GENERALIZED FOURIER TRANSFORMS $\mathscr{F}_{k,a}$

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Abstract. We construct a two-parameter family of actions $\omega_{k,a}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ by differential-difference operators on $\mathbb{R}^N$. Here, k is a multiplicity-function for the Dunkl operators, and $a > 0$ arises from the interpolation of the Weil representation and the minimal unitary representation of the conformal group. We prove that this action $\omega_{k,a}$ lifts to a unitary representation of the universal covering of $SL(2,\mathbb{R})$, and can even be extended to a holomorphic semigroup $\Omega_{k,a}$. Our semigroup generalizes the Hermite semigroup studied by R. Howe ($k \equiv 0$, $a = 2$) and the Laguerre semigroup by T. Kobayashi and G. Mano ($k \equiv 0$, $a = 1$). The boundary value of our semigroup $\Omega_{k,a}$ provides us with (k,a)-*generalized Fourier transforms* $\mathscr{F}_{k,a}$, which includes the Dunkl transform $\mathscr{D}_k$ ($a = 2$) and a new unitary operator $\mathscr{H}_k$ ($a = 1$) as a Dunkl-type generalization of the classical Hankel transform. We establish a generalization of the Plancherel theorem, and the Heisenberg uncertainty principle for $\mathscr{F}_{k,a}$. We also find explicit kernel functions for $\Omega_{k,a}$ and $\mathscr{F}_{k,a}$ for $a = 1, 2$ by means of Bessel functions and the Dunkl intertwining operator.

1. Introduction

The classical Fourier transform is one of the most basic objects in analysis; it may be understood as belonging to a one-parameter group of unitary operators on $L^2(\mathbb{R}^N)$, and this group may even be extended holomorphically to a semigroup (the *Hermite semigroup*) $I(z)$ generated by the self-adjoint operator $\Delta - \|x\|^2$. This is a holomorphic semigroup of bounded operators depending on a complex variable z in the complex right half-plane, viz. $I(z + w) = I(z)I(w)$.

The primary aim of our study is to analyze the Dunkl Laplacian Δ_k and to construct a deformation of the classical situation, namely, a generalization $\mathscr{F}_{k,a}$ of the Fourier transform, and the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ with infinitesimal generator $\|x\|^{2-a}\Delta_k - \|x\|^a$, acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^N)$. We investigate these operators $\mathscr{F}_{k,a}$ and $\mathscr{I}_{k,a}(z)$ in the context of integral operators as well as representation theory.

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The deformation parameters in our setting consist of a real parameter a and a parameter k coming from Dunkl's theory of differential-difference operators associated to a finite Coxeter group; also the dimension N and the complex variable z may be considered as parameters of the theory. We point out, that already deformations with $k \equiv 0$ are new. They might provide interesting generalizations of classical pseudo-differential calculus, and possibly new generators of stochastic processes. Also our study could provide new insights about the analysis of Dunkl operators and Dunkl Laplacians, since we in particular construct a new Hermite–Dunkl semigroup $\mathscr{I}_{k,2}(z)$ and a new Hankel–Dunkl transform $\mathscr{F}_{k,1}$.

In the diagram below we have summarized some of the deformation properties by indicating the limit behaviour of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$; it is seen how various previous integral transforms fit in our picture. In particular we obtain as special cases the Dunkl transform $\mathscr{D}_k$ [2] ($a = 2$, $z = \frac{\pi i}{2}$ and k arbitrary), the Hermite semigroup $I(z)$ [5] ($a = 2$, $k = 0$ and z arbitrary), and the Laguerre semigroup [7] ($a = 1$, $k \equiv 0$ and z arbitrary).

The 'boundary value' of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ from $\operatorname{Re} z > 0$ to the imaginary axis gives rise to a one-parameter subgroup of unitary operators. The specialization $\mathscr{I}_{k,a}(\frac{\pi i}{2})$ will be called as a (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ (up to a phase factor), which reduces to the Fourier transform ($a = 2$ and $k \equiv 0$), the Dunkl transform $\mathscr{D}_k$ ($a = 2$ and k arbitrary), and the Hankel transform ($a = 1$ and $k \equiv 0$).

The basic machinery of the present article is to construct triples of differential-difference operators generating the Lie algebra of $SL(2, \mathbb{R})$, and see how they are integrated to unitary representations of the universal covering group.

One further aspect of our constructions is the link to minimal unitary representations of two reductive groups.

