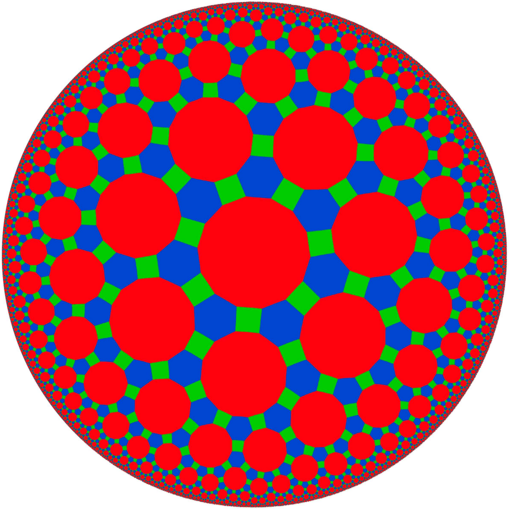
Throughout (W, S) will denote a Coxeter system:

$$\begin{aligned} W &= \langle s \in S \mid s^2 = 1, (st)^{m_{st}} = 1 \rangle \\ &= \langle s \in S \mid s^2 = 1, \underbrace{st\ldots}_{m_{st}\text{terms}} = \underbrace{ts\ldots}_{m_{st}\text{terms}} \rangle \end{aligned}$$

(where $m_{st} \in \{2, 3, \ldots, \infty\}$).

For example, we could take W to be a real reflection group...

...or the symmetries of this tesselation of the hyperbolic plane:



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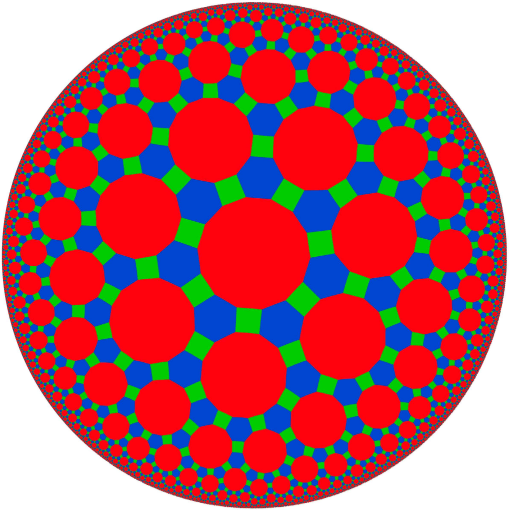

To a Coxeter system (W, S) one may associate a simplicial complex $|(W, S)|$ called the Coxeter complex of W.

Let $n = |S|$ denote the rank of W. Its construction is as follows:

- colour the n faces of the standard $n-1$-simplex by the set S,
- take one such simplex for each element $w \in W$ (from now on we will call these simplices *alcoves*).
- glue the alcove corresponding to w to that corresponding to ws along the wall coloured by s.

For example, consider $W = S_3$:

$$W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$

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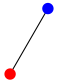

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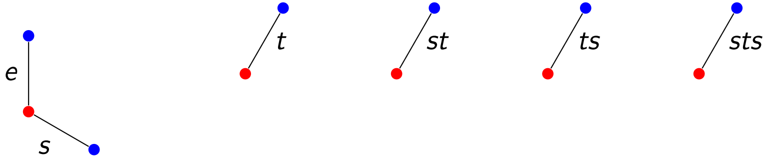

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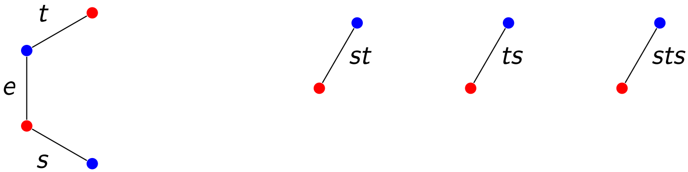

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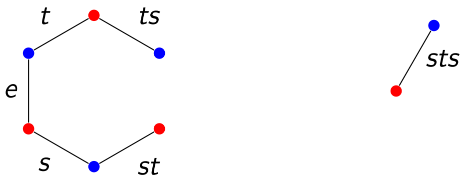

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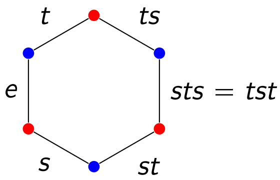
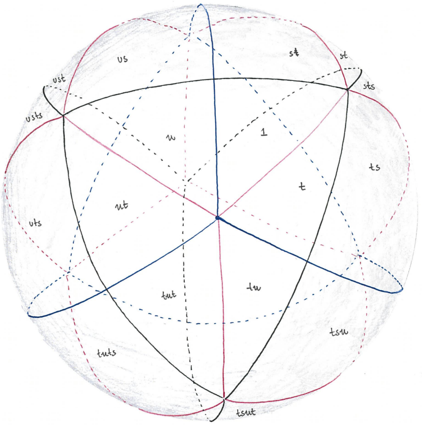
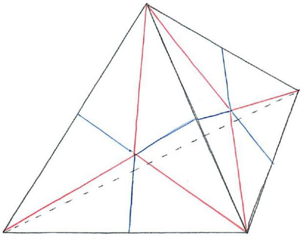
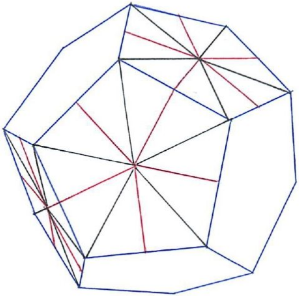


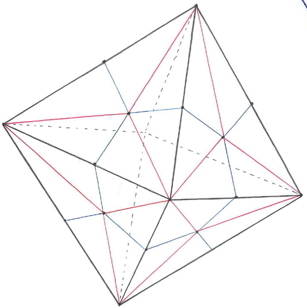
The Coxeter complex of $S_4 =$ • —— • —— • :



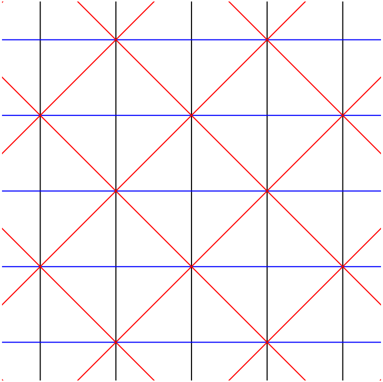
(barycentric subdivision of the tetrahedron).

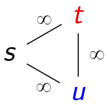






$$s \overset{4}{\text{———}} t \overset{4}{\text{———}} u$$





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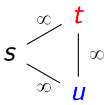

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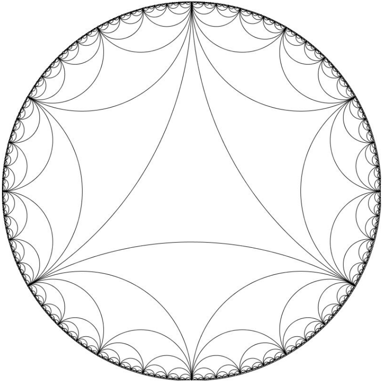

;

By construction $|(W, S)|$ has a left action of W.

W also acts on the alcoves of $|(W, S)|$ on the right by

$$\Delta_w \cdot s = \Delta_{ws}.$$

This action is *not* simplicial, but is "local": cross the wall coloured by s.

The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W. Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W. The product in W induces a multipliction in $\mathbb{Z}W$.

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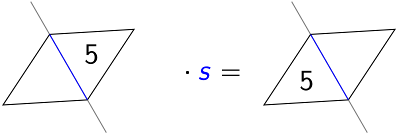
Hence we can picture an element of $\mathbb{Z}W$ as the assignment of integers to each alcove, such that only finitely many are non-zero. If we view $\mathbb{Z}W$ as a right module over itself it is easy to picture the action of the elements of S:



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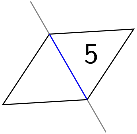
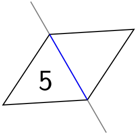
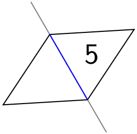


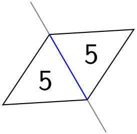
$\cdot\, s =$



Similarly ("s averaging operator")



$\cdot\, (1 + s) =$



Let $\ell : W \to \mathbb{N}$ denote the length function on W:

$$\begin{aligned}\ell(w) &= \text{length of a minimal expression for } w \text{ in the generators } s \\ &= \text{number of walls crossed in a minimal path } id \to w \text{ in } |(W, S)|.\end{aligned}$$

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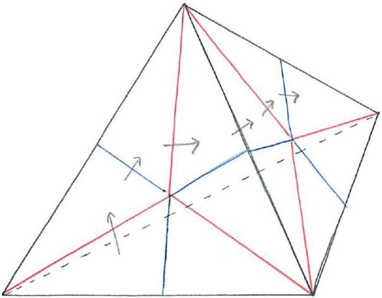

The Hecke algebra H is a quantization of $\mathbb{Z}W$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{H_x \mid x \in W\}$ parametrised by W. If we write $\underline{H}_s := H_s + vH_{id}$ then the multiplication in H is determined by

$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1} H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

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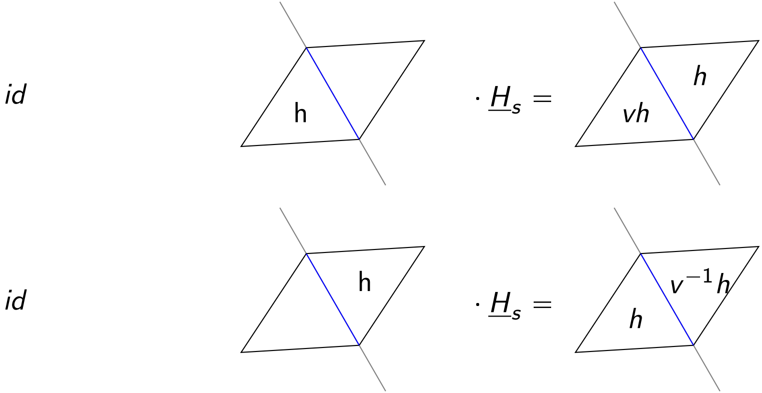
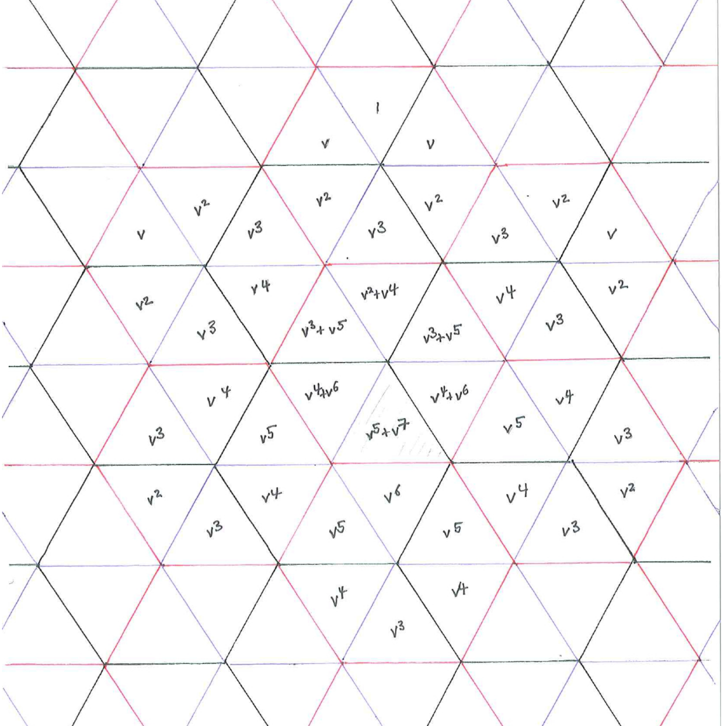
We can visualise this as follows: ("quantized averaging operator")



In 1979 Kazhdan and Lusztig defined a new basis for the Hecke algebra using the combinatorial structure of W. We denote this new basis by $\{\underline{H}_x \mid x \in W\}$. It satisfies

$$\underline{H}_x := H_x + \sum_{\substack{y \in W \\ \ell(y) < \ell(x)}} h_{y,x} H_y$$

with $h_{y,x} \in v\mathbb{Z}[v]$. These polynomials are the *Kazhdan-Lusztig polynomials*.


