On the Geometry of Affine Immersions

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Our purpose is to offer a new approach to affine differential geometry based on the notion of affine immersion of an affinely connected manifold (M^n, ∇) into an ambiant manifold $(\tilde{M}^m, \tilde{\nabla})$. In the present paper we are mostly concerned with the case where $m = n + 1$ and particularly $\tilde{M}^{n+1}$ is the ordinary affine space $\mathbb{R}^{n+1}$ and prove several theorems on affine immersions which are closely related to known results on isometric immersions in Riemannian or pseudo-Riemannian geometry.

In Sections 1 and 2 we define the notion of affine immersion, develop several formulas, reformulate some of the basic notions in classical affine differential geometry and discuss several examples. In Section 3 we study affine immersions of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ and prove Theorem 1 which is an analogue of the cylinder theorem for complete flat hypersurfaces in euclidean and Lorentzian spaces. In Section 4 we prove Theorem 2 concerning affine immersions of a metric connection which gives a precise statement of the result hinted at by Cartan [1] and indicated by Norden in the Appendix of [6]. We obtain a few corollaries concerning rigidity of affine immersions. In Section 5 we prove Theorem 3 on the non-existence of affine immersion into $\mathbb{R}^{n+1}$ of a compact manifold with an equiaffine connection with strictly negative-definite Ricci tensor.

1. Affine immersions.

Throughout this paper, we deal with affine connections without torsion so this condition will not be mentioned each time.

Let M be an n-dimensional differentiable manifold with an affine connection ∇, and let $\tilde{M}$ be an $(n+1)$-dimensional differentiable manifold with an affine connection $\tilde{\nabla}$. By an affine immersion $f : (M,\nabla) \to (\tilde{M},\tilde{\nabla})$ we mean an immersion $M \to \tilde{M}$ for which there exists locally (that is, around each point of M) a transversal vector field ξ along f which has the following property: if X and Y are arbitrary vector fields on M, we have

$$\tilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi ,$$

where the left-hand side denotes the covariant derivative with respect to X of the vector field $f_*(Y)$ along f and the first term of the right-hand side is the tangential component and the second term is the transversal component. It is easy to check that h is a symmetric bilinear form on each tangent space $T_x(M)$. We may simplify the equation by dropping f_* and write

$$(1) \qquad \tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi.$$

In particular, if h is 0 at x (that is, $\tilde{\nabla}_X Y$ is tangent to M), then we say that f is <u>totally geodesic at</u> x. Obviously, this condition is independent of the choice of ξ. We have

<u>Proposition</u> 1. <u>Let</u> $f:(M,\nabla)\to(\tilde{M},\tilde{\nabla})$ <u>be an affine immersion and</u> ξ_1 <u>and</u> ξ_2 <u>two associated transversal fields. Then the directions</u> $[\xi_1]$ <u>and</u> $[\xi_2]$ <u>can differ only on the interior of the set where h vanishes (i.e. on totally geodesic pieces).</u>

Proof. Write

$$(2)\qquad \xi_2 = Z + \varphi\,\xi_1 ,$$

where Z is a vector field tangent to M and φ is a function on M. We have then

$$\tilde{\nabla}_X Y = \nabla_X Y + h_2(X,Y)\xi_2 = \nabla_X Y + h_2(X,Y)Z + \varphi\, h_2(X,Y)\xi_1 .$$

Comparing it with (1), we have

$$h_2(X,Y)Z = 0 \qquad \text{and} \qquad \varphi\, h_2(X,Y) = h_1(X,Y).$$

If f is not totally geodesic at x, then there exists $X,Y \in T_x(M)$ such that $h_2(X,Y) \neq 0$. Then $Z=0$ at x. Thus $\xi_2 = \varphi\xi_1$. □

It also follows that whether h is nondegenerate is independent of the choice of ξ. We say f is nondegenerate if h is.

For an affine immersion $f:(M,\nabla)\to(\tilde{M},\tilde{\nabla})$ we also write

$$(3)\qquad \tilde{\nabla}_X\,\xi = -S(X) + \tau(X)\xi ,$$

where $-S(X)$ denotes the tangential component. It is easily verified that S is a tensor field of type (1,1) and τ is a 1-form. We call S the <u>shape operator</u> and τ the <u>transversal connection form</u> for f.

Following the standard routine for geometry of hypersurfaces, we may now compute

the tangential components $\tan[\tilde{R}(X,Y)Z]$ and $\tan[\tilde{R}(X,Y)\xi]$

and

the transversal components $\operatorname{trans}[\tilde{R}(X,Y)Z]$ and $\operatorname{trans}[\tilde{R}(X,Y)\xi]$

in terms of the curvature tensor R of (M,∇), h, S, τ etc. We obtain

Proposition 2.

I $\tan[\tilde{R}(X,Y)Z] = R(X,Y)Z - [h(Y,Z)SX - h(X,Z)SY]$

II $\operatorname{trans}[\tilde{R}(X,Y)Z] = (\nabla_X h)(Y,Z) + \tau(X)\, h(Y,Z) - (\nabla_Y h)(X,Z) - \tau(X)h(X,Z)$

III $\tan[\tilde{R}(X,Y)\xi] = -(\nabla_X S)(Y) + \tau(X)SY + (\nabla_Y S)(X) - \tau(Y)SX$

IV $\operatorname{trans}[\tilde{R}(X,Y)\xi] = -h(X,SY) + h(SX,Y) + 2\, d\tau(X,Y)$.

We now consider certain important special cases.

For an affine connection ∇ on M, the Ricci tensor Ric is defined by

(4) $$\mathrm{Ric}(Y,Z) = \operatorname{trace}\{X \to R(X,Y)Z\}.$$

Ric may not be symmetric. It is known that Ric is symmetric if and only if around each point there is a parallel volume element, namely, a nonzero n-form ω such that $\nabla\omega = 0$. If M is simply connected, it follows that Ric is symmetric if and only if M admits a volume element ω parallel relative to ∇, that is, if and only if (M,∇) is equiaffine. (M,∇,ω) is called an equiaffine structure.

If $(\tilde{M},\tilde{\nabla},\tilde{\omega})$ is an equiaffine structure and $f\colon (M,\nabla) \to (\tilde{M},\tilde{\nabla})$ an affine immersion and ξ an associated transversal field, then we define a volume element ω on M by

(5) $$\omega(X_1,\ldots,X_n) = \tilde{\omega}(X_1,\ldots,X_n,\xi)$$

where $\{X_1,\ldots,X_n\}$ is any basis in $T_x(M)$. Using (1), (3) and (5) we see that

(6) $$\nabla_X\omega = \tau(X)\,\omega\ .$$

It follows that (M,∇,ω) is an equiaffine structure if and only if $\tau = 0$.

If (M,∇,ω) and $(\tilde{M},\tilde{\nabla},\tilde{\omega})$ are equiaffine structures, $f:(M,\nabla)\to(\tilde{M},\tilde{\nabla})$ an affine immersion, then an associated transversal field is called <u>equiaffine</u> if (5) holds for any basis $\{X_1,\ldots,X_n\}$ in $T_x(M)$. We have $\tau = 0$. Assuming that f is totally geodesic nowhere, the associated transversal field ξ is now uniquely determined because of (5).

<u>Remark</u>. The study of affine immersion of an equiaffine connection into flat affine space is equivalent to what is called relative geometry, see [6], [7], [8].

We have

<u>Proposition 3. If</u> (M,∇,ω) <u>and</u> $(\tilde{M},\tilde{\nabla},\tilde{\omega})$ <u>are equiaffine structures and if f is an affine immersion:</u> $(M,\nabla)\to(\tilde{M},\tilde{\nabla})$<u>, then an associated transversal vector field</u> ξ <u>can be chosen to be equiaffine</u>.

Proof. Simply multiply ξ by $\varphi = \omega(X_1,\ldots,X_n)/\,\tilde{\omega}(X_1,\ldots,X_n,\xi)$. □

Recall that two affine connections ∇ and $\overline{\nabla}$ (both with zero torsion) on a manifold M are <u>projectively related</u> if there is a 1-form ρ on M such that

(7) $\overline{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X$

for all vector fields X and Y. See, for example, [5].

A change from ∇ to $\overline{\nabla}$ is called a projective change. An affine connection ∇ is said to be projectively flat if it can be changed projectively to a flat affine connection $\overline{\nabla}$ (i.e. zero curvature tensor $\overline{R}$).

Suppose an affine connection ∇ on a differentiable manifold M has symmetric Ricci tensor (in particular, suppose it is equiaffine). For dim M ≥ 3, ∇ is projectively flat if and only if the projective curvature tensor

(8) $W(X,Y)Z = R(X,Y)Z - [\gamma(Y,Z)X - \gamma(X,Z)Y]$, where $\gamma = \mathrm{Ric}/(n-1)$

is identitcally 0. For dim M = 2, ∇ is projectively flat if and only if γ satisfies Codazzi's equation: $(\nabla_X \gamma)(Y,Z) = (\nabla_Y \gamma)(X,Z)$. If dim M $\geq$ 3 and if W = 0, then γ satisfies Codazzi's equation. On the other hand, if dim M = 2, then W is automatically 0.

If (M,∇) is projectively flat, then

(9) $R(X,Y)Z = \gamma(Y,Z)X - \gamma(X,Z)Y$.

We now consider the formulas I - IV in certain special cases.

a. Case where $(\tilde{M},\tilde{\nabla})$ is projectively flat:

$\tilde{R}(X,Y)Z = \tilde{\gamma}(Y,Z)X - \tilde{\gamma}(X,Z)Y$ is tangential. Thus

Ia. $R(X,Y)Z = \tilde{\gamma}(Y,Z)X - \tilde{\gamma}(X,Z)Y + h(Y,Z)SX - h(X,Z)SY$ - Gauss-

From this, we get

$$\mathrm{Ric}(Y,Z) = (n-1)\tilde{\gamma}(Y,Z) + h(Y,Z)\,\mathrm{tr}\,S - h(SY,Z).$$

In particular, if $\tilde{\nabla}$ is flat, we have

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$$

$$\mathrm{Ric}\,(Y,Z) = h(Y,Z)\,\mathrm{tr}\,S - h(SY,Z).$$

IIa. $(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z)$ -Codazzi-

We set

(10) $C(X,Y,Z) = (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z)$,

which is symmetric in Y and Z like h, as well as in X and Y by virtue of IIa, thus symmetric in X,Y, and Z. We call C the cubic form of the affine immersion. This is a generalization of the classical cubic form in affine differential geometry.

b. Case where (M,∇,ω), $(\tilde{M},\tilde{\nabla},\tilde{\omega})$ are equiaffine and the transversal field ξ is equiaffine:

Since $\tau = 0$, we get

IIb. $(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$ - Codazzi for h -

IIIb. $(\nabla_Y S)(X) - \tilde{\gamma}(Y,\xi)X = (\nabla_X S)(Y) - \tilde{\gamma}(X,\xi)Y$

In particular, if $\tilde{\nabla}$ is flat, $(\nabla_Y S)(X) = (\nabla_X S)(Y)$ - Codazzi for S-

IVb. $h(SX,Y) = h(X,SY)$ - Ricci-

2. Examples.

We discuss some examples of affine immersions.

Example 1 - Isometrically immersed hypersurface. Let (M,g) be a

Riemannian manifold of dimension n with Levi-Civita connection ∇. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold of dimension n+1 with Levi-Civita connection $\tilde{\nabla}$. If f: $(M,g) \to (\tilde{M}, \tilde{g})$ is an isometric immersion, then $f:(M,\nabla) \to (\tilde{M}, \tilde{\nabla})$ is an affine immersion with a transversal vecotr field ξ given locally as a unit normal vector field.

