ON THE MUMFORD - TATE CONJECTURE FOR ABELIAN VARIETIES

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Abstract. In this paper we prove the Mumford - Tate conjecture for absolutely simple abelian variety J of non-exceptional dimension over a number field k under the following assumption: J has many ordinary reductions, $\mathrm{Cent}(\mathrm{End}(J\otimes\overline{k}))=\mathbb{Z}$, ($\dim_k J$=2(mod 4) and $\mathrm{End}(J\otimes\overline{k})=\mathbb{Z}$) or ($\dim_k J$=4(mod 8) and $\mathrm{End}^0(J\otimes\overline{k})$ is a quaternion division algebra over $\mathbb{Q}$).

§0. Introduction

0.1. Let J be an abelian variety over a number field $k\subset\mathbb{C}, [k:\mathbb{Q}]<\infty$. Suppose that l is a prime number,

$$\rho_l:\mathrm{Gal}(\overline{k}/k)\to\mathrm{GL}(H^1_{et}(J\otimes\overline{k},\mathbb{Q}_l))$$

is the natural l-adic representation.

It is well known that ρ_l is unramified outside a finite set T of non-Archimedean places of k. We denote by $F_{\overline{v}}\in\mathrm{Gal}(\overline{k}/k)$ the Frobenius element associated with a place $\overline{v}$ of $\overline{\mathbb{Q}}$ lying over an unramified place v of k. It is well known that the conjugacy class of $\rho_l(F_{\overline{v}}^{-1})$ depends only on v, the characteristic polynomial of $\rho_l(F_{\overline{v}}^{-1})$ lies in $\mathbb{Z}[t]\subset\mathbb{Q}_l[t]$, and all its roots are of absolute value $(\mathrm{Norm}_{k/\mathbb{Q}}(v))^{1/2}$.

Let S be a set of non-Archimedean places of k. We recall that the Dirichlet density of S in the set of all non-Archimedean places of k is defined as

$$\lim_{x\to\infty}\frac{\log x}{x}\mathrm{Card}\{v\in S\mid\mathrm{Norm}_{k/\mathbb{Q}}(v)\le x\}$$

(if such limit exists). It is well known that the density of $\{v\mid\mathrm{Norm}_{k/\mathbb{Q}}(v)=p_v\}$ equals 1 [4, ch.8, sect.2.4].

The following result is well known.

0.2. **J.-P. Serre theorem**[5,sect.6]. *Let J be a simple abelian variety over a number field k . If $\dim_k J$ is an odd integer and $\mathrm{End}(J\otimes\overline{k})=\mathbb{Z}$, then the Hodge* [8],[9], *Tate* [18] *and Mumford - Tate conjectures* [10] *hold for J.*

The survey of Serre's technique is contained in [5].

We want to extend Serre theorem into the area of even dimensions.

Let Δ be the set of all eigenvalues of $\rho_l(F_{\overline{v}}^{-1})$ (without counting multiplicities). The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts in a natural way on Δ and on $\Delta\cdot\Delta$. For each

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element $\eta \in \Delta \cdot \Delta$ we define a map $T_\eta : \Delta \to \overline{\mathbb{Q}}^\times$ by the formula $T_\eta(\delta) = \eta\delta^{-1}$. This map is a modification of the corresponding map $T^0_\gamma : \Delta \to \overline{\mathbb{Q}}^\times$ in [5,sect.5.2], which is defined by the formula $T^0_\gamma(\delta) = \gamma^2\delta^{-1}$ for $\gamma \in \Delta$. It is evident that for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\mathrm{Card}(T_{\sigma(\eta)}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\sigma(\eta)}(\sigma(\Delta)) \cap \sigma(\Delta)) = \mathrm{Card}(T_\eta(\Delta) \cap \Delta),$$

and hence for any constant c the set

$$\{\eta \in \Delta \cdot \Delta | \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = c\} \textit{ is } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) - \textit{invariant}. \tag{0.2.1}$$

So we have a good instrument of computing the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subsets of $\Delta \cdot \Delta$.

0.3. We recall that J has an ordinary reduction at a non-Archimedean place v of k with a residue field $k(v) = \mathbb{F}_{q_v}$ of characteristic p_v $\Leftrightarrow$ the special fibre J_v of the Neron minimal model of J is an abelian variety and the following equivalent conditions hold:

(0.3.1) p_v-rank of J_v equals $\dim_{k(v)} J_v$;

(0.3.2) for any eigenvalue δ of the Frobenius endomorphism of l-adic Tate module $T_l(J_v \otimes_{k(v)} \overline{k(v)})(l \neq p_v)$ and for any place w of $\overline{\mathbb{Q}}$ over p_v the following relation holds:

$$\frac{w(\delta)}{w(q_v)} \in \{0, 1\}$$

[6,sect.2].

0.4. **Definition**. An abelian variety J over a number field k has many ordinary reductions $\Leftrightarrow$ there exists a set S of non-Archimedean places of k such that J has an ordinary reduction at each place $v \in S$ and the density of S is *positive*.

It is well known that an abelian variety J of dimension ≤ 2 has many ordinary reductions. Moreover, in this case we may assume that the density of S is equal to 1 [12].

0.5. **Yu.G.Zarhin theorem**[19,th.4.2]. *Suppose that an abelian variety J over a number field k has many ordinary reductions. Then each simple factor g of the reductive Lie algebra* Lie $\mathrm{Im}(\rho_l) \otimes \overline{\mathbb{Q}_l}$ *is a classical Lie algebra of type A_m, B_m, C_m or D_m, and the highest weight of any irreducible g-submodule $V \subset V_l \otimes \overline{\mathbb{Q}_l}$ is a minuscule weight(microweight) in Bourbaki's terminology* [3].

This theorem is proved in [19] under the assumption that S has density 1. We have remarked that the positivity of the density is sufficient [17,th.1.13].

0.6. We denote by $\mathbb{N}^+ = \{1, 2, 3, ...\}$ the set of all *positive* natural numbers. We also define the binomial coefficient by the usual formula

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

We introduce some sets of *exceptional* numbers

$$Ex(1) = \{4^l, \frac{1}{2}\binom{4l+2}{2l+1}^{2m-1}, 2^{8lm+4l-4m-3}, 4^l(m+1)^{2l+1} \mid l, m \in \mathbb{N}^+\} =$$

={4,10,16,32,64,108,126,256,500,512,864,1024,1372,1716,2048,2916,3888, 4000,4096,5324,6912,8192,8788,10976,13500,16384,19652,23328,24310,27436, 32000,37044,42592,48668,50000,55296,62500,65536,70304,78732,87808,97556, 108000,119164,124416,131072,139968,143748,157216,171500,186624,202612, 219488,237276,256000,262144,268912,275684,296352,318028,340736,352716, 364500,389344,415292,442368,470596,500000,524288,530604,562432,595508, 629856,665500,702464,740772,780448,821516,864000,907924,944784,953312, ...},

$$Ex(3) = \{4^{l+1}, 6^{l+1}, \binom{4m+4}{2m+2}^{l}, \binom{4m+2}{2m+1}^{2l},$$

$$2^{(4m-1)l}, 4^l(m+2)^{2l}, 2^{l+1}(m+4)^{l+1} \mid l,m \in \mathbb{N}^+\} =$$

={ 8,16,36,64,70,100,128,144,196,216,256,324,400,484,512,576,676,784, 900,924,1000,1024,1156,1296,1444,1600,1728,1764,1936,2048,2116,2304, 2500,2704,2744,2916,3136,3364,3600,3844,4096,... }.

It is evident that the density of $Ex(1) \cup Ex(3)$ in the set $\mathbb{N}$ is equal to zero.

According to Albert's classification [11] the division algebra $\mathrm{End}^0(J_{\mathbf{C}})$ belongs to one of the following types:

Type 1. $\mathrm{End}^0(J_{\mathbf{C}}) = K = K_0$ is a totally real field of algebraic numbers, $e = [K:\mathbb{Q}]$ divides $\dim J_{\mathbf{C}}$.

Type 2. $K = K_0$, $\mathrm{End}^0(J_{\mathbf{C}})$ is a quaternion division algebra over K such that for any embedding $\sigma : K \to \mathbb{R}$

$$\mathrm{End}^0(J_{\mathbf{C}}) \otimes_{K,\sigma} \mathbb{R} = M_2(\mathbb{R}).$$

Type 3. $K = K_0$, $\mathrm{End}^0(J_{\mathbf{C}})$ is a quaternion division algebra over K such that for any embedding $\sigma : K \to \mathbb{R}$

$$\mathrm{End}^0(J_{\mathbf{C}}) \otimes_{K,\sigma} \mathbb{R} = \mathbb{K}$$

is the algebra of classical quaternions.

Type 4. K is an imaginary quadratic extension of a totally real field K_0 and for any embedding $\sigma : K_0 \to \mathbb{R}$

$$\mathrm{End}^0(J_{\mathbf{C}}) \otimes_{K_0,\sigma} \mathbb{R} = M_n(\mathbb{C}).$$

0.7. We introduce here some new sets of exceptional numbers:

$$Ex_1^{gen}(1) = \{2(2l+1)^2 \mid l \in \mathbb{N}^+\} =$$

={18,50,98,162,242,338,450,578,722,882,1058,1250,1458,1682,1922,2178,2450,2738, 3042,3362,3698,4050,4418,4802,5202,5618,6050,6498,...},

$$Ex_1^{sp}(1) = \{\binom{2^{r+2}}{2^{r+1}}, m\binom{2^{r+2}}{2^{r+1}} \mid r \in \mathbb{N}^+,$$

$$m = \frac{\binom{2^{r+2}}{2^{r+1}}}{\binom{2^{r+2}-2n}{2^{r+1}-n} + \binom{2^{r+2}-2n}{2^{r+1}-n-1}}$$

is an odd integer for some natural number $n \in [2, 2^{r+1}]$ or

$$m = \frac{\binom{2^{r+2}}{2^{r+1}}}{2\binom{2^{r+2}-2n-1}{2^{r+1}-n}}$$

is an odd integer for some natural number $n \in [1, 2^{r+1}-2]\} =$
$=\{70,490,12870,16563690,27606150,601080390,...\}$.

