v,$$

then

$$Cl_A(G) = G(\mathbf{A}(\infty)) \setminus G(\mathbf{A})/G(k),$$

(cf. [Bo], [PlR], [Ro]).

Especially in the case $G = \mathbf{G}_m$, the class set is exactly the ideal class group of the global field k. Many other information related with the class number can be found in [PlR] and reference therein. In general, class sets contain lot of arithmetic information of the groups under consideration, and it is an important arithmetic invariant for group schemes over A. This was one of the main motivations for Nisnevich to introduce a new Grothendieck topology, which was originally called completely decomposed topology and now is called Nisnevich topology. A site with Nisnevich topology is called a Nisnevich site and the corresponding cohomology is called Nisnevich cohomology, denoted by $\mathrm{H}^i_{Nis}(X, G)$, where G is a sheaf of groups over a scheme X (see [Ni1] - [Ni4]). The following theorem records most basic properties of Nisnevich cohomology that we need in this paper.

Theorem 11. *Let X be a noetherian scheme of finite Krull dimension d.*
1) (Kato - Saito, [KS]) *For any sheaf F of abelian groups over X_{Nis}, we have $\mathrm{H}^n_{Nis}(X, F) = 0$, for all $n > d$.*
2) [Ni3], [Ni4] *We have the following exact sequence of cohomology sets for any sheaf of groups G over X*

$$1 \to \mathrm{H}^1_{Nis}(X, G) \to \mathrm{H}^1_{et}(X, G) \to \mathrm{H}^0_{Nis}(X, R^1 f_* G)) \to 1,$$

where $f : X_{et} \to X_{Nis}$ is canonical projection.
3) ([Ni1], [Ni3], [Ni4]) *Let G be a flat affine group scheme over X with smooth generic fiber. If X is, moreover, a Dedekind scheme $Spec(A)$ in above notation, then we have the following bijections*

$$Cl_v(G) \simeq \mathrm{H}^1_{Nis}(A_v, G),$$

$$\mathrm{H}^1_{Zar}(A, G) \simeq Cl_A(G) \simeq \mathrm{H}^1_{Nis}(A, G),$$

$$Cl(S, G) \simeq \mathrm{H}^1_{Nis}(A_S, G),$$

for all v and finite set of primes S. ∎

4.1.1. Remark. Regarding Theorem 11, 3), it was shown in [Ha1], prior to [Ni1], [Ni3], [Ni4], that there always exists an injection $\mathrm{H}^1_{Zar}(A, G) \hookrightarrow$

$Cl_A(G)$.

4.2. Formally, all results above and their methods of proof for (Weak) Corestriction principle for the case of Nisnevich cohomology also hold true. The main points to check are the following. First notice that Proposition 8.4 of [SGA 3], Exp. XXIV still holds if we replace the étale topology by the Nisnevich one. In fact, the covering in the Nisnevich topology is also one in the étale topology, and the Nisnevich cohomology can be computed by using Čech cocycles (see [Ni3], of [MV]). Since the argument is short, so we repeat it here.

Proposition $11'$**.** *Let $S' \to S$ be finite étale morphism of Noetherian schemes. Let G' be a sheaf of groups over S', $G = \prod_{S'/S} G'$, the restriction of scalars of G' from S' to S. Then in the Nisnevich topology, the functors*

$$P \mapsto P \times_S S',\ P' \mapsto \prod_{S'/S} P'$$

induces a bijection $\mathrm{H}^1_{Nis}(S, G) \to \mathrm{H}^1_{Nis}(S', G')$.

We need the following

Lemma 1. ([SGA3, Exp. XXIV, Prop. 8.2.]) *Let $\mathcal{C}$ be a category with fiber products, equipped with a topology, which is weaker than the canonical one, $S' \to S$ a morphism in $\mathcal{C}$, G' a sheaf of groups over S', $G = \prod_{S'/S} G'$ the sheaf of groups over S. Let $\mathrm{H}^1_S(S', G') \subset \mathrm{H}^1(S', G')$ be the set of classes of principal homogeneous G'-spaces trivialized by some sieve of S', which is obtained by base change from a suitable covering sieve of S. The canonical map $\mathrm{H}^1(S, G) \to \mathrm{H}^1(S', G'), P \mapsto P \times_S S'$ defines a bijection $\mathrm{H}^1(S, G) \simeq \mathrm{H}^1_S(S', G')$.*