1
v v
v^2 v^2 v^2 v^2
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v^3 v^3+v^5 v^3+v^5 v^3
v^4 v^4+v^6 v^4+v^6 v^4
v^3 v^5 v^5+v^7 v^5 v^3
v^2 v^4 v^6 v^4 v^2
v^3 v^5 v^5 v^3
v^4 v^4
v^3

The definition is inductive. The first few Kazhdan-Lusztig basis elements are easily defined:

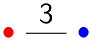
$$\underline{H}_{id} := H_{id}, \quad \underline{H}_s := H_s + vH_{id} \quad \text{for } s \in S.$$

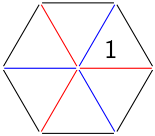
Now the work begins. Suppose that we have calculated $\underline{H}_y$ for all y with $\ell(y) \leqslant \ell(x)$. Choose $s \in S$ with $\ell(xs) > \ell(x)$ and write

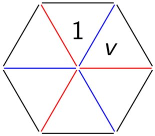
$$\underline{H}_x \underline{H}_s = H_{xs} + \sum_{\ell(y) < \ell(xs)} g_y H_y.$$

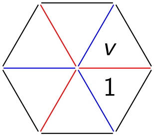
The formula for the action of $\underline{H}_s$ shows that $g_y \in \mathbb{Z}[v]$ for all $y < \ell(xs)$. If all $g_y \in v\mathbb{Z}[v]$ then $\underline{H}_{xs} := \underline{H}_x \underline{H}_s$. Otherwise we set

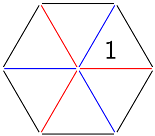
$$\underline{H}_{xs} = \underline{H}_x \underline{H}_s - \sum_{\substack{y \\ \ell(y) < \ell(x)}} g_y(0) \underline{H}_y.$$

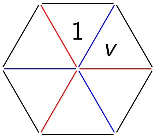


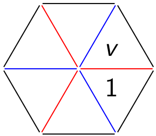
$\underline{H}_{id} =$ 

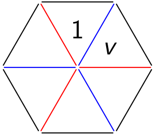
$\underline{H}_{t} =$ 

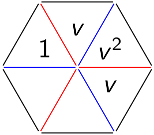
$\underline{H}_{s} =$ 

$\underline{H}_{id} =$ 

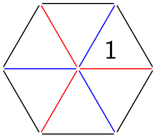
$\underline{H}_{t} =$ 

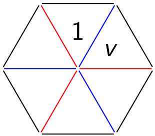
$\underline{H}_{s} =$ 

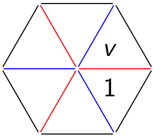
$\underline{H}_{t}\underline{H}_{s} =$ 

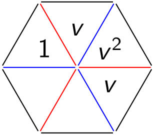
$\cdot\, \underline{H}_{s} =$ 

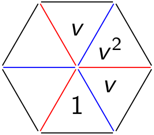
$= \underline{H}_{ts}$

$\underline{H}_{id} =$ 

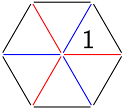
$\underline{H}_{t} =$ 

$\underline{H}_{s} =$ 

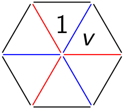
$\underline{H}_{ts} =$ 

$\underline{H}_{st} =$ 

$\underline{H}_{id} =$



$\underline{H}_{t} =$



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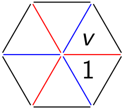

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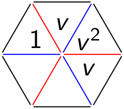

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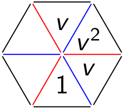
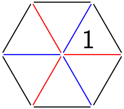
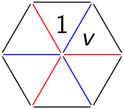
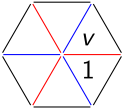
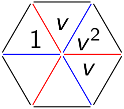
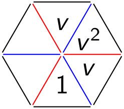


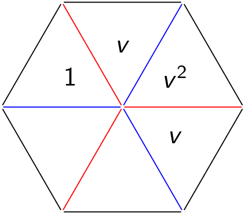
$\underline{H}_{id} =$ 

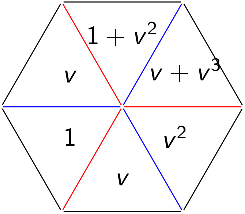
$\underline{H}_{t} =$ 

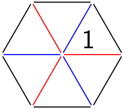
$\underline{H}_{s} =$ 

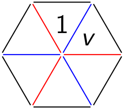
$\underline{H}_{ts} =$ 

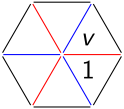
$\underline{H}_{st} =$ 

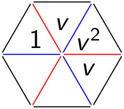
$\underline{H}_{ts}\underline{H}_{t} =$ 

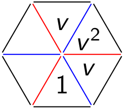
$\cdot \underline{H}_{t} =$ 

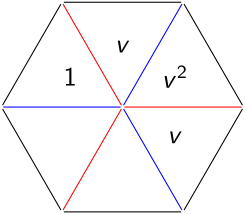
$\underline{H}_{id} =$ 

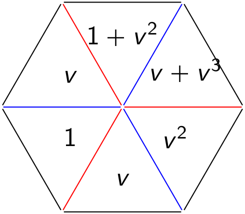
$\underline{H}_{t} =$ 

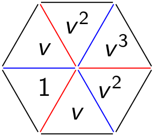
$\underline{H}_{s} =$ 

$\underline{H}_{ts} =$ 

$\underline{H}_{st} =$ 

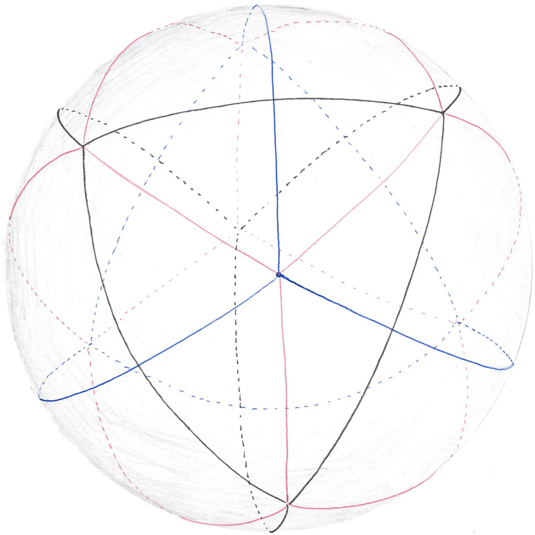
$\underline{H}_{ts}\underline{H}_{t} =$ 

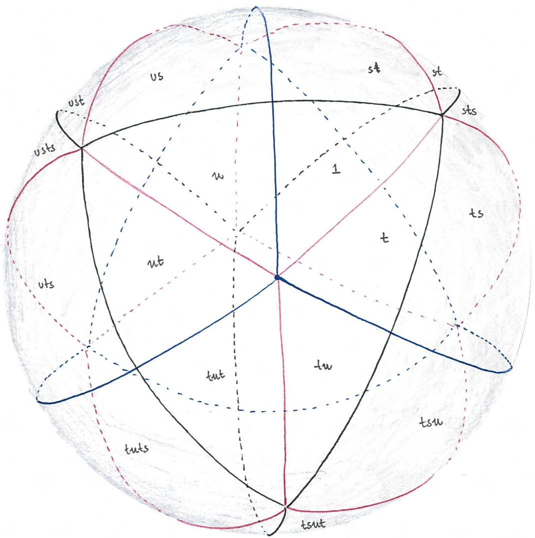
$\cdot \underline{H}_{t} =$ 

Hence: $\underline{H}_{tst} = \underline{H}_{ts}\underline{H}_{t} - \underline{H}_{t} =$ 

For dihedral groups (rank 2) we always have $h_{y,x} = v^{\ell(x)-\ell(y)}$ (Kazhdan-Lusztig basis elements are *smooth*.)

However in higher rank the situation quickly becomes more interesting...