Example 2 - Affine cylinder. Roughly speaking, an affine cylinder in $\mathbb{R}^{n+1}$ is a hypersurface generated by a parallel family of affine (n-1)-spaces $\mathbb{R}^{n-1}(t)$, each through a point of γ in $\mathbb{R}^{n+1}$. We define an affine cylinder immersion precisely as follows.

Let $\gamma(t)$ be a smooth curve in $\mathbb{R}^{n+1}$ and $\xi(t)$ a vector field along $\gamma(t)$. Let $\mathbb{R}^{n-1}$ be an affine (n-1)-space in $\mathbb{R}^{n+1}$ and consider all parallel (n-1)-spaces and denote by $\mathbb{R}^{n-1}(p)$ the one through p. We assume that

(i) $\gamma'(t)$, $\xi(t)$ and $\mathbb{R}^{n-1}(\gamma(t))$ are linearly independent;

(ii) $\gamma''(t) = \rho(t)\xi(t)$, where $\rho = \rho(t)$ is a certain differentiable function.

Now we define a mapping $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ as follows. Write $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ so every point of $\mathbb{R}^n$ is written as (t,y), $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$. Let

$$f(t,y) = \gamma(t) + y.$$

For this immersion f, we take a transversal field

$$\xi(t,y) = \xi(t) \text{ translated to } f(t,y)$$

by virtue of condition (i). It is easy to verify that f is an affine immersion of $\mathbb{R}^n \to \mathbb{R}^{n+1}$. For the curve $x(t) = (t,0)$ in $\mathbb{R}^n$, we have

$$\tilde{\nabla}_t f(\vec{x_t}) = \gamma''(t) = \rho(t)\xi(t) \quad \text{so} \quad h(\partial/\partial t, \partial/\partial t) = \rho(t).$$

In the special case where we can take $\xi = \gamma''$ and furthermore γ'' and γ''' are linearly independent, we call it a <u>proper affine cylinder</u>. In this case, we see from $\tilde{\nabla}_t \xi = \gamma''' = f_*(S(\partial/\partial t)) + \tau(\partial/\partial t)\gamma''$ that S never vanishes. We also see that h never vanishes.

<u>Example</u> 3 - <u>Graph immersion</u>. Let (M^n, ∇) be a manifold with a flat affine connection and φ: $(M^n, \nabla) \to \mathbb{R}^n$ an affine immersion. Thus φ is an immersion such that every point p of M^n has a neighborhood U on which φ is an affine-connection preserving diffeomorphism with an open neighborhood V of $\varphi(p)$ in $\mathbb{R}^n$. Consider $\mathbb{R}^n$ as a hyperplane H in $\mathbb{R}^{n+1}$ and let ξ be a parallel vector field transveral to H. For any differentiable function F: $M^n \to \mathbb{R}$, we define $f: M^n \to \mathbb{R}^{n+1}$ by $f(x) = \varphi(x) + F(x)\xi$, for $x \in M^n$.

We have

$$f_*(Y) = \varphi_*(Y) + (dF)(Y)\,\xi \qquad \text{for } Y \in T_x(M^n)$$

so f is an immersion. For vector fields X and Y on M^n, we have

$$\tilde{\nabla}_X f_*(Y) = \tilde{\nabla}_X \varphi_*(Y) + \tilde{\nabla}_X(YF\,\xi) = \varphi_*(\nabla_X Y) + (XYF)\xi$$

$$= f_*(\nabla_X Y) + (XYF - (\nabla_X Y)F)\,\xi.$$

Thus f is an affine immersion with $h(X,Y) = XY\,F - (\nabla_X Y)F$, which coincides with the Hessian of F. Thus f is nondegenerate if the Hessian H is nondegenerate. We have also $S = 0$.

Conversely, we may prove

Proposition 4. Suppose (M^n, ∇) is a flat connection and $f : (M^n, \nabla) \to \mathbb{R}^{n+1}$ an affine immersion such that $S = 0$. Then it is affinely equivalent to the graph immersion for a certain function $F: M^n \to \mathbb{R}$.

Proof. By assuming a transveral field ξ to be equiaffine, $S = 0$ implies that $\tilde{\nabla}_X \xi = 0$, that is, ξ is a constant (parallel) vector field. Let $H = \mathbb{R}^n$ be a hyperplane in $\mathbb{R}^{n+1}$ which is transversal to ξ. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection along the direction of ξ so that $\pi \circ f : M^n \to \mathbb{R}^n$ is an affine immersion with image W, an open subset of $\mathbb{R}^n$. We can find a differentiable function $F: M^n \to \mathbb{R}$ such that $f(x) = (\pi \circ f)(x) + F(x)\xi$. Thus f is a graph immersion. □

Example 4 - Centro-affine hypersurface. Suppose $f: M \to \mathbb{R}^{n+1} - \{o\}$ is an immersed hypersurface such that relative to o in $\mathbb{R}^{n+1}$ the position vector $\overrightarrow{o\,f(x)}$ is always transversal to $f(M)$ at $f(x)$. Take $\xi = -\overrightarrow{of(x)}$ as a transversal vector field for f. Then $\tilde{\nabla}_X \xi = -X$ so that $\tau = 0$ and $S = I$ (identity). By writing $\tilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi$, we see that $\nabla_X Y$ is indeed an affine connection (with zero torsion) on M.Thus $f :(M, \nabla) \to \mathbb{R}^{n+1}$ is an affine immersion. This is called a centro-affine hypersurface. From the formula (Ib) we get

$$(11) \qquad R(X,Y)Z = h(Y,Z)X - h(X,Z)Y, \quad \gamma(Y,Z) = h(Y,Z).$$

Proposition 5. For a centro-affine hypersurface $f: (M, \nabla) \to (\mathbb{R}^{n+1} - \{o\}, \tilde{\nabla})$ *and for any function* $\lambda: M \to \mathbb{R}+$, *the mapping* $x \mapsto \lambda(x) f(x)$ *defines a centro-affine hypersurface* $\lambda f: (M, \nabla') \to (\mathbb{R}^{n+1} - \{o\}, \tilde{\nabla})$ *where* ∇' *is projectivelty related to* ∇ *by*

$$\nabla'_X Y = \nabla_X Y + \rho(X) Y + \rho(Y) X, \quad \text{where} \quad \rho = d \log \varphi.$$

Conversely, any projective change of (M, ∇) *can be locally obtained in this manner*.

The proof is straightforward and omitted. □

Corollary. Let (M, ∇, ω) *be a differentiable manifold with a projectively flat equiaffine connecton. Then* (M, ∇) *can be locally realized as a centro-affine hypersurface in* $\mathbb{R}^{n+1} - \{o\}$.

Proof. If (M, ∇') is flat, then it can be locally realized as a piece of a hyperplane with induced volume element ω_0 in $\mathbb{R}^{n+1} - \{o\}$. Now we can make a projective change back to ∇ by modifying this hyperplane by a suitable function λ, namely, $\lambda = \omega / \omega_0$. □

Example 5- *Conormal immersion* .

Let $f: (M, \nabla, \omega) \to \mathbb{R}^{n+1}$ is a nondegenerate affine immersion of an equiaffine structure with an equiaffine transversal field ξ. We denote by $\mathbb{R}_{n+1}$ the vector space dual to the vector space $\mathbb{R}^{n+1}$ underlying the affine space $\mathbb{R}^{n+1}$. We define $v: M \to \mathbb{R}_{n+1} - \{0\}$ as follows.

For $x \in M$, v_x is an element of $\mathbb{R}_{n+1}$ such that

(12) $\qquad v_x(Y) = 0 \quad$ for $\; Y \in T_x(M) \quad$ and $\quad v_x(\xi_x) = 1$,

where Y and ξ_x are considered as elements of the vector space $\mathbb{R}^{n+1}$ naturally identified with $T_x(\mathbb{R}^{n+1})$. Denoting by $\tilde{\nabla}$ the usual flat connection in $\mathbb{R}_{n+1}$, we have

(13) $\qquad (\tilde{\nabla}_Y v)(\xi) = 0 \quad$ and $\quad (\tilde{\nabla}_Y v)(f_* X) = -h(Y,X)\;$ for all $X, Y \in T_x(M)$.

Since h is nondegenerate, we see that if $(\tilde{\nabla}_Y v)(f_* X) = 0$ for all X, then $Y = 0$. Since $\tilde{\nabla}_Y v = v_*(Y)$, it follows that the mapping v is nonsingular. Hence we may consider $v\colon M \to \mathbb{R}_{n+1} - \{0\}$ as a centro-affine hypersurface, called the <u>conormal immersion</u> for f.

Taking $-v$ as the transveral vector field as in Example 4 we write

(14) $$\tilde{\nabla}_X(v_*(Y)) = v_*(\nabla^*{}_X Y) - h^*(X,Y)v,$$

where ∇^* is an affine connection on M and h^* the second fundamental form. These are related to the affine connection ∇, the affine metric h and the affine shape operator S fot the original hypersurface $f\colon M \to \mathbb{R}^{n+1}$ in the following way:

(15) $\qquad h*(X,Y) = h(SX,Y)\quad$ (also equal to $\gamma_*(X,Y)$ as in Example 4)

(16) $\qquad X\,h(Y,Z) = h(\nabla^*{}_X Y, Z) + h(\nabla_X Z, Y)$

and

(17) $\quad \hat{\nabla}_X Y = (\nabla_X Y + \nabla^*{}_X Y)/2,$

where $\hat{\nabla}$ denotes the Levi-Civita connection for the affine metric h.

The formulas (15) and (16) are consequences of basic formulas for f and (12),(13) and (14). (17) follows from (16). They can be found, in different notations, in [6], p.127-129. It is a classical fact that the cubic form C for f vanishes if and only if $\nabla = \hat{\nabla} = \nabla^*$.

Example 6 - Blaschke immersion. Suppose f: $(M, \nabla, \omega) \to (\tilde{M}, \tilde{\nabla}, \tilde{\omega})$ is an affine immersion with equiaffine transversal field. If, furthermore, f is nondegenerate and if ω coincides with the volume element ω_h of the nondegenerate metric h, then we say that f is a Blaschke immersion. For the case where $(\tilde{M}, \tilde{\nabla}, \tilde{\omega})$ is an ordinary affine space $\mathbb{R}^{n+1}$ with the flat affine connection and the standard volume element given by the determinant, this is exactly the kind of affine immersion which has been the primary object of study in affine differential geometry developed by Blaschke and his school in the period 1910-40. The first step in the subject is to prove, for the standard equiaffine structure in $\mathbb{R}^{n+1}$, the following basic result.

Let M be a hypersurface immersed in $\mathbb{R}^{n+1}$. For any choice of a transversal vector field ξ, define an affine connection ∇ and the bilinear form h by equation (1). Whether h is nondegenerate or not is independent of the choice of

ξ, and we say that M is nondegenerate if h is. Denote by ω_h the volume element for h.

<u>Proposition</u> 6. <u>If</u> M <u>is a nondegenerate hypersurface immersed in</u> $\mathbb{R}^{n+1}$, <u>there is a unique choice of</u> ξ <u>such that</u>

i) ω_h <u>coincides with</u> ω <u>defined by</u> $\omega(X_1, \ldots, X_n) = \tilde{\omega}(X_1, \ldots, X_n, \xi)$;

ii) (M, ∇, ω) <u>is equiaffine</u>.

This unique ξ is called the <u>affine normal</u> and the corresponding h the <u>affine metric</u>.

The proof of Proposition 6 can be found in [4].

3. Affine immersions $\mathbb{R}^n \to \mathbb{R}^{n+1}$

In this section we are interested in classifying all affine immersions: $M = \mathbb{R}^n \to \mathbb{R}^{n+1}$. We always choose an equiaffine transversal field ξ as we may. From Section 1 we have the formulas

$h(Y,Z)\,SX = h(X,Z)\,SY$ - Gauss equation in case $R = 0$ -

$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$ - Codazzi equation for h -

$(\nabla_X S)(Y) = (\nabla_Y S)(X)$ - Codazzi equation for S -

$h(SX,Y) = h(X,SY)$ - Ricci equation -.