It is evident that the density of the set $Ex(1) \cup Ex_1^{gen}(1) \cup Ex_1^{sp}(1)$ in $\mathbb{N}^+$ is equal to zero.

Now we are able to extend Serre theorem 0.2 into the area of even dimensions.

0.8. **Main Theorem**. *Suppose that J is an absolutely simple abelian variety over a number field k, $[k : \mathbb{Q}] < \infty$. Assume that J has many ordinary reductions and* $\mathrm{Cent}(\mathrm{End}\,(J \otimes \overline{k})) = \mathbb{Z}$.

1) *If $J \otimes \overline{k}$ is an abelian variety of the 1st type by Albert's classification,* $\dim_k J = 2 \pmod 4$ *and*

$$\dim_k J \notin Ex(1) \cup Ex_1^{gen}(1) \cup Ex_1^{sp}(1)$$

then the Hodge, Tate and Mumford - Tate conjectures hold for J.

2) *If $J \otimes \overline{k}$ is an abelian variety of the 2nd type by Albert's classification,* $\dim_k J = 4 \pmod 8$ *and*

$$\dim_k J \notin 2\{Ex(1) \cup Ex_1^{gen}(1) \cup Ex_1^{sp}(1)\}$$

then the Hodge, Tate and Mumford - Tate conjectures hold for J.

3) *If $J \otimes \overline{k}$ is an abelian variety of the 3d type by Albert's classification,* $\dim_k J = 4 \pmod 8$ *and*

$$\dim_k J \notin Ex(3) \cup 2\{Ex_1^{sp}(1)\}$$

then the Mumford - Tate conjecture holds for J.

§1. Some properties of l-adic representations

1.1. We start to prove the main theorem.

First of all we recall some facts from the theory of linear representations of simple Lie algebras over an algebraically closed field of characteristic zero.

If g is a simple Lie algebra of type $A_m (m \geq 1)$, then in N.Bourbaki's notations

$$\omega_i = \epsilon_1 + ... + \epsilon_i - \frac{i}{m+1}(\epsilon_1 + ... + \epsilon_{m+1}),$$

Weyl group $W(R)$ is the group of all permutations of $\{\epsilon_1, ..., \epsilon_{m+1}\}$ [3,ch.6,sect.4.7],
$\dim E(\omega_r) = \binom{m+1}{r}$ [3,ch.8,table 2],

$E(\omega_r)$ is symplectic or orthogonal $\Leftrightarrow r = \frac{m+1}{2}$ [3,ch.8,table 1] and in this case

$$\omega_r = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{(m+1)/2} - \epsilon_{(m+3)/2} - \dots - \epsilon_{m+1}),$$

$$chE(\omega_r) = chE(\omega_{(m+1)/2}) = \sum_{\substack{a_i \in \{\pm 1\} \\ a_1 + \dots + a_{m+1} = 0}} e^{a_1 \frac{\epsilon_1}{2} + \dots + a_{m+1} \frac{\epsilon_{m+1}}{2}} =$$

$$= \sum_{\substack{a_i \in \{\pm 1\} \\ a_1 + \dots + a_m \in \{\pm 1\}}} e^{a_1 \frac{\epsilon_1 - \epsilon_{m+1}}{2} + \dots + a_m \frac{\epsilon_m - \epsilon_{m+1}}{2}}.$$

If g is a simple Lie algebra of type $B_m (m \geq 2)$, then dim $E(\omega_m) = 2^m$ [3,ch.8, table 2],

$$\omega_m = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_m),$$

$$chE(\omega_m) = \sum_{a_i \in \{\pm 1\}} e^{a_1 \frac{\epsilon_1}{2} + \dots + a_m \frac{\epsilon_m}{2}}$$

[3,ch.6,sect.4.5].

If g is a simple Lie algebra of type $C_m (m \geq 2)$, then dim $E(\omega_1) = 2m$ [3,ch.8, table 2], $\omega_1 = \epsilon_1$,

$$chE(\omega_1) = \sum_{\substack{a_i \in \{\pm 1\} \\ i \in \{1, \dots, m\}}} e^{a_i \epsilon_i}$$

[3,ch.6,sect.4.6].

If g is a simple Lie algebra of type $D_m (m \geq 3)$, then dim $E(\omega_1) = 2m$ [3,ch.8, table 2], $\omega_1 = \epsilon_1$,

$$chE(\omega_1) = \sum_{\substack{a_i \in \{\pm 1\} \\ i \in \{1, \dots, m\}}} e^{a_i \epsilon_i},$$

$$\dim E(\omega_{m-1}) = \dim E(\omega_m) = 2^{m-1},$$

$$chE(\omega_{m-1}) = \sum_{\substack{a_i \in \{\pm 1\} \\ \mathrm{Card}\{i | a_i = -1\} \equiv 1 (\mathrm{mod}\ 2)}} e^{a_1 \frac{\epsilon_1}{2} + \dots + a_m \frac{\epsilon_m}{2}},$$

$$chE(\omega_m) = \sum_{\substack{a_i \in \{\pm 1\} \\ \mathrm{Card}\{i | a_i = -1\} = 0 (\mathrm{mod}\ 2)}} e^{a_1 \frac{\epsilon_1}{2} + \dots + a_m \frac{\epsilon_m}{2}}$$

[3,ch.6,sect.4.8].

1.2. **Lemma**[15,sect.4.8.1]. *Let $v_2 : \mathbb{Q}^\times \to \mathbb{Z}$ be the standard 2-adic valuation of the field $\mathbb{Q}$. Assume that g is a simple Lie algebra of type A_l and the highest weight of an irreducible representation $\rho : g \to \mathrm{End}\, E$ is the minuscule weight. Then:*

1) *if ρ is an orthogonal representation, then $l = 4m - 1$,*

$$\deg(\rho) = \binom{4m}{2m},$$

$$v_2(\deg(\rho)) = \chi(m)$$

is the number of units in the binary representation of m; therefore $v_2(deg(\rho)) \geq 1$ *and* $v_2(\deg(\rho)) = 1 \Leftrightarrow l = 2^{r+1} - 1 (r \in \mathbb{N}^+) \Leftrightarrow \deg(\rho) \in \{\binom{2^{r+1}}{2^r} \mid r \in \mathbb{N}^+\} = \{6, 70, 12870, 601080390, ...\}$;

2) *if* ρ *is a symplectic representation and* $l \geq 2$, *then* $l = 4m + 1$,

$$\deg(\rho) = \binom{4m+2}{2m+1},$$

$$v_2(\deg(\rho)) = \chi(m) + 1 \geq 2,$$

therefore deg (ρ)=0(mod 4).

1.3. Let G_{V_l} be the algebraic envelope of Im(ρ_l) $\subset$GL(V_l), where

$$V_l = H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l).$$

By F.A.Bogomolov theorem [1] Lie Im(ρ_l)=Lie(G_{V_l}) and G_{V_l} containes the group G_m of homotheties. By G.Faltings theorems [7] G_{V_l} is reductive and

$$\mathrm{End}_{G_{V_l}}(V_l) = \mathrm{End}(J) \otimes \mathbb{Q}_l.$$

Let g_l=Lie Im(ρ_l). We shall denote by g_l^{ss} the semisimple part of g_l. By J.-P.Serre theorem [5,th.3.10] the rank of G_{V_l} (resp.g_l) is independent of l. In the case under consideration we may assume that G_{V_l}=$S_{V_l}$$\cdot$$\mathrm{G}_m$, where S_{V_l}=$[G_{V_l}, G_{V_l}]$ is the commutator subgroup of G_{V_l} [5, sect.1.2.2b].

1.4. Assume that v is a non-Archimedean place of k at which J has a good reduction. Let $\overline{v}$ be any extension of v to $\overline{k}$ and let $F_{\overline{v}} \in \mathrm{Gal}(\overline{k}/k)$ be the corresponding Frobenius element. It is well known that the characteristic polynomial of $\rho_l(F_{\overline{v}}^{-1})$ coincides with the characteristic polynomial of the Frobenius endomorphism π_v of the reduction J_v of J at v. We denote by Δ the set of all eigenvalues of $\rho_l(F_{\overline{v}}^{-1})$ (without counting multiplicities). Let Γ_v be a multiplicative subgroup of $\overline{\mathbb{Q}}^\times$ generated by Δ.

It is well known that $\mathbb{Q}[\pi_v]$=$\prod K_i$, K_i are number fields. The multiplicative group $\mathbb{Q}[\pi_v]^\times$ defines a $\mathbb{Q}$-torus $T_{\pi_v} = \prod R_{K_i/\mathbb{Q}}(\mathrm{G}_{mK_i})$, where $R_{K_i/\mathbb{Q}}$ are the Weil restrictions of scalar functors. Let H_v be the smallest algebraic subgroup of T_{π_v} defined over $\mathbb{Q}$, such that $\pi_v \in H_v(\mathbb{Q})$. As is well-known, H_v is a group of multiplicative type. The connected component of the identity in H_v is called the *Frobenius torus* T_v. It can be regarded as the $\mathbb{Q}$-model of the connected component of 1 in the Zariski closure of the set $\{\rho_l(F_{\overline{v}}^{-1})^n | n \in \mathbb{Z}\}$ in G_{V_l} [5, sect.3b].