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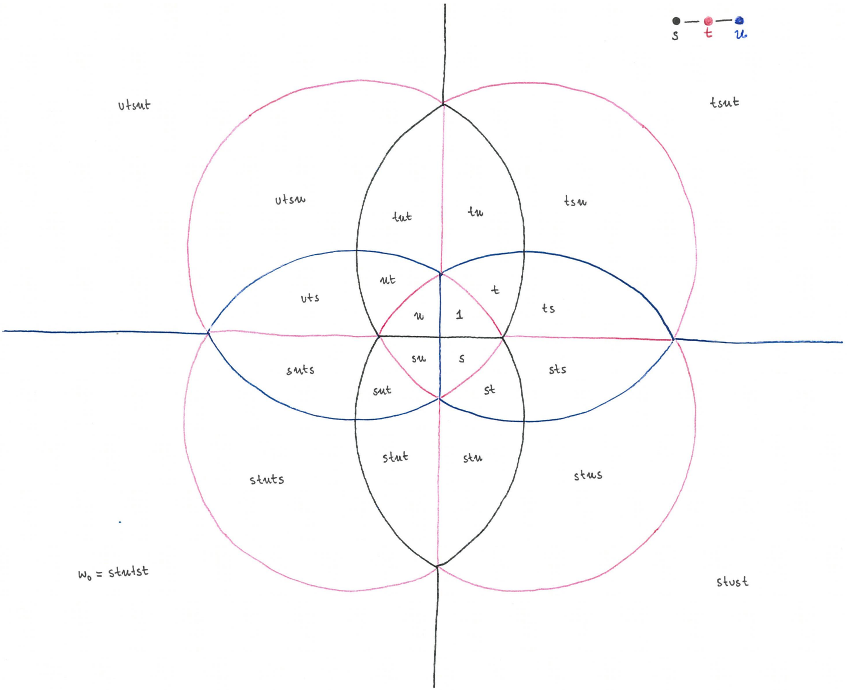
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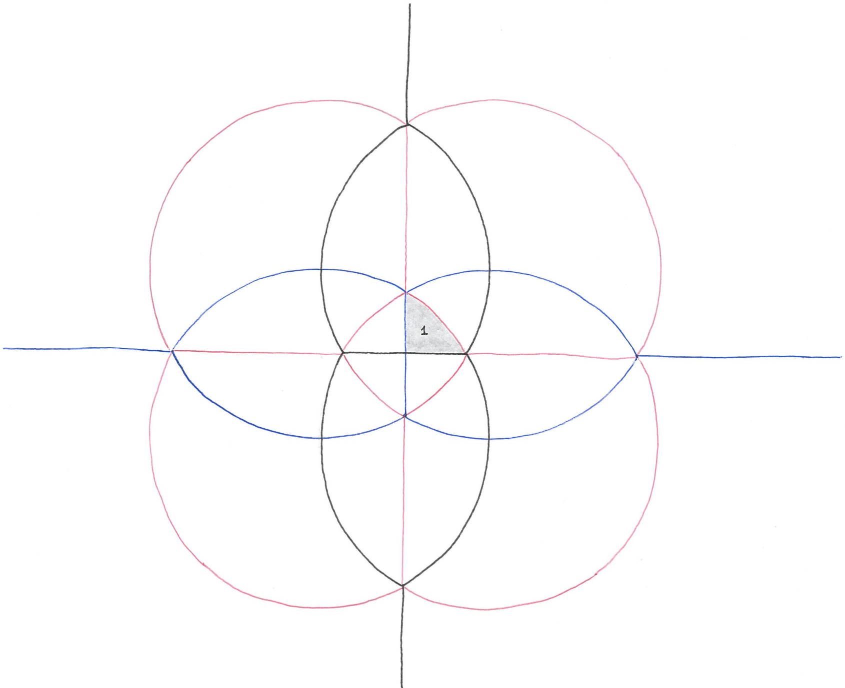
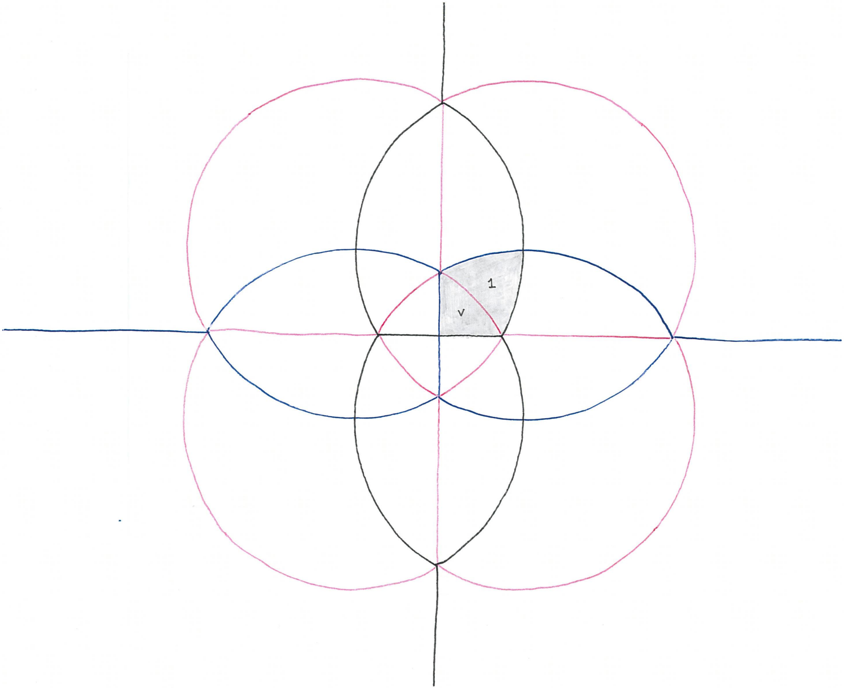
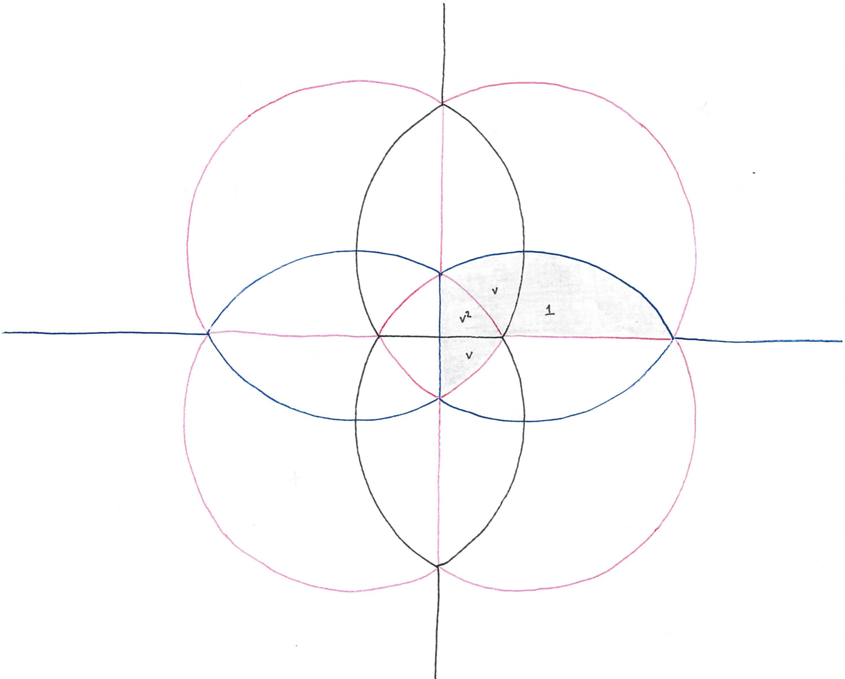
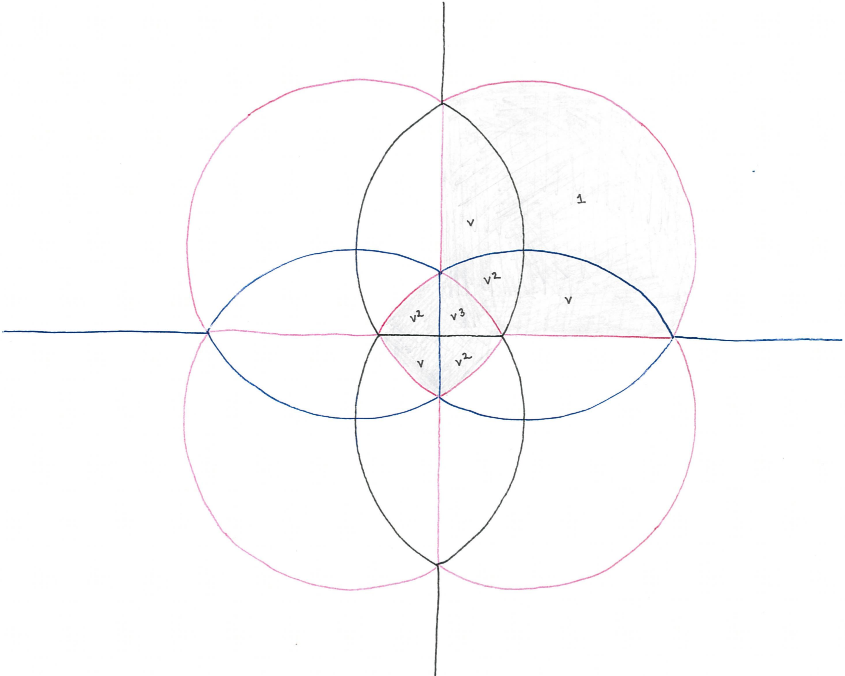
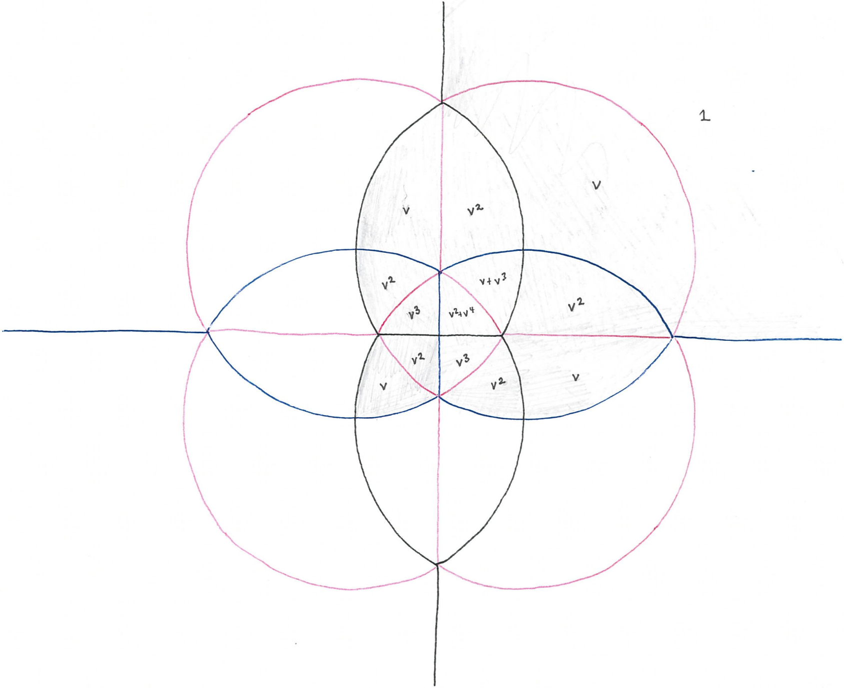
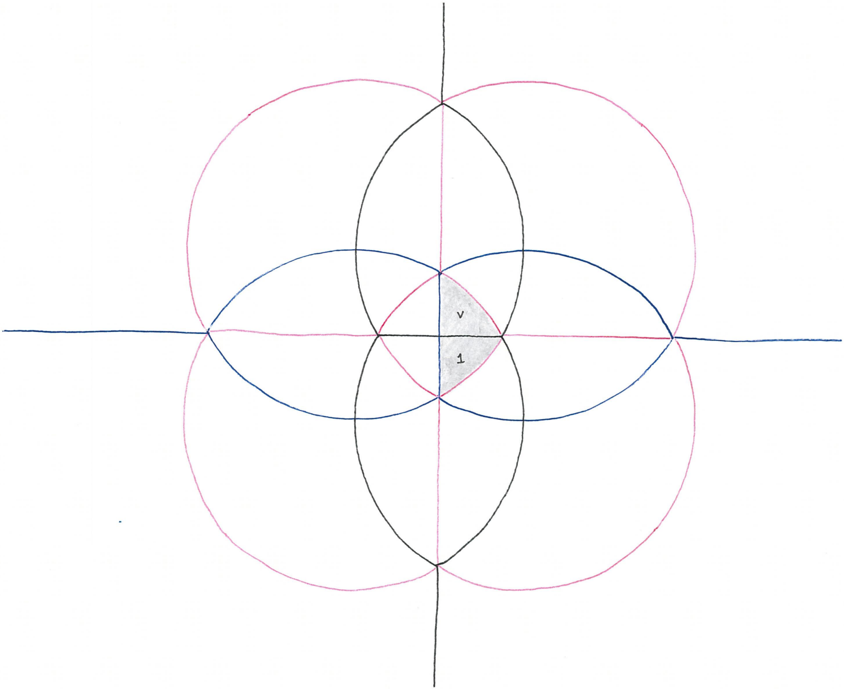