Lemma 2. ([SGA 3, Exp. XXIV, Lemme 8.3]) *With notation as in Lemma 1, the assertion $\mathrm{H}^1_S(S', G') = \mathrm{H}^1(S', G')$ is local over S, i.e., if there is a covering $\{S_i \to S\}_i$ such that for all i we have $\mathrm{H}^1_{S_i}(S' \times_S S_i, G') = \mathrm{H}^1(S' \times_S S_i, G')$ then we have $\mathrm{H}^1_S(S', G') = \mathrm{H}^1(S', G')$.*

Proof of Proposition $11'$. With above notation, by Lemma 1, we need only show that $\mathrm{H}^1_{Nis,S}(S', G') = \mathrm{H}^1_{Nis}(S', G')$. By Lemma 2, it suffices to verify this for a specific Nisnevich covering. We just take the finite covering $\{S_i \to S\}$, $i = 1, ..., n$, consisting of n copies of S, and let S' be their direct sum. Thus the sheaf G' is given by a collection of n sheaves G_i over S, and $\mathrm{H}^1_{Nis}(S', G') \simeq \prod_{1 \leq i \leq n} \mathrm{H}^1_{Nis}(S, G_i)$. On the other hand, $G = \prod_{S'/S} G' = \prod_i G_i$, so $\mathrm{H}^1_{Nis}(S, G) = \prod_i \mathrm{H}^1_{Nis}(S, G_i)$, hence we have $\mathrm{H}^1_{Nis,S}(S', G') = \mathrm{H}^1_{Nis}(S', G')$ as required. ∎

Hence Theorem 2 still holds in the new setting. Second, we have a natural map with trivial kernel $\mathrm{H}^1_{Nis}(X, G) \to \mathrm{H}^1_{et}(X, G)$. Using these facts, one can show that all main results in previous sections remain valid in the setting of Nisnevich topology.

4.3. Now let notation be as in Proposition 8, and let $\pi : G \to T$ be a morphism of A-group schemes, where G is A-reductive group scheme and T is an A-torus. It is obvious that π induces a map of class sets $\pi_v : Cl_v(G) \to Cl_v(T)$, $\pi_A : Cl_A(G) \to Cl_A(T)$, where $Cl_v(T), Cl_A(T)$ have natural group structure. If $A' \in \mathcal{C}_A$ then one has a norm maps $N_{A'/A} : T(A') \to T(A)$, $N_{A'_v/A_v} : T(A'_v) \to T(A_v)$, $N_{k'/k} : T(k') \to T(k)$, $N_{k'_v/k_v} : T(k'_v) \to T(k_v)$. Then for an extension $w|v$ in k', we have

Proposition 12. *In the two diagrams*

$$\begin{array}{ccc} Cl_w(G) & \stackrel{\pi_w}{\to} & Cl_w(T) \\ & & \downarrow N_{k'_v/k_v} \\ Cl_v(G) & \stackrel{\pi_v}{\to} & Cl_v(T) \end{array}$$

and

$$\begin{array}{ccc} Cl_{A'}(G) & \stackrel{\pi_{A'}}{\to} & Cl_{A'}(T) \\ & & \downarrow N_{A'/A} \\ Cl_A(G) & \stackrel{\pi_A}{\to} & Cl_A(T) \end{array}$$

the following inclusions hold

$$N_{k'_v/k_v}(\pi'_v(Cl_w(G))) \subset \pi_v(Cl_v(G)),$$

$$N_{A'/A}(\pi'_A(Cl'_A(G))) \subset \pi_A(Cl_A(G)).$$

Proof. The proof follows from Proposition 8. Another proof follows from Remarks 4.2 combined with Theorem 11. ∎

4.4. Let k be a global field, A a Dedekind ring with quotient field k, ∞ the set of infinite primes of A, $\mathbf{A}(\infty)$ the set of integral adèles of $\mathbf{A}$. The problem of computing class sets for a given linear algebraic group G defined over k is a non-trivial one, and depends on the matrix realization (i.e., embedding) of $G \hookrightarrow \mathbf{GL}_n$. In this case, we write $G(B) = G(\bar{k}) \cap \mathbf{GL}_n(B)$ for any overring B/A. One of the most interesting cases is when the class set has a natural group structure, which then becomes the class group of G (denoted by $\mathcal{G}Cl(G)$ as in [PlR]). Recall that for a finite set S of primes of A, G has weak approximation with respect to S if $G(k)$ is dense in the product of v-adic topologies on $\prod_{v \in S} G(k_v)$, and G has strong approximation with respect to S with $S \supset \infty$, if, for a given matrix realization $G \hookrightarrow \mathbf{GL}_n$, $G(A_S)$ is dense in $\prod_{v \in S} G(k_v)$. Equivalently, the subset $G(k)$ is dense in $G(\mathbf{A}_S)$, or the same, $G(k)G_S$ is dense in $G(\mathbf{A})$, where $G_S := \prod_{v \in S} G(k_v)$. In the case $S = \infty$, G is said to have absolute strong approximation over k. Then it follows that $Cl_A(G) = 1$. Also, it has been shown ([PlR], Prop. 8.8. p. 451) that if G is a semisimple algebraic group defined over a number field k, such that the simply connected covering $\tilde{G}$ of G has the absolute strong approximation, then $Cl_A(G)$ has a natural structure of finite abelian group, and its order is the class number of G. In the case of connected reductive k-groups G, we have the following similar property characterizing $Cl_A(G)$ as a finite abelian group, slightly extending Proposition 8.8 of [PlR]. The method of proof is a slight modification of (loc. cit.) and [Kn3]. The following statements (Proposition 13, Theorem 14) will finish the proof of Theorem III mentioned in Introduction.

