CERTAIN GROUP-THEORETICAL APPLICATIONS OF THE HILTON-MILNOR THEOREM

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Abstract. We consider certain group theoretical applications of the well-known Hilton-Milnor Theorem. For $n \geq 1$, let F_{2n} be a free group of rank $2n$ with generators $\{x_1, y_1, \dots, x_n, y_n\}$. Consider the following normal subgroups of F_{2n} : $R_i = \langle x_i, y_i\rangle^{F_{2n}},\ i = 1, \dots, n,\ R_{n+1} = \langle x_1 \dots x_n, y_1 \dots y_n\rangle^{F_{2n}}$. We prove the following isomorphisms of abelian groups:

$$\frac{R_1 \cap \dots \cap R_{n+1}}{R_1 \cap \dots \cap R_{n+1} \cap \gamma_{n+1}(F_{2n})} \simeq \gamma_{n+1}(F_2)/\gamma_{n+2}(F_2),\ \text{for } n \neq 4k-2,$$

$$\frac{R_1 \cap \dots \cap R_{n+1}}{R_1 \cap \dots \cap R_{n+1} \cap \gamma_{n+1}(F_{2n})} \simeq \gamma_{n+1}(F_2)/\gamma_{n+2}(F_2) \oplus \gamma_{2k}(F_2)/\gamma_{2k+1}(F_2),\ \text{for } n = 4k-2,$$

where $\{\gamma_i(F_{2n})\}_{i\geq 1}$ is the lower central series of F_{2n}. The 4-periodicity in the formulation of the above statement comes naturally from homotopy theory, namely from the description of torsion-free components of the homotopy groups of spheres.

1. Introduction

Given a free group F and normal subgroups $(n \geq 2)$

$$R_1, \dots, R_n \trianglelefteq F,$$

consider the quotient group

$$I_n(F, R_1, \dots, R_n) = \frac{R_1 \cap \dots \cap R_n}{\prod_{I\cup J=\{1,\dots,n\},\ I\cap J=\emptyset}[\bigcap_{i\in I} R_i, \bigcap_{j\in J} R_j]}.$$

Here $\bigcap$ denotes the intersection of subgroups in the free group F and $\prod$ is the product of commutator subgroups as indicated. Let $n \geq 1$, F_{2n} be a free group with basis $\{x_1, y_1, \dots, x_n, y_n\}$. Consider the following normal subgroups of F_{2n}:

$$R_i = \langle x_i, y_i\rangle^{F_{2n}},\ i = 1, \dots, n,$$
$$R_{n+1} = \langle x_1 \dots x_n, y_1 \dots y_n\rangle^{F_{2n}}.$$

It follows from [4] that, for $n \geq 1$, there is the following isomorphism of abelian groups:

$$\pi_{n+1}(S^2 \vee S^2) \simeq I_{n+1}(F_{2n}, R_1, \dots, R_{n+1}), \tag{1.1}$$

where $S^2 \vee S^2$ is the wedge of two 2-spheres. From the other hand, the Hilton-Milnor Theorem (see [3]) implies the following isomorphism:

$$\pi_{n+1}(S^2 \vee S^2) \simeq \bigoplus_{i=2}^{n+1} \pi_{n+1}(S^i)^{\oplus\tau(i-1)}, \tag{1.2}$$

where

$$\tau(i) = \frac{1}{i}\sum_{d|i} \mu(d)2^{i/d},$$

(the number of basic commutators of the fixed length in the free (super) Lie ring) $\mu(d)$ being the Möbius function. For example, the first nontrivial homotopy groups of $S^2\vee S^2$ are:

$$\begin{aligned}
&\pi_2(S^2\vee S^2)\simeq \pi_2(S^2)^{\oplus 2}\simeq \mathbb{Z}^{\oplus 2},\\
&\pi_3(S^2\vee S^2)\simeq \pi_3(S^2)^{\oplus 2}\oplus \pi_3(S^3)\simeq \mathbb{Z}^{\oplus 3},\\
&\pi_4(S^2\vee S^2)\simeq \pi_4(S^2)^{\oplus 2}\oplus \pi_4(S^3)\oplus \pi_4(S^4)^{\oplus 2}\simeq \mathbb{Z}_2^{\oplus 3}\oplus \mathbb{Z}^{\oplus 2},\\
&\dots .
\end{aligned}$$