1

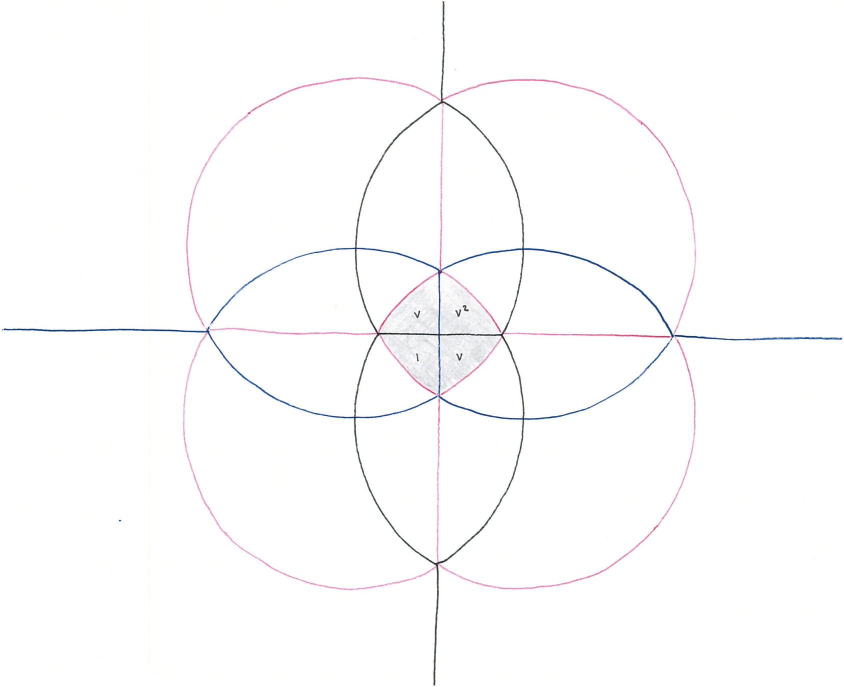

1
v


v
v²
1
v

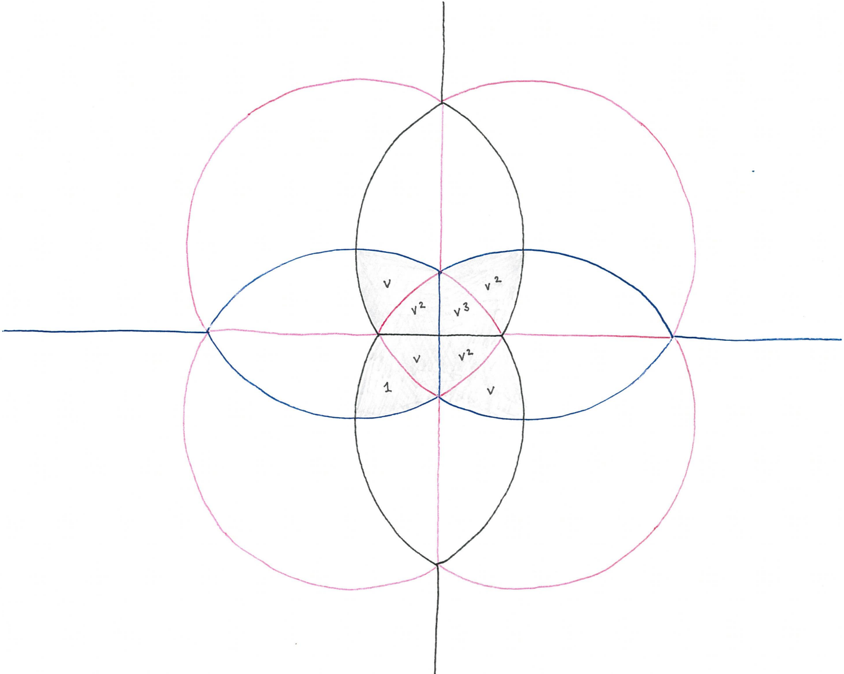

1
v
v2
v
v2
v3
v
v2


1
v
v2
v
v2
v+v3
v3
v2+v4
v2
v2
v3
v
v2
v

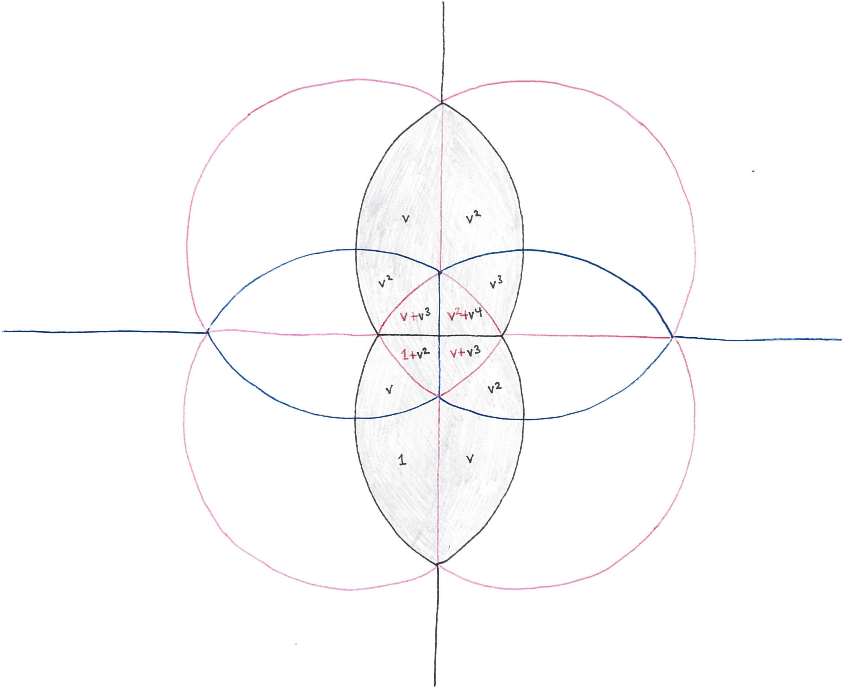

v
1

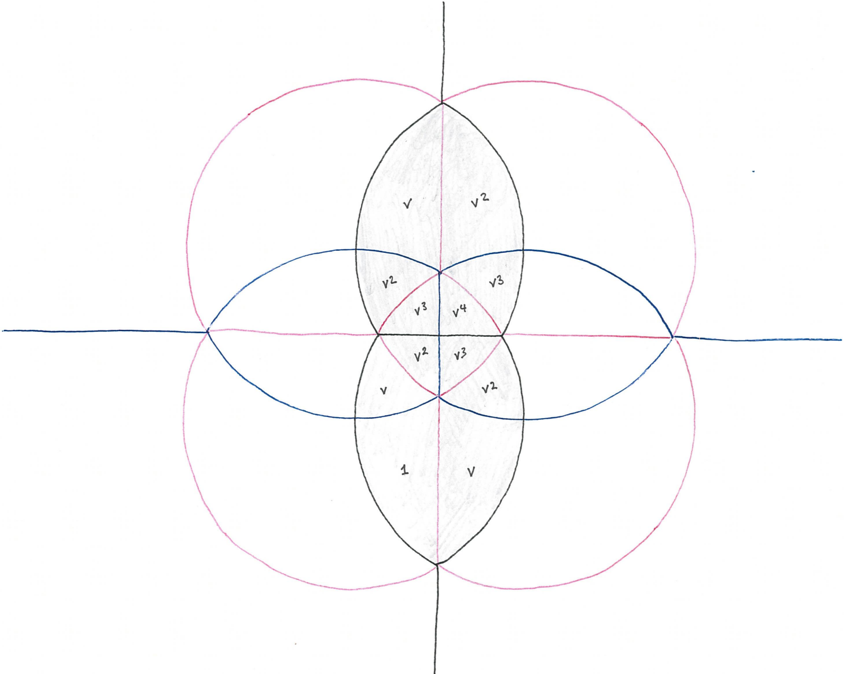


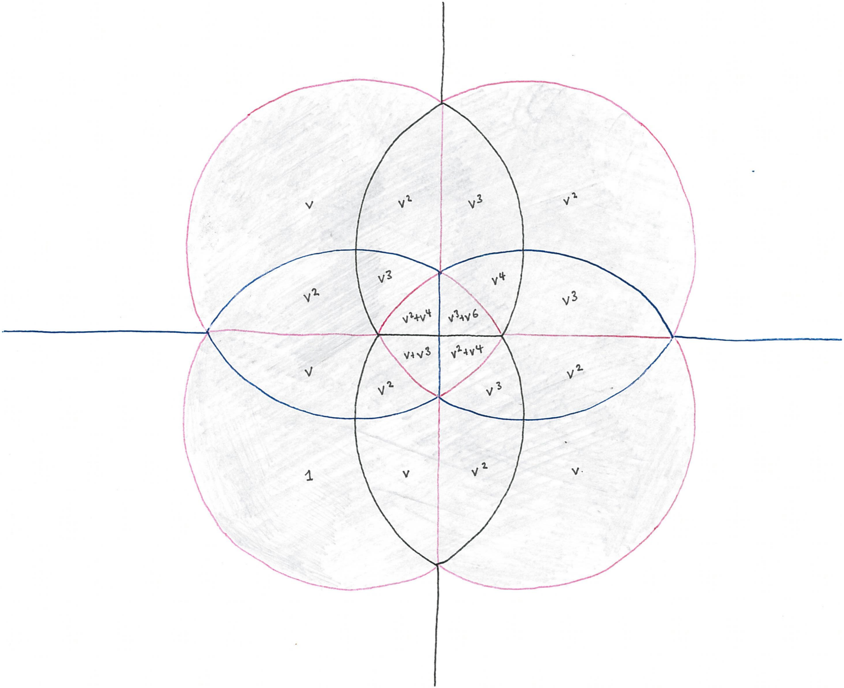
v
v²
1
v



v
v2
v2
v3
v
v2
1
v


v
v²
v²
v³
v+v³
v²+v⁴
1+v²
v+v³
v
v²
1
v


v
v2
v2
v3
v3
v4
v2
v3
v
v2
1
v


v
v^2
v^3
v^2
v^2
v3
v4
v3
v^2+v^4
v^3+v^6
v+v3
v^2+v4
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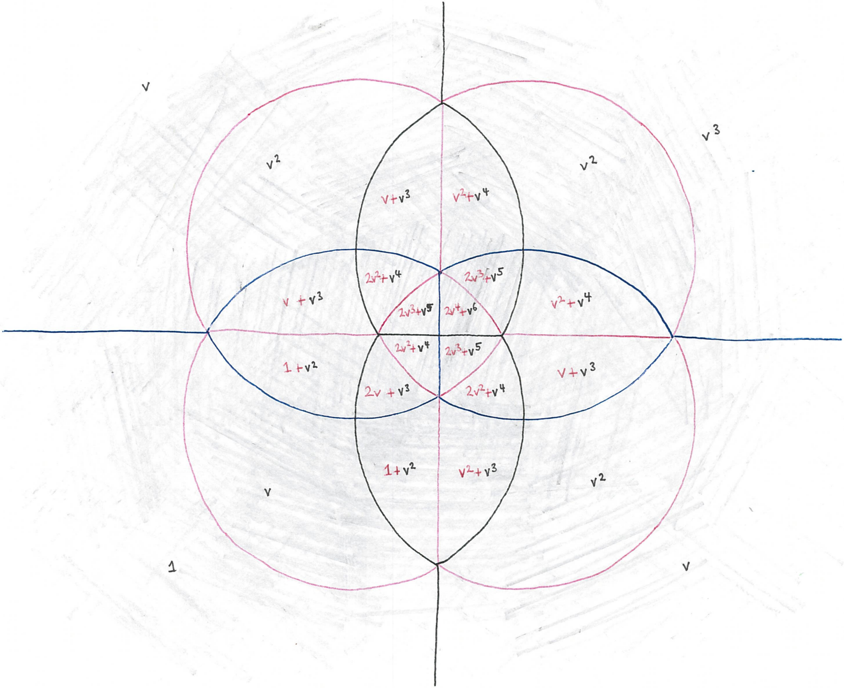
v^2
v3
v^2
1
v
v^2
v.



v
v2
v3
v2
v+v3
v2+v4
2v2+v4
2v3+v5
v+v3
2v3+v5
2v4+v6
v2+v4
2v2+v4
2v3+v5
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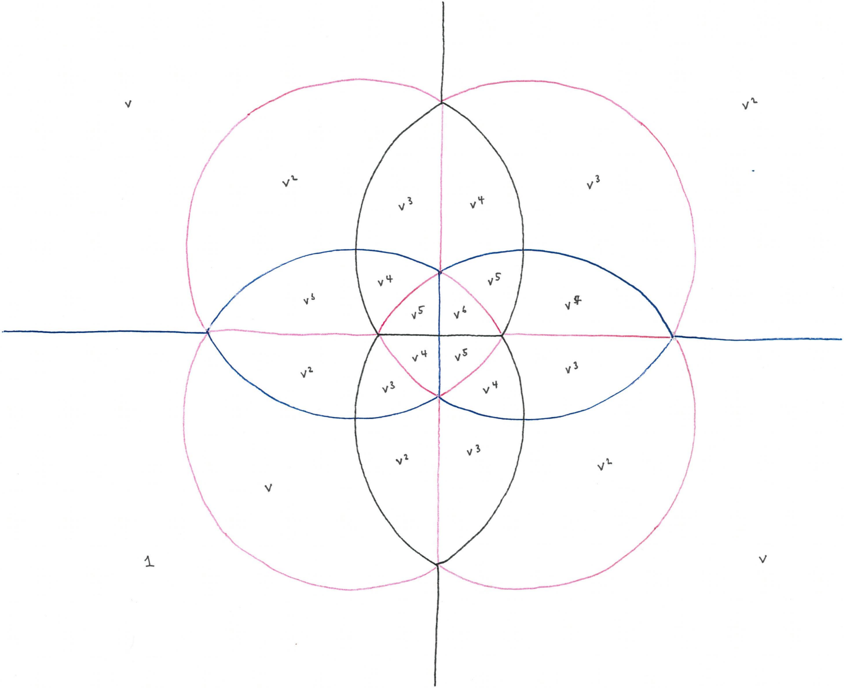
v+v3
2v+v3
2v2+v4
1+v2
v2+v3
v
v2
1
v