1.5. As an easy consequence of [5, prop.3.6, 5.2.1, lemma 2.1, cor.3.8] we have the following result.

After replacing k by some finite extension we may assume that for some set S of density 1 in the set of all non-Archimedean places of k and for each $v \in S$ *the following conditions hold:*

1) *for a fixed integer* $n \geq 2$ *such that* $l^n > (2\dim_k J)^2$, *the* l^n*-torsion points of* $J(\overline{k})$ *are rational points over* k ;

2) p_v =char$(k(v)) > (2\dim_k J)^2$;

3) $\mathrm{Norm}_{k/\mathbf{Q}}(v) = p_v$;
4) *the Frobenius trace* $\mathrm{Tr}(\rho_l(F_{\overline{v}}^{-1}))$ *is not divisible by* p_v;
5) Γ_v *is torsion-free,* G_{V_l} *is connected and* $\rho_l(F_{\overline{v}}^{-1}) \in T_v(\overline{\mathbf{Q}_l})$;
6) *the Frobenius torus* T_v *is a maximal torus of* G_{V_l} *and*

$$\mathrm{rank}(\Gamma_v) = \dim(T_v) = \mathrm{rank}(G_{V_l}).$$

By the condition of the theorem J has many ordinary reductions. Hence, we may assume that the following additional condition holds:

7) for each element $\delta \in \Delta$ and for any place w of $\overline{\mathbf{Q}}$ over p_v we have

$$\frac{w(\delta)}{w(p_v)} \in \{0, 1\}$$

(because in virtue of the condition (3) above $k(v) = \mathbb{F}_{p_v}$). In this case we have the following important relation

$$\frac{w(\Delta \cdot \Delta)}{w(p_v^2)} \subset \{0, \frac{1}{2}, 1\}. \tag{1.5.8}$$

§2. Proof of the main theorem for abelian variety without complex multiplication

2.1. We assume that $J \otimes \overline{k}$ is an abelian variety of the 1st type by Albert's classification. It is well known that $V_l \otimes \overline{\mathbf{Q}_l}$ is an irreducible symplectic $g_l^{ss} \otimes \overline{\mathbf{Q}_l}$-module. Let $d = \dim_k J$.

Assume that the Lie algebra $g_l^{ss} \otimes \overline{\mathbf{Q}_l}$ is simple. Theorem 0.5 and the relation $\dim_k J \notin Ex(1)$ imply that $g_l^{ss} \otimes \overline{\mathbf{Q}_l}$ is the Lie algebra of type C_d [16,sect.1.3-1.8]. On the other hand, Lie $\mathrm{Hg}(J_{\mathbf{C}}) \otimes \overline{\mathbf{Q}_l} \subset sp(V_l \otimes \overline{\mathbf{Q}_l})$. By Piatetski-Shapiro -Deligne - Borovoi theorem [13],[2] there exists a canonical embedding

$$\mathrm{Lie}\ \mathrm{Im}(\rho_l) \subset \mathrm{Lie}[\mathrm{MT}(J_{\mathbf{C}})(\mathbf{Q}_l)] = \mathbf{Q}_l \times \mathrm{Lie}[\mathrm{Hg}(J_{\mathbf{C}})(\mathbf{Q}_l)].$$

So there exists a canonical isomorphism of Lie algebras

$$\mathrm{Lie}\ \mathrm{Im}(\rho_l) \simeq \mathrm{Lie}[\mathrm{MT}(J_{\mathbf{C}})(\mathbf{Q}_l)].$$

2.2. Now we may assume that the Lie algebra $g_l^{ss} \otimes \overline{\mathbf{Q}_l}$ is not simple.

Let $f : S \to S_{V_l} \otimes \overline{\mathbf{Q}_l}$ be the universal covering, where $S = S_1 \times S_2 \times \ldots \times S_q$ is a product of simple simply connected algebraic $\overline{\mathbf{Q}_l}$-groups. An isogeny f extends to an isogeny

$$f : \mathbf{G}_m \times S_1 \times \ldots \times S_q \to \mathbf{G}_m \cdot (S_{V_l} \otimes \overline{\mathbf{Q}_l}) = G_{V_l} \otimes \overline{\mathbf{Q}_l},$$

defined by the formula $f((a, s)) = a \cdot f(s)$ for $a \in \mathbf{G}_m, s \in S_1 \times \ldots \times S_q$.

By (1.5.6) the Frobenius torus T_v is a maximal torus of G_{V_l}. Hence

$$T = (f^{-1}(T_v \otimes \overline{\mathbf{Q}_l}))^0 \subset \mathbf{G}_m \times S_1 \times \ldots \times S_q$$

is a maximal subtorus. Consider the canonical projections

$$\mathrm{pr}_0 : \mathrm{G}_m \times S_1 \times ... \times S_q \to \mathrm{G}_m$$

$$\mathrm{pr}_i : \mathrm{G}_m \times S_1 \times ... \times S_q \to S_i.$$

It is evident that $T = \mathrm{pr}_0(T) \times \mathrm{pr}_1(T) \times ... \times \mathrm{pr}_q(T)$.

On the other hand,

$$V_l \otimes \overline{\mathbb{Q}_l} = W_1 \otimes ... \otimes W_q,$$

where W_1 is an irreducible $\mathrm{G}_m \times S_1$-module, W_2 is an irreducible S_2-module,...,W_q is an irreducible S_q-module. Let

$$\rho_1 : \mathrm{G}_m \times S_1 \to \mathrm{GL}(W_1),$$

$$\rho_i : S_i \to \mathrm{GL}(W_i)(i \geq 2)$$

are the corresponding representations. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{G}_m \times S_1 \times ... \times S_q & \stackrel{\rho_1 \otimes ... \otimes \rho_q}{\longrightarrow} & \mathrm{GL}(W_1 \otimes ... \otimes W_q) \\ \downarrow f & & \| \\ G_{V_l} \otimes \overline{\mathbb{Q}_l} & \subset & \mathrm{GL}(W_1 \otimes ... \otimes W_q) \end{array}$$

By (1.5.5) $\rho_l(F_{\overline{v}}^{-1}) \in T_v(\overline{\mathbb{Q}_l})$, hence there exists an element

$$\tau_{\overline{v}} = (\tau_0, \tau_1, ..., \tau_q) \in \mathrm{pr}_0(T) \times \mathrm{pr}_1(T) \times ... \times \mathrm{pr}_q(T)$$

such that

$$(\rho_1 \otimes ... \otimes \rho_q)(\tau_{\overline{v}}) = f(\tau_{\overline{v}}) = \rho_l(F_{\overline{v}}^{-1}).$$

We see that each eigenvalue of $\rho_l(F_{\overline{v}}^{-1})$ is of the form $\chi_0^{(0)}(\tau_0) \cdot \chi_i^{(1)}(\tau_1)...\chi_j^{(q)}(\tau_q)$, where $\chi_k^{(m)} \in \mathrm{X}(\mathrm{pr}_m(T))$ are some characters.

2.3. By (1.5.1) $\mathrm{Im}(\rho_l) \subset \{x \in \mathrm{End}\ T_l(J \otimes \overline{k}) \mid x \in 1 + l^n \mathrm{End}\, T_l(J \otimes \overline{k})\}$. Hence for any $x \in \mathrm{Im}(\rho_l)$ the l-adic logarithm $\log x$ is defined.

Let μ be the Haar measure on $\mathrm{Im}(\rho_l)$ normalized by the equality $\mu(\mathrm{Im}(\rho_l)) = 1$. It is well known that $X = \{x \in \mathrm{Im}(\rho_l) \mid \log x$ is a regular element in Lie $\mathrm{Im}(\rho_l)\}$ is open and everywhere dense in $\mathrm{Im}(\rho_l)$. Its boundary ∂X is a closed analytic subset. So $\mu(\partial X) = 0$ [14, sect.2.2]. Moreover, the set X is invariant under conjugation in $\mathrm{Im}(\rho_l)$. By Chebotarev theorem the density of $\{v | \rho_l(F_{\overline{v}}^{-1}) \in X\}$ is equal to $\mu(X) = 1 - \mu(\partial X) = 1$ [14, sect.2.2, corollary 2]. Hence we may assume that for v the conditions (1.5.1)-(2.5.7) hold and $\log \rho_l(F_{\overline{v}}^{-1})$ is a regular element in Lie $\mathrm{Im}(\rho_l)$.

Let $\lambda = \chi_0^{(0)}(\tau_0)$. According to the results of sections 0.5, 1.1, 2.2 we may assume that for Lie algebra LieS_1 we have:

for type $A_m (m \geq 1) : \chi_i^{(1)}(\tau_1) = \alpha_1^{a_1}...\alpha_m^{a_m} (a_j \in \{\pm 1\}, a_1 + ... + a_m \in \{\pm 1\})$;

for type $B_m (m \geq 2) : \chi_i^{(1)}(\tau_1) = \alpha_1^{a_1}...\alpha_m^{a_m} (a_j \in \{\pm 1\})$;

for type $C_m(m \geq 2)$:$\chi_i^{(1)}(\tau_1) = \alpha_j^{a_j}(a_j \in \{\pm 1\}, j \in \{1, ..., m\})$;
for type $D_m(m \geq 3)$:$\chi_i^{(1)}(\tau_1) = \alpha_j^{a_j}(a_j \in \{\pm 1\}, j \in \{1, ..., m\})$
or $\chi_i^{(1)}(\tau_1) = \alpha_1^{a_1} ... \alpha_m^{a_m}(a_j \in \{\pm 1\}, \mathrm{Card}\{j \mid a_j = -1\} = 1 (\mathrm{mod}\ 2))$
or $\chi_i^{(1)}(\tau_1) = \alpha_1^{a_1} ... \alpha_m^{a_m}(a_j \in \{\pm 1\}, \mathrm{Card}\{j \mid a_j = -1\} = 0 (\mathrm{mod}\ 2))$,
where $\alpha_1, ..., \alpha_m \in \overline{\mathbb{Q}}^\times$ are multiplicatively independent (in other words,these numbers generate the multiplicative subgroup in $\overline{\mathbb{Q}}^\times$ of rank m).

