On smooth rational curves on a polarized K3 surface

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Introduction.

In this paper, we shall prove the following existence theorem concerning with a pair of a K3 surface and a smooth rational curve with given degree on it. Besides its own interest, this theorem may have some application to the existence problem of smooth rational curves with given degree on a polarized Calabi-Yau 3-fold (cf. [Ka, Theorem 2.1], [C]).

Main Theorem. *Let $N \geq 3$ and $d \geq 1$ be arbitrarily chosen integers. Then, there exists a pair consisting of a non-singular rational curve $C \subset \mathbb{P}^N$ of degree d and a non-singular primitively embedded K3 surface $S \subset \mathbb{P}^N$ such that $C \subset S$.*

Here, by the words primitively embedded, we mean that the embedding $S \subset \mathbb{P}^N$ is given by the complete linear system of a primitive (in the Picard group) very ample line bundle on S.

In the case when $N = 3$, Mori proved the existence of a pair (S, C) as in the Main Theorem by making use of some special Kummer surface ([MO]). We shall construct such a pair by applying Torelli's theorem for an algebraic K3 surface (cf. [PS], [BPV], [MR]) and Saint-Donat's theory on projective models of K3 surfaces (cf. [SD], [MM], [MO]).

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Proof of the Main Theorem.

First of all, we recall the following lemma.

Lemma 1 ([SD], [MM], and [MO, Theorem 5]). *Let S be a projective K3 surface and H be a nef divisor on S with $H^2 \geq 4$. Then, H is very ample if and only if the following 3 conditions are satisfied:*

(1) *there are no irreducible curves E such that $E^2 = 0$ and $E.H = 1, 2$,*
(2) *there are no irreducible curves E such that $E^2 = 2$ and $H \sim 2E$,*
(3) *there are no irreducible curves E such that $E^2 = -2$ and $E.H = 0$.*

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Lemma 2. *Let $n \geq 2$, d be positive integers. Then, there exist a projective K3 surface S, a primitive very ample line bundle H on S, and a smooth rational curve C on S such that $H^2 = 2n$ and $H.C = d$.*

Proof. Let us consider the 2-dimensional lattice $L = \mathbf{Z}h \oplus \mathbf{Z}c$ with intersection form

$$\begin{pmatrix} (h.h) & (c.h) \\ (h.c) & (c.c) \end{pmatrix} = \begin{pmatrix} 2n & d \\ d & -2 \end{pmatrix}.$$

Since L is an even integral lattice of rank 2 and of signature $(1,1)$, by the primitive embedding theorem of even lattice ([N]) and by Torelli's theorem for an algebraic K3 surface ([PS], [BPV]), we know that there exists a projective K3 surface S with $Pic\, S \cong \mathbf{Z}h \oplus \mathbf{Z}c$ (See [MR, Corollary 2.9]). Moreover, since the nef big cone of S is a fundamental domain of the action of the reflection group generated by the integrally defined reflections $v \mapsto v + (v.b)b$ for $b \in Pic\, S$ with $b^2 = -2$ on the positive cone in $Pic\, S \otimes \mathbf{R}$ by [PS] or [BPV, VIII, Proposition 3.9], and since $h^2 > 0$, we may assume without loss of generality that h represents a nef big line bundle on S. In what follows, by abuse of notation, we consider h and c as elements of $Pic\, S$.

Claim (2.1). *h is very ample on S.*

Proof of Claim (2.1). Since h is nef and $h^2 \geq 4$, it is enough to check the condition (1), (2), (3) of Lemma 1. The condition (2) is obvious because h is a part of $\mathbf{Z}$-basis of $Pic\, S$. We shall check the condition (1). Assume the contrary that there exists an element E of $Pic\, S$ such that $E^2 = 0$ and $E.h = 1, 2$. Then, we have:

$$\begin{pmatrix} (h.h) & (h.E) \\ (h.E) & (E.E) \end{pmatrix} = \begin{pmatrix} 2n & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2n & 2 \\ 2 & 0 \end{pmatrix}$$

and cosequently, $|det(h, E)| = 1$ or 4. On the other hand, since $\{h, c\}$ is a $\mathbf{Z}$-basis of $Pic\, S$, $|det(h, c)|$ must divide the integer $|det(h, E)|$. But this is impossible because $|det(h, c)| = 4n + d^2 > 4$. Next, we check the condition (3). Assume the contrary that there exists an element D of $Pic\, S$ such that $h.D = 0$ and $D^2 = -2$. Then, $|det(h, D)| = 4n$. But this is absurd by the same reason as before because $|det(h, c)| = 4n + d^2 > 4n = |det(h, D)| > 0$.

Now, in order to finish the proof of Lemma 2, it is enough to show the next claim.

Claim (2.2). *The complete linear system $|c|$ contains a smooth rational curve as its element.*

Proof of Claim (2.2). Since $c^2 = -2$, by Riemann-Roch theorem, we see that either $|c|$ or $|-c|$ contains an effective member. But, since $-c.h = -d < 0$ and since h is very ample, we see that $|-c|$ can not contain an effective member. Thus $|c|$ contains an effective member. If this member is irreducible and reduced, then we get the desired result by the genus formula. We shall assume the contrary that $|c|$ contains no irreducible and reduced members and shall derive a contradiction.

Since $c^2 = -2 < 0$, we can write $c = aR + D$ in $Pic\,S$, where R is an irreducible and reduced curve with $R^2 = -2$, a is a positive integer, and D is an effective divisor such that $R \not\subset Supp\,D$. If $D = 0$, we have $a \geq 2$ by our assumption. But, in this case, we have $-2 = c^2 = a^2R^2 \leq -8$, which is absurd. Next, we treat the case when $D \neq 0$. Then, we have $d = c.h = aR.h + D.h$. Since h is very ample and D is not zero, we have $0 < r := R.h < d$. Then, we have $0 < |det(h, R)| = 4n + r^2 < 4n + d^2 = |det(h, c)|$. But this is impossible because $|det(h, c)|$ must divide $|det(h, R)|$.

Now, we can finish the proof of our main theorem. Let us put $n := N - 1 \geq 2$. Then, for arbitrarily chosen integers $N \geq 3$ and $d \geq 1$, we can find a projective K3 surface S, a primitive very ample line bundle H on S, and a smooth rational curve C on S with $H^2 = 2n$ and $H.C = d$ by Lemma 2. Since we know that $h^0(\mathcal{O}_S(H)) = n + 2$ for an ample line bundle H on a K3 surface S with $H^2 = 2n$, we get the desired result. Q.E.D.

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