If h is identically 0, then f is totally geodesic and $f(\mathbb{R}^n)$ is an affine hyperplane in $\mathbb{R}^{n+1}$. If S is identically 0, then by Proposition 4 f is a graph immersion.

In the general case, let $\Omega = \{x \in M;\ S_x \neq 0,\ h_x \neq 0\}$. We prove

Lemma 1. For each $x \in \Omega$, Ker h = Ker S and its dimension is $n-1$.

Proof. For each $x \in \Omega$ the equality Ker h = Ker S follows directly from the definition and the Gauss equation. If for some $x \in \Omega$ we had rank $S \geq 2$, then there would be tangent vectors X and Y such that SX and SY are linearly independent. The Gauss equation then would imply $X, Y \in \operatorname{Ker} h = \operatorname{Ker} S$, a contradiction. □

For $x \in \Omega$, the subspace $N_x = \operatorname{Ker} h_x = \operatorname{Ker} S_x \subset T_x(M)$ is called the relative nullity space at x.

Lemma 2. The distribution $N: x \to N_x$ on Ω is involutive and totally geodesic.

Proof. It is sufficient to show that N is totally geodesic, that is, for vector fields Y,Z belonging to N, $\nabla_X Y \in N$. In the equation of Codazzi for h : $(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$ take $Y,Z \in N$. Then we get

$$X\, h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = Y\, h(X,Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)$$

and hence $h(X, \nabla_Y Z) = 0$. This being valid for all X, we have $\nabla_Y Z \in N$. □

Now if L is a leaf of the relative nullity foliation N, L is totally geodesic in $M = \mathbb{R}^n$. Indeed, f(L) is totally geodesic in $\mathbb{R}^{n+1}$. Our goal is to show that each leaf L is complete. Let x_t be a geodesic starting at x_0 in the leaf L. To show that x_t extends for all values of t in L, first extend it as a geodesic in M. It is sufficient to show that x_t lies in Ω, because then it lies in L. So suppose there is b > 0 such that $x_b \notin \Omega$ and $x_t \in \Omega$ for all $t < b$.

We need

Lemma 3. *Let X be a vector field on some open subset W of Ω containing the geodesic x_t, $0 \le t < b$, such that $\nabla_X X = 0$, $X \in N$, and X at x_t equals the tangent vector $\overrightarrow{x_t}$ for $0 \le t < b$. Let U be a parallel vector field on $M = \mathbb{R}^n$ which is transversal to the hyperplane $H = \mathbb{R}^{n-1}$ of $M = \mathbb{R}^n$ that contains L.*

(i) *Write $\nabla_U X = \mu U + Z$ at each point $p \in \Omega \cap H$, where $Z_p \in N_p$. Then the function μ satisfies $X\mu = -\mu^2$ along x_t, $0 \le t < b$.*

(ii) <u>Write</u> $SU = \lambda U + W$ <u>at each point</u> $p \in V$, <u>where</u> $W \in T_p(H)$. <u>Then the function</u> λ <u>satisfies</u> $X\lambda = -\mu\lambda$ <u>along</u> x_t, $0 \le t < b$.

(iii) <u>Let</u> $\rho = h(U,U)$ <u>on</u> V. <u>Then</u> $X\rho = -\mu\rho$ <u>along</u> x_t, $0 \le t < b$.

Proof.

(i) $\nabla_X(\nabla_U X) = \nabla_X(\mu U + Z) = (X\mu)U + \mu \nabla_X Z + \nabla_X U = (X\mu)U \mod N.$

Since $R = 0$, we have along x_t, $0 \le t < b$

$$\nabla_X(\nabla_U X) = \nabla_{[X,U]}X = -\nabla_{\nabla_U X}X = -\nabla_{\mu U + Z}X$$

$$= -\mu\nabla_U X - \nabla_Z X = -\mu^2 U \mod N.$$

Hence $(X\mu)U = -\mu^2 U \mod N$ and $X\mu = -\mu^2$.

(ii) From the Codazzi equation for S

$$\nabla_X(SU) - S(\nabla_X U) = \nabla_U(SX) - S(\nabla_U X),$$

we get along x_t, $0 \le t < b$

$$(X\lambda)U + \lambda(\nabla_X U) + \nabla_X W = -\mu SU = -\mu(\lambda Y + W)$$

and $(X\lambda)U = -\mu\lambda U \mod N$. Thus $X\lambda = -\mu\lambda$ along x_t.

(iii) We have along x_t, $0 \le t < b$

$$X\rho = Xh(U,U) = (\nabla_X h)(U,U) - 2h(\nabla_X U,U) = (\nabla_U h)(X,U)$$

$$= U\,h(X,U) - h(\nabla_U X,U) - h(X,\nabla_U U) = -\mu\,h(U,U) = -\mu\rho. \qquad \square$$

Now we can conclude the proof that $x_b \in \Omega$ as follows. The equations in (i), (ii) and (iii) are

$d\mu/dt = -\mu^2$, $\quad d\lambda/dt = -\lambda\mu$, $\quad d\rho/dt = -\rho\mu$ for $0 \leq t < b$.

Thus μ is identically 0 or $\mu = 1/(t+a)$ for some a. It follows that λ = constant or $\lambda = 1/c(t+a)$ and the same for ρ. In all cases, neither λ nor ρ approaches 0 as $t \to b$. Now at the point $p = x_b$, this means $SU \neq 0$ as well as $h(U,U) \neq 0$. Thus $p \in \Omega$.

With completeness of L established, we know $x_t \in L$ for all t. Thus the possibility of $\mu = 1/(t+a)$ is excluded. Hence $\mu = 0$ and thus λ and ρ are equal to constants on the leaf L.

We can now prove

<u>Proposition</u> 5. <u>Let</u> $f: \mathbf{R}^n \to \mathbf{R}^{n+1}$ <u>be an affine immersion such that</u> S <u>and</u> h <u>vanish nowhere</u>. <u>Then</u> f <u>is affine-equivalent to a proper affine cylinder immersion</u>.

Proof. In the foregoing discussions, we now have $\Omega = \mathbf{R}^n$. We have already proved that each leaf of the relative nullity foliation is complete. Thus each leaf is a hyperplane in $\mathbf{R}^n$, and all leaves are parallel hyperplanes because they are disjoint from each other.

We take a vector U transversal to all these hyperplanes and consider a line x_t in the direction of of U. Write $\mathbf{R}^{n-1}(t)$ for the leaf through the point x_t. Since each leaf is mapped totally geodesically, $f(\mathbf{R}^{n-1}(t))$ is an affine (n-1)-space in $\mathbf{R}^{n+1}$. Also, if Y_t is a parallel vector field along x_t such that $Y_t \in T_x(\mathbf{R}^{n-1}(t))$, then

$$\tilde{\nabla}_t f_*(Y_t) = f_*(\nabla_t Y_t) + h(U, Y_t) = 0.$$

Thus $f_*(Y_t)$ is parallel in $\mathbb{R}^{n+1}$. This shows that all subspaces $f(\mathbb{R}^{n-1}(t))$ are parallel to each other.

Now it is easy to verify that f is affinely equivalent to a proper affine cylinder immersion based on the parallel family $f(\mathbb{R}^{n-1}(t))$ and the curve $\gamma(t) = f(x_t)$. The original transversal field ξ_t is in the direction of $\gamma'(t)$.

We can now state

Theorem 1. *Let* $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ *be an affine immersion. Then* $\Omega = \{x \in \mathbb{R}^n ; S_x \neq 0, \ h_x \neq 0\}$, *if not empty, is the union of parallel hyperplanes. Each connected component* Ω_α *of* Ω *is a strip consisting of parallel hyperplanes and* $f : \Omega_\alpha \to \mathbb{R}^{n+1}$ *is affinely equivalent to a proper affine cylinder immersion.*

Remark. On each component of $\mathbb{R}^n - \cup \overline{\Omega_\alpha}$ is a mixture of graph immersions and totally geodesic immersions. One can easily construct examples piecing together different types of affine immersions, but proving a general description is not easy.

Corollary. *An analytic immersion* $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$ *is either totally goedesic or affinely equivalent to a graph immersion or affinely equivalent to an affine cylinder immersion.*

Proof. If h or S is identically 0, we know that f is totally geodesic or a graph immersion. Otherwise, the open subset Ω is dense. On each connected component Ω_α, f is a proper affine cylinder immersion. Since Ω is dense, all these immersions of the components extend to an affine cylinder immersion f. □

Remark. It is not difficult to construct a C^∞ affine immersion $M^2 \to \mathbb{R}^3$ of the affine Möbius band $M^2 = \mathbb{R}^2/\varphi$, where φ is the affine map : $(x,y) \to (x+1,-y)$. By the corollary, however, there can be no analytic immersion of this kind.

4. Affine immersions of pseudo-riemannian manifolds

We prove the following theorem which is a precise statement for the result of Cartan and Norden mentioned in the introduction.

Theorem 2. Let (M^n,g) be a pseudo-riemannian manifold, ∇ its Levi-Civita connection and $f: (M^n,\nabla) \to \mathbb{R}^{n+1}$ an affine immersion with a transversal field ξ. If f is nondegenerate, we have either

(i) ∇ is flat and f is a graph immersion;

or

(ii) ∇ is not flat and $\mathbb{R}^{n+1}$ admits a parallel pseudo-riemannian metric relative to which f is an isometric immersion and ξ is perpendicular to $f(M^n)$.

Proof. We first establish

Lemma. Let (M,h) be a pseudo-riemannian manifold and let ∇ and ∇^* be two affine connections with zero torsion on M which are conjugate relative to h, that is,

$$X\, h(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z)$$

for all vector fields X, Y and Z. Let B be a nonsingular $(1,1)$ tensor field which is symmetric relative to h and define pseudo-riemannian metrics g and g^* by

$$g(X,Y) = h(BX,Y) \quad \text{and} \quad g^*(X,Y) = h(B^{-1}X,Y).$$

Then $(\nabla_X g)(Y,Z) + (\nabla^*_X g^*)(Y,Z) = 0$ for all vector fields X, Y and Z. In particular, ∇ is the Levi-Civita connection for g if and only if ∇^* is the Levi-Civita connection for g^*.

Proof. We have

$$(\nabla^*_X g^*)(Y,Z) = X\, g^*(Y,Z) - g^*(\nabla^*_X Y, Z) - g^*(Y, \nabla^*_X Z)$$

$$= Xh(B^{-1}Y,Z) - h(\nabla^*_X Y, B^{-1}Z) - h(B^{-1}Y, \nabla^*_X Z)$$

$$= Xh(B^{-1}Y,Z) - \{X\, h(Y, B^{-1}Z) - h(Y, \nabla_X B^{-1}Z)\}$$

$$- \{X\, h(Z, B^{-1}Y) - h(Z, \nabla_X B^{-1}X)\}$$

$$= h(Z, \nabla_X B^{-1}Y) + h(Y, \nabla_X B^{-1}Z) - X\, h(Y, B^{-1}Z).$$

Replacing Y, Z by BY, BZ we get

$$(\nabla^*_X g^*)(BY,BZ) = h(BZ, \nabla_X Y) + h(BY, \nabla_X Z) - Xh(BY,Z)$$

$$= g(Z, \nabla_X Y) + g(Y, \nabla_X Z) - Xg(Y,Z) = -(\nabla_X g)(Y,Z). \qquad \square$$

To prove the theorem, we may assume that ξ is equiaffine and we consider the conormal immersion $v\colon (M^n, \nabla^*) \to \mathbb{R}_{n+1}$. We recall that the affine connection ∇^* is conjugate to ∇ relative to the form h for f; cf. equation

(16).