v
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v2
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v3
v4
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v5
v5
v5
v6
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v3
v2
v
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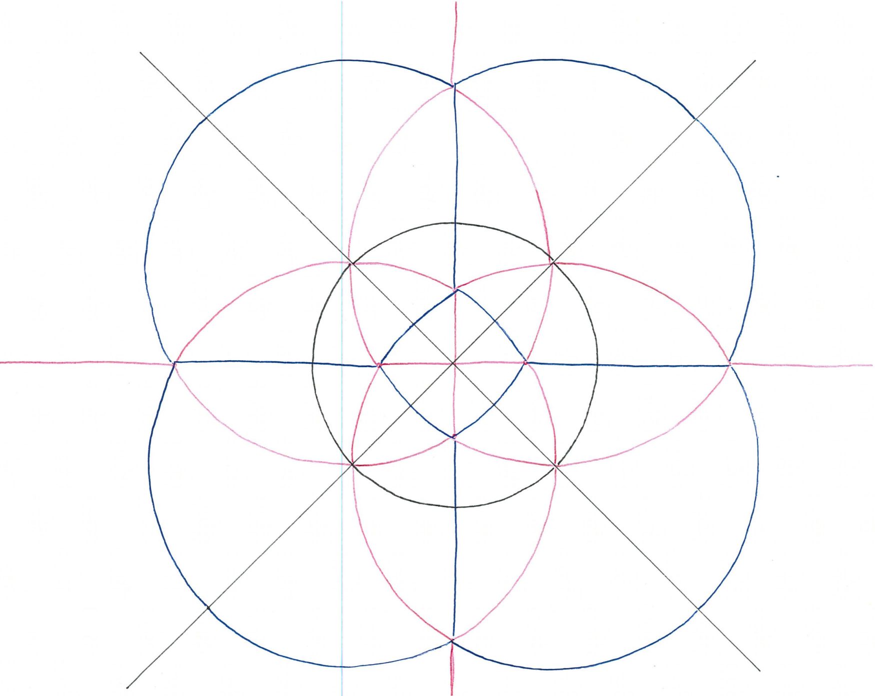
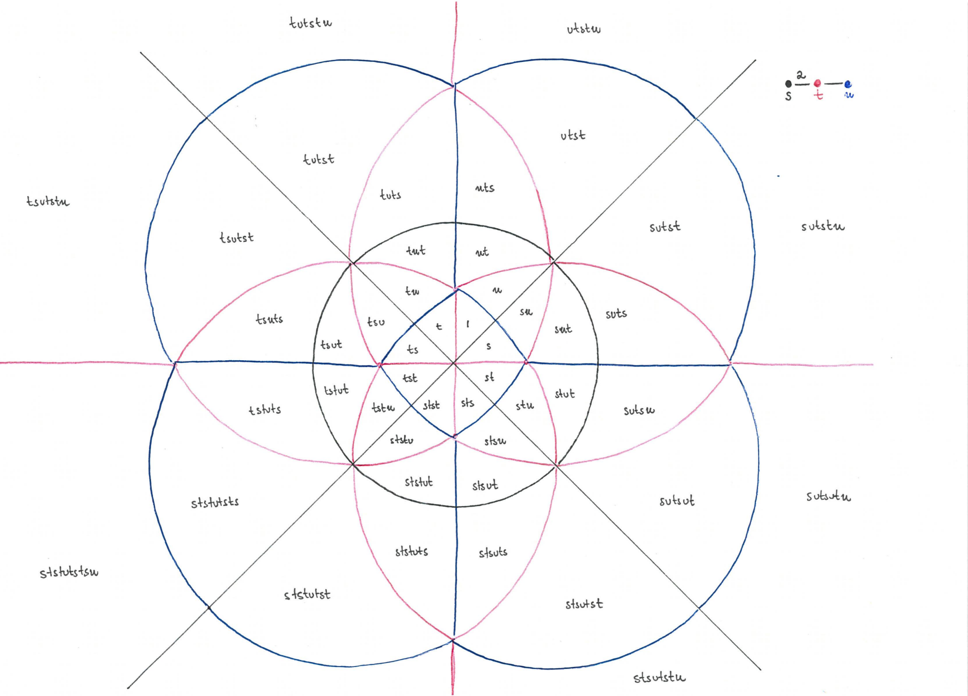
v





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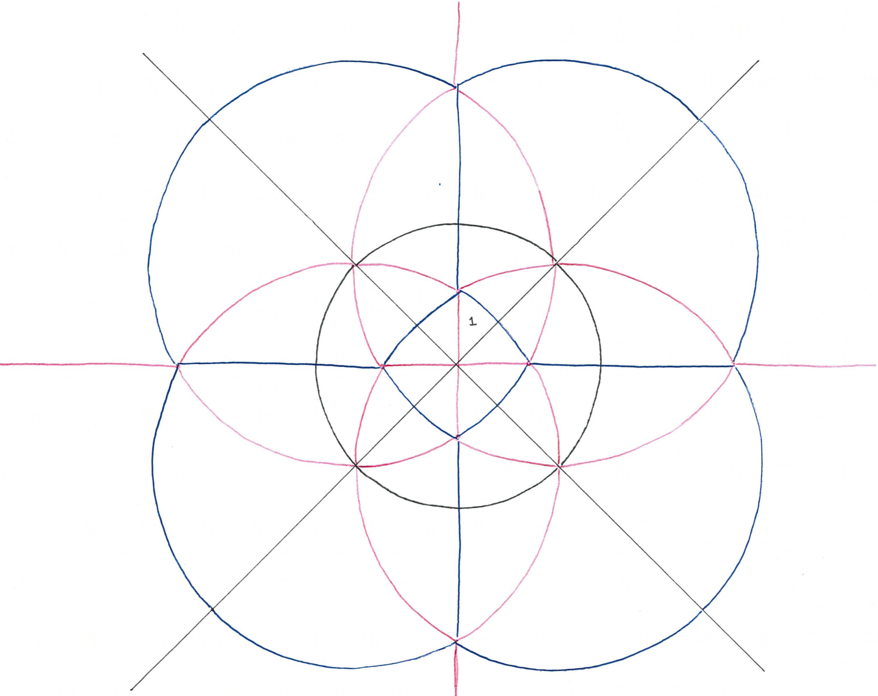
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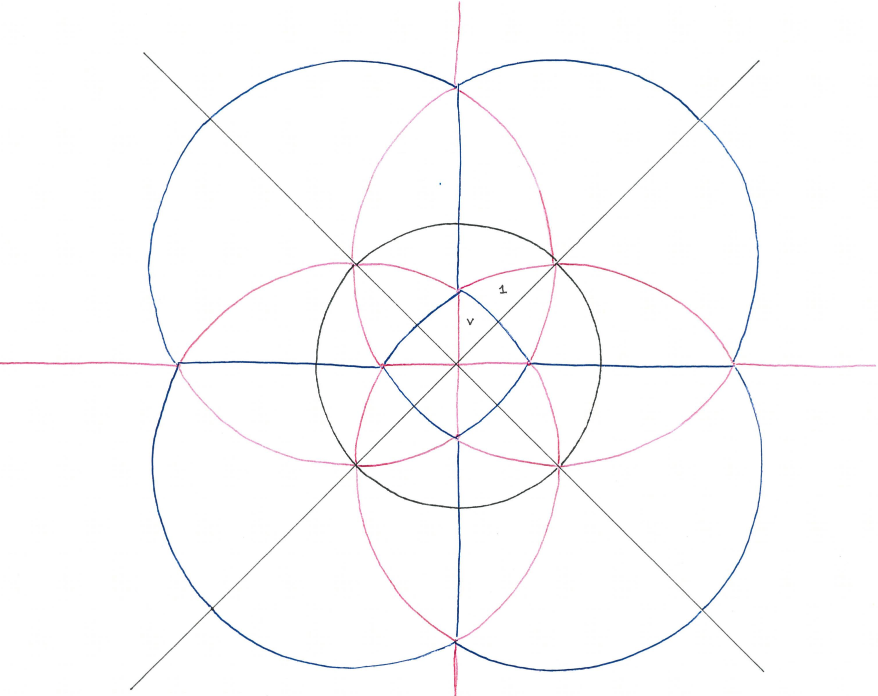
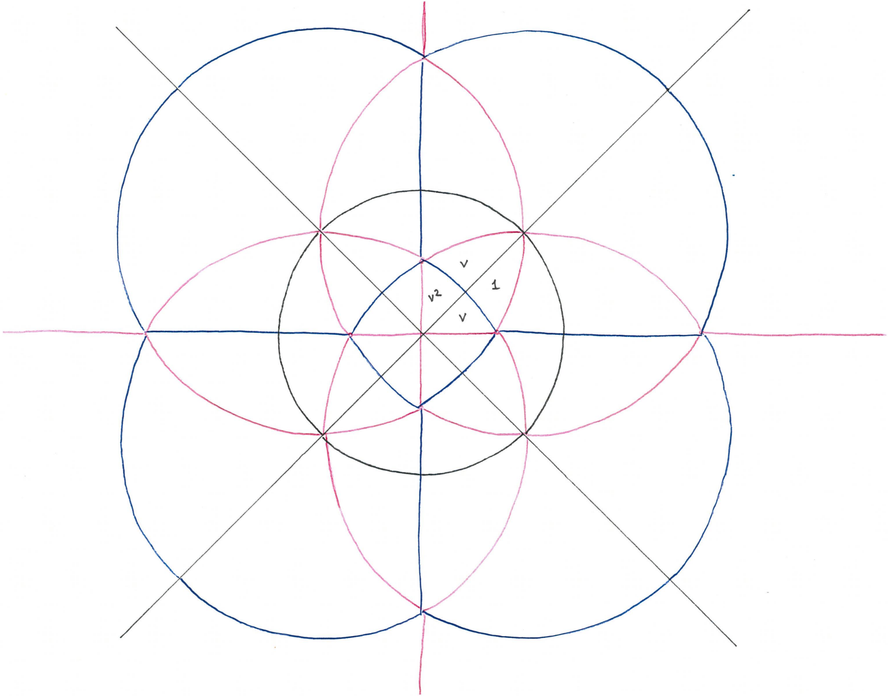
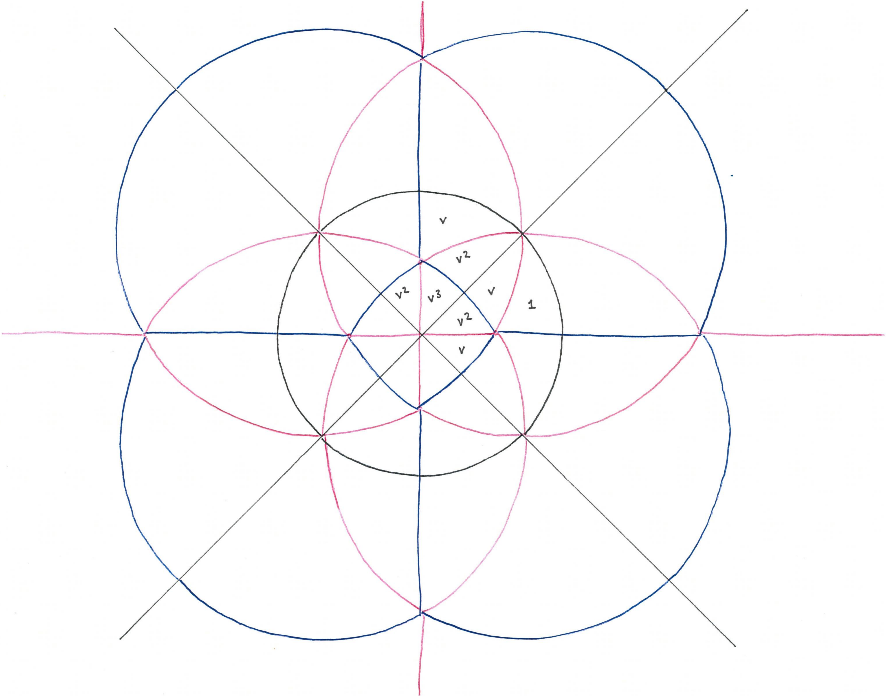
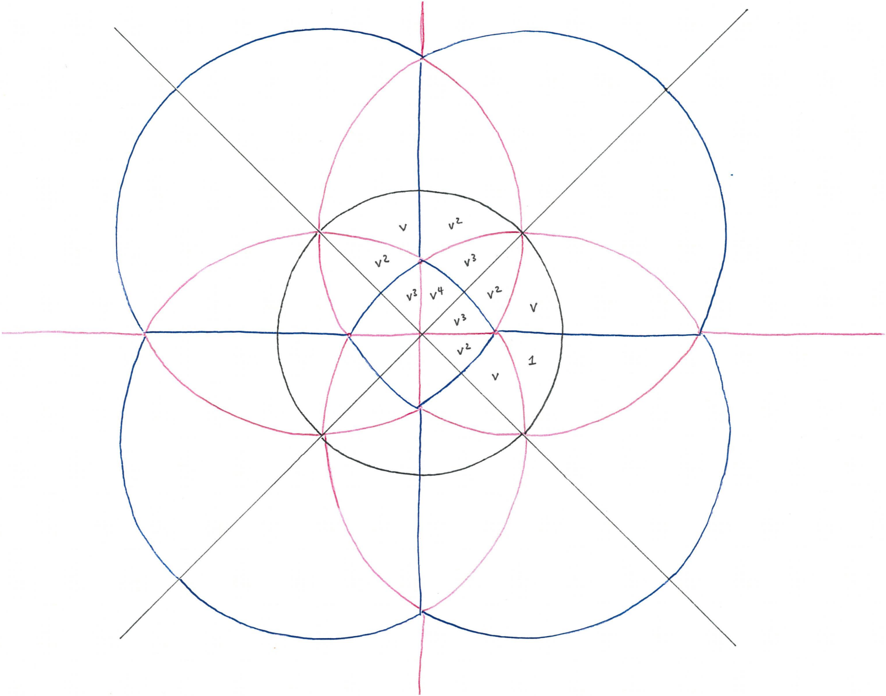
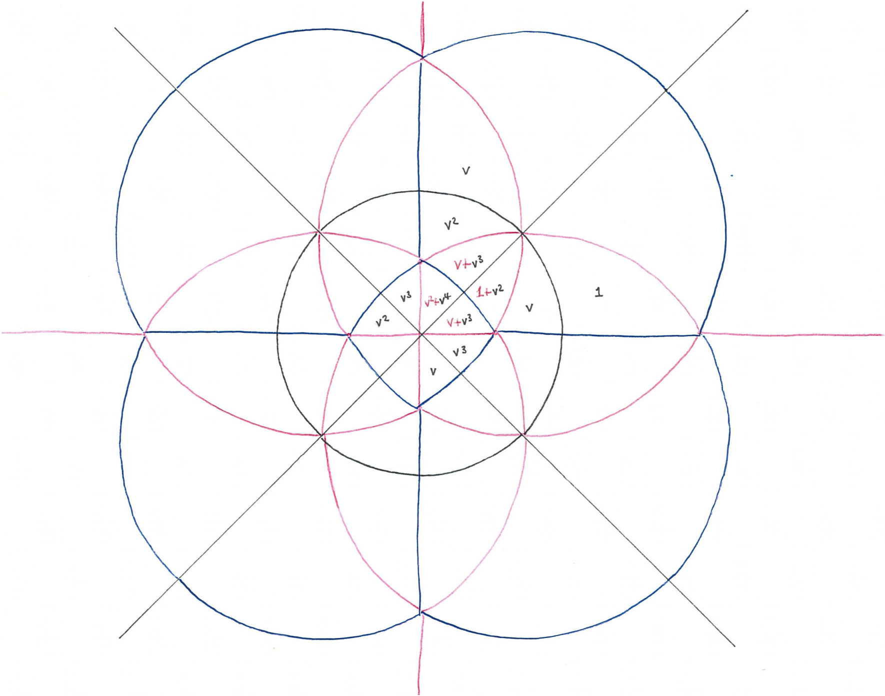
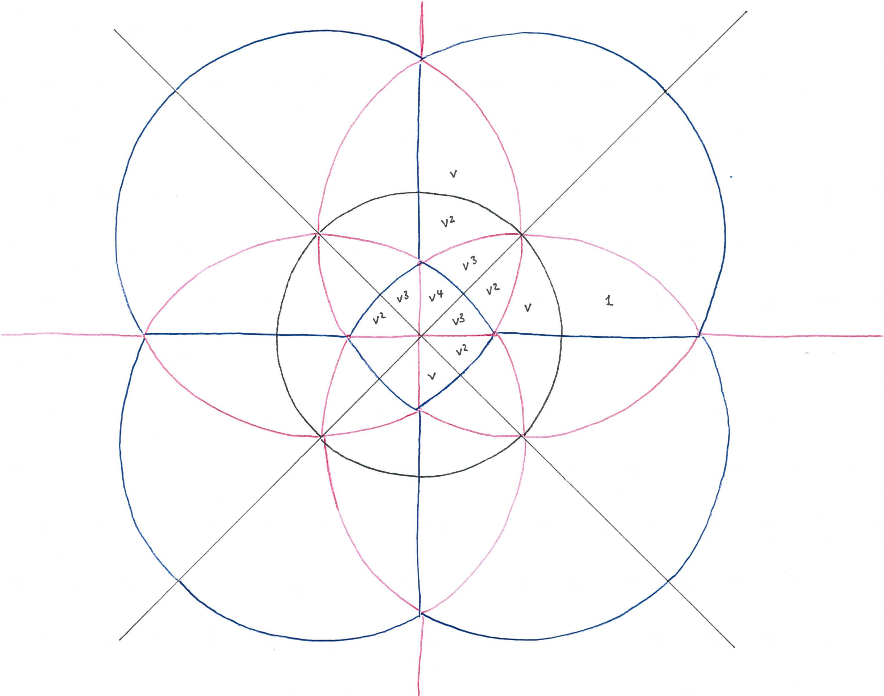