Proposition 13. *Let k be a global field, A a Dedekind ring with quotient*

field k. Let G be a connected reductive k-group such that the simply connected covering $\tilde{G}$ of the derived subgroup $[G,G]$ of G has absolute strong approximation. Then in any matrix realization
a) the principal double class $G(\mathbf{A}(\infty))G(k)$ contains the derived subgroup $[G(\mathbf{A}),G(\mathbf{A})]$;
b) the principal double class $G(\mathbf{A}(\infty))G(k)$ is a normal subgroup of $G(\mathbf{A})$;
c) the class set $Cl_A(G)$ has a natural structure of a finite abelian group, and we have

$$Cl_A(G) = \mathcal{G}Cl(G) \simeq G(\mathbf{A})/G(\mathbf{A}(\infty))G(k).$$

Proof. 1) Let $G = G'T$, where the product is almost direct, G' is semisimple, T is a central k-subtorus of G and there is a k-isogeny

$$1 \to F \to \tilde{G} \times S \xrightarrow{\pi} G = G'T \to 1,$$

where $\tilde{G}$ is the simply connected covering of G'.

2) It is an important observation by Deligne [De], Sec. 2.0.2, that in the above exact sequence, $\pi(\tilde{G})$ is a normal subgroup of $G(k)$ with abelian quotient group. In particular,

$$[G(k),G(k)] \subset \pi(\tilde{G}(k)).$$

Moreover, this is true for G considered as a sheaf of groups over some site. Since $\mathbf{A}$ is a k-algebra, the above exact sequence can be considered as an exact sequence of $\mathbf{A}$-group schemes, therefore, by considering the flat cohomology we have an exact sequence

$$1 \to F(\mathbf{A}) \to \tilde{G}(\mathbf{A}) \times S(\mathbf{A}) \xrightarrow{\pi_A} G(\mathbf{A}) \xrightarrow{\delta_A} \mathrm{H}^1_{flat}(\mathbf{A},F).$$

Since the above sequence is exact, and the cohomology group $\mathrm{H}^1_{flat}(\mathbf{A},F)$ is commutative, it follows that $\mathrm{Im}\,(\pi_A)$ is a normal subgroup of $G(\mathbf{A})$, containing $[G(\mathbf{A}),G(\mathbf{A})]$. Also, from what has been said, we have

$$[G(\mathbf{A}),G(\mathbf{A})] \subset \pi_A(\tilde{G}(\mathbf{A})) \subset \mathrm{Im}\,(\pi_A).$$

(This has been proved by Kneser in the case of number fields. One may also use the arguments given in [Oe], Chap. II, related with the cohomology of adelic groups.)

3) By assumption, $\tilde{G}$ has absolute strong approximation, hence we have $\tilde{G}(\mathbf{A}(\infty))\tilde{G}(k) = \tilde{G}(\mathbf{A})$. We show that

$$\pi_A(\tilde{G}(\mathbf{A})) \subset G(\mathbf{A}(\infty))G(k)$$

by showing that

$$\pi_A(\tilde{G}(\mathbf{A})) \subset G'(\mathbf{A}(\infty))G'(k).$$

Indeed, let W be the finite set of all finite primes v of k such that π is not defined over A_v. It is well-known that W is finite. If W is empty, we are done. Assume W is non-empty. It is clear that we have

$$\pi_A(\prod_{v \notin W} \tilde{G}(A_v) \times \prod_{v \in W \cup \infty} \{1\}) \subset G'(\mathbf{A}(\infty)),$$

$$\pi_A(\prod_{v \in \infty} \tilde{G}(A_v) \times \prod_{v \notin \infty} \{1\}) \subset G'(\mathbf{A}(\infty)).$$