2.4. By the condition of the theorem $\dim_k J = 2 (\mathrm{mod}\ 4)$. It follows from the results of sections 1.1-1.2 that 2 divides $\dim_{\overline{\mathbb{Q}}_l} W_i$. Hence $S = S_1 \times S_2$ is a product of two simple simply connected algebraic $\overline{\mathbb{Q}_l}$-groups,

$$v_2(\dim_{\overline{\mathbb{Q}}_l} W_i) = 1\, (i = 1, 2). \tag{2.4.1}$$

We may assume that W_1 is a symplectic S_1-module and W_2 is an orthogonal S_2-module. From (2.4.1) it follows that Lie S_i is not an algebra of type $B_n(n \geq 2)$. If Lie S_i is an algebra of type $D_n(n \geq 3)$ then $W_i = E(\omega_1), i = 2$. In virtue of lemma 1.2 Lie S_1 is not an algebra of type $A_n(n \geq 2)$. Hence a pair (type of $g_l^{ss} \otimes \overline{\mathbb{Q}_l}, V_l \otimes \overline{\mathbb{Q}_l}$) assumes one of the following values:

$$(C_m \times D_n, E(\omega_1^{(1)} + \omega_1^{(2)}))(m \geq 1, n \geq 3),$$

$$(C_m \times A_{2^{q+1}-1}, E(\omega_1^{(1)} + \omega_{2^q}^{(2)}))(m \geq 1, q \geq 1),$$

where an index (i) shows that the corresponding fundamental weight relates to the i-th factor.

2.5. Assume that $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is a semisimple Lie algebra of type $C_m \times D_n(m \geq 1, n \geq 3), V_l \otimes \overline{\mathbb{Q}_l} = E(\omega_1^{(1)} + \omega_1^{(2)})$(we recall that $A_1 = C_1$). In this case $\dim_k J = 2mn$, where m, n are *odd* integers. We may assume that each element $\delta \in \Delta$ is of the form $\lambda \alpha_i^{a_i} \beta_j^{b_j}$, where $a_i, b_j \in \{\pm 1\}$ and $\lambda, \alpha_1, ... \alpha_m, \beta_1, ... \beta_n$ are multiplicatively independent. This structure of Δ does not distinguish the cases $C_m \times D_n$ and $C_n \times D_m$. So,we may assume that $m \leq n$. On the other hand, we have to assume that $m \neq n$ (hence $\dim_k J \notin Ex_1^{gen}(1)$), because we want to use the following lemma.

2.6. **Lemma**. *Suppose that* $m < n$. *Then*

$$\eta \in \{\lambda^2 \alpha_i^{\pm 2} \mid i = 1, ..., m\} \Leftrightarrow \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = 2n.$$

Proof. Let $\eta \in \Delta \cdot \Delta$. We may assume that

$$\eta \in \{\lambda^2, \lambda^2\alpha_1^2, \lambda^2\alpha_1\alpha_2, \lambda^2\beta_1^2, \lambda^2\alpha_1^2\beta_1^2, \lambda^2\alpha_1\alpha_2\beta_1^2, \lambda^2\beta_1\beta_2, \lambda^2\alpha_1^2\beta_1\beta_2, \lambda^2\alpha_1\alpha_2\beta_1\beta_2\}.$$

If $\delta \in \Delta$, then $T_\eta(\delta) = \eta\delta^{-1}$, hence

$$T_{\lambda^2}(\delta) \in \Delta \Leftrightarrow \delta \in \Delta,$$

$$T_{\lambda^2\alpha_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\beta_j^{\pm 1} \mid j = 1, ..., n\},$$

$$T_{\lambda^2\alpha_1\alpha_2}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_{1,2}\beta_j^{\pm1}\mid j=1,...,n\},$$

$$T_{\lambda^2\beta_1^2}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_i^{\pm1}\beta_1\mid i=1,...,m\},$$

$$T_{\lambda^2\alpha_1^2\beta_1^2}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_1\beta_1\},$$

$$T_{\lambda^2\alpha_1\alpha_2\beta_1^2}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_{1,2}\beta_1\},$$

$$T_{\lambda^2\beta_1\beta_2}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_i^{\pm1}\beta_{1,2}\mid i=1,...,m\},$$

$$T_{\lambda^2\alpha_1^2\beta_1\beta_2}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_1\beta_{1,2}\},$$

$$T_{\lambda^2\alpha_1\alpha_2\beta_1\beta_2}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_{1,2}\beta_{1,2}\}.$$

Note that m,n are odd integers, $1\le m<n$. So the statement of the lemma follows from the relations above.

2.7. Lemma 2.6 and (0.2.1) imply that the set $\{\lambda^2\alpha_i^{\pm2}\mid i=1,...,m\}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. From (1.5.8) it follows that for any place w of $\overline{\mathbb{Q}}$ over p_v we have

$$\frac{w(\lambda^2\alpha_1^2)}{w(p_v^2)}\in\{0,\frac{1}{2},1\}.$$

Suppose that

$$\frac{w(\lambda^2\alpha_1^2)}{w(p_v^2)}=0$$

for some place w. Then for each $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\frac{(\sigma w)(\sigma(\lambda^2\alpha_1^2))}{(\sigma w)(p_v^2)}=0,$$

hence from the relation

$$\sigma(\lambda^2\alpha_1^2)\in\{\lambda^2\alpha_i^{\pm2}\mid i=1,...,m\}$$

obtained above and from the transitivity of a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\{\ w\mid w$ is a place of $\overline{\mathbb{Q}}$ over $p_v\ \}$ it follows that $\forall w\mid p_v\ \exists\lambda^2\alpha_i^{2a_i}(a_i\in\{\pm1\},i\in\{1,...,m\})$ such that $w(\lambda^2\alpha_i^{2a_i})=0$.

So, $\forall w\mid p_v$

$$0=w(\lambda^2\alpha_i^{2a_i})=\frac{1}{2}\{w(\lambda^2\alpha_i^{2a_i}\beta_1^2)+w(\lambda^2\alpha_i^{2a_i}\beta_1^{-2})\}.$$

Since both summands in the last brackets are nonnegative, we have the relation

$$w(\lambda^2\alpha_i^{2a_i}\beta_1^2)=w(\lambda^2\alpha_i^{2a_i}\beta_1^{-2})=0.$$

So $w(\beta_1)=0$ for *all* $w\mid p_v$. It follows that β_1 is a root of 1 [19,sublemma 3.4.0] contrary to the assumption that $\lambda,\alpha_1,...,\alpha_m,\beta_1,...,\beta_n$ are multiplicatively independent.

Suppose that

$$\frac{w(\lambda^2\alpha_1^2)}{w(p_v^2)} = 1$$

for some place w. Let ρ be a complex conjugation defined by some fixed embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$. It is well known that

$$\frac{w(\lambda^2\alpha_1^2)}{w(p_v^2)} + \frac{(\rho w)(\lambda^2\alpha_1^2)}{(\rho w)(p_v^2)} = 1$$

[16,(3.16.2)]. So in our situation we have the impossible relation

$$\frac{(\rho w)(\lambda^2\alpha_1^2)}{(\rho w)(p_v^2)} = 0.$$

Hence

$$\frac{w(\lambda^2\alpha_1^{\pm 2})}{w(p_v^2)} = \frac{1}{2}$$

for all places $w \mid p_v$. It follows that α_1 is a root of 1 [19,sublemma 3.4.0] contrary to the assumption that $\lambda, \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n$ are multiplicatively independent. So $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is not a Lie algebra of type $C_m \times D_n$.

2.8. Suppose that $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is a Lie algebra of type $C_m \times A_{2^{q+1}-1}$,

$$V_l \otimes \overline{\mathbb{Q}_l} = E(\omega_1^{(1)} + \omega_{2^q}^{(2)}))(m \geq 1, q \geq 1).$$

It is well known that $A_3 \simeq D_3$. So we may assume that $q = r+1 \geq 2, r \in \mathbb{N}^+, m$ is an odd integer, each element $\delta \in \Delta$ is of the form $\lambda\alpha_i^{a_i}\beta_1^{b_1}...\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}}$,where $a_i, b_j \in \{\pm 1\}$,$b_1 + ... + b_{2^{r+2}-1} \in \{\pm 1\}$ and $\lambda, \alpha_1, ...\alpha_m, \beta_1, ...\beta_{2^{r+2}-1}$ are multiplicatively independent.