1
v


v
1
v2
v

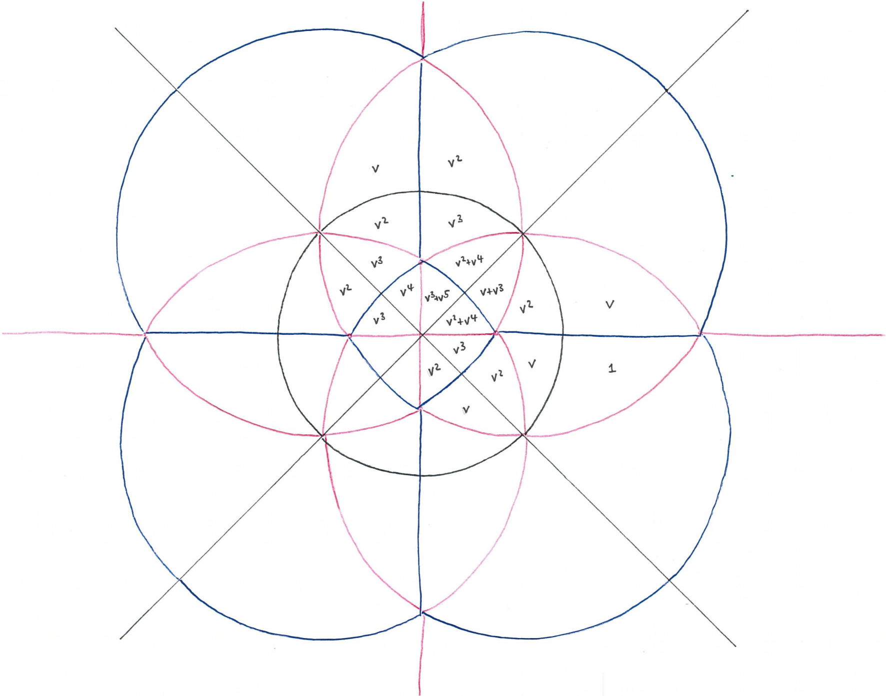

v
v2
v2
v3
v
1
v2
v

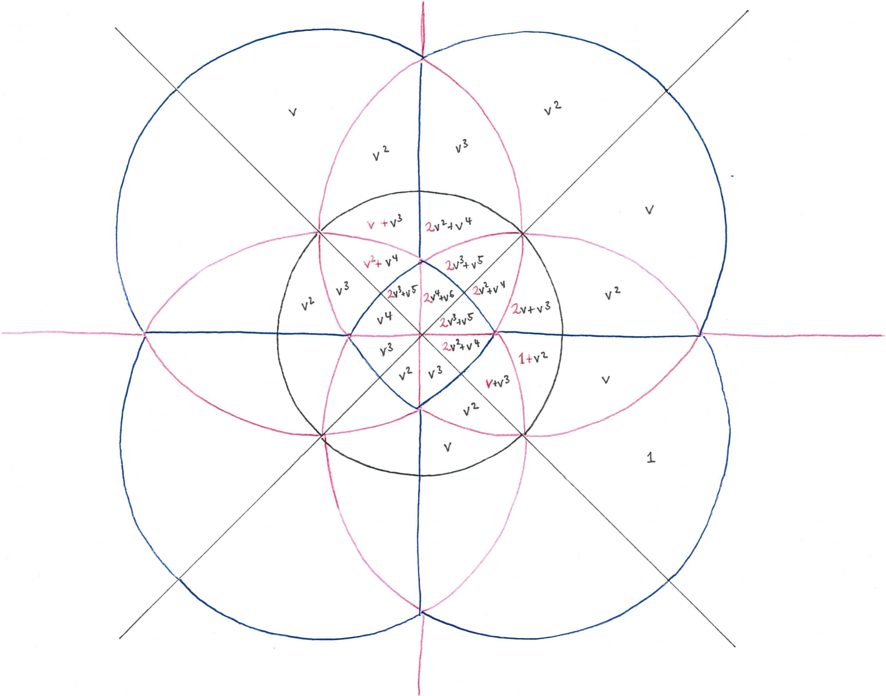

v
v2
v2
v3
v3
v4
v2
v
v3
v2
1
v


v
v^2
$v+v^3$
v^3
v^2+v^4
$1+v^2$
v
1
v^2
$v+v^3$
$v3$
v

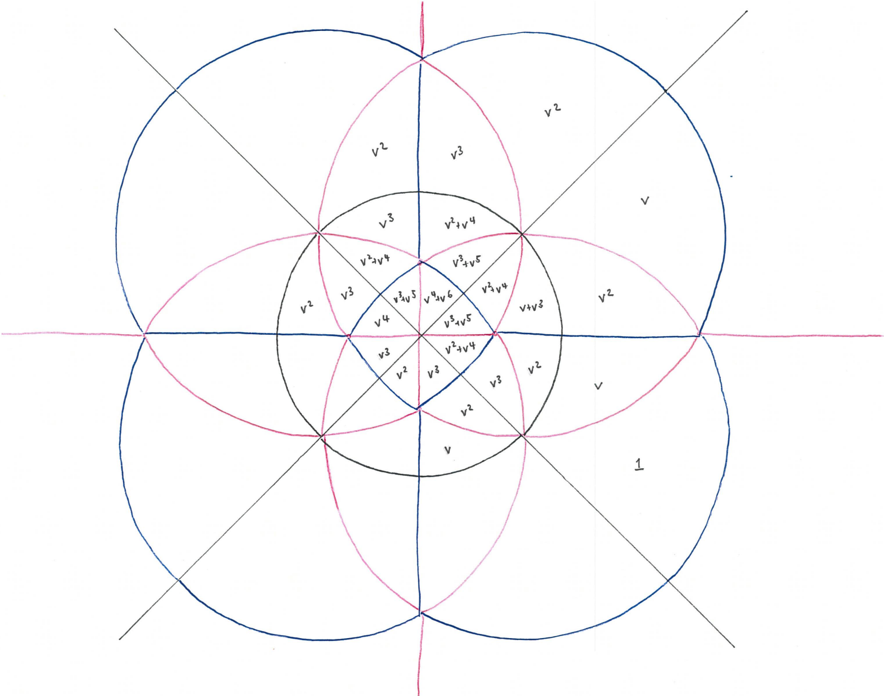


v
v2
v3
v2
1
v3
v4
v2
v
v2
v3
v2
v


v
v²
v²
v³
v³
v²+v⁴
v²
v⁴
v³+v⁵
v+v³
v²
v
v³
v²+v⁴
v³
v
v²
v²
1
v



v
v^2
v^3
v^2
v
$v+v^3$
$2v^2+v^4$
v^2+v^4
$2v^3+v^5$
v^2
v^3
$2v^3+v^5$
$2v^4+v^6$
$2v^2+v^4$
$2v+v^3$
v^2
v^4
$2v^3+v^5$
v^3
$2v^2+v^4$
$1+v^2$
v^2
v^3
$v+v^3$
v
v^2
v
1



v2
v3
v2
v
v3
v2+v4
v2+v4
v3+v5
v2
v3
v3+v5
v4+v6
v3+v4
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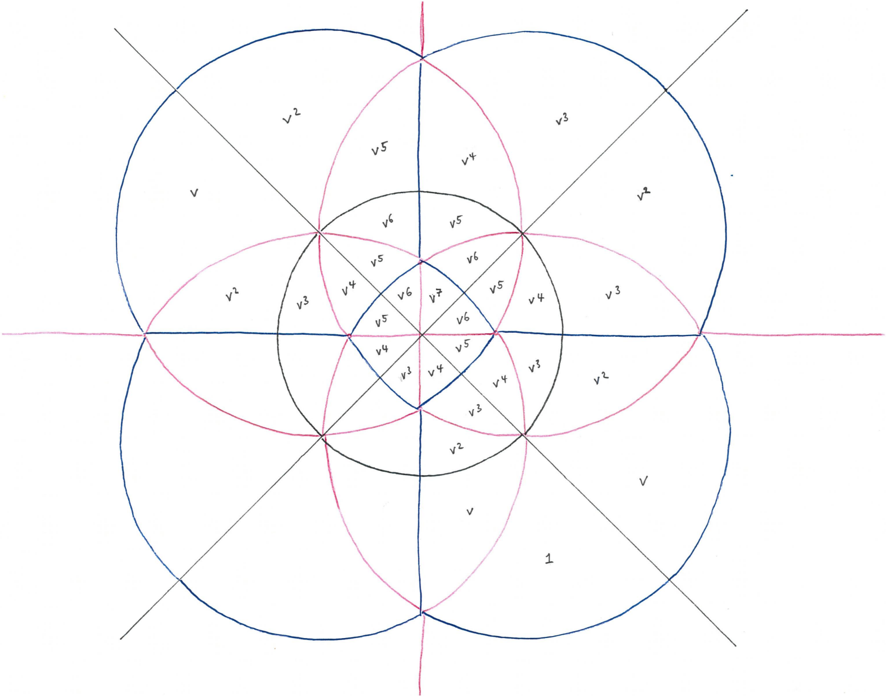
v2
v
v2
v
1