Therefore it remains to show that

$$\pi_A(\prod_{v \in W} \tilde{G}(A_v) \times \prod_{v \notin W} \{1\}) \subset G'(\mathbf{A}(\infty)).$$

Denote by $Cl(.)$ the operation of taking closure. Since $\tilde{G}$ has absolute strong approximation over k, $\tilde{G}(k)$ is dense in the adèle topology in the restricted product $\prod'_{v \notin \infty}(\tilde{G}(k_v), \tilde{G}(A_v))$, hence

$$\prod_{v \in W} \tilde{G}(A_v) \times \prod_{v \notin W} \{1\} \subset Cl(\tilde{G}(k)),$$

where the closure is taken in $\tilde{G}(\mathbf{A})$. Therefore

$$\pi_A(\prod_{v \in W} \tilde{G}(A_v) \times \prod_{v \notin W} \{1\}) \subset \pi_A(Cl(\tilde{G}(k))).$$

Since π_A is continuous in the adèle topology, which has a countable basis, it follows easily that

$$\pi_A(Cl(\tilde{G}(k))) \subset Cl(\pi_A(\tilde{G}(k)))$$

$$\subset Cl(G'(k))$$

$$
\begin{aligned}
&\subset Cl(G(k))\\
&\subset G(\mathbf{A}(\infty))G(k),
\end{aligned}
$$

since the latter is an open subset of $G(\mathbf{A})$ containing $G(k)$. Therefore we have

$$\pi_A(\tilde{G}(\mathbf{A})) \subset G'(\mathbf{A}(\infty))G'(k)$$

as required. It follows from above that

$$
\begin{aligned}
[G(\mathbf{A}), G(\mathbf{A})] \subset \pi_A(\tilde{G}(\mathbf{A})) &= \pi_A(\tilde{G}(\mathbf{A}(\infty))\tilde{G}(k))\\
&\subset G(\mathbf{A}(\infty)))G(k).
\end{aligned}
$$

4) We show that $G(\mathbf{A}(\infty))G(k)$ is a normal subgroup of $G(\mathbf{A})$. Let $g, g_1 \in G(\mathbf{A}(\infty)), h, h_1 \in G(k)$. Then

$$
\begin{aligned}
(gh)(g_1h_1) &= g.g_1(g_1^{-1}.h.g_1.h^{-1})h.h_1\\
&= (g.g_1)[g_1^{-1}, h]h.h_1\\
&\in G(\mathbf{A}(\infty))(G(\mathbf{A}(\infty))G(k))G(k) \qquad \text{(by 3))}\\
&= G(\mathbf{A}(\infty))G(k); \qquad (*)\\
(g.h)^{-1} &= g^{-1}.h^{-1}(h.g.h^{-1}.g^{-1})\\
&= (g^{-1}.h^{-1})(g_2.h_2) \qquad \text{(by 3))}\\
&\in G(\mathbf{A}(\infty))G(k) \qquad \text{(by (*))}.
\end{aligned}
$$

Hence $G(\mathbf{A}(\infty))G(k)$ is a subgroup of $G(\mathbf{A})$, and since it contains $[G(\mathbf{A}), G(\mathbf{A})]$, it is a normal subgroup of $G(\mathbf{A})$.
5) In [Kn3], it has been proved that over a number field k, for any $g \in G(\mathbf{A})$, we have

$$G(\mathbf{A}(\infty)).g.G(k) = g.G(\mathbf{A}(\infty))G(k).$$

One checks without difficulty that the same argument works in the case $char.k > 0$ (by using 2)). From above we see that $G(\mathbf{A}(\infty))G(k)$ is a normal subgroup of $G(\mathbf{A})$, and the double class set

$$Cl_A(G) = G(\mathbf{A}(\infty)) \setminus G(\mathbf{A})/G(k) = G(\mathbf{A})/G(\mathbf{A}(\infty))G(k) = \mathcal{G}Cl_A(G)$$

is naturally the class group of G. ∎

4.4.1. Remarks. If we replace the condition that $\tilde{G}$ has absolute strong approximation over k by the (obviously weaker) condition

$$[G(\mathbf{A}), G(\mathbf{A})] \subset G(\mathbf{A}(\infty))G(k),$$

then all the statements of Proposition 13 still holds and the proof remains the same.