2.9. **Lemma**. *Suppose that* $\dim_k J \notin Ex_1^{gen}(1) \cup Ex_1^{sp}(1)$. *Then*

$$\eta \in \{\lambda^2\alpha_i^{\pm 1}\alpha_j^{\pm 1} \mid i,j \in \{1, ..., m\}, i \neq j\} \Leftrightarrow \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = 2\binom{2^{r+2}}{2^{r+1}}.$$

Proof. Let $\eta \in \Delta \cdot \Delta$. We may assume that

$$\eta \in \{\lambda^2, \lambda^2\alpha_1^2, \lambda^2\alpha_1\alpha_2; \lambda^2\beta_1^2, \lambda^2\alpha_1^2\beta_1^2, \lambda^2\alpha_1\alpha_2\beta_1^2; \lambda^2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n-1}^{-2},$$

$$\lambda^2\alpha_1^2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n-1}^{-2}, \lambda^2\alpha_1\alpha_2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n-1}^{-2}(2 \leq n \leq 2^{r+1});$$

$$\lambda^2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n}^{-2}, \lambda^2\alpha_1^2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n}^{-2}, \lambda^2\alpha_1\alpha_2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n}^{-2}$$

$$(1 \leq n \leq 2^{r+1} - 1)\}.$$

If $\delta \in \Delta$, then $T_\eta(\delta) = \eta\delta^{-1}$, hence

$$T_{\lambda^2}(\delta) \in \Delta \Leftrightarrow \delta \in \Delta,$$

$$\mathrm{Card}(T_{\lambda^2}(\Delta)\cap\Delta) = \mathrm{Card}(\Delta) = 2\dim_k J = 2m\binom{2^{r+2}}{2^{r+1}} \neq 2\binom{2^{r+2}}{2^{r+1}}$$

in virtue of the relations

$$\dim_k J \notin \{\binom{2^{r+2}}{2^{r+1}} \mid r \in \mathbb{N}^+\} \subset Ex_1^{sp}(1)$$

(in particular, $m \neq 1$). On the other hand,

$$T_{\lambda^2\alpha_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\beta_1^{b_1}...\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}} \mid b_j \in \{\pm1\}, b_1 + ... + b_{2^{r+2}-1} \in \{\pm1\}\},$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1^2}(\Delta)\cap\Delta) = \dim_{\overline{\mathbb{Q}}} E(\omega_{2^{r+1}}^{(2)}) = \binom{2^{r+2}}{2^{r+1}} \neq 2\binom{2^{r+2}}{2^{r+1}};$$

$$T_{\lambda^2\alpha_1\alpha_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_{1,2}\beta_1^{b_1}...\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}} \mid b_j \in \{\pm1\}, b_1 + ... + b_{2^{r+2}-1} \in \{\pm1\}\},$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1\alpha_2}(\Delta)\cap\Delta) = 2\dim_{\overline{\mathbb{Q}}} E(\omega_{2^{r+1}}^{(2)}) = 2\binom{2^{r+2}}{2^{r+1}}.$$

It is clear that

$$T_{\lambda^2\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta = \lambda\alpha_i^{\pm1}\beta_1\beta_2^{b_2}...\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

where $i \in \{1, ..., m\}, b_j \in \{\pm1\}, 1 + b_2 + ... + b_{2^{r+2}-1} \in \{\pm1\}$. It is evident that we can get $(b_2, ..., b_{2^{r+2}-1})$ from $(1, .., 1, -1, ..., -1)$ ($2^{r+1}-1$ times 1 and $2^{r+1}-1$ times -1) or from $(1, .., 1, -1, ..., -1)$ ($2^{r+1}-2$ times 1 and 2^{r+1} times -1) by some permutation of coordinates. Hence we have

$$\mathrm{Card}(T_{\lambda^2\beta_1^2}(\Delta)\cap\Delta) = 2m\left\{\binom{2^{r+2}-2}{2^{r+1}-1} + \binom{2^{r+2}-2}{2^{r+1}-2}\right\} =$$

$$2m\cdot\frac{1}{2}\binom{2^{r+2}}{2^{r+1}} = m\binom{2^{r+2}}{2^{r+1}} \neq 2\binom{2^{r+2}}{2^{r+1}}$$

because m is an odd integer.

On the other hand,

$$T_{\lambda^2\alpha_1^2\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta = \lambda\alpha_1\beta_1\beta_2^{b_2}...\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1^2}(\Delta)\cap\Delta) = \binom{2^{r+2}-2}{2^{r+1}-1} + \binom{2^{r+2}-2}{2^{r+1}-2} =$$

$$\frac{1}{2}\binom{2^{r+2}}{2^{r+1}} \neq 2\binom{2^{r+2}}{2^{r+1}};$$

$$T_{\lambda^2\alpha_1\alpha_2\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta = \lambda\alpha_{1,2}\beta_1\beta_2^{b_2}...\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1\alpha_2\beta_1^2}(\Delta)\cap\Delta)=2\left\{\binom{2^{r+2}-2}{2^{r+1}-1}+\binom{2^{r+2}-2}{2^{r+1}-2}\right\}=$$

$$\binom{2^{r+2}}{2^{r+1}}\neq 2\binom{2^{r+2}}{2^{r+1}}.$$

It is clear that

$$T_{\lambda^2\beta_1^2\ldots\beta_n^2\beta_{n+1}^{-2}\ldots\beta_{2n-1}^{-2}}(\delta)\in\Delta\Leftrightarrow\delta=\lambda\alpha_i^{\pm1}\beta_1\ldots\beta_n\beta_{n+1}^{-1}\ldots\beta_{2n-1}^{-1}\beta_{2n}^{b_{2n}}\ldots\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

where $i\in\{1,\ldots,m\}, b_j\in\{\pm1\}, 1+b_{2n}+\ldots+b_{2^{r+2}-1}\in\{\pm1\}$. It is evident that we can get $(b_{2n},\ldots,b_{2^{r+2}-1})$ from $(1,..,1,-1,\ldots,-1)$ ($2^{r+1}-n$ times 1 and $2^{r+1}-n$ times -1) or from $(1,..,1,-1,\ldots,-1)$ ($2^{r+1}-n-1$ times 1 and $2^{r+1}-n+1$ times -1) by some permutation of coordinates, $2\le n\le 2^{r+1}$. Hence we have

$$\mathrm{Card}(T_{\lambda^2\beta_1^2\ldots\beta_n^2\beta_{n+1}^{-2}\ldots\beta_{2n-1}^{-2}}(\Delta)\cap\Delta)=2m\left\{\binom{2^{r+2}-2n}{2^{r+1}-n}+\binom{2^{r+2}-2n}{2^{r+1}-n-1}\right\}\neq$$

$$2\binom{2^{r+2}}{2^{r+1}},$$

because $\dim_k J\notin Ex_1^{sp}(1)$.

It is clear that

$$T_{\lambda^2\alpha_1^2\beta_1^2\ldots\beta_n^2\beta_{n+1}^{-2}\ldots\beta_{2n-1}^{-2}}(\delta)\in\Delta\Leftrightarrow\delta=\lambda\alpha_1\beta_1\ldots\beta_n\beta_{n+1}^{-1}\ldots\beta_{2n-1}^{-1}\beta_{2n}^{b_{2n}}\ldots\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

where $b_j\in\{\pm1\}, 1+b_{2n}+\ldots+b_{2^{r+2}-1}\in\{\pm1\}$. So,

$$\mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1^2\ldots\beta_n^2\beta_{n+1}^{-2}\ldots\beta_{2n-1}^{-2}}(\Delta)\cap\Delta)=\binom{2^{r+2}-2n}{2^{r+1}-n}+\binom{2^{r+2}-2n}{2^{r+1}-n-1}\neq$$

$$2\binom{2^{r+2}}{2^{r+1}},$$

because $n\ge 2$.

On the other hand,

$$T_{\lambda^2\alpha_1\alpha_2\beta_1^2\ldots\beta_n^2\beta_{n+1}^{-2}\ldots\beta_{2n-1}^{-2}}(\delta)\in\Delta\Leftrightarrow\delta=\lambda\alpha_{1,2}\beta_1\ldots\beta_n\beta_{n+1}^{-1}\ldots\beta_{2n-1}^{-1}\beta_{2n}^{b_{2n}}\ldots\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1\alpha_2\beta_1^2\ldots\beta_n^2\beta_{n+1}^{-2}\ldots\beta_{2n-1}^{-2}}(\Delta)\cap\Delta)=2\left\{\binom{2^{r+2}-2n}{2^{r+1}-n}+\binom{2^{r+2}-2n}{2^{r+1}-n-1}\right\}$$

$$\le 4\binom{2^{r+2}-2n}{2^{r+1}-n}\le 4\binom{2^{r+2}-2}{2^{r+1}-1}=2\binom{2^{r+2}}{2^{r+1}}\frac{1}{(2^{r+2}-1)(2^{r+1}+1)}<2\binom{2^{r+2}}{2^{r+1}}.$$

It is clear that

$$T_{\lambda^2\beta_1^2\ldots\beta_n^2\beta_{n+1}^{-2}\ldots\beta_{2n}^{-2}}(\delta)\in\Delta\Leftrightarrow\delta=\lambda\alpha_i^{\pm1}\beta_1\ldots\beta_n\beta_{n+1}^{-1}\ldots\beta_{2n}^{-1}\beta_{2n+1}^{b_{2n+1}}\ldots\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

where $i \in \{1, ..., m\}, b_j \in \{\pm 1\}, b_{2n+1} + ... + b_{2^{r+2}-1} \in \{\pm 1\}$. It is evident that we can get $(b_{2n+1}, ..., b_{2^{r+2}-1})$ from $(1, .., 1, -1, ..., -1)$ ($2^{r+1}-n$ times 1 and $2^{r+1}-n-1$ times -1) or from $(1, .., 1, -1, ..., -1)$ ($2^{r+1}-n-1$ times 1 and $2^{r+1}-n$ times -1) by some permutation of coordinates, $1 \leq n \leq 2^{r+1}-1$. Hence we have

$$\mathrm{Card}(T_{\lambda^2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n}^{-2}}(\Delta) \cap \Delta) = 2m\left\{\binom{2^{r+2}-2n-1}{2^{r+1}-n} + \binom{2^{r+2}-2n-1}{2^{r+1}-n-1}\right\} =$$

$$4m\binom{2^{r+2}-2n-1}{2^{r+1}-n} \neq 2\binom{2^{r+2}}{2^{r+1}},$$

otherwise for $n = 2^{r+1}-1$ we would have:

$$4m = 2\binom{2^{r+2}}{2^{r+1}},$$

$$m = \frac{1}{2}\binom{2^{r+2}}{2^{r+1}}$$

is an odd integer,

$$\dim_k J = m\binom{2^{r+2}}{2^{r+1}} = 2m^2 \in Ex_1^{gen}(1)$$

contrary to the assumption of lemma; for $1 \leq n \leq 2^{r+1}-2$ we would have:

$$m = \frac{\binom{2^{r+2}}{2^{r+1}}}{2\binom{2^{r+2}-2n-1}{2^{r+1}-n}}$$

is an odd integer for some natural number $n \in [1, 2^{r+1}-2]$,

$$\dim_k J = m\binom{2^{r+2}}{2^{r+1}} \in Ex_1^{sp}(1)$$

contrary to the assumption of lemma.