v2
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v5
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v5
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v5
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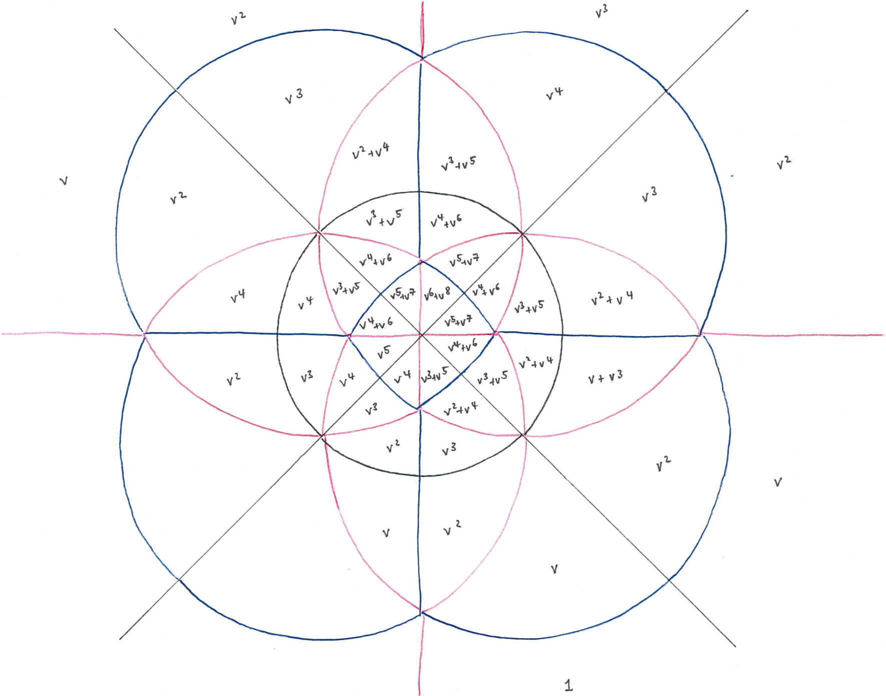
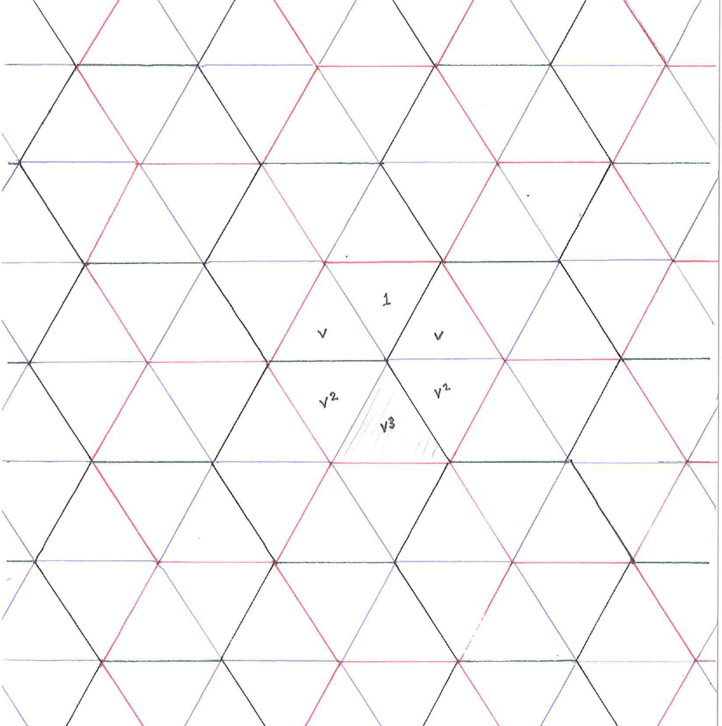
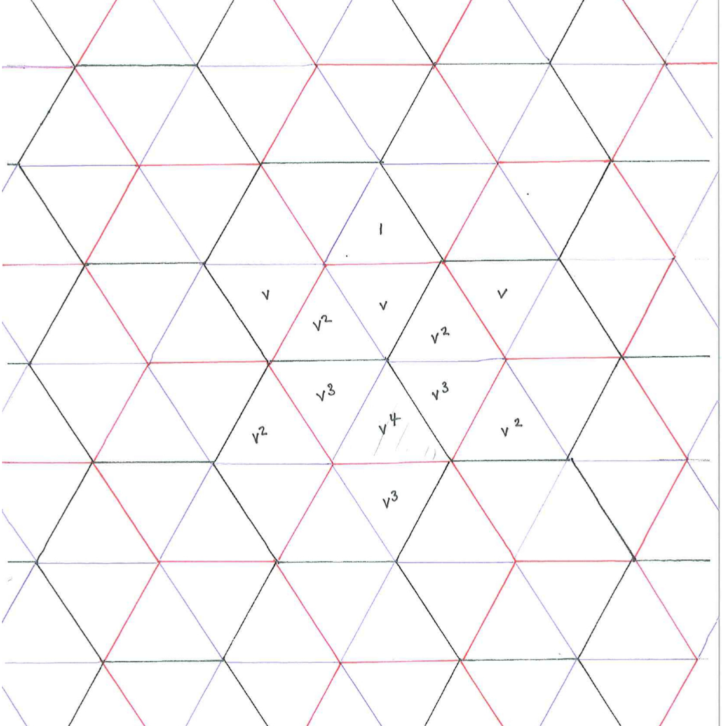
v3
v2
v
v
1



v^2
v^3
$v\,3$
v^4
v^2+v^4
v^3+v^5
v
v^2
v^2
$v\,3$
v^3+v^5
v^4+v^6
v^4+v^6
v^5+v^7
v^4
$v\,4$
v^3+v^5
v^5+v^7
v^6+v^8
v^4+v^6
v^3+v^5
v^2+v^4
$v^4+v\,6$
v^5+v^7
$v\,5$
v^4+v^6
v^2+v^4
v^2
$v3$
$v\,4$
$v\,4$
v^3+v^5
v^3+v^5
$v+v\,3$
v^3
v^2+v^4
v^2
$v\,3$
v^2
v
v
v^2
v


1
v
v
v2
v2
v3


1
v
v
v
v2
v2
v3
v3
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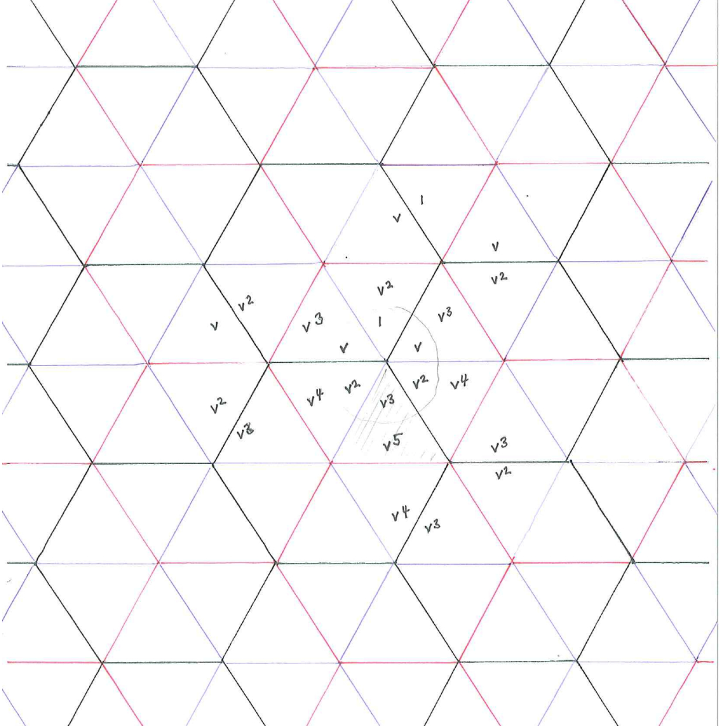
v4
v2
v3


v
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v
v2
v
v2
v3
v2
1
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v
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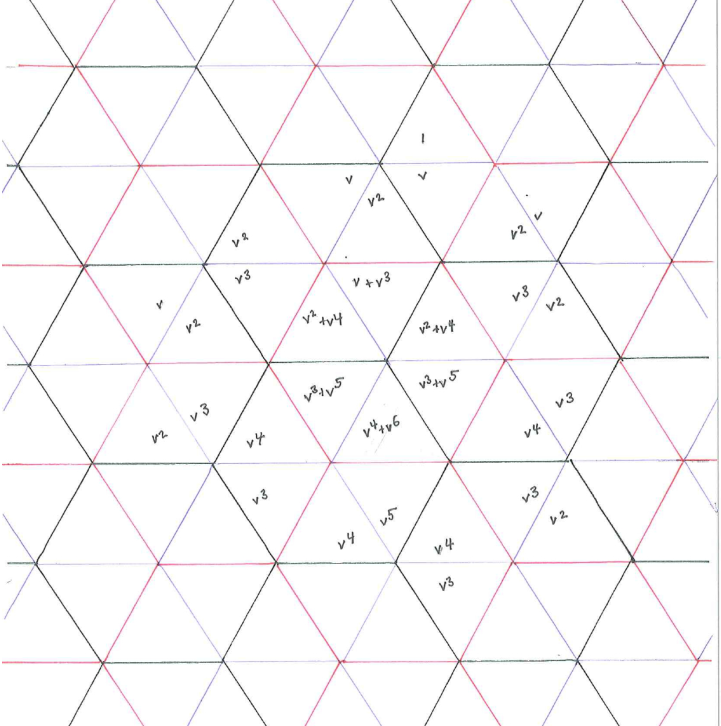
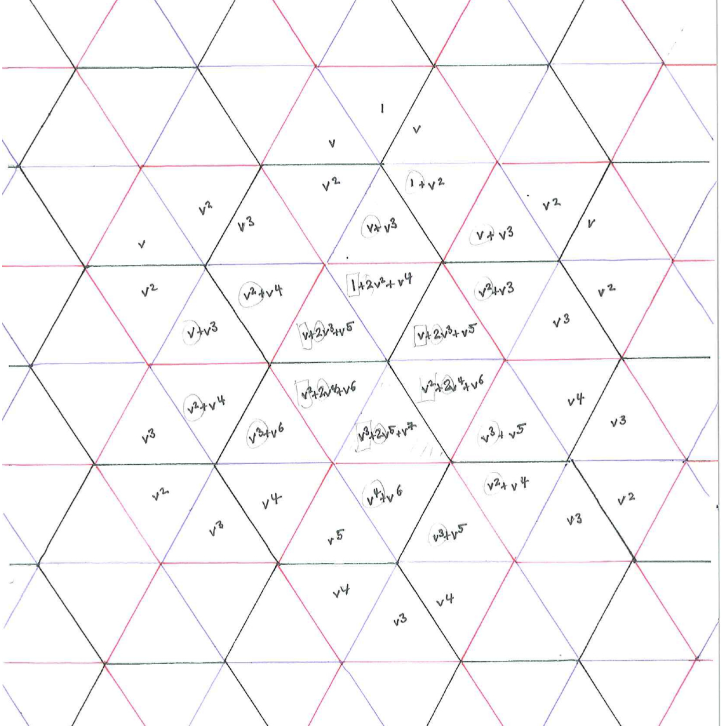
v3
v2
v4
v3



1
v
v
v2
v2
v
v2
v3
v + v3
v3
v2
v
v2
v2+v4
v2+v4
v3+v5
v3+v5
v3
v2
v3
v4
v4+v6
v4
v3
v3
v5
v2
v4
v4
v3