4.5. Assume that the natural group structure exists on the class set of a connected reductive group G defined over a global field k, and the same also holds for $G_{k'}$ for all finite extension k'/k. In this case, one may ask if $\mathcal{G}Cl(G)$ possesses certain norm map. More precisely, if k'/k is a finite separable extension of fields, we ask whether there is a norm homomorphism

$$N_{k'/k} : \mathcal{G}Cl(G_{k'}) \to \mathcal{G}Cl(G),$$

which is functorial in k'/k. With notation as above, in [De], Deligne has introduced the group

$$\Pi(G) := G(\mathbf{A})/\pi(\tilde{G}(\mathbf{A}))G(k)$$

for a connected reductive group G defined over a global field k. It is an abelian quotient group of $G(\mathbf{A})$, and it was shown to have a norm homomorphism $N_{k'/k} : \Pi(G_{k'}) \to \Pi(G)$ ([De], Sec. 2.4), which plays a role in the study of reciprocity law for canonical models of Shimura varieties. If $\tilde{G}$ has absolute strong appoximation, then the class group $\mathcal{G}Cl(G)$ is a factor group of $\Pi(G)$ and it is quite possible that in this case, we also have a norm homomorphism $\mathcal{G}Cl(G_{k'}) \to \mathcal{G}Cl(G)$. In the case of reductive A-group schemes we have a property, similar to Proposition 13, for reductive A-group schemes, and, under the same assumption, also a norm homomorphism as follows.

Theorem 14. *Let k be a global field, A a Dedekind ring with quotient field k, G a reductive A-group scheme such that the simply connected covering $\tilde{G}_k$ of the derived subgroup $G' := [G, G]$ of G has absolute strong approximation over k. Then for any $A' \in \mathcal{C}_A$, we have a norm homomorphism*

$$N_{A'/A} : \mathcal{G}Cl_{A'}(G_{A'}) \to \mathcal{G}Cl_A(G),$$

such that if $A'' \in \mathcal{C}_{A'}$, then

$$N_{A''/A} = N_{A'/A} \circ N_{A''/A'}.$$

Before proving Theorem 14, we need the following result. Let k be a global field, G a smooth connected reductive k-group, H a z-extension of $G, T = H/[H,H]$. Denote by

$$\mathrm{A}(P) = \prod_v P(k_v)/Cl(P(k))$$

the obstruction to weak approximation over k for a k-group P, where $P(k)$ is embedded diagonally into the direct product $\prod_v P(k_v)$. Then

Proposition 15. *We have canonical isomorphisms of finite abelian groups*

$$\mathrm{A}(G) \simeq \mathrm{A}(H) \simeq \mathrm{A}(T).$$

Proof. In the case k is a number field, these isomorphisms were established in [T5], [T6] (see also [BKG] for some further extensions). In the case of global function field, we use similar arguments (see the proof of Lemma 3.8, [T6]) in combination with the existence of z-extensions proved in Section 1 (see also [T4]) for the case of characteristic $p > 0$, the triviality of Galois cohomology of simply connected semisimple k-groups (Harder's Theorem, [Ha2], Satz A), and their analog over local function fields (Bruhat - Tits Theorem, [BrT]). ∎

Proof. (of Theorem 14). We present two proofs of this theorem.

First proof. Claim 1. Assume that $[G,G]$ is simply connected. Consider the following exact sequence of reductive A-group schemes

$$1 \to \tilde{G} \to G \xrightarrow{\pi} T \to 1,$$

where $T = G/\tilde{G}$ is a A-torus. Then we have canonical isomorphism of finite abelian groups

$$\mathcal{G}Cl_A(G) \simeq \mathcal{G}Cl_A(T).$$

We know that π induces a continuous homomorphism $\pi_A : G(\mathbf{A}) \to T(\mathbf{A})$. We notice that since π is defined over A, and the class set of G is a class group $\mathcal{G}Cl_A(G)$, π induces a homomorphism between class groups

$$\pi' : \mathcal{G}Cl_A(G) \to \mathcal{G}Cl_A(T).$$

Let $t = (t_v) \in T(\mathbf{A})$. Let S be a finite set of finite primes of A, such that for $v \notin S$ we have $t_v \in T(\mathcal{O}_v)$. We may take S sufficiently large such that for $S' := \infty \cup S$, we have

$$\mathrm{A}(G) \simeq \mathrm{A}(S', G) := \prod_{v \in S'} G(k_v)/Cl(G(k)) \simeq$$

$$\simeq \mathrm{A}(T) \simeq \mathrm{A}(S', T) := \prod_{v \in S'} T(k_v)/Cl(T(k))$$