It is evident that

$$T_{\lambda^2\alpha_1^2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n}^{-2}}(\delta) \in \Delta \Leftrightarrow \delta = \lambda\alpha_1\beta_1...\beta_n\beta_{n+1}^{-1}...\beta_{2n}^{-1}\beta_{2n+1}^{b_{2n+1}}...\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

where $b_j \in \{\pm 1\}, b_{2n+1} + ... + b_{2^{r+2}-1} \in \{\pm 1\}$. Hence,

$$\mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n}^{-2}}(\Delta) \cap \Delta) = 2\binom{2^{r+2}-2n-1}{2^{r+1}-n} \neq 2\binom{2^{r+2}}{2^{r+1}},$$

because $n \geq 1$.

On the other hand,

$$T_{\lambda^2\alpha_1\alpha_2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n}^{-2}}(\delta) \in \Delta \Leftrightarrow \delta = \lambda\alpha_{1,2}\beta_1...\beta_n\beta_{n+1}^{-1}...\beta_{2n}^{-1}\beta_{2n+1}^{b_{2n+1}}...\beta_{2^{r+2}-1}^{b_{2^{r+2}-1}},$$

where $b_j \in \{\pm 1\}, b_{2n+1} + ... + b_{2^{r+2}-1} \in \{\pm 1\}$. Hence,

$$\mathrm{Card}(T_{\lambda^2\alpha_1\alpha_2\beta_1^2...\beta_n^2\beta_{n+1}^{-2}...\beta_{2n}^{-2}}(\Delta) \cap \Delta) = 4\binom{2^{r+2}-2n-1}{2^{r+1}-n} \leq 4\binom{2^{r+2}-3}{2^{r+1}-1} =$$

$$\binom{2^{r+2}}{2^{r+1}}\frac{2^{r+1}}{2^{r+2}-1} < \binom{2^{r+2}}{2^{r+1}}.$$

Lemma 2.9 is proved.

2.10. By lemma 2.9 and (0.2.1) the set $\{\lambda^2\alpha_i^{\pm 1}\alpha_j^{\pm 1} \mid i,j \in \{1,...,m\}, i \neq j\}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant.

From (1.5.8) it follows that for any place w of $\overline{\mathbb{Q}}$ over p_v we have

$$\frac{w(\lambda^2\alpha_1\alpha_2)}{w(p_v^2)} \in \{0, \frac{1}{2}, 1\}.$$

Suppose that

$$\frac{w(\lambda^2\alpha_1\alpha_2)}{w(p_v^2)} = 0$$

for some place w. Then for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\frac{(\sigma w)(\sigma(\lambda^2\alpha_1\alpha_2))}{(\sigma w)(p_v^2)} = 0,$$

hence from the relation

$$\sigma(\lambda^2\alpha_1\alpha_2) \in \{\lambda^2\alpha_i^{\pm 1}\alpha_j^{\pm 1} \mid i,j \in \{1,...,m\}, i \neq j\}$$

obtained above and from the transitivity of a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set $\{w \mid w$ is a place of $\overline{\mathbb{Q}}$ over $p_v\}$ it follows that $\forall w \mid p_v\ \exists \lambda^2\alpha_i^{a_i}\alpha_j^{a_j} (a_i, a_j \in \{\pm 1\}, i,j \in \{1,...,m\}, i \neq j)$ such that $w(\lambda^2\alpha_i^{a_i}\alpha_j^{a_j}) = 0$.

So, $\forall w \mid p_v$

$$0 = w(\lambda^2\alpha_i^{a_i}\alpha_j^{a_j}) = \frac{1}{2}\{w(\lambda^2\alpha_i^{a_i}\alpha_j^{a_j}\beta_1^2) + w(\lambda^2\alpha_i^{a_i}\alpha_j^{a_j}\beta_1^{-2})\}.$$

On the other hand,

$$\lambda^2\alpha_i^{a_i}\alpha_j^{a_j}\beta_1^{2b_1} \in \Delta \cdot \Delta.$$

Consequently both summands in the last brackets are nonnegative and we have the relation

$$w(\lambda^2\alpha_i^{a_i}\alpha_j^{a_j}\beta_1^2) = w(\lambda^2\alpha_i^{a_i}\alpha_j^{a_j}\beta_1^{-2}) = 0.$$

So $w(\beta_1) = 0$ for *all* $w \mid p_v$. It follows that β_1 is a root of 1 [19,sublemma 3.4.0] contrary to the assumption that $\lambda, \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_{2^{r+2}-1}$ are multiplicatively independent.

If

$$\frac{w(\lambda^2\alpha_1\alpha_2)}{w(p_v^2)} = 1,$$

then we have the impossible relation

$$\frac{(\rho w)(\lambda^2\alpha_1\alpha_2)}{(\rho w)(p_v^2)} = 0.$$

Hence,

$$\frac{w(\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1})}{w(p_v^2)} = \frac{1}{2}$$

for all places $w \mid p_v$. It follows that $\alpha_1\alpha_2$ is a root of 1 [19,sublemma 3.4.0] contrary to the assumption that $\lambda, \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_{2^{r+2}-1}$ are multiplicatively independent. So $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is not a Lie algebra of type $C_m \times A_{2^{q+1}-1}$.

§3. Proof of the main theorem for abelian variety of the 2nd or the 3d type by Albert's classification

3.1. We assume that $J \otimes \overline{k}$ is an abelian variety of the 2nd type by Albert's classification. From the well known relations

$$M_2(\overline{\mathbb{Q}_l}) \simeq \mathrm{End}^0(J \otimes \overline{k}) \otimes \overline{\mathbb{Q}_l} \simeq \mathrm{End}_{g_l \otimes \overline{\mathbb{Q}_l}} V_l \otimes \overline{\mathbb{Q}_l}, \qquad (3.1.1)$$

$$\mathrm{NS}(J \otimes \overline{k}) \otimes \overline{\mathbb{Q}_l} \simeq (\bigwedge^2 H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l) \otimes \overline{\mathbb{Q}_l})^{g_l^{ss} \otimes \overline{\mathbb{Q}_l}} \qquad (3.1.2)$$

and from Schur's lemma it follows that $V_l \otimes \overline{\mathbb{Q}_l}$ is the direct sum of two copies of an irreducible symplectic $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$-module. Since each eigenvalue $\delta \in \Delta$ has multiplicity 2 we can deduce the statement of the theorem by the same procedure as above.

3.2. We assume that $J \otimes \overline{k}$ is an abelian variety of the 3d type by Albert's classification. From the relations (3.1.1)-(3.1.2) and from Schur's lemma it follows that $V_l \otimes \overline{\mathbb{Q}_l}$ is the direct sum of two copies of an irreducible orthogonal $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$-module W.

Assume that a Lie algebra $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is simple. From the relation $\dim_k J \notin Ex(3)$ it follows that $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is a Lie algebra of type $D_{d/2}$,where $d = \dim_k J$ [16,sect.1.3-1.8]. On the other hand, Lie $\mathrm{Hg}(J_{\mathbb{C}}) \otimes \overline{\mathbb{Q}_l} \subset so(W)$. By Piatetski-Shapiro -Deligne - Borovoi theorem [13],[2] there exists a canonical embedding

$$\mathrm{Lie\ Im}(\rho_l) \subset \mathrm{Lie}[\mathrm{MT}(J_{\mathbb{C}})(\mathbb{Q}_l)] = \mathbb{Q}_l \times \mathrm{Lie}[\mathrm{Hg}(J_{\mathbb{C}})(\mathbb{Q}_l)].$$

So there exists a canonical isomorphism of Lie algebras

$$\mathrm{Lie\ Im}(\rho_l) \simeq \mathrm{Lie}[\mathrm{MT}(J_{\mathbb{C}})(\mathbb{Q}_l)].$$

3.3. Now we may assume that a Lie algebra $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is not simple. By the condition of the theorem $\dim_k J = 4 (\mathrm{mod}\ 8)$. It follows from the results of sections 1.1-1.2 that $W = W_1 \otimes W_2$ and 2 divides $\dim_{\overline{\mathbb{Q}_l}} W_i$. Hence $S = S_1 \times S_2$ is a product of two simple simply connected algebraic $\overline{\mathbb{Q}_l}$-groups,

$$v_2(\dim_{\overline{\mathbb{Q}_l}} W_i) = 1\ (i = 1, 2). \qquad (3.3.1)$$

From (3.3.1) it follows that Lie S_i is not an algebra of type $B_n (n \geq 2)$. If Lie S_i is an algebra of type $D_n (n \geq 3)$ then $W_i = E(\omega_1^{(i)})$. Hence a pair (type of $g_l^{ss} \otimes \overline{\mathbb{Q}_l}, V_l \otimes \overline{\mathbb{Q}_l}$) assumes one of the following values:

$$(C_m \times C_n, E(\omega_1^{(1)} + \omega_1^{(2)})^{\oplus 2})(m \geq 1, n \geq 1),$$

$$(D_m \times D_n, E(\omega_1^{(1)} + \omega_1^{(2)})^{\oplus 2})(m \geq 3, n \geq 3),$$

$$(D_m \times A_{2^{q+1}-1}, E(\omega_1^{(1)} + \omega_{2^q}^{(2)})^{\oplus 2})(m \geq 3, q \geq 1),$$

$$(A_{2^{p+1}-1} \times A_{2^{q+1}-1}, E(\omega_{2^p}^{(1)} + \omega_{2^q}^{(2)})^{\oplus 2})(p \geq 1, q \geq 1),$$

where m, n are *odd* integers.

On the other hand, it is easy to see that $2 \cdot Ex_1^{gen}(1) \subset Ex(3)$. Hence $\dim_k J \notin 2 \cdot Ex_1^{gen}(1)$.