Since h is nondegenerate, we may write $g(X,Y) = h(BX,Y)$, where B is a certain nonsingular (1,1) tensor symmetric relative to h. We define a pseudo-riemannian metric g^* by $g^*(X,Y) = h(B^{-1}X,Y)$. By the lemma, we see that ∇^* is the Levi-Civita connection for g^*.

Now the conormal immersion being a centro-affine immersion, we know that ∇^* is projectively flat. Since ∇^* is the Levi-civita connection for g^*, it follows by a theorem of Dini-Beltrami that g^* has constant sectional curvature, say, c. The form $h*$ for the conormal immersion is, by equation (11), equal to the normalized Ricci tensor γ^*, which is in this case equal to $c\,g^*$. Thus $h* = c\,g^*$, in particular, $\nabla^* h^* = 0$.

Case (i): $c = 0$. Then ∇^* is flat. Since $h* = 0$, by (15) the shape operator S for f is 0 and by the Gauss equation ∇ is flat. By Proposition 4 we conclude that f is a graph immersion.

Case (ii): $c \neq 0$. We shall show that $\mathbb{R}_{n+1}$ admits a parallel pseudo-riemannian metric $\langle\ ,\ \rangle^*$ such that

$$\langle v_*(X), v_*(Y)\rangle^* = g^*(X,Y) \quad \text{for} \quad X,Y \in T_x(M)$$

$$\langle v, v_*(X)\rangle^* = 0 \quad \text{for} \quad X \in T_x(M)$$

$$\langle v,v\rangle^* = -1/c.$$

For this purpose, we define $\langle\ ,\ \rangle^*$ in each $T_{v(x)}(\mathbb{R}_{n+1})$ using exactly the above three equations and show that this metric tensor field along v is parallel in $\mathbb{R}_{n+1}$. Thus we wish to verify

(*) $X \langle U, V \rangle^* = \langle \tilde{\nabla}_X U, V \rangle^* + \langle U, \tilde{\nabla}_X V \rangle^*$

for all vector fields U and V along v and a vector field X on M.

If U and V are of the form $v_*(Y)$ and $v_*(Z)$, where Y and Z are vector fields on M, the equation (*) reduces to $(\nabla^*_X g^*)(Y,Z) = 0$.

If $U = v_*(Y)$ and $V = v$, then $X \langle v_*(Y), v \rangle^* = 0$ and

$$\langle \tilde{\nabla}_X U, V \rangle^* = \langle \tilde{\nabla}_X v_*(Y), v \rangle^* = \langle v_*(\nabla_X Y), v \rangle^* + \langle h^*(X,Y)v, v \rangle^*$$

$$= h^*(X,Y) \langle v, v \rangle^* = - h^*(X,Y)/c$$

as well as $\langle U, \tilde{\nabla}_X V \rangle = \langle v_*(X), v_*(Y) \rangle = g^*(X,Y)$. Thus (*) is satisfied.

Finally, if $U = V = v$, (*) is obvious.

Now it remains to show that $\mathbb{R}^{n+1}$ admits a parallel pseudo-riemannian metric $\langle \ , \ \rangle$ such that

$$\langle f_*(X), f_*(Y) \rangle = g(X,Y), \quad \langle f_*(X), \xi \rangle = 0, \quad \langle \xi, \xi \rangle = -1/c$$

for all vector fields X and Y on M. Indeed, using the nondegenerate form $\langle \ , \ \rangle^*$ in $\mathbb{R}_{n+1}$, we identify $\mathbb{R}_{n+1}$ with $\mathbb{R}^{n+1}$ (both as vector spaces) by $u \in \mathbb{R}_{n+1} \mapsto \theta(u) \in \mathbb{R}^{n+1}$ with $w(\theta(u)) = \langle u, w \rangle^*$ for all $w \in \mathbb{R}_{n+1}$. We then define $\langle \ , \ \rangle$ in $\mathbb{R}^{n+1}$ as the dual inner product, namely,

$$\langle X, Y \rangle = \langle \theta^{-1}(X), \theta^{-1}(Y) \rangle^* \quad \text{for } X, Y \in \mathbb{R}^{n+1} .$$

In order to show that this inner product $\langle \ , \ \rangle$ is the desired one, we first remark the following fact. Let $u = v_*(X)$ for $X \in T_x(M)$. Then for any $Y \in T_x(M)$ we have $v_*(Y)\ (\theta(u)) = \langle v_*(Y), v_*(X) \rangle^* = g^*(X,Y)$. On the other hand, $v(\theta(u)) = 0$. It follows that $\theta(u) = -f_*(B^{-1}X)$, where B is a certain nonsingular (1,1) tensor. We have

$$g^*(X,Y) = v_*(Y)\theta(u) = -v_*(Y)(f_*(B^{-1}X)) = h(B^{-1}X,Y),$$

where we use the relation (13). Now for X,Y we have

$$f_*(B^{-1}X) = -\theta(v_*(X)), \quad f_*(B^{-1}Y) = -\theta(v_*(Y))$$

and

$$\langle f_*(B^{-1}X), f_*(B^{-1}Y)\rangle = \langle v_*(X), v_*(Y)\rangle^* = g^*(X,Y).$$

Replacing X,Y by BX, BY in this equation we obtain

$$\langle f_*(X), f_*(Y)\rangle = g^*(BX,BY) = h(B^{-1}BX,BY) = h(X,BY).$$

But as in the lemma, $h(X,BY) = g(X,Y)$. Hence

$$g(X,Y) = \langle f_*(X), f_*(Y)\rangle.$$

The other identities are obvious from $\theta(v) = \xi$. The proof of the theorem is now complete. □

We state a few corollaries.

Corollary 1. Let (M^n,g) be a pseudo-riemannian manifold, ∇ its Levi-Civita connection, and $f: (M^n,\nabla) \to \mathbb{R}^{n+1}$ an affine immersion. If the Ricci tensor of g is nondegenerate, then $\mathbb{R}^{n+1}$ admits a parallel pseudo-riemannian metric such that f is an isometric immersion and the transversal field is perpendicular to $f(M^n)$.

Proof. From $Ric(Y,Z) = h(Y,Z)\,trS - h(SY,Z)$, it follows that h is nondegenerate if the Ricci tensor is nondegenerate.

Corollary 2. Let g be a riemannian metric on S^2 with Gaussian curvature

$K > 0$ and Levi-Civita connection ∇. Then there exists an affine immersion $f: (S^2, \nabla) \to \mathbb{R}^3$ which is unique up to an affine transformation of $\mathbb{R}^3$.

Proof. By the solution to Weyl's problem (see, for example, [9], p.226) (S^2, g) has an isometric imbedding f into euclidean space $\mathbb{R}^3$ with standard metric and it is rigid. So $f : (S^2, \nabla) \to \mathbb{R}^3$ is an affine imbedding. Suppose $f_1 : (S^2, \nabla) \to \mathbb{R}^3$ is another affine immersion. Theorem 2 implies that it is isometric relative to a certain parallel pseudo-riemannian metric $\langle \ , \ \rangle$ in $\mathbb{R}^3$. This metric must be Euclidean to accommodate a compact surface with positive definite metric induced on it. Since one can find an affine transformation A of $\mathbb{R}^3$ which transforms the metric $\langle \ , \ \rangle$ into the standard metric, it follows that $A \circ f_1$ is an isometric immersion into $\mathbb{R}^3$ with standard euclidean metric, and as such, congruent to f. This means that f_1 differs from f by an affine transformation. □

Corollary 3. _Let_ g _be the standard riemannian metric on_ S^n _with constant sectional curvature_ 1. _For every affine immersion_ $f: (S^n, \nabla) \to \mathbb{R}^{n+1}$, _the image_ $f(S^n)$ _is an ellipsoid_ (_relative to a Euclidean metric_).

Corollary 4. _Let_ (H^n, g) _be the hyperbolic space with standard riemannian metric of constant sectional curvature_ -1. _Then every affine transformation_ $f: (H^n, \nabla) \to \mathbb{R}^{n+1}$ _is an isometric immersion of_ (H^n, g) _into_ $\mathbb{R}^{n+1}$ _with flat Lorentz metric_. _If_ $n \geq 3$, $f(M^n)$ _is affinely congruent to one component of the two-sheeted hyperboloid_ $-x_0^2 + x_1^2 + \cdots + x_n^2 = -1, \quad x_0 > 0$.

Remark 2. In the proof of Theorem 2, the sign of c generally depends on the affine immersion f.

5. Equiaffine immersions of compact manifolds

It is a standard theorem in euclidean differential geometry that a compact riemannian manifold (M_n, g) with negative-definite Ricci tensor cannot be isometrically immersed in a euclidean space $\mathbb{R}^{n+1}$: any compact immersed hypersurface has to be locally strictly convex somewhere and the Ricci tensor is positive-definite at convex points. For affine immersions this argument does not apply, because convexity does not imply positivity of the Ricci tensor. For example, the hyperbolic space H^n can be affinely imbedded as one component of a two-sheeted hyperboloid.

We can still prove

Theorem 3. *Let* (M^n, ∇, ω) *be a compact equiaffine manifold with negative-definite Ricci tensor (or more generally, with nondegenerate, but not positive-definite, Ricci tensor). Then* (M^n, ∇) *does not admit an affine immersion into* $\mathbb{R}^{n+1}$.

Proof. Let $f : (M^n, \nabla) \to \mathbb{R}^{n+1}$ be an affine immersion. We choose a transversal field to be equiaffine. As in Corollary 1 in Section 4, h is nondegenerate with the Ricci tensor. Thus viewing M^n as a hypersurface in euclidean space $\mathbb{R}^{n+1}$, the usual second fundamental form is proportional to h and thus nondegenerate. It follows that M^n is diffeomorphic to S^n, h is definite, and $f(M^n)$ is a strictly convex hypersurface (fo example, see [4], p.41). By diagonalizing S relative to h, we see that Ric for ∇ is positive-definite at a point where the bilinear form $B(Y,Z) = h(SY,Z)$ is positve-definite. We shall show that there is such a point, contradicting the assumption on Ric and thus concluding the proof of Theorem 3.

From Example 5 recall that $(n-1)B$ is equal to the Ricci tensor of the conormal connection ∇^* on M, which is equiaffine and projectively flat. Thus our assertion will follow from the next lemma.

Lemma. Let $\tilde{\nabla}$ be a projectively flat equiaffine connection on S^n with volume element $\tilde{\omega}$. Then there are points on S^n where the Ricci tensor of $\tilde{\nabla}$ is positive-definite.

Proof . Recall that $(S^n, \tilde{\nabla})$ is projectively equivalent to (S^n, ∇_0), where ∇_0 is the standard affine connection (Levi-Civita connection) on S^n (see, for example [3]). Consider S^n as a unit sphere in $\mathbb{R}^{n+1}$. We may obtain a centro-affine immersion $\varphi: S^n \to \mathbb{R}^{n+1}$ so that the induced volume element coincides with $\tilde{\omega}$. The induced connection ∇^* is projectively flat and coincides with $\tilde{\nabla}$, since they have the same volume element. See, for example, [5], Proposition 2.

Thus we may consider $\varphi: S^n \to \mathbb{R}^{n+1}$, where the image $\varphi(S^n)$ is star-shaped with respect to the origin. Let p be a point where a nondegenerate height function has a maximum. Then $\varphi(S^n)$ is strictly convex towards the origin at p, and thus by (11) Ric is positive-definite at p. □

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On the Geometry of Affine Immersions

Katsumi Nomizu and Ulrich Pinkall

Our purpose is to offer a new approach to affine differential geometry based on the notion of affine immersion of an affinely connected manifold (M^n, ∇) into an ambiant manifold $(\tilde{M}^m, \tilde{\nabla})$. In the present paper we are mostly concerned with the case where $m = n + 1$ and particularly $\tilde{M}^{n+1}$ is the ordinary affine space $\mathbb{R}^{n+1}$ and prove several theorems on affine immersions which are closely related to known results on isometric immersions in Riemannian or pseudo-Riemannian geometry.