(see the proof of [T6], Lemma 3.8). Then π induces an isomorphism

$$\pi_{S'} : \mathrm{A}(S', G) \simeq \mathrm{A}(S', T),$$

such that $\pi_{S'}^{-1}(Cl(T(k))) = Cl(G(k)))$. We can write

$$t = t_\infty . t_S . t'_S,$$

where

$$t_\infty \in T(k_\infty) \times \prod_{v \notin \infty} \{1\}, t_S \in \prod_{v \in S} T(k_v) \times \prod_{v \notin S} \{1\}, t'_S \in \prod_{v \notin S} T(\mathcal{O}_v) \times \prod_{v \in S} \{1\}.$$

By Tits result (Theorem 3 a)), and Kneser - Bruhat - Tits about the triviality of the H^1 of simply connected groups above, it is clear that $t'_S \in \mathrm{Im}\ (\pi_A)$. From the isomorphism above, we can choose $g_S \in \prod_{v \in S} G(k_v)$ such that $\pi_S(g_S) = t_S$ ($mod.\ Cl(T(k))$). All these facts show that π induces a surjective homomorphism

$$\pi' : \mathcal{G}Cl_A(G) \to \mathcal{G}Cl_A(T).$$

Next we show that π' is a monomorphism. Let $g = (g_v) \in G(\mathbf{A})$ such that $\pi_A(g) = t_\infty t_f t_k \in T(\mathbf{A}(\infty))T(k)$, the principal double class of $T(\mathbf{A})$, where

$$t_k \in T(k), t_\infty \in T(k_\infty) \times \prod_{v \notin \infty} \{1\}, t_f \in \prod_{v \notin \infty} T(\mathcal{O}_v) \times \prod_{v \in \infty} \{1\}.$$

As we notice above, $t_f \in \mathrm{Im}\ \pi_A$, say $t_f = \pi_A(h_f)$, where

$$h_f \in \prod_{v \notin \infty} G(\mathcal{O}_v) \times \prod_{v \in \infty} \{1\}.$$

By replacing $g = (g_v)$ by $h_f^{-1} g$, we may assume that $t_f = 1$. Thus we have

$$\pi_A(g) = t_\infty t_k.$$

Let W be a finite set of finite primes of A such that for $v \notin W$ then $g_v \in G(\mathcal{O}_v)$. We may enlarge W so that $W = S$ satisfies the same condition regarding weak approximation presented above. Since, as it is well-known, the weak approximation holds with respect to archimedean primes, it follows that $t_\infty t_k \in Cl(T(k))$, where the closure is being taken in $\prod_{v \in S \cup \infty} T(k_v)$. We can write $g = g_\infty g_S g'_S$, where

$$g_\infty \in G(k_\infty) \times \prod_{v \notin \infty} \{1\}, g_S \in \prod_{v \in S} G(k_v) \times \prod_{v \notin S} \{1\}, g'_S \in \prod_{v \notin S \cup \infty} G(\mathcal{O}_v) \times \prod_{v \in S \cup \infty} \{1\}.$$

On one hand, the image of g in the class group is the same as that of $g_\infty g_S$. On the other hand, the image of $g_\infty g_S$ in $\prod_{v \in S \cup \infty} T(k_v)$ is equal to $t_\infty t_S \in Cl(T(k))$. It follows from above that $g_\infty g_S \in Cl(G(k))$, the closure being taken in $\prod_{v \in S \cup \infty} G(k_v)$. Hence $g_\infty g_S \in Cl(G(k))$, where the closure is taken in $G(\mathbf{A})$. Since $G(\mathbf{A}(\infty))$ is an open subgroup of $G(\mathbf{A})$, it follows that

$$Cl(G(k)) \subset G(\mathbf{A}(\infty)) G(k),$$

hence

$$\begin{aligned} g &= g_\infty g_S g'_S \\ &\in Cl(G(k)) \textstyle\prod_{v \notin S \cup \infty} G(\mathcal{O}_v) \times \prod_{v \in S \cup \infty} \{1\} \\ &\subset G(\mathbf{A}(\infty)) G(k) \textstyle\prod_{v \notin S \cup \infty} G(\mathcal{O}_v) \times \prod_{v \in S \cup \infty} \{1\} \\ &\subset G(\mathbf{A}(\infty)) G(k), \end{aligned}$$

where the last inclusion follows from the proof of Proposition 13. Thus g has trivial image in the class group as required. (To prove the last inclusion, one may also use the strong approximation assumption and also a result due to Deligne [De], Corollary 2.0.9.)