The following problem rises naturally: *for every $i = 2,\dots,n+1$, identify the summand $\pi_{n+1}(S^i)^{\oplus\tau(i)}$ as a subgroup of $I_{n+1}(F_{2n},R_1,\dots,R_{n+1})$.* As a contribution to this problem, we analyze here the torsion-free part of the homotopy group $\pi_{n+1}(S^2\vee S^2)$ as a summand of $I_{n+1}(F_{2n},R_1,\dots,R_{n+1})$. As a natural group-theoretical application of this analysis, we have the following

Theorem 1. *There is a natural isomorphism of abelian groups*

$$\frac{R_1\cap\dots\cap R_{n+1}}{R_1\cap\dots\cap R_{n+1}\cap\gamma_{n+1}(F_{2n})}\simeq \gamma_{n+1}(F_2)/\gamma_{n+2}(F_2),\ \ \textit{for } n\neq 4k-2,$$
$$\frac{R_1\cap\dots\cap R_{n+1}}{R_1\cap\dots\cap R_{n+1}\cap\gamma_{n+1}(F_{2n})}\simeq \gamma_{n+1}(F_2)/\gamma_{n+2}(F_2)\oplus\gamma_{2k}(F_2)/\gamma_{2k+1}(F_2),\ \ \textit{for } n= 4k-2,$$

where $\{\gamma_i(F_{2n})\}_{i\geq 1}$ is the lower central series of F_{2n}.

The abelian groups in Theorem 1 are exactly the torsion-free parts of the homotopy groups $\pi_{n+1}(S^2\vee S^2)$. The *4-periodicity* from Theorem 1 is not very clear from the group-theoretical point of view, however comes naturally from the point of view of homotopy theory.

2. Simplicial groups and Hilton-Milnor Theorem

2.1. **Milnor's construction.** Recall that, for a given pointed simplicial set K, the $F[K]$-construction is the simplicial group with $F[K]_n = F(K_n\setminus *)$, where $F(-)$ is the free group functor. Then there is a weak homotopy equivalence

$$|F[K]|\simeq \Omega\Sigma|K|.$$

For the n-sphere, the simplicial group $F[S^n]$ can be defined as a certain simplicial group with $F[S^n]_k=\{1\},\ k\leq n-1$ and $F[S^n]_{n+k}$ is a free group of rank $\binom{n+k}{k}$ for $k\geq 0$. Furthermore, it is shown in [5] that for every simplicial group G with $G_k=\{1\},\ k\leq n-1$ and G_{n+k} a free group of rank $\binom{m+k}{k}$, $k\geq 0$, there is a simplicial monomorphism $F[S^n]\to G$, which induces the homotopy equivalence and an isomorphism of their nilpotent completions. We will use the standard notation $K(\mathbb{Z}\oplus\mathbb{Z},1)$ for the abelianization of $F[S^1\vee S^1]$.

The following result due to Curtis (see [1]) is the main ingredient in the proof of Theorem 1:

Theorem 2. *Let F be a connected simplicial group or Lie algebra (over $\mathbb{Z}$) then $\gamma_r(F)$ is $log_2 r$-connected.*

2.2. **Samuelson product.** The Whitehead product can be described with the help of simplicial language. Let G be a simplicial group.

For $p, q \geq 1$, let

$$(a; b) = (a_1, \dots, a_p; b_1, \dots, b_q)$$

be a permutation of $(0, \dots, p+q-1)$, such that $a_1 < \dots < a_p,\ b_1 < \dots < b_q$. We will refer to such $(a; b)$ as a $(p; q)$-shuffle. Denote by $sign(a; b)$ the sign of the permutation $(a; b)$.