In Sections 1 and 2 we define the notion of affine immersion, develop several formulas, reformulate some of the basic notions in classical affine differential geometry and discuss several examples. In Section 3 we study affine immersions of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ and prove Theorem 1 which is an analogue of the cylinder theorem for complete flat hypersurfaces in euclidean and Lorentzian spaces. In Section 4 we prove Theorem 2 concerning affine immersions of a metric connection which gives a precise statement of the result hinted at by Cartan [1] and indicated by Norden in the Appendix of [6]. We obtain a few corollaries concerning rigidity of affine immersions. In Section 5 we prove Theorem 3 on the non-existence of affine immersion into $\mathbb{R}^{n+1}$ of a compact manifold with an equiaffine connection with strictly negative-definite Ricci tensor.

1. Affine immersions.

Throughout this paper, we deal with affine connections without torsion so this condition will not be mentioned each time.

Let M be an n-dimensional differentiable manifold with an affine connection ∇, and let $\tilde{M}$ be an $(n+1)$-dimensional differentiable manifold with an affine connection $\tilde{\nabla}$. By an affine immersion $f:(M,\nabla) \to (\tilde{M},\tilde{\nabla})$ we mean an immersion $M \to \tilde{M}$ for which there exists locally (that is, around each point of M) a transversal vector field ξ along f which has the following property: if X and Y are arbitrary vector fields on M, we have

$$\tilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi ,$$

where the left-hand side denotes the covariant derivative with respect to X of the vector field $f_*(Y)$ along f and the first term of the right-hand side is the tangential component and the second term is the transversal component. It is easy to check that h is a symmetric bilinear form on each tangent space $T_x(M)$. We may simplify the equation by dropping f_* and write

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi. \tag{1}$$

In particular, if h is 0 at x (that is, $\tilde{\nabla}_X Y$ is tangent to M), then we say that f is <u>totally geodesic at</u> x. Obviously, this condition is independent of the choice of ξ. We have

<u>Proposition 1.</u> <u>Let</u> $f : (M, \nabla) \to (\tilde{M}, \tilde{\nabla})$ <u>be an affine immersion and</u> ξ_1 <u>and</u> ξ_2 <u>two associated transversal fields. Then the directions</u> $[\xi_1]$ <u>and</u> $[\xi_2]$ <u>can differ only on the interior of the set where h vanishes (i.e. on totally geodesic pieces).</u>

Proof. Write

$$(2) \qquad \xi_2 = Z + \varphi\, \xi_1 \,,$$

where Z is a vector field tangent to M and φ is a function on M. We have then

$$\tilde{\nabla}_X Y = \nabla_X Y + h_2(X,Y)\xi_2 = \nabla_X Y + h_2(X,Y)Z + \varphi\, h_2(X,Y)\xi_1 .$$

Comparing it with (1), we have

$$h_2(X,Y)Z = 0 \qquad \text{and} \qquad \varphi\, h_2(X,Y) = h_1(X,Y).$$

If f is not totally geodesic at x, then there exists $X, Y \in T_x(M)$ such that $h_2(X,Y) \neq 0$. Then $Z = 0$ at x. Thus $\xi_2 = \varphi \xi_1$. □

It also follows that whether h is nondegenerate is independent of the choice of ξ. We say f is nondegenerate if h is.

For an affine immersion $f: (M, \nabla) \to (\tilde{M}, \tilde{\nabla})$ we also write

$$(3) \qquad \tilde{\nabla}_X \xi = -S(X) + \tau(X)\xi \,,$$

where $-S(X)$ denotes the tangential component. It is easily verified that S is a tensor field of type (1,1) and τ is a 1-form. We call S the <u>shape operator</u> and τ the <u>transversal connection form</u> for f.

Following the standard routine for geometry of hypersurfaces, we may now compute

the tangential components $\tan[\tilde{R}(X,Y)Z]$ and $\tan[\tilde{R}(X,Y)\xi]$

and

the transversal components $\mathrm{trans}[\tilde{R}(X,Y)Z]$ and $\mathrm{trans}[\tilde{R}(X,Y)\xi]$

in terms of the curvature tensor R of (M,∇), h, S, τ etc. We obtain

<u>Proposition</u> 2.

I $\tan[\tilde{R}(X,Y)Z] = R(X,Y)Z - [h(Y,Z)SX - h(X,Z)SY]$

II $\mathrm{trans}[\tilde{R}(X,Y)Z] = (\nabla_X h)(Y,Z) + \tau(X)\, h(Y,Z) - (\nabla_Y h)(X,Z) - \tau(X)h(X,Z)$

III $\tan[\tilde{R}(X,Y)\xi] = -(\nabla_X S)(Y) + \tau(X)SY + (\nabla_Y S)(X) - \tau(Y)SX$

IV $\mathrm{trans}[\tilde{R}(X,Y)\xi] = -h(X,SY) + h(SX,Y) + 2\, d\tau(X,Y)$.

We now consider certain important special cases.

For an affine connection ∇ on M, the Ricci tensor Ric is defined by

$$\mathrm{Ric}(Y,Z) = \mathrm{trace}\ \{X \mapsto R(X,Y)Z\}. \tag{4}$$

Ric may not be symmetric. It is known that Ric is symmetric if and only if around each point there is a parallel volume element, namely, a nonzero n-form ω such that $\nabla\omega = 0$. If M is simply connected, it follows that Ric is symmetric if and only if M admits a volume element ω parallel relative to ∇, that is, if and only if (M,∇) is equiaffine. (M,∇,ω) is called an <u>equiaffine structure</u>.

If $(\tilde{M},\tilde{\nabla},\tilde{\omega})$ is an equiaffine structure and $f\colon (M,\nabla) \to (\tilde{M},\tilde{\nabla})$ an affine immersion and ξ an associated transversal field, then we define a volume element ω on M by

(5) $$\omega(X_1,\ldots,X_n) = \tilde{\omega}(X_1,\ldots,X_n,\xi)$$

where $\{X_1,\ldots,X_n\}$ is any basis in $T_x(M)$. Using (1), (3) and (5) we see that

(6) $$\nabla_X\omega = \tau(X)\,\omega .$$

It follows that (M,∇,ω) is an equiaffine structure if and only if $\tau = 0$.

If (M,∇,ω) and $(\tilde{M},\tilde{\nabla},\tilde{\omega})$ are equiaffine structures, $f : (M,\nabla) \to (\tilde{M},\tilde{\nabla})$ an affine immersion, then an associated transversal field is called <u>equiaffine</u> if (5) holds for any basis $\{X_1,\ldots,X_n\}$ in $T_x(M)$. We have $\tau = 0$. Assuming that f is totally geodesic nowhere, the associated transversal field ξ is now uniquely determined because of (5).

<u>Remark</u>. The study of affine immersion of an equiaffine connection into flat affine space is equivalent to what is called relative geometry, see [6], [7], [8].

We have

<u>Proposition 3. If (M,∇,ω) and $(\tilde{M},\tilde{\nabla},\tilde{\omega})$ are equiaffine structures and if f is an affine immersion: $(M,\nabla) \to (\tilde{M},\tilde{\nabla})$, then an associated transversal vector field ξ can be chosen to be equiaffine</u>.

Proof. Simply multiply ξ by $\varphi = \omega(X_1,\ldots,X_n)/\,\tilde{\omega}(X_1,\ldots,X_n,\xi)$. □

Recall that two affine connections ∇ and $\bar{\nabla}$ (both with zero torsion) on a manifold M are <u>projectively related</u> if there is a 1-form ρ on M such that

(7) $\quad \overline{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X$

for all vector fields X and Y. See, for example, [5].

A change from ∇ to $\overline{\nabla}$ is called a projective change. An affine connection ∇ is said to be <u>projectively flat</u> if it can be changed projectively to a flat affine connection $\overline{\nabla}$ (i.e. zero curvature tensor $\overline{R}$).

Suppose an affine connection ∇ on a differentiable manifold M has symmetric Ricci tensor (in particular, suppose it is equiaffine). For dim M ≥ 3, ∇ is projectively flat if and only if the projective curvature tensor

(8) $W(X,Y)Z = R(X,Y)Z - [\gamma(Y,Z)X - \gamma(X,Z)Y]$, where $\gamma = \mathrm{Ric}/(n-1)$

is identitcally 0. For dim M = 2, ∇ is projectively flat if and only if γ satisfies Codazzi's equation: $(\nabla_X \gamma)(Y,Z) = (\nabla_Y \gamma)(X,Z)$. If dim M $\geq$ 3 and if W = 0, then γ satisfies Codazzi's equation. On the other hand, if dim M = 2, then W is automatically 0.

If (M, ∇) is projectively flat, then

(9) $\quad R(X,Y)Z = \gamma(Y,Z)X - \gamma(X,Z)Y.$

We now consider the formulas I - IV in certain special cases.

a. <u>Case where</u> $(\tilde{M}, \tilde{\nabla})$ <u>is projectively flat</u>:

$\tilde{R}(X,Y)Z = \tilde{\gamma}(Y,Z)X - \tilde{\gamma}(X,Z)Y$ is tangential. Thus

Ia. $R(X,Y)Z = \tilde{\gamma}(Y,Z)X - \tilde{\gamma}(X,Z)Y + h(Y,Z)SX - h(X,Z)SY$ - Gauss-

From this, we get

$$\mathrm{Ric}(Y,Z) = (n-1)\tilde{\gamma}(Y,Z) + h(Y,Z)\,\mathrm{tr}\,S - h(SY,Z).$$

In particular, if $\tilde{\nabla}$ is flat, we have

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$$

$$\mathrm{Ric}\,(Y,Z) = h(Y,Z)\,\mathrm{tr}\,S - h(SY,Z).$$

IIa. $(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z)$ -Codazzi-

We set

(10) $C(X,Y,Z) = (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z)$,

which is symmetric in Y and Z like h, as well as in X and Y by virtue of IIa, thus symmetric in X,Y, and Z. We call C the <u>cubic form</u> of the affine immersion. This is a generalization of the classical cubic form in affine differential geometry.

b. <u>Case where</u> $(M,\nabla,\omega),(\tilde{M},\tilde{\nabla},\tilde{\omega})$ <u>are equiaffine and the transversal field ξ is equiaffine</u>:

Since $\tau = 0$, we get

IIb. $(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$ - Codazzi for h -

IIIb. $(\nabla_Y S)(X) - \tilde{\gamma}(Y,\xi)X = (\nabla_X S)(Y) - \tilde{\gamma}(X,\xi)Y$

In particular, if $\tilde{\nabla}$ is flat, $(\nabla_Y S)(X) = (\nabla_X S)(Y)$ - Codazzi for S-

IVb. $h(SX,Y) = h(X,SY)$ - Ricci-

2. Examples.

We discuss some examples of affine immersions.

<u>Example</u> 1 - <u>Isometrically immersed hypersurface</u>. Let (M,g) be a

Riemannian manifold of dimension n with Levi-Civita connection ∇. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold of dimension n+1 with Levi-Civita connection $\tilde{\nabla}$. If $f: (M,g) \to (\tilde{M}, \tilde{g})$ is an isometric immersion, then $f:(M,\nabla) \to (\tilde{M}, \tilde{\nabla})$ is an affine immersion with a transversal vecotr field ξ given locally as a unit normal vector field.