1
v
v
v2
1+v2
v2
v3
v+v3
v+v3
v2
v
v
v2
v2+v4
1+2v2+v4
v2+v3
v2
v+v3
v+2v3+v5
v+2v3+v5
v3
v2+v4
v2+2v4+v6
v2+2v4+v6
v4
v3
v3+v6
v3+2v5+v7
v3+v5
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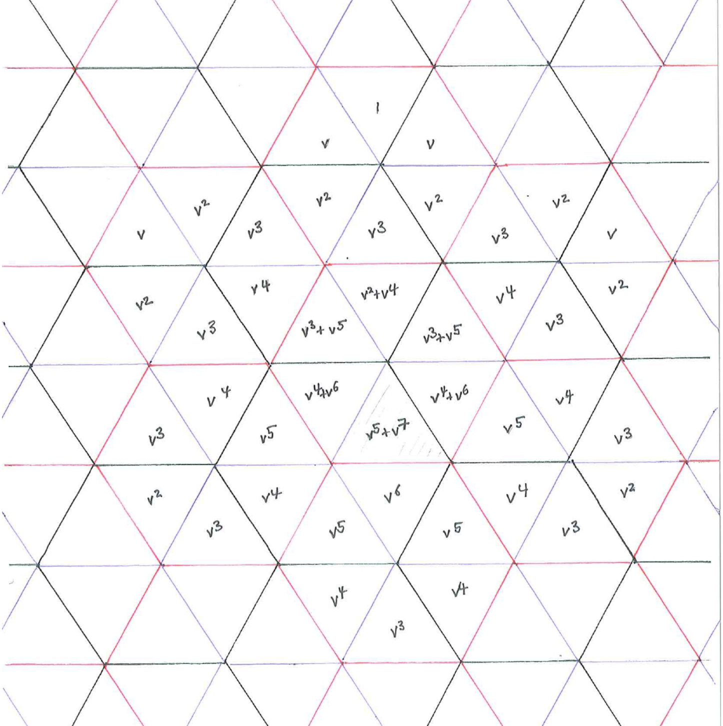
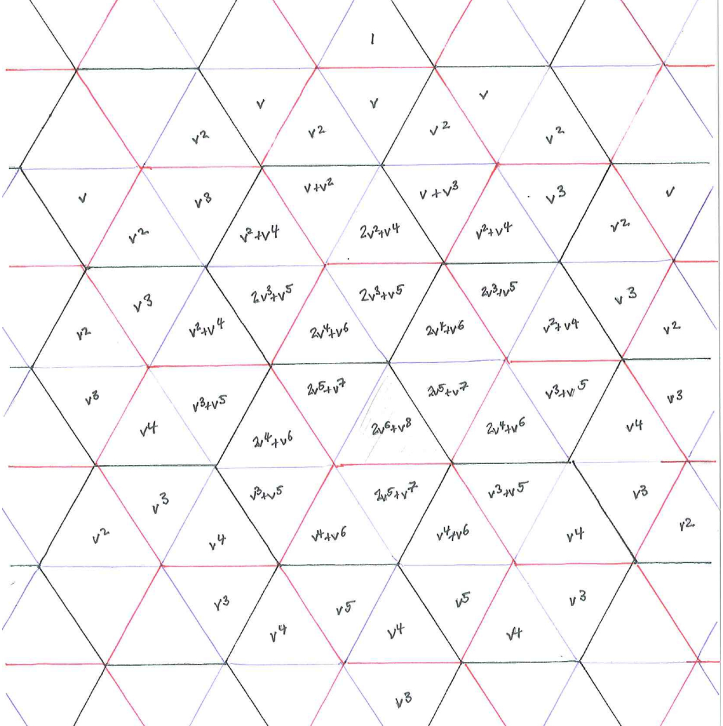
v3+v5
v3
v4
v4
v3


1
v v
v^2 v^2 v^2 v^2
v v^3 v^3 v^3 v
v^2 v^4 v^2+v^4 v^4 v^2
v^3 v^3+v^5 v^3+v^5 v^3
v^4 v^4+v^6 v^4+v^6 v^4
v^3 v^5 v^5+v^7 v^5 v^3
v^2 v^4 v^6 v^4 v^2
v^3 v^5 v^5 v^3
v^4 v^4
v^3



1
v
v
v
v2
v2
v2
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v
v+v2
v+v3
v3
v
v2
v3
v2+v4
2v2+v4
v2+v4
v2
v3
2v3+v5
2v3+v5
2v3+v5
v3
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v2+v4
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2v4+v6
v2+v4
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v3+v5
2v5+v7
2v5+v7
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v3
v4
2v4+v6
2v6+v8
2v4+v6
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v3+v5
2v5+v7
v3+v5
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v4+v6
v4+v6
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v3

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geqslant 0}[v]$$

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geqslant 0}[v]$$

Established for crystallographic W by Kazhdan and Lusztig in 1980, using Deligne's proof of the Weil conjectures.

Crystallographic: $m_{st} \in \{2, 3, 4, 6, \infty\}$.

Why are Kazhdan-Lusztig polynomials hard?

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Roughly: all positive polynomials are Kazhdan-Lusztig polynomials!

The most complicated Kazhdan-Lusztig-Vogan polynomial computed by the *Atlas of Lie groups and Representations* project:

$$152q^{22}+3\,472q^{21}+38\,791q^{20}+293\,021q^{19}+1\,370\,892q^{18}+$$
$$+4\,067\,059q^{17}+7\,964\,012q^{16}+11\,159\,003q^{15}+$$
$$+11\,808\,808q^{14}+9\,859\,915q^{13}+6\,778\,956q^{12}+$$
$$+3\,964\,369q^{11}+2\,015\,441q^{10}+906\,567q^{9}+$$
$$+363\,611q^{8}+129\,820q^{7}+41\,239q^{6}+$$
$$+11\,426q^{5}+2\,677q^{4}+492q^{3}+61q^{2}+3q$$

(This polynomial is associated to the reflection group of type E_8. See `www.liegroups.org`.)

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Kazhdan-Lusztig character formula (conjectured in 1979):

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(A major generalisation of the Weyl character formula.)

The Kazhdan-Lusztig character formula was proved 1981 by Beilinson-Bernstein and Brylinski-Kashiwara using every trick in the book: algebraic differential equations "D-modules"; the Riemann-Hilbert correspondence (monodromy of differential equations); the theory of perverse sheaves (algebraic topology of singular varieties); Deligne's theory of weights (arithmetic geometry):

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"The amazing feature of the proof is that it does not try to solve the problem but just keeps translating it in languages of different areas of mathematics (further and further away from the original problem) until it runs into Deligne's method of weight filtrations which is capable to solve it.

So have a seat; it is going to be a long journey."

– Joseph Bernstein.

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iii) Kazhdan-Lusztig polynomials might end up helping us understand the HOMFLYPT polynomial of a link...

Theorem (Elias-W)

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Using results of Soergel we obtain an algebraic proof of the Kazhdan-Lusztig character formula.

The idea (going back to Soergel) is to find a vector space which behaves like the intersection cohomology of a Schubert variety, even if this variety does not exist. (Much like the coinvariant algebra for a non Weyl group should be regarded as the cohomology of a flag variety, even if no such flag variety extists.)

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The key property of intersection cohomology is the "decomposition theorem": the intersection cohomology of a variety is a summand of the cohomology of any resolution.

For Schubert varieties there exist resolutions of singularities (so called Bott-Samelson resolutions) whose cohomology admit elementary algebraic descriptions.

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For any word $(s, t, \dots, u)$ in S the cohomology of the corresponding Bott-Samelson variety is:

$$BS(s, t, \dots, u) := R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} \mathbb{R}.$$

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Theorem (Soergel)

If W is a Weyl group then the intersection cohomology of the Schubert variety $\overline{BxB/B}$ is the unique largest indecomposable R-module summand of $BS(s, t, \ldots, u)$.

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie}\, T$ to define the "geometric representation" $\mathfrak{h}$ of W. Let R denote the regular functions on $\mathfrak{h}$, a polynomial ring over $\mathbb{R}$.

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Let H_x denote the largest R-module direct summand of $BS(s, t, \dots, u)$ where $(s, t, \dots, u)$ is any reduced expression for x. Soergel shows:

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Examples:

1. If W is a Weyl group then $H_x = IH^*(\overline{BxB/B}; \mathbb{R})$, the intersection cohomology of a Schubert variety.
2. If W is finite, with longest element w_0, then H_{w_0} is the coinvariant algebra.

Conjecture (Soergel)

The graded dimension of

$$\Gamma_{\leqslant y} H_x / \Gamma_{< y} H_x$$

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If W is a Weyl group, then Soergel's conjecture follows from the Kazhdan and Lusztig's theorem relating intersection cohomology and Kazhdan-Lusztig polynomials.

Soergel's conjecture obviously implies that Kazhdan-Lusztig polynomials have positive coefficients.

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Since then Soergel modules and bimodules have popped up throughout representation theory, and have even been used by Khovanov to construct HOMFLY-PT homology.

A key idea in our proof of Soergel's conjecture is to show that each H_x "looks like the cohomology of a smooth projective variety".

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In 2006 de Cataldo and Migliorini gave Hodge theoretic proofs of the decomposition theorem, a deep result about the topology of algebraic maps between algebraic varieties.

The modules $BS(s, t, \ldots, u)$ are equipped with an intersection form, using a combinatorial analogue of the fundamental class. In a complicated induction over the length of x we show that this intersection form restricts to a non-degenerate "intersection form" on $H_x \subset BS(s, t, \ldots, u)$ and that analogues of the hard Lefschetz theorem and the Hodge-Riemann bilinear relations inductively, following the ideas of de Cataldo and Migliorini.

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As is the case for de Cataldo and Migliorini, one needs the whole package of statements for the induction to work.

Theorem (Elias-W)

For any $\rho \in V^*$ in the interior of the fundamental alcove:

i) *(Hard-Lefschetz theorem) left multiplication by ρ^i gives an isomorphism*
$$(H_x)^{\ell(x)-i} \to (H_x)^{\ell(x)+i}$$

1. *(Hodge-Riemann bilinear relations) The restriction of the form $(\alpha, \beta) := \langle \alpha, \rho^i \beta \rangle$ to the kernel of ρ^{i+1} in $(H_x)^{\ell(x)-i}$ is definite.*

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Example: If $m_{st} = 5$ then the role of the integral lattice is replaced by $\mathbb{Z}[\phi]$, where ϕ denotes the golden ratio!

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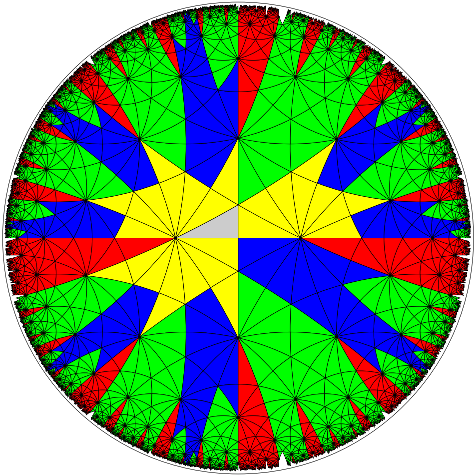
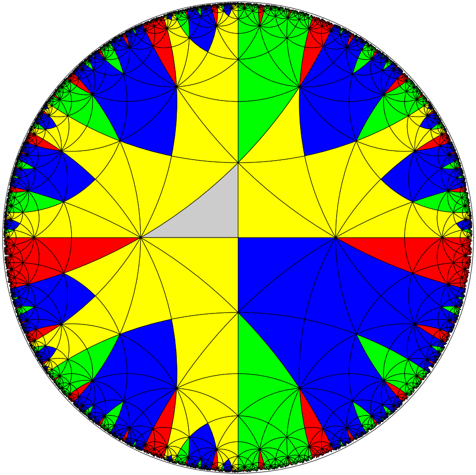
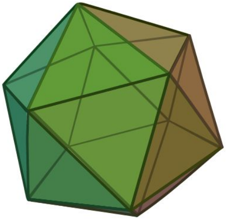
ii) What does Kazhdan-Lusztig theory mean in the non-crystallographic case?





For more images of two-sided cells in hyperbolic groups see Paul Gunnell's web page.



Thanks for listening!