For $x \in G_p, y \in G_q$ define the bracket

$$\langle x, y \rangle := \prod_{(a;b)} [s_b x, s_a y]^{sign(a;b)}, \tag{2.1}$$

where the product is taken over the set of all (p, q)-shuffles $(a; b)$ and

$$s_b = s_{b_q} \dots s_{b_1},\ \ s_a = s_{a_p} \dots s_{a_1}.$$

It is easy to show (see, for example, [1]), that the definition of the bracket $\langle , \rangle$ can be extended to the homotopy classes of G, i.e. the product 2.1 induces the product

$$\langle x, y \rangle \in \pi_{p+q}(G),\ \ x \in \pi_p(G),\ \ y \in \pi_q(G),$$

called the *Samuelson product* in G. Now, for a given topological space (simplicial set) X, the Whitehead product

$$[u, v] \in \pi_{p+q+1}(X),\ \ u \in \pi_{p+1}(X),\ \ v \in \pi_{q+1}(X),$$

can be defined as

$$[u, v] = (-1)^p \partial^{-1} \langle \partial u, \partial v \rangle,$$

where $\partial : \pi_*(X) \to \pi_*(GX)$ and GX is the Kan's loop construction of X.

For every $i = 2, \dots, n+1$, we have $\tau(i)$ maps

$$c_j : \Omega\Sigma(S^1 \wedge \dots \wedge S^1) \to \Omega\Sigma(S^1 \vee S^1),\ \ j = 1, \dots, \tau(i)$$

indexed by basic commutators of weight i in two variables. Hilton-Milnor Theorem implies that for every such map and $k \geq 2$, the induced homomorphism

$$c_j^* : \pi_k(\Omega\Sigma(S^1 \wedge \dots \wedge S^1)) = \pi_{k+1}(S^i) \to \pi_k(\Omega\Sigma(S^1 \vee S^1)) = \pi_{k+1}(S^2 \vee S^2)$$

is a splitting monomorphism. Furthermore, all the homotopy classes of $\pi_*(S^2 \vee S^2)$ consist of the images of such maps. The maps c_j can be written simplicialy with the help of the Milnor $F[K]$-construction. The maps c_j can be written as certain simplicial maps

$$c_j : F[S^{i-1}] \to F[S^1 \vee S^1],$$

which induce monomorphisms of the homotopy groups. For the analog of Hilton-Milnor Theorem for simplicial algebras see [2].

3. Proof of Theorem 1

For a finitely generated abelian group A, let $\mathsf{TF}(A)$ be the torson-free subgroup of A. We will use the notation $\mathcal{L}^n$ and $T^n(A)$ for the n-th Lie functor and the n-th tensor power respectively. Since for every abelian group A, the composition

$$\mathcal{L}^n(A) \to T^n(A) \to \mathcal{L}^n(A)$$

is the same as n-multiplication, and $T^nK(\mathbb{Z}\oplus\mathbb{Z},1)$ is $K((\mathbb{Z}\oplus\mathbb{Z})^{\otimes n},n)$, we conclude that the group

$$\pi_n\mathcal{L}^sK(\mathbb{Z}\oplus\mathbb{Z},1)\otimes\mathbb{Q}=0,\ \ s>n.$$

Since

$$R_1\cap\dots\cap R_n\subseteq\gamma_n(F_{2n}), \tag{3.1}$$

we have

$$\pi_n\mathcal{L}^sK(\mathbb{Z}\oplus\mathbb{Z},1)=0,\ \ s<n.$$

Now Theorem 2 implies that

$$\mathsf{TF}(\pi_{n+1}(S^2\vee S^2))=\mathsf{TF}(\pi_nK(\mathbb{Z}\oplus\mathbb{Z},1))=\mathsf{TF}(\pi_n\mathcal{L}^nK(\mathbb{Z}\oplus\mathbb{Z},1)). \tag{3.2}$$