<u>Example</u> 2 - <u>Affine cylinder</u>. Roughly speaking, an affine cylinder in $\mathbb{R}^{n+1}$ is a hypersurface generated by a parallel family of affine (n-1)-spaces $\mathbb{R}^{n-1}(t)$, each through a point of γ in $\mathbb{R}^{n+1}$. We define an <u>affine cylinder immersion</u> precisely as follows.

Let $\gamma(t)$ be a smooth curve in $\mathbb{R}^{n+1}$ and $\xi(t)$ a vector field along $\gamma(t)$. Let $\mathbb{R}^{n-1}$ be an affine (n-1)-space in $\mathbb{R}^{n+1}$ and consider all parallel (n-1)-spaces and denote by $\mathbb{R}^{n-1}(p)$ the one through p. We assume that

(i) $\gamma'(t)$, $\xi(t)$ and $\mathbb{R}^{n-1}(\gamma(t))$ are linearly independent;

(ii) $\gamma''(t) = \rho(t)\xi(t)$, where $\rho = \rho(t)$ is a certain differentiable function.

Now we define a mapping $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$ as follows. Write $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ so every point of $\mathbb{R}^n$ is written as (t,y), $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$. Let

$$f(t,y) = \gamma(t) + y.$$

For this immersion f, we take a transversal field

$$\xi(t,y) = \xi(t) \text{ translated to } f(t,y)$$

by virtue of condition (i). It is easy to verify that f is an affine immersion of $\mathbb{R}^n \to \mathbb{R}^{n+1}$. For the curve $x(t) = (t,0)$ in $\mathbb{R}^n$, we have

$$\tilde{\nabla}_t f(\vec{x_t}) = \gamma''(t) = \rho(t)\xi(t) \quad \text{so} \quad h(\partial/\partial t, \partial/\partial t) = \rho(t).$$

In the special case where we can take $\xi = \gamma''$ and furthermore γ'' and γ''' are linearly independent, we call it a <u>proper affine cylinder</u>. In this case, we see from $\tilde{\nabla}_t \xi = \gamma''' = f_*(S(\partial/\partial t)) + \tau(\partial/\partial t)\gamma''$ that S never vanishes. We also see that h never vanishes.

<u>Example</u> 3 - <u>Graph immersion</u>. Let (M^n, ∇) be a manifold with a flat affine connection and φ: $(M^n, \nabla) \to \mathbb{R}^n$ an affine immersion. Thus φ is an immersion such that every point p of M^n has a neighborhood U on which φ is an affine-connection preserving diffeomorphism with an open neighborhood V of $\varphi(p)$ in $\mathbb{R}^n$. Consider $\mathbb{R}^n$ as a hyperplane H in $\mathbb{R}^{n+1}$ and let ξ be a parallel vector field transveral to H. For any differentiable function F: $M^n \to \mathbb{R}$, we define $f: M^n \to \mathbb{R}^{n+1}$ by $f(x) = \varphi(x) + F(x)\xi$, for $x \in M^n$.

We have

$$f_*(Y) = \varphi_*(Y) + (dF)(Y)\,\xi \quad \text{for } Y \in T_x(M^n)$$

so f is an immersion. For vector fields X and Y on M^n, we have

$$\tilde{\nabla}_X f_*(Y) = \tilde{\nabla}_X \varphi_*(Y) + \tilde{\nabla}_X(YF\,\xi) = \varphi_*(\nabla_X Y) + (XYF)\xi$$

$$= f_*(\nabla_X Y) + (XYF - (\nabla_X Y)F)\,\xi.$$

Thus f is an affine immersion with $h(X,Y) = XY\,F - (\nabla_X Y)F$, which coincides with the Hessian of F. Thus f is nondegenerate if the Hessian H is nondegenerate. We have also $S = 0$.

Conversely, we may prove

<u>Proposition</u> 4. <u>Suppose</u> (M^n, ∇) is a flat connection and $f : (M^n, \nabla) \to \mathbb{R}^{n+1}$ <u>an affine immersion such that</u> $S = 0$. <u>Then it is affinely equivalent to the graph immersion for a certain function</u> $F: M^n \to \mathbb{R}$.

Proof. By assuming a transveral field ξ to be equiaffine, $S = 0$ implies that $\tilde{\nabla}_X \xi = 0$, that is, ξ is a constant (parallel) vector field. Let $H = \mathbb{R}^n$ be a hyperplane in $\mathbb{R}^{n+1}$ which is transversal to ξ. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection along the direction of ξ so that $\pi \circ f : M^n \to \mathbb{R}^n$ is an affine immersion with image W, an open subset of $\mathbb{R}^n$. We can find a differentiable function $F: M^n \to \mathbb{R}$ such that $f(x) = (\pi \circ f)(x) + F(x)\xi$. Thus f is a graph immersion. □

<u>Example</u> 4 - <u>Centro-affine hypersurface</u>. Suppose $f: M \to \mathbb{R}^{n+1} - \{o\}$ is an immersed hypersurface such that relative to o in $\mathbb{R}^{n+1}$ the position vector $\overrightarrow{o\, f(x)}$ is always transversal to $f(M)$ at $f(x)$. Take $\xi = -\overrightarrow{of(x)}$ as a transversal vector field for f. Then $\tilde{\nabla}_X \xi = -X$ so that $\tau = 0$ and $S = I$ (identity). By writing $\tilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi$, we see that $\nabla_X Y$ is indeed an affine connection (with zero torsion) on M. Thus $f : (M, \nabla) \to \mathbb{R}^{n+1}$ is an affine immersion. This is called a centro-affine hypersurface. From the formula (Ib) we get

(11) $$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y, \quad \gamma(Y,Z) = h(Y,Z).$$

Proposition 5. For a centro-affine hypersurface $f: (M, \nabla) \to (\mathbb{R}^{n+1} - \{o\}, \tilde{\nabla})$ and for any function $\lambda: M \to \mathbb{R}+$, the mapping $x \mapsto \lambda(x)f(x)$ defines a centro-affine hypersurface $\lambda f: (M, \nabla') \to (\mathbb{R}^{n+1} - \{o\}, \tilde{\nabla})$ where ∇' is projectivelty related to ∇ by

$$\nabla'_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X, \quad \text{where} \quad \rho = d \log \varphi.$$

Conversely, any projective change of (M, ∇) can be locally obtained in this manner.

The proof is straightforward and omitted. □

Corollary. Let (M, ∇, ω) be a differentiable manifold with a projectively flat equiaffine connecton. Then (M, ∇) can be locally realized as a centro-affine hypersurface in $\mathbb{R}^{n+1} - \{o\}$.

Proof. If (M, ∇') is flat, then it can be locally realized as a piece of a hyperplane with induced volume element ω_0 in $\mathbb{R}^{n+1} - \{o\}$. Now we can make a projective change back to ∇ by modifying this hyperplane by a suitable function λ, namely, $\lambda = \omega/\omega_0$. □

Example 5- Conormal immersion.

Let $f: (M, \nabla, \omega) \to \mathbb{R}^{n+1}$ is a nondegenerate affine immersion of an equiaffine structure with an equiaffine transversal field ξ. We denote by $\mathbb{R}_{n+1}$ the vector space dual to the vector space $\mathbb{R}^{n+1}$ underlying the affine space $\mathbb{R}^{n+1}$. We define $v: M \to \mathbb{R}_{n+1} - \{0\}$ as follows.

For $x \in M$, v_x is an element of $\mathbb{R}_{n+1}$ such that

(12) $$v_x(Y) = 0 \quad \text{for} \quad Y \in T_x(M) \quad \text{and} \quad v_x(\xi_x) = 1,$$

where Y and ξ_x are considered as elements of the vector space $\mathbb{R}^{n+1}$ naturally identified with $T_x(\mathbb{R}^{n+1})$. Denoting by $\tilde{\nabla}$ the usual flat connection in $\mathbb{R}_{n+1}$, we have

(13) $$(\tilde{\nabla}_Y v)(\xi) = 0 \quad \text{and} \quad (\tilde{\nabla}_Y v)(f_*X) = -h(Y,X) \quad \text{for all } X, Y \in T_x(M).$$

Since h is nondegenerate, we see that if $(\tilde{\nabla}_Y v)(f_*X) = 0$ for all X, then $Y = 0$. Since $\tilde{\nabla}_Y v = v_*(Y)$, it follows that the mapping v is nonsingular. Hence we may consider $v\colon M \to \mathbb{R}_{n+1} - \{0\}$ as a centro-affine hypersurface, called the <u>conormal immersion</u> for f.

Taking $-v$ as the transveral vector field as in Example 4 we write

(14) $$\tilde{\nabla}_X(v_*(Y)) = v_*(\nabla^*_X Y) - h^*(X,Y)v,$$

where ∇^* is an affine connection on M and h^* the second fundamental form. These are related to the affine connection ∇, the affine metric h and the affine shape operator S fot the original hypersurface $f\colon M \to \mathbb{R}^{n+1}$ in the following way:

(15) $$h^*(X,Y) = h(SX,Y) \quad \text{(also equal to } \gamma_*(X,Y) \text{ as in Example 4)}$$

(16) $$X\,h(Y,Z) = h(\nabla^*_X Y, Z) + h(\nabla_X Z, Y)$$

and

(17) $\hat{\nabla}_X Y = (\nabla_X Y + \nabla^*{}_X Y)/2,$

where $\hat{\nabla}$ denotes the Levi-Civita connection for the affine metric h.

The formulas (15) and (16) are consequences of basic formulas for f and (12),(13) and (14). (17) follows from (16). They can be found, in different notations, in [6], p.127-129. It is a classical fact that the cubic form C for f vanishes if and only if $\nabla = \hat{\nabla} = \nabla^*$.

<u>Example</u> 6 - <u>Blaschke immersion</u>. Suppose f: $(M,\nabla,\omega) \to (\tilde{M},\tilde{\nabla},\tilde{\omega})$ is an affine immersion with equiaffine transversal field. If, furthermore, f is nondegenerate and if ω coincides with the volume element ω_h of the nondegenerate metric h, then we say that f is a Blaschke immersion. For the case where $(\tilde{M},\tilde{\nabla},\tilde{\omega})$ is an ordinary affine space $\mathbb{R}^{n+1}$ with the flat affine connection and the standard volume element given by the determinant, this is exactly the kind of affine immersion which has been the primary object of study in affine differential geometry developed by Blaschke and his school in the period 1910-40. The first step in the subject is to prove, for the standard equiaffine structure in $\mathbb{R}^{n+1}$, the following basic result.

Let M be a hypersurface immersed in $\mathbb{R}^{n+1}$. For any choice of a transversal vector field ξ, define an affine connection ∇ and the bilinear form h by equation (1). Whether h is nondegenerate or not is independent of the choice of

ξ, and we say that M is nondegenerate if h is. Denote by ω_h the volume element for h.

Proposition 6. If M is a nondegenerate hypersurface immersed in $\mathbb{R}^{n+1}$, there is a unique choice of ξ such that

i) ω_h coincides with ω defined by $\omega(X_1,\ldots,X_n) = \tilde{\omega}(X_1,\ldots,X_n, \xi)$;

ii) (M,∇,ω) is equiaffine.

This unique ξ is called the affine normal and the corresponding h the affine metric.

The proof of Proposition 6 can be found in [4].

3. <u>Affine immersions</u> $\mathbb{R}^n \to \mathbb{R}^{n+1}$

In this section we are interested in classifying all affine immersions: $M = \mathbb{R}^n \to \mathbb{R}^{n+1}$. We always choose an equiaffine transversal field ξ as we may. From Section 1 we have the formulas

$h(Y,Z)\,SX = h(X,Z)\,SY$ - Gauss equation in case $R = 0$ -

$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$ - Codazzi equation for h -

$(\nabla_X S)(Y) = (\nabla_Y S)(X)$ - Codazzi equation for S -

$h(SX,Y) = h(X,SY)$ - Ricci equation -.