Claim 2. With above notation and assumptions, we have the following exact sequence of finite abelian groups

$$1 \to \mathcal{G}Cl_A(Z) \to \mathcal{G}Cl_A(H) \to \mathcal{G}Cl_A(G) \to 1.$$

Indeed, from the exact sequence

$$1 \to Z \to H \to G \to 1$$

we derive without difficulty the exact sequence on adelic points

$$1 \to Z(\mathbf{A}) \to H(\mathbf{A}) \to G(\mathbf{A}) \to 1,$$

$$1 \to Z(\mathbf{A}(\infty)) \to H(\mathbf{A}(\infty)) \to G(\mathbf{A}(\infty)) \to 1,$$

$$1 \to Z(k) \to H(k) \to G(k) \to 1.$$

and from this the corresponding class groups. (One may also invoke results on Nisnevich cohomology to deduce this (simple) fact. See [Ni4].)

Due to the functoriality of étale cohomology of tori (or just use the results proved in Sections 2- 3), the corestriction (i.e., the norm) homomorphism exist for the class group $\mathcal{G}Cl_A(Z)$ of Z (denoted by N_1), and for the class group $\mathcal{G}Cl_A(T)$ of T, hence also for $\mathcal{G}Cl_A(H)$ (denoted by N_2). The following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{G}Cl_{A'}(Z_{A'}) & \to & \mathcal{G}Cl_{A'}(H_{A'}) & \to & \mathcal{G}Cl_{A'}(G_{A'}) & \to & 1 \\ & & \downarrow N_1 & & \downarrow N_2 & & \downarrow N_3 & & \\ 1 & \to & \mathcal{G}Cl_A(Z) & \to & \mathcal{G}Cl_A(H) & \to & \mathcal{G}Cl_A(G) & \to & 1 \end{array}$$

resulting from this functoriality, with exact rows, shows the existence of the corestriction (norm) map N_3 for $\mathcal{G}Cl_A(G)$ as required. ■

Second proof. For simplicity, we denote

$B = G(\mathbf{A}), C = G(\mathbf{A}(\infty)), D = G(k), E = \tilde{G}(\mathbf{A}), F = \pi(\tilde{G}(\mathbf{A})), J = \tilde{G}(\mathbf{A}(\infty)),$

where $\pi : \tilde{G} \to G' = [G, G]$ denotes the canonical projection from simply connected covering $\tilde{G}$ of the semisimple part G' of G. We prove the following

Claim 3. There exists a norm homomorphism

$$N : \mathcal{G}Cl_{A'}(G_{A'}) \to \mathcal{G}Cl_A(G)$$

which is compatible with the Deligne's norm homomorphism in the sense that the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Ker}\,(q') & \to & B'/F' & \xrightarrow{q'} & \mathcal{G}Cl_{A'}(G_{A'}) & \to & 1 \\ & & \downarrow q_1 & & \downarrow q_2 & & \downarrow N_{A'/A} & & \\ 1 & \to & \mathrm{Ker}\,(f) & \to & B/F & \xrightarrow{q} & \mathcal{G}Cl_A(G) & \to & 1 \end{array}$$

where $(.)'$ *means an object is obtained if we pass from* k *to a finite extension* k'/k*, i.e., considered over a finite separable extension* k'/k*.*

With our assumption on the strong approximation, we know from the proof of Proposition 13, that CD is a normal subgroup of finite index of B, and $\mathcal{G}Cl_A(G) = B/CD$. From [T2], we know that there is a norm homomorphism for the quotient group B/F. (In fact, from results of Sections 2, 3, under our assumption on absolute strong approximation, it follows also that in the case of local or global fields, the Corestriction principle holds for the canonical map $ab^0_G : \mathrm{H}^0_{et}(A, G) \to \mathrm{H}^0_{ab,et}(A, G)$. From this fact, one deduces without difficulty the above mentioned norm homomorphism.) This norm homomorphism is compatible with the Deligne's norm homomorphism for the group $\Pi(G)$, i.e., the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Ker}\,(f') & \to & B'/F' & \xrightarrow{f'} & \Pi(G_{A'}) & \to & 1 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ 1 & \to & \mathrm{Ker}\,(f) & \to & B/F & \xrightarrow{f} & \Pi(G) & \to & 1 \end{array}$$