We know from the well-known Theorem due to Serre that $\mathsf{TF}(\pi_k(S^l))=\mathbb{Z}$ if $k=l$ or $l=2s,k=4s-1,\ s\geq 1$ and $\mathsf{TF}(\pi_k(S^l))=0$ otherwise. Therefore, the isomorphism (1.2) implies the following isomorphism of abelian groups

$$\mathsf{TF}(\pi_{n+1}(S^2\vee S^2))=\begin{cases}\pi_{n+1}(S^{n+1})^{\oplus\tau(n+1)}\oplus\mathsf{TF}((\pi_{n+1}(S^{2s})^{\oplus\tau(2s)})\simeq\mathbb{Z}^{\oplus(\tau(n+1)+\tau(2s))}, \\ \qquad\qquad\text{if } n=4s-2,\ s\geq 1\\ \pi_{n+1}(S^{n+1})^{\oplus\tau(n+1)}\simeq\mathbb{Z}^{\oplus\tau(n+1)}\ \text{otherwise.}\end{cases}$$

Since $\prod_{I\cup J=\{1,\dots,n+1\},\ I\cap J=\emptyset}[\bigcap_{i\in I}R_i,\bigcap_{j\in J}R_j]\subseteq\gamma_{n+1}(F_{2n})$, we have the following isomorphism

$$\pi_n\mathcal{L}^nL(\mathbb{Z}\oplus\mathbb{Z},1)\simeq\frac{R_1\cap\dots\cap R_{n+1}}{R_1\cap\dots\cap R_{n+1}\cap\gamma_{n+1}(F_{2n})}. \tag{3.3}$$

The inclusion (3.1) implies that the right-hand group in (3.3) is a subgroup in $\gamma_n(F_{2n})/\gamma_{n+1}(F_{2n})$ and, therefore, is torsion-free. The isomorpisms (3.3) and (3.2) imply the following isomorphism:

$$\mathsf{TF}(\pi_{n+1}(S^2\vee S^2))\simeq\frac{R_1\cap\dots\cap R_{n+1}}{R_1\cap\dots\cap R_{n+1}\cap\gamma_{n+1}(F_{2n})} \tag{3.4}$$

and the Theorem 1 is proved.

Remark. It is easy to see that the same proof works for the case of the wedge of r 2-spheres $\bigvee_{i=1}^r S^2,\ r\geq 2$. For every $n\geq 3$, we have the following isomorphisms of abelian groups:

$$\mathsf{TF}(\pi_{n+1}(\bigvee_{i=1}^r S^2))\simeq\frac{R_1\cap\dots\cap R_{n+1}}{R_1\cap\dots\cap R_{n+1}\cap\gamma_{n+1}(F_{rn})}\simeq\gamma_n(F_r)/\gamma_{n+1}(F_r),\ \ n\neq 4k-2,$$

$$\mathsf{TF}(\pi_{n+1}(\bigvee_{i=1}^r S^2))\simeq\frac{R_1\cap\dots\cap R_{n+1}}{R_1\cap\dots\cap R_{n+1}\cap\gamma_{n+1}(F_{rn})}\simeq$$
$$\gamma_n(F_r)/\gamma_{n+1}(F_r)\oplus\gamma_{2k}(F_r)/\gamma_{2k+1}(F_r),\ \ n=4k-2,$$

where F_{nr} is a free group with generator set $\{x_1^{(1)},\dots,x_1^{(r)},\dots,x_n^{(1)},\dots,x_n^{(r)}\}$ with normal subgroups $R_i=\langle x_i^{(1)},\dots,x_i^{(r)}\rangle^{F_{rn}},\ R_{n+1}=\langle x_1^{(1)}\dots x_n^{(1)},\dots,x_1^{(r)}\dots x_n^{(r)}\rangle^{F_{rn}}$.

Group-theoretical description of the isomorphism (3.4). The isomorphisms of abelian groups (3.4), can be defined combinatorially using the simplicial language. Define the monomorphism

$$\mathbb{Z}^{\oplus\tau(n)} \hookrightarrow \frac{R_1 \cap \dots \cap R_{n+1}}{R_1 \cap \dots \cap R_{n+1} \cap \gamma_{n+1}(F_{2n})}, \ n \geq 3. \tag{3.5}$$