If h is identically 0, then f is totally geodesic and $f(\mathbb{R}^n)$ is an affine hyperplane in $\mathbb{R}^{n+1}$. If S is identically 0, then by Proposition 4 f is a graph immersion.

In the general case, let $\Omega = \{x \in M;\ S_x \neq 0,\ \ h_x \neq 0\}$. We prove

<u>Lemma</u> 1. <u>For each</u> $x \in \Omega$, $\operatorname{Ker} h = \operatorname{Ker} S$ <u>and its dimension is</u> $n-1$.

Proof. For each $x \in \Omega$ the equality $\operatorname{Ker} h = \operatorname{Ker} S$ follows directly from the definition and the Gauss equation. If for some $x \in \Omega$ we had $\operatorname{rank} S \geq 2$, then there would be tangent vectors X and Y such that SX and SY are linearly independent. The Gauss equation then would imply $X, Y \in \operatorname{Ker} h = \operatorname{Ker} S$, a contradiction. □

For $x \in \Omega$, the subspace $N_x = \operatorname{Ker} h_x = \operatorname{Ker} S_x \subset T_x(M)$ is called the <u>relative nullity space</u> at x.

<u>Lemma</u> 2. <u>The distribution</u> $N: x \to N_x$ <u>on</u> Ω <u>is involutive and totally</u>

geodesic.

Proof. It is sufficient to show that N is totally geodesic, that is, for vector fields Y, Z belonging to N, $\nabla_X Y \in N$. In the equation of Codazzi for h: $(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$ take $Y, Z \in N$. Then we get

$$X\,h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = Y\,h(X,Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)$$

and hence $h(X, \nabla_Y Z) = 0$. This being valid for all X, we have $\nabla_Y Z \in N$. □

Now if L is a leaf of the relative nullity foliation N, L is totally geodesic in $M = \mathbb{R}^n$. Indeed, f(L) is totally geodesic in $\mathbb{R}^{n+1}$. Our goal is to show that each leaf L is complete. Let x_t be a geodesic starting at x_0 in the leaf L. To show that x_t extends for all values of t in L, first extend it as a geodesic in M. It is sufficient to show that x_t lies in Ω, because then it lies in L. So suppose there is $b > 0$ such that $x_b \notin \Omega$ and $x_t \in \Omega$ for all $t < b$.

We need

Lemma 3. Let X be a vector field on some open subset W of Ω containing the geodesic x_t, $0 \le t < b$, such that $\nabla_X X = 0$, $X \in N$, and X at x_t equals the tangent vector $\overrightarrow{x_t}$ for $0 \le t < b$. Let U be a parallel vector field on $M = \mathbb{R}^n$ which is transversal to the hyperplane $H = \mathbb{R}^{n-1}$ of $M = \mathbb{R}^n$ that contains L.

(i) Write $\nabla_U X = \mu U + Z$ at each point $p \in \Omega \cap H$, where $Z_p \in N_p$. Then the function μ satisfies $X\mu = -\mu^2$ along x_t, $0 \le t < b$.

(ii) *Write* $SU = \lambda U + W$ *at each point* $p \in V$, *where* $W \in T_p(H)$. *Then the function* λ *satisfies* $X\lambda = -\mu\lambda$ *along* x_t, $0 \le t < b$.

(iii) *Let* $\rho = h(U,U)$ *on* V. *Then* $X\rho = -\mu\rho$ *along* x_t, $0 \le t < b$.

Proof.

(i) $\nabla_X(\nabla_U X) = \nabla_X(\mu U + Z) = (X\mu)U + \mu \nabla_X Z + \nabla_X U = (X\mu)U \mod N.$

Since $R = 0$, we have along x_t, $0 \le t < b$

$$\nabla_X(\nabla_U X) = \nabla_{[X,U]}X = -\nabla_{\nabla_U X}X = -\nabla_{\mu U + Z}X$$

$$= -\mu\nabla_U X - \nabla_Z X \equiv -\mu^2 U \mod N.$$

Hence $(X\mu)U \equiv -\mu^2 U \mod N$ and $X\mu = -\mu^2$.

(ii) From the Codazzi equation for S

$$\nabla_X(SU) - S(\nabla_X U) = \nabla_U(SX) - S(\nabla_U X),$$

we get along x_t, $0 \le t < b$

$$(X\lambda)U + \lambda(\nabla_X U) + \nabla_X W = -\mu\, SU = -\mu(\lambda Y + W)$$

and $(X\lambda)U \equiv -\mu\lambda U \mod N$. Thus $X\lambda = -\mu\lambda$ along x_t.

(iii) We have along x_t, $0 \le t < b$

$$X\rho = Xh(U,U) = (\nabla_X h)(U,U) - 2h(\nabla_X U,U) = (\nabla_U h)(X,U)$$

$$= U h(X,U) - h(\nabla_U X,U) - h(X,\nabla_U U) = -\mu\, h(U,U) = -\mu\rho. \qquad \square$$

Now we can conclude the proof that $x_b \in \Omega$ as follows. The equations in (i), (ii) and (iii) are

$d\mu/dt = -\mu^2$, $\quad d\lambda/dt = -\lambda\mu$, $\quad d\rho/dt = -\rho\mu$ for $0 \le t < b$.

Thus μ is identically 0 or $\mu = 1/(t+a)$ for some a. It follows that $\lambda =$ constant or $\lambda = 1/c(t+a)$ and the same for ρ. In all cases, neither λ nor ρ approaches 0 as $t \to b$. Now at the point $p = x_b$, this means $SU \neq 0$ as well as $h(U,U) \neq 0$. Thus $p \in \Omega$.

With completeness of L established, we know $x_t \in L$ for all t. Thus the possibility of $\mu = 1/(t+a)$ is excluded. Hence $\mu = 0$ and thus λ and ρ are equal to constants on the leaf L.

We can now prove

<u>Proposition</u> 5. <u>Let</u> $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$ <u>be an affine immersion such that</u> S <u>and</u> h <u>vanish nowhere</u>. <u>Then</u> f <u>is affine-equivalent to a proper affine cylinder immersion</u>.

Proof. In the foregoing discussions, we now have $\Omega = \mathbb{R}^n$. We have already proved that each leaf of the relative nullity foliation is complete. Thus each leaf is a hyperplane in $\mathbb{R}^n$, and all leaves are parallel hyperplanes because they are disjoint from each other.

We take a vector U transversal to all these hyperplanes and consider a line x_t in the direction of of U. Write $\mathbb{R}^{n-1}(t)$ for the leaf through the point x_t. Since each leaf is mapped totally geodesically, $f(\mathbb{R}^{n-1}(t))$ is an affine (n-1)-space in $\mathbb{R}^{n+1}$. Also, if Y_t is a parallel vector field along x_t such that $Y_t \in T_x(\mathbb{R}^{n-1}(t))$, then

$$\tilde{\nabla}_t f_*(Y_t) = f_*(\nabla_t Y_t) + h(U, Y_t) = 0.$$

Thus $f_*(Y_t)$ is parallel in $\mathbb{R}^{n+1}$. This shows that all subspaces $f(\mathbb{R}^{n-1}(t))$ are parallel to each other.

Now it is easy to verify that f is affinely equivalent to a proper affine cylinder immersion based on the parallel family $f(\mathbb{R}^{n-1}(t))$ and the curve $\gamma(t) = f(x_t)$. The original transversal field ξ_t is in the direction of $\gamma'(t)$.

We can now state

<u>Theorem1</u>. <u>Let</u> $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ <u>be an affine immersion</u>. <u>Then</u> $\Omega = \{x \in \mathbb{R}^n ; S_x \neq 0,\ h_x \neq 0\}$, <u>if not empty</u>, <u>is the union of parallel hyperplanes</u>. <u>Each connected component</u> Ω_α <u>of</u> Ω <u>is a strip consisting of parallel hyperplanes and</u> $f : \Omega_\alpha \to \mathbb{R}^{n+1}$ <u>is affinely equivalent to a proper affine cylinder immersion</u>.

<u>Remark</u>. On each component of $\mathbb{R}^n - \cup\ \overline{\Omega_\alpha}$ is a mixture of graph immersions and totally geodesic immersions. One can easily construct examples piecing together different types of affine immersions, but proving a general description is not easy.

<u>Corollary</u>. <u>An analytic immersion</u> $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ <u>is either totally goedesic or affinely equivalent to a graph immersion or affinely equivalent to an affine cylinder immersion</u>.

Proof. If h or S is identically 0, we know that f is totally geodesic or a graph immersion. Otherwise, the open subset Ω is dense. On each connected component Ω_α, f is a proper affine cylinder immersion. Since Ω is dense, all these immersions of the components extend to an affine cylinder immersion f. □

Remark. It is not difficult to construct a C^∞ affine immersion $M^2 \to \mathbb{R}^3$ of the affine Möbius band $M^2 = \mathbb{R}^2/\varphi$, where φ is the affine map : $(x,y) \mapsto (x+1,-y)$. By the corollary, however, there can be no analytic immersion of this kind.

4. Affine immersions of pseudo-riemannian manifolds

We prove the following theorem which is a precise statement for the result of Cartan and Norden mentioned in the introduction.

Theorem 2. Let (M^n,g) be a pseudo-riemannian manifold, ∇ its Levi-Civita connection and $f: (M^n,\nabla) \to \mathbb{R}^{n+1}$ an affine immersion with a transversal field ξ. If f is nondegenerate, we have either

(i) ∇ is flat and f is a graph immersion;

or

(ii) ∇ is not flat and $\mathbb{R}^{n+1}$ admits a parallel pseudo-riemannian metric relative to which f is an isometric immersion and ξ is perpendicular to $f(M^n)$.

Proof. We first establish

Lemma. Let (M,h) be a pseudo-riemannian manifold and let ∇ and ∇^* be two affine connections with zero torsion on M which are conjugate relative to h, that is,

$$X\,h(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*{}_X Z)$$

for all vector fields X, Y and Z. Let B be a nonsingular $(1,1)$ tensor field which is symmetric relative to h and define pseudo-riemannian metrics g and g^* by

$$g(X,Y) = h(BX,Y) \quad \text{and} \quad g^*(X,Y) = h(B^{-1}X, Y).$$

Then $(\nabla_X g)(Y,Z) + (\nabla^*{}_X g^*)(Y,Z) = 0$ for all vector fields X, Y and Z. In particular, ∇ is the Levi-Civita connection for g if and only if ∇^* is the Levi-Civita connection for g^*.

Proof. We have

$$(\nabla^*{}_X g^*)(Y,Z) = X\,g^*(Y,Z) - g^*(\nabla^*{}_X Y, Z) - g^*(Y, \nabla^*{}_X Z)$$

$$= Xh(B^{-1}Y, Z) - h(\nabla^*{}_X Y, B^{-1}Z) - h(B^{-1}Y, \nabla^*{}_X Z)$$

$$= Xh(B^{-1}Y, Z) - \{X\,h(Y, B^{-1}Z) - h(Y, \nabla_X B^{-1}Z)\}$$

$$- \{X\,h(Z, B^{-1}Y) - h(Z, \nabla_X B^{-1}X)\}$$

$$= h(Z, \nabla_X B^{-1}Y) + h(Y, \nabla_X B^{-1}Z) - X\,h(Y, B^{-1}Z).$$

Replacing Y, Z by BY, BZ we get

$$(\nabla^*{}_X g^*)(BY, BZ) = h(BZ, \nabla_X Y) + h(BY, \nabla_X Z) - Xh(BY, Z)$$

$$= g(Z, \nabla_X Y) + g(Y, \nabla_X Z) - Xg(Y,Z) = -(\nabla_X g)(Y,Z). \qquad \square$$

To prove the theorem, we may assume that ξ is equiaffine and we consider the conormal immersion $\nu: (M^n, \nabla^*) \to \mathbb{R}_{n+1}$. We recall that the affine connection ∇^* is conjugate to ∇ relative to the form h for f; cf. equation

(16).