Indeed, we just need to show that f_1 is induced from corestriction (norm) homomorphisms previously obtained for algebraic groups over local and global

fields as in [T2]. Take a z-extension $1 \to Z \to H \to G \to 1$. By using the surjectivity of the homomorphisms $H(\mathbf{A}) \to G(\mathbf{A})$ and $H(k) \to G(k)$, we are reduced to proving the same assertion for H, i.e., we may assume $H = G$. But one checks that in this case Ker $(f) = G(k) \cap \tilde{G}(\mathbf{A}) = \tilde{G}(k)$, and the norm homomorphism for Ker (f) is nothing else than the Deligne's norm homomorphism constructed in [De], Sec. 2.4.
We have the following exact sequence of groups

$$1 \to \text{Ker } (g) \to B/FD \to \mathcal{G}Cl_A(G) \to 1.$$

Since there exists a norm homomorphism of $\Pi(G) = B/FD$ compatible with Deligne' norm homomorphism, the proof of the existence of a norm homomorphism of $\mathcal{G}Cl_A(G)$ compatible with Deligne' norm homomorphism is reduced to that of Ker (g). Again, as in the previous part, we may assume that $H = G$. In this case one checks that Ker $(g) = CD/ED$. Since $\tilde{G}$ has absolute strong approximation over k, we have

$$\begin{aligned} CD/ED &= CD/JD \\ &= C.JD/JD \\ &= C/C \cap JD \\ &= C/J(C \cap D) \\ &= C/JG(A). \end{aligned}$$

Therefore we are reduced to proving the exsitence of a norm homomorphism for $C/JG(A)$ which is compatible with Deligne' norm homomorphism. We notice that J is a normal subgroup of C, and that there exists a norm homomorphism of C/J compatible with Deligne' norm homomorphism (which, for finite primes, follows from Sections 2 - 3, and for infinite primes follows from [De] and/or [T2]). By considering the exact sequence

$$1 \to \text{Ker } (h) \to C/J \xrightarrow{h} C/JG(A) \to 1$$

we are reduced to proving the same assertion for

$$\text{Ker } (h) = JG(A)/J = G(A)/J \cap G(A) = G(A)/\tilde{G}(A),$$

which has been already proved in Section 3. ∎

The proof of Theorem III in the Introduction now follows from above results.

As a consequence of the proof of Theorem 14, we derive the following result, which can be considered as a complement to a description of the class groups given by Nisnevich in the case of semisimple group schemes, or the case of group schemes with semisimple groups as generic fibers) (see [Ni4], Theorem 4.3).

Corollary 16. *With notation and assumption as in Theorem 14, there exist well-defined A-tori Z, T, where Z is an induced A-torus, satisfying the following exact sequence of finite abelian groups*

$$1 \to \mathcal{G}Cl_A(Z) \to \mathcal{G}Cl_A(T) \to \mathcal{G}Cl_A(G) \to 1.$$

Proof. Take any z-extension

$$1 \to Z \to H \to G \to 1$$

for the reductive A-group G. Denote by $\tilde{G}$ the derived subgroup of H, which is a semisimple simply connected A-group scheme, and let $T = H/\tilde{G}$, the A-torus quotient. Since Z is an induced A-torus, as in Claim 2 of the second proof, we have the corresponding exact sequence for class groups

$$1 \to \mathcal{G}Cl_A(Z) \to \mathcal{G}Cl_A(H) \to \mathcal{G}Cl_A(G) \to 1.$$

Also, by Claim 1, we have canonical isomorphism of finite abelian groups

$$\mathcal{G}Cl_A(H) \simeq \mathcal{G}Cl_A(T).$$

Thus we obtain the exact sequence desired. ∎

4.6. Remarks. 1) It is worth of noticing that the restriction map for the class sets of linear algebraic groups over number fields has been studied before by Rohlfs [Ro], Satz 3.1, in a very general setting. In particular, he studied the map

$$Res : G(\mathbf{A}(\infty)) \setminus G(\mathbf{A})/G(k) \to G(\mathbf{A}_l(\infty)) \setminus G(\mathbf{A}_l)/G(l),$$

where l is a finite Galois extension of k, $\mathbf{A}_l$ denotes the adèle ring of l, and obtained a beautiful expression of the kernel (in the category of pointed sets) of the restriction map Res via Galois cohomology of G. Theorem 14 can be considered as a complement to this result. It would be nice to extend the results obtained above to the case considered by Rohlfs [Ro], Satz 3.1 and Korollar 3.2.

2) In most of results above, which are proved under the assumption of absolute strong approximation, we may relax this condition by assuming only that the class number of $\tilde{G}$ is equal to 1. (It would be nice to verify the "Kottwitz principle" ([Ko]) in this case.) Also, one may also reformulate the results for the case of S-class groups in an appropriate way, for a finite set S of primes containing ∞.

3) It would be nice to have norm homomorphism for the class group (still under the condition on abosolute strong approximation assumption) for any connected reductive k-group (i. e. without assuming that G is a reductive A-group scheme).

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