3.4. Assume that $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is a semisimple Lie algebra of type $C_m \times C_n, (m \geq 1, n \geq 1), V_l \otimes \overline{\mathbb{Q}_l} = E(\omega_1^{(1)} + \omega_1^{(2)})^{\oplus 2}$. In this case $\dim_k J = 4mn$, where m, n are odd integers. We may assume that $m \leq n$. On the other hand, we have to assume that $\dim_k J = 4$ or $m \neq n$ because $\dim_k J \notin 2 \cdot Ex_1^{gen}(1)$. If $\dim_k J = 4$ then $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is a semisimple Lie algebra of type $C_1 \times C_1 = D_2$ and the Mumford - Tate conjecture holds for J.

We assume that $1 \leq m < n$. Then we can exclude the variant $(C_m \times C_n, E(\omega_1^{(1)} + \omega_1^{(2)})^{\oplus 2})(m \geq 1, n \geq 1)$ by the arguments of sections 2.6-2.7.

3.5. Assume that $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is a semisimple Lie algebra of type $D_m \times D_n (m \geq 3, n \geq 3), V_l \otimes \overline{\mathbb{Q}_l} = E(\omega_1^{(1)} + \omega_1^{(2)})^{\oplus 2}$. The structure of Δ does not distinguish the cases $D_m \times D_n$ and $C_m \times C_n$. Hence we may exclude this variant as above.

3.6. The variant $(D_m \times A_{2^{q+1}-1}, E(\omega_1^{(1)} + \omega_{2^q}^{(2)})^{\oplus 2})(m \geq 3, q \geq 1)$ can be excluded by the arguments of sections 2.8-2.10 because $\dim_k J \notin 2 \cdot \{Ex_1^{gen}(1) \cup Ex_1^{sp}(1)\}$.

3.7. Consider the variant $(A_{2^{p+1}-1} \times A_{2^{q+1}-1}, E(\omega_{2^p}^{(1)} + \omega_{2^q}^{(2)})^{\oplus 2})(p \geq 1, q \geq 1)$. From the relation $\dim_k J \notin 2 \cdot Ex_1^{gen}(1)$ it follows that $p \neq q$. So we may assume that $p < q$. Since $A_3 \simeq D_3$ we may assume that $2 \leq p = r + 1 < q = s + 1$, where $r, s \in \mathbb{N}^+$, $r < s$. Each element $\delta \in \Delta$ has the form

$$\lambda \alpha_1^{a_1} \ldots \alpha_{2^{r+2}-1}^{a_{2^{r+2}-1}} \cdot \beta_1^{b_1} \ldots \beta_{2^{s+2}-1}^{b_{2^{s+2}-1}},$$

where $a_i, b_j \in \{\pm 1\}$, $a_1 + \ldots + a_{2^{r+2}-1} \in \{\pm 1\}$, $b_1 + \ldots + b_{2^{s+2}-1} \in \{\pm 1\}$ and $\lambda, \alpha_1, \ldots, \alpha_{2^{r+2}-1}, \beta_1, \ldots, \beta_{2^{s+2}-1}$ are multiplicatively independent.

3.8. **Lemma**. *Let $\eta \in \Delta \cdot \Delta$. We have $\eta \in \{\lambda^2 (\alpha_i^2 \alpha_j^{-2})^{\pm 1} | i \neq j\} \Leftrightarrow$*

$$\mathrm{Card}(T_\eta(\Delta) \cap \Delta) = 2\binom{2^{r+2}-3}{2^{r+1}-1}\binom{2^{s+2}}{2^{s+1}} = \frac{2^r}{2^{r+2}-1} \cdot \binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}}.$$

Proof. We may assume that $\eta = \lambda^2 \nu_1 \nu_2$, where

$$\nu_1 \in \{1, \alpha_1^2, \alpha_1^2 \ldots \alpha_n^2 \alpha_{n+1}^{-2} \ldots \alpha_{2n-1}^{-2} (2 \leq n \leq 2^{r+1}), \alpha_1^2 \ldots \alpha_n^2 \alpha_{n+1}^{-2} \ldots \alpha_{2n}^{-2}$$

$$(1 \leq n \leq 2^{r+1} - 1)\},$$

$$\nu_2 \in \{1, \beta_1^2, \beta_1^2 \dots \beta_m^2 \beta_{m+1}^{-2} \dots \beta_{2m-1}^{-2} (2 \le m \le 2^{s+1}), \beta_1^2 \dots \beta_m^2 \beta_{m+1}^{-2} \dots \beta_{2m}^{-2} \\ (1 \le m \le 2^{s+1} - 1)\}.$$

It is easy to see that $\mathrm{Card}(T_\eta(\Delta) \cap \Delta) = c_1(\nu_1) \cdot c_2(\nu_2)$, where

$$c_1(1) = \binom{2^{r+2}}{2^{r+1}},$$

$$c_1(\alpha_1^2) = \binom{2^{r+2}-2}{2^{r+1}-1} + \binom{2^{r+2}-2}{2^{r+1}-2} = \frac{1}{2}\binom{2^{r+2}}{2^{r+1}}$$

$$c_1(\alpha_1^2 \dots \alpha_n^2 \alpha_{n+1}^{-2} \dots \alpha_{2n-1}^{-2}) = \binom{2^{r+2}-2n}{2^{r+1}-n} + \binom{2^{r+2}-2n}{2^{r+1}-n-1},$$

$$c_1(\alpha_1^2 \dots \alpha_n^2 \alpha_{n+1}^{-2} \dots \alpha_{2n}^{-2}) = 2\binom{2^{r+2}-2n-1}{2^{r+1}-n}$$

$$c_2(1) = \binom{2^{s+2}}{2^{s+1}},$$

$$c_2(\beta_1^2) = \binom{2^{s+2}-2}{2^{s+1}-1} + \binom{2^{s+2}-2}{2^{s+1}-2} = \frac{1}{2}\binom{2^{s+2}}{2^{s+1}}$$

$$c_2(\beta_1^2 \dots \beta_m^2 \beta_{m+1}^{-2} \dots \beta_{2m-1}^{-2}) = \binom{2^{s+2}-2m}{2^{s+1}-m} + \binom{2^{s+2}-2m}{2^{s+1}-m-1},$$

$$c_2(\beta_1^2 \dots \beta_m^2 \beta_{m+1}^{-2} \dots \beta_{2m}^{-2}) = 2\binom{2^{s+2}-2m-1}{2^{s+1}-m}.$$

For example,

$$\mathrm{Card}(T_{\lambda^2 \alpha_1^2 \dots \alpha_n^2 \alpha_{n+1}^{-2} \dots \alpha_{2n-1}^{-2} \beta_1^2}(\Delta) \cap \Delta) =$$

$$\left\{ \binom{2^{r+2}-2n}{2^{r+1}-n} + \binom{2^{r+2}-2n}{2^{r+1}-n-1} \right\} \cdot \frac{1}{2}\binom{2^{s+2}}{2^{s+1}}.$$

Using these calculations, the relations

$$\binom{2^{r+2}-2}{2^{r+1}-1} + \binom{2^{r+2}-2}{2^{r+1}-2} = \frac{1}{2}\binom{2^{r+2}}{2^{r+1}},$$

$$\binom{2^{r+2}-3}{2^{r+1}-1} = \frac{2^{r-1}}{2^{r+2}-1} \cdot \binom{2^{r+2}}{2^{r+1}},$$

$$\binom{2^{r+2}-4}{2^{r+1}-2} = \frac{2^{r-1}}{2^{r+2}-1} \cdot \frac{2^{r+1}-1}{2^{r+2}-3} \cdot \binom{2^{r+2}}{2^{r+1}}$$

and similar relations with s instead of r we can deduce the statement of lemma from the inequality $r < s$ and elementary properties of decreasing function

$$f(r) = \frac{2^r}{2^{r+2}-1} (\text{note that } \frac{\partial f(r)}{\partial r} < 0).$$

Indeed, for $n \geq 2$

$$c_1(\alpha_1^2 \ldots \alpha_n^2 \alpha_{n+1}^{-2} \ldots \alpha_{2n-1}^{-2}) = \binom{2^{r+2}-2n}{2^{r+1}-n} + \binom{2^{r+2}-2n}{2^{r+1}-n-1} \leq 2\binom{2^{r+2}-2n}{2^{r+1}-n}$$

$$\leq 2\binom{2^{r+2}-4}{2^{r+1}-2} = 2\binom{2^{r+2}-3}{2^{r+1}-1} \cdot \frac{2^{r+1}-1}{2^{r+2}-3} < 2\binom{2^{r+2}-3}{2^{r+1}-1},$$

$$c_1(\alpha_1^2 \ldots \alpha_n^2 \alpha_{n+1}^{-2} \ldots \alpha_{2n}^{-2}) = 2\binom{2^{r+2}-2n-1}{2^{r+1}-n} \leq 2\binom{2^{r+2}-5}{2^{r+1}-2} < 2\binom{2^{r+2}-3}{2^{r+1}-1},$$

hence we may assume that $\nu_1 \in \{1, \alpha_1^2, \alpha_1^2\alpha_2^{-2}\}$.