Let $n \geq 3$. Consider $F[S^{i-1}]$ for $i = 2, \dots, n+1$. The element $\sigma \in \pi_{n+1}(S^i)$ defines a certain element $e(\sigma) \in F[S^{i-1}]_n/\mathcal{B}_n(S^{i-1})$. Consider the simplicial group $F[S^1 \vee S^1]$ with $F[S^1 \vee S^1]_1$ a free group with generators x_1, x_2. For $k \geq 2$, define

$$\hat{s}_j = s_k \dots s_{j+1} s_{j-1} \dots s_0 : F[S^1 \vee S^1]_1 \to F[S^1 \vee S^1]_k, \ j = 0, \dots, k$$

For example, $\hat{s}_0 = s_2 s_1, \ \hat{s}_1 = s_2 s_0, \ \hat{s}_2 = s_1 s_0$, for $k = 3$. The set of homomorphisms $\{\hat{s}_0, \dots, \hat{s}_k\}$ can be naturally ordered as follows:

$$\hat{s}_0 < \hat{s}_1 < \dots < \hat{s}_k.$$

Let c be a basic commutator of weight n in two symbols x_1, x_2. That is, $c/\gamma_{n+1}F_2$ is an element from a Hall basis of $\gamma_n(F_2)/\gamma_{n+1}(F_2)$. Write symbolically c as

$$c = [x_{j_1}, \dots, x_{j_n}], \ j_l = 1, 2.$$

(we remember that the configuration of brackets in c can be non-left-orientable). Define then the element

$$\hat{c} = \prod_{(i_1,\dots,i_n)\in S_n} [\hat{s}_{i_1} x_{j_1}, \dots, \hat{s}_{i_n} x_{j_n}]^{sign(i_1,\dots,i_n)} \in F[S^1 \vee S^1]_n.$$

For example, for $c = [x_1, x_2, x_1]$, we have

$$\hat{c} = [s_2 s_1 x_1, s_2 s_0 x_2, s_1 s_0 x_1][s_1 s_0 x_1, s_2 s_1 x_2, s_2 s_0 x_1][s_2 s_0 x_1, s_1 s_0 x_2, s_2 s_1 x_1]$$
$$[s_2 s_0 x_1, s_2 s_1 x_2, s_1 s_0 x_1]^{-1}[s_2 s_1 x_1, s_1 s_0 x_2, s_2 s_0 x_1]^{-1}[s_1 s_0 x_1, s_2 s_0 x_2, s_2 s_1 x_1]^{-1}.$$

Now consider the abelian group $\mathbb{Z}^{\oplus\tau(n)}$ as a quotient $\gamma_n(F_2)/\gamma_{n+1}(F_2)$ with a Hall basis $\{c_*\}_{*\in I}$. The definition of the Samuelson product implies that the j_{n+1}-image of the element c_* in (3.5) is the element $\hat{c}_*$ in $\frac{R_1 \cap \dots \cap R_{n+1}}{R_1 \cap \dots \cap R_{n+1} \cap \gamma_{n+1}(F_{2n})}$ (here the basis of F_{2n} is written as $\{\hat{s}_0 x_1, \hat{s}_0 x_2, \dots, \hat{s}_{n-1} x_0, \hat{s}_{n-1} x_1\}$).

Analogically one can define the monomorphism

$$\mathbb{Z}^{\oplus\tau(2s)} \hookrightarrow \frac{R_1 \cap \dots \cap R_{n+1}}{R_1 \cap \dots \cap R_{n+1} \cap \gamma_{n+1}(F_{2n})}, \ n = 4s - 2. \tag{3.6}$$

For $n = 4s - 2$, $F[S^{2s-1}]_{2s-1}$ an infinite cyclic group with generator σ. The element of infinite order in $\pi_{4s-1}(S^{2s})$ defines an element $\kappa_s \in F[S^{2s-1}]_{4s-2}/\mathcal{B}_{4s-2}$. The direct computations show that the element κ_s can be presented modulo $\mathcal{B}_{4s-2}$ as

$$\bar{\kappa}_s = \langle \sigma, \sigma \rangle = \prod_{(a;b)} [s_b \sigma, s_a \sigma]^{sign(a;b)},$$

where $(a; b)$ runs over the set of all $(2s-1; 2s-1)$-shuffles. Now the map (3.6) can be constructed with the help of $(2s - 1)$-Samuelson products and the structure of the element $\bar{\kappa}_s$.

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