Since h is nondegenerate, we may write $g(X,Y) = h(BX,Y)$, where B is a certain nonsingular (1,1) tensor symmetric relative to h. We define a pseudo-riemannian metric g^* by $g^*(X,Y) = h(B^{-1}X,Y)$. By the lemma, we see that ∇^* is the Levi-Civita connection for g^*.

Now the conormal immersion being a centro-affine immersion, we know that ∇^* is projectively flat. Since ∇^* is the Levi-civita connection for g^*, it follows by a theorem of Dini-Beltrami that g^* has constant sectional curvature, say, c. The form $h*$ for the conormal immersion is, by equation (11), equal to the normalized Ricci tensor γ^*, which is in this case equal to $c\, g^*$. Thus $h* = c\, g^*$, in particular, $\nabla^* h^* = 0$.

Case (i): $c = 0$. Then ∇^* is flat. Since $h* = 0$, by (15) the shape operator S for f is 0 and by the Gauss equation ∇ is flat. By Proposition 4 we conclude that f is a graph immersion.

Case (ii): $c \neq 0$. We shall show that $\mathbb{R}_{n+1}$ admits a parallel pseudo-riemannian metric $\langle\ ,\ \rangle^*$ such that

$$\langle v_*(X),\, v_*(Y)\rangle^* = g^*(X,Y) \quad \text{for} \quad X,Y \in T_x(M)$$

$$\langle v,\, v_*(X)\rangle^* = 0 \quad \text{for} \quad X \in T_x(M)$$

$$\langle v,v\rangle^* = -1/c.$$

For this purpose, we define $\langle\ ,\ \rangle^*$ in each $T_{v(x)}(\mathbb{R}_{n+1})$ using exactly the above three equations and show that this metric tensor field along v is parallel in $\mathbb{R}_{n+1}$. Thus we wish to verify

$(*) \quad X \langle U, V \rangle^* = \langle \tilde{\nabla}_X U, V \rangle^* + \langle U, \tilde{\nabla}_X V \rangle^*$

for all vector fields U and V along v and a vector field X on M.

If U and V are of the form $v_*(Y)$ and $v_*(Z)$, where Y and Z are vector fields on M, the equation (*) reduces to $(\nabla^*{}_X g^*)(Y,Z) = 0$.

If $U = v_*(Y)$ and $V = v$, then $X \langle v_*(Y), v \rangle^* = 0$ and

$$\langle \tilde{\nabla}_X U, V \rangle^* = \langle \tilde{\nabla}_X v_*(Y), v \rangle^* = \langle v_*(\nabla_X Y), v \rangle^* + \langle h^*(X,Y)v, v \rangle^*$$

$$= h^*(X,Y) \langle v, v \rangle^* = - h^*(X,Y)/c$$

as well as $\langle U, \tilde{\nabla}_X V \rangle = \langle v_*(X), v_*(Y) \rangle = g^*(X,Y)$. Thus (*) is satisfied. Finally, if $U = V = v$, (*) is obvious.

Now it remains to show that $\mathbb{R}^{n+1}$ admits a parallel pseudo-riemannian metric $\langle \; , \; \rangle$ such that

$$\langle f_*(X), f_*(Y) \rangle = g(X,Y), \quad \langle f_*(X), \xi \rangle = 0, \quad \langle \xi, \xi \rangle = -1/c$$

for all vector fields X and Y on M. Indeed, using the nondegenerate form $\langle \; , \; \rangle^*$ in $\mathbb{R}_{n+1}$, we identify $\mathbb{R}_{n+1}$ with $\mathbb{R}^{n+1}$ (both as vector spaces) by $u \in \mathbb{R}_{n+1} \mapsto \theta(u) \in \mathbb{R}^{n+1}$ with $w(\theta(u)) = \langle u, w \rangle^*$ for all $w \in \mathbb{R}_{n+1}$. We then define $\langle \; , \; \rangle$ in $\mathbb{R}^{n+1}$ as the dual inner product, namely,

$$\langle X, Y \rangle = \langle \theta^{-1}(X), \theta^{-1}(Y) \rangle^* \quad \text{for } X, Y \in \mathbb{R}^{n+1}.$$

In order to show that this inner product $\langle \; , \; \rangle$ is the desired one, we first remark the following fact. Let $u = v_*(X)$ for $X \in T_x(M)$. Then for any $Y \in T_x(M)$ we have $v_*(Y)\,(\theta(u)) = \langle v_*(Y), v_*(X) \rangle^* = g^*(X,Y)$. On the other hand, $v(\theta(u)) = 0$. It follows that $\theta(u) = -f_*(B^{-1}X)$, where B is a certain nonsingular (1,1) tensor. We have

$$g^*(X,Y) = v_*(Y)\theta(u) = -v_*(Y)(f_*(B^{-1}X)) = h(B^{-1}X,Y),$$

where we use the relation (13). Now for X,Y we have

$$f_*(B^{-1}X) = -\theta(v_*(X)), \quad f_*(B^{-1}Y) = -\theta(v_*(Y))$$

and

$$\langle f_*(B^{-1}X), f_*(B^{-1}Y)\rangle = \langle v_*(X), v_*(Y)\rangle^* = g^*(X,Y).$$

Replacing X,Y by BX, BY in this equation we obtain

$$\langle f_*(X), f_*(Y)\rangle = g^*(BX,BY) = h(B^{-1}BX,BY) = h(X,BY).$$

But as in the lemma, $h(X,BY) = g(X,Y)$. Hence

$$g(X,Y) = \langle f_*(X), f_*(Y)\rangle.$$

The other identities are obvious from $\theta(v) = \xi$. The proof of the theorem is now complete. □

We state a few corollaries.

Corollary 1. *Let (M^n,g) be a pseudo-riemannian manifold, ∇ its Levi-Civita connection, and $f: (M^n,\nabla) \to \mathbb{R}^{n+1}$ an affine immersion. If the Ricci tensor of g is nondegenerate, then $\mathbb{R}^{n+1}$ admits a parallel pseudo-riemannian metric such that f is an isometric immersion and the transversal field is perpendicular to $f(M^n)$.*

Proof. From $Ric(Y,Z) = h(Y,Z)\,trS - h(SY,Z)$, it follows that h is nondegenerate if the Ricci tensor is nondegenerate.

Corollary 2. Let g be a riemannian metric on S^2 with Gaussian curvature

$K > 0$ and Levi-Civita connection ∇. Then there exists an affine immersion f: $(S^2, \nabla) \to \mathbb{R}^3$ which is unique up to an affine transformation of $\mathbb{R}^3$.

Proof. By the solution to Weyl's problem (see, for example, [9], p.226) (S^2, g) has an isometric imbedding f into euclidean space $\mathbb{R}^3$ with standard metric and it is rigid. So $f : (S^2, \nabla) \to \mathbb{R}^3$ is an affine imbedding. Suppose $f_1 : (S^2, \nabla) \to \mathbb{R}^3$ is another affine immersion. Theorem 2 implies that it is isometric relative to a certain parallel pseudo-riemannian metric $\langle\ ,\ \rangle$ in $\mathbb{R}^3$. This metric must be Euclidean to accommodate a compact surface with positive definite metric induced on it. Since one can find an affine transformation A of $\mathbb{R}^3$ which transforms the metric $\langle\ ,\ \rangle$ into the standard metric, it follows that $A \circ f_1$ is an isometric immersion into $\mathbb{R}^3$ with standard euclidean metric, and as such, congruent to f. This means that f_1 differs from f by an affine transformation. □

<u>Corollary</u> 3. <u>Let</u> g <u>be the standard riemannian metric on</u> S^n <u>with constant sectional curvature</u> 1. <u>For every affine immersion</u> $f: (S^n, \nabla) \to \mathbb{R}^{n+1}$, <u>the image</u> $f(S^n)$ <u>is an ellipsoid</u> (<u>relative to a Euclidean metric</u>).

<u>Corollary</u> 4. <u>Let</u> (H^n, g) <u>be the hyperbolic space with standard riemannian metric of constant sectional curvature</u> -1. <u>Then every affine transformation</u> $f: (H^n, \nabla) \to \mathbb{R}^{n+1}$ <u>is an isometric immersion of</u> (H^n, g) <u>into</u> $\mathbb{R}^{n+1}$ <u>with flat Lorentz metric</u>. <u>If</u> $n \geq 3$, $f(M^n)$ <u>is affinely congruent to one component of the two-sheeted hyperboloid</u> $-x_0^2 + x_1^2 + \cdots + x_n^2 = -1$, $x_0 > 0$.

<u>Remark</u> 2. In the proof of Theorem 2, the sign of c generally depends on

the affine immersion f.

5. Equiaffine immersions of compact manifolds

It is a standard theorem in euclidean differential geometry that a compact riemannian manifold (M_n, g) with negative-definite Ricci tensor cannot be isometrically immersed in a euclidean space $\mathbb{R}^{n+1}$: any compact immersed hypersurface has to be locally strictly convex somewhere and the Ricci tensor is positive-definite at convex points. For affine immersions this argument does not apply, because convexity does not imply positivity of the Ricci tensor. For example, the hyperbolic space H^n can be affinely imbedded as one component of a two-sheeted hyperboloid.

We can still prove

Theorem 3. *Let (M^n, ∇, ω) be a compact equiaffine manifold with negative-definite Ricci tensor (or more generally, with nondegenerate, but not positive-definite, Ricci tensor). Then (M^n, ∇) does not admit an affine immersion into $\mathbb{R}^{n+1}$.*

Proof. Let $f: (M^n, \nabla) \to \mathbb{R}^{n+1}$ be an affine immersion. We choose a transversal field to be equiaffine. As in Corollary 1 in Section 4, h is nondegenerate with the Ricci tensor. Thus viewing M^n as a hypersurface in euclidean space $\mathbb{R}^{n+1}$, the usual second fundamental form is proportional to h and thus nondegenerate. It follows that M^n is diffeomorphic to S^n, h is definite, and $f(M^n)$ is a strictly convex hypersurface (fo example, see [4], p.41). By diagonalizing S relative to h, we see that Ric for ∇ is positive-definite at a point where the bilinear form $B(Y,Z) = h(SY,Z)$ is positve-definite. We shall show that there is such a point, contradicting the assumption on Ric and thus concluding the proof of Theorem 3.

From Example 5 recall that $(n-1)B$ is equal to the Ricci tensor of the conormal connection ∇^* on M, which is equiaffine and projectively flat. Thus our assertion will follow from the next lemma.

Lemma. Let $\tilde{\nabla}$ be a projectively flat equiaffine connection on S^n with volume element $\tilde{\omega}$. Then there are points on S^n where the Ricci tensor of $\tilde{\nabla}$ is positive-definite.

Proof . Recall that $(S^n, \tilde{\nabla})$ is projectively equivalent to (S^n, ∇_0), where ∇_0 is the standard affine connection (Levi-Civita connection) on S^n (see, for example [3]). Consider S^n as a unit sphere in $\mathbb{R}^{n+1}$. We may obtain a centro-affine immersion $\varphi: S^n \to \mathbb{R}^{n+1}$ so that the induced volume element coincides with $\tilde{\omega}$. The induced connection ∇^* is projectively flat and coincides with $\tilde{\nabla}$, since they have the same volume element. See, for example, [5], Proposition 2.

Thus we may consider $\varphi: S^n \to \mathbb{R}^{n+1}$, where the image $\varphi(S^n)$ is star-shaped with respect to the origin. Let p be a point where a nondegenerate height function has a maximum. Then $\varphi(S^n)$ is strictly convex towards the origin at p, and thus by (11) Ric is positive-definite at p. $\square$

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