It is evident that $\eta \neq \lambda^2$. On the other hand, if $\eta = \lambda^2\beta_1^2$ then

$$\frac{2^r}{2^{r+2}-1} \cdot \binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}} = \binom{2^{r+2}}{2^{r+1}} \cdot \frac{1}{2} \cdot \binom{2^{s+2}}{2^{s+1}}$$

and we get the impossible relation

$$\frac{2^r}{2^{r+2}-1} = \frac{1}{2}.$$

If $\eta = \lambda^2\beta_1^2 \ldots \beta_m^2 \beta_{m+1}^{-2} \ldots \beta_{2m-1}^{-2}$ then we have

$$\binom{2^{r+2}}{2^{r+1}}\left\{\binom{2^{s+2}-2m}{2^{s+1}-m} + \binom{2^{s+2}-2m}{2^{s+1}-m-1}\right\} = \frac{2^r}{2^{r+2}-1} \cdot \binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}},$$

$$\frac{2^r}{2^{r+2}-1} \cdot \binom{2^{s+2}}{2^{s+1}} = \binom{2^{s+2}-2m}{2^{s+1}-m} + \binom{2^{s+2}-2m}{2^{s+1}-m-1} \leq 2 \cdot \binom{2^{s+2}-2m}{2^{s+1}-m} \leq$$

$$2 \cdot \binom{2^{s+2}-4}{2^{s+1}-2} = \frac{2^s}{2^{s+2}-1} \cdot \frac{2^{s+1}-1}{2^{s+2}-3} \cdot \binom{2^{s+2}}{2^{s+1}},$$

$$f(r) = \frac{2^r}{2^{r+2}-1} \leq \frac{2^s}{2^{s+2}-1} \cdot \frac{2^{s+1}-1}{2^{s+2}-3} < \frac{2^s}{2^{s+2}-1} = f(s)$$

contrary to the assumption that $r < s$.

If $\eta = \lambda^2\beta_1^2 \ldots \beta_m^2 \beta_{m+1}^{-2} \ldots \beta_{2m}^{-2}$ then we have

$$\binom{2^{r+2}}{2^{r+1}} \cdot 2 \cdot \binom{2^{s+2}-2m-1}{2^{s+1}-m} = \frac{2^r}{2^{r+2}-1} \cdot \binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}},$$

$$\frac{2^r}{2^{r+2}-1} \cdot \binom{2^{s+2}}{2^{s+1}} = 2 \cdot \binom{2^{s+2}-2m-1}{2^{s+1}-m} \leq 2 \cdot \binom{2^{s+2}-3}{2^{s+1}-1} = \frac{2^s}{2^{s+2}-1}\binom{2^{s+2}}{2^{s+1}},$$

$$f(r) = \frac{2^r}{2^{r+2}-1} \leq \frac{2^s}{2^{s+2}-1} = f(s)$$

contrary to the assumption that $r < s$.

If $\eta = \lambda^2\alpha_1^2$ then

$$\frac{1}{2}\binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}} = \frac{2^r}{2^{r+2}-1}\cdot\binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}},$$

and we get the impossible relation

$$\frac{2^r}{2^{r+2}-1} = \frac{1}{2}.$$

If $\eta = \lambda^2\alpha_1^2\beta_1^2$ then

$$\frac{1}{2}\binom{2^{r+2}}{2^{r+1}}\frac{1}{2}\binom{2^{s+2}}{2^{s+1}} = \frac{2^r}{2^{r+2}-1}\cdot\binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}},$$

and we get the impossible relation

$$\frac{2^r}{2^{r+2}-1} = \frac{1}{4}.$$

If $\eta = \lambda^2\alpha_1^2\beta_1^2\ldots\beta_m^2\beta_{m+1}^{-2}\ldots\beta_{2m-1}^{-2}$ then we have

$$\frac{1}{2}\binom{2^{r+2}}{2^{r+1}}\left\{\binom{2^{s+2}-2m}{2^{s+1}-m}+\binom{2^{s+2}-2m}{2^{s+1}-m-1}\right\} = \frac{2^r}{2^{r+2}-1}\cdot\binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}},$$

$$\frac{2^r}{2^{r+2}-1}\cdot\binom{2^{s+2}}{2^{s+1}} = \frac{1}{2}\left\{\binom{2^{s+2}-2m}{2^{s+1}-m}+\binom{2^{s+2}-2m}{2^{s+1}-m-1}\right\} \le \binom{2^{s+2}-2m}{2^{s+1}-m}$$

$$\le \binom{2^{s+2}-4}{2^{s+1}-2} = \frac{2^{s-1}}{2^{s+2}-1}\cdot\frac{2^{s+1}-1}{2^{s+2}-3}\cdot\binom{2^{s+2}}{2^{s+1}} \le \frac{2^{s-1}}{2^{s+2}-1}\binom{2^{s+2}}{2^{s+1}},$$

and we get the impossible relation

$$f(r) = \frac{2^r}{2^{r+2}-1} \le \frac{2^{s-1}}{2^{s+2}-1} = \frac{1}{2}f(s) < \frac{1}{2}f(r).$$

If $\eta = \lambda^2\alpha_1^2\beta_1^2\ldots\beta_m^2\beta_{m+1}^{-2}\ldots\beta_{2m}^{-2}$ then

$$\frac{1}{2}\binom{2^{r+2}}{2^{r+1}}\cdot 2\cdot\binom{2^{s+2}-2m-1}{2^{s+1}-m} = \frac{2^r}{2^{r+2}-1}\cdot\binom{2^{r+2}}{2^{r+1}}\binom{2^{s+2}}{2^{s+1}},$$

$$\frac{2^r}{2^{r+2}-1}\cdot\binom{2^{s+2}}{2^{s+1}} = \binom{2^{s+2}-2m-1}{2^{s+1}-m} \le \binom{2^{s+2}-3}{2^{s+1}-1} = \frac{2^{s-1}}{2^{s+2}-1}\binom{2^{s+2}}{2^{s+1}},$$

and we get the impossible relation

$$f(r) = \frac{2^r}{2^{r+2}-1} \le \frac{2^{s-1}}{2^{s+2}-1} = \frac{1}{2}f(s) < \frac{1}{2}f(r).$$

If $\nu_2 \in \{\beta_1^2, \beta_1^2 ... \beta_m^2 \beta_{m+1}^{-2} ... \beta_{2m-1}^{-2}, \beta_1^2 ... \beta_m^2 \beta_{m+1}^{-2} ... \beta_{2m}^{-2}\}$ then

$$c_2(\nu_2) < \binom{2^{s+2}}{2^{s+1}}$$

and $\mathrm{Card}(T_{\lambda^2\alpha_1^2\alpha_2^{-2}\nu_2}(\Delta) \cap \Delta) = c_1(\alpha_1^2\alpha_2^{-2}) \cdot c_2(\nu_2) =$

$$2\binom{2^{r+2}-3}{2^{r+1}-1} \cdot c_2(\nu_2) < 2\binom{2^{r+2}-3}{2^{r+1}-1}\binom{2^{s+2}}{2^{s+1}}.$$

Lemma 3.8 is proved.

3.9. By lemma 3.8 the set $\{\lambda^2(\alpha_i^2\alpha_j^{-2})^{\pm 1} \mid i \neq j\}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant.

From (1.5.8) it follows that for any place w of $\overline{\mathbb{Q}}$ over p_v we have

$$\frac{w(\lambda^2\alpha_1^2\alpha_2^{-2})}{w(p_v^2)} \in \{0, \frac{1}{2}, 1\}.$$

Suppose that

$$\frac{w(\lambda^2\alpha_1^2\alpha_2^{-2})}{w(p_v^2)} = 0$$

for some place w. Then for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\frac{(\sigma w)(\sigma(\lambda^2\alpha_1^2\alpha_2^{-2}))}{(\sigma w)(p_v^2)} = 0,$$

hence from the relation

$$\sigma(\lambda^2\alpha_1^2\alpha_2^{-2}) \in \{\lambda^2(\alpha_i^2\alpha_j^{-2})^{\pm 1} \mid i \neq j\}$$

obtained above and from the transitivity of a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set $\{w \mid w \text{ is a place of } \overline{\mathbb{Q}} \text{ over } p_v\}$ it follows that $\forall w \mid p_v\ \exists \lambda^2(\alpha_i^2\alpha_j^{-2})^a (a \in \{\pm 1\}, i \neq j)$ such that $w(\lambda^2(\alpha_i^2\alpha_j^{-2})^a) = 0$.

So, $\forall w \mid p_v$

$$0 = w(\lambda^2(\alpha_i^2\alpha_j^{-2})^a) = \frac{1}{2}\{w(\lambda^2(\alpha_i^2\alpha_j^{-2})^a\beta_1^2) + w(\lambda^2(\alpha_i^2\alpha_j^{-2})^a)\beta_1^{-2})\}.$$

On the other hand,

$$\lambda^2(\alpha_i^2\alpha_j^{-2})^a\beta_1^{2b_1} \in \Delta \cdot \Delta.$$

Consequently both summands in the last brackets are nonnegative and we have the relation

$$w(\lambda^2(\alpha_i^2\alpha_j^{-2})^a\beta_1^2) = w(\lambda^2(\alpha_i^2\alpha_j^{-2})^a)\beta_1^{-2}) = 0$$

So $w(\beta_1) = 0$ for *all* $w \mid p_v$. It follows that β_1 is a root of 1 [19,sublemma 3.4.0] contrary to the assumption that $\lambda, \alpha_1, ..., \alpha_{2^{r+2}-1}, \beta_1, ..., \beta_{2^{s+2}-1}$ are multiplicatively independent.

If

$$\frac{w(\lambda^2\alpha_1^2\alpha_2^{-2})}{w(p_v^2)}=1,$$

then we have the impossible relation

$$\frac{(\rho w)(\lambda^2\alpha_1^2\alpha_2^{-2})}{(\rho w)(p_v^2)}=0.$$

Hence,

$$\frac{w(\lambda^2(\alpha_1^2\alpha_2^{-2})^{\pm 1})}{w(p_v^2)}=\frac{1}{2}$$

for all places $w \mid p_v$. It follows that $\alpha_1^2\alpha_2^{-2}$ is a root of 1 [19,sublemma 3.4.0] contrary to our assumptions. So $g_l^{ss}\otimes\overline{\mathbb{Q}_l}$ is not a Lie algebra of type $A_{2^{p+1}-1}\times A_{2^{q+1}-1}$. Theorem 0.8 is proved.

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