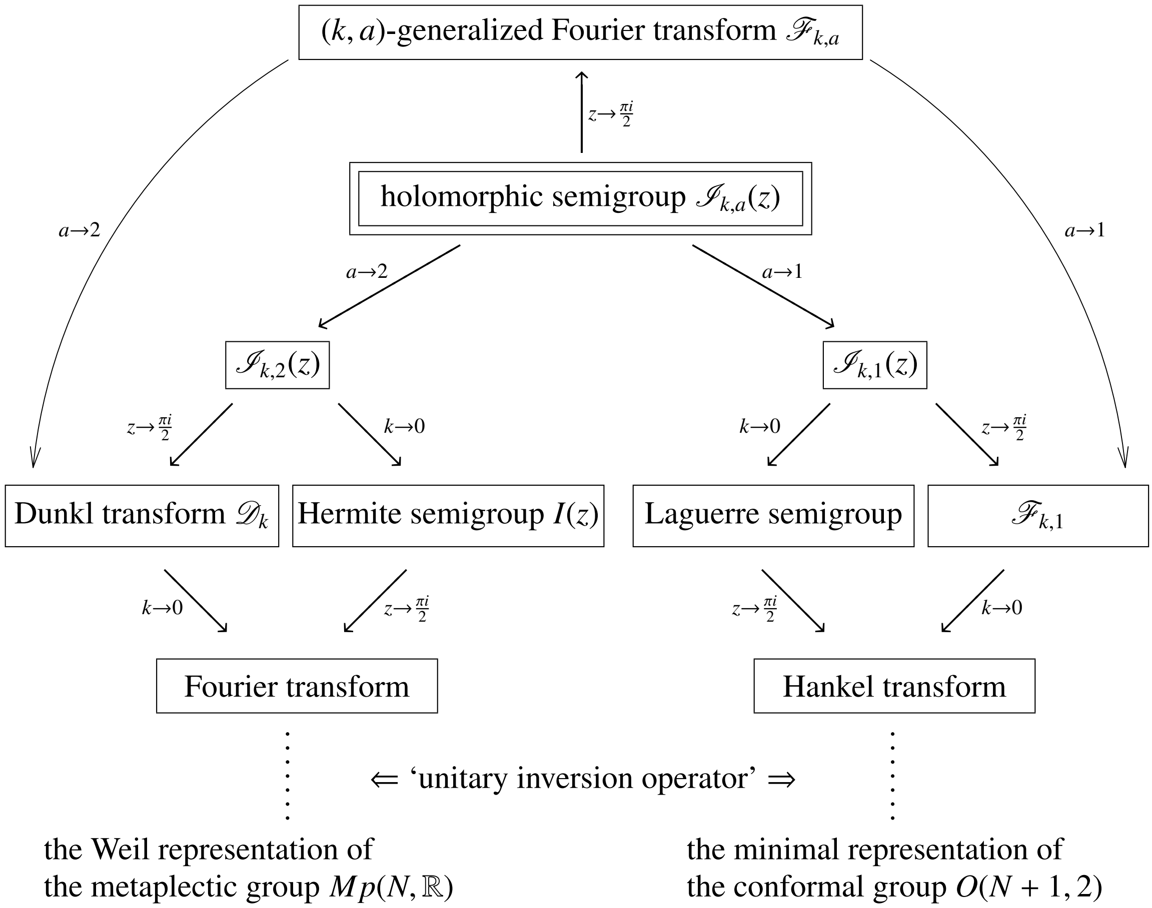

Diagram 1. Special values of holomorphic semigroup $\mathscr{I}_{k,a}(z)$

2. Holomorphic semigroup $\mathscr{I}_{k,a}(z)$ with two parameters k and a

Our holomorphic semigroup $\mathscr{I}_{k,a}(z)$ is built on Dunkl operators [1]. To fix notation, let $\mathfrak{C}$ be the Coxeter group associated with a reduced root system $\mathscr{R}$ in $\mathbb{R}^N$. For a $\mathfrak{C}$-invariant function $k \equiv (k_\alpha)$ (*multiplicity function*) on $\mathscr{R}$, we set $\langle k \rangle := \frac{1}{2} \sum_{\alpha \in \mathscr{R}} k_\alpha$, and write Δ_k for the Dunkl Laplacian on $\mathbb{R}^N$. This is a differential-difference operator, which reduces to the Euclidean Laplacian when $k \equiv 0$.

We take $a > 0$ to be yet another deformation parameter, and introduce the following differential-difference operator

$$\Delta_{k,a} := \|x\|^{2-a} \Delta_k - \|x\|^a. \tag{2.1}$$

Here, $\|x\|$ is the standard norm of $x \in \mathbb{R}^N$. We define a density function $\vartheta_{k,a}(x)$ on $\mathbb{R}^N$ by

$$\vartheta_{k,a}(x) := \|x\|^{a-2} \prod_{\alpha \in \mathscr{R}} |\langle \alpha, x \rangle|^{k_\alpha}. \tag{2.2}$$

In the case $a = 2$ and $k \equiv 0$, we have $\vartheta_{0,2}(x) \equiv 1$ and recover the classical setting where

$$\Delta_{0,2} = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \sum_{j=1}^{N} x_j^2, \text{ the Hermite operator on } L^2(\mathbb{R}^N).$$

Here are remarkable properties of our differential-difference operator $\Delta_{k,a}$:

Theorem A. *Suppose $a > 0$ and $a + 2\langle k \rangle + N - 2 > 0$.*
1) *$\Delta_{k,a}$ extends to a self-adjoint operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*
2) *There is no continuous spectrum of $\Delta_{k,a}$.*
3) *All the discrete spectra of $\Delta_{k,a}$ are negative.*

The (k, a)*-generalized Laguerre semigroup* $\mathscr{I}_{k,a}(z)$ is defined to be the semigroup with infinitesimal generator $\frac{1}{a}\Delta_{k,a}$, that is,

$$\mathscr{I}_{k,a}(z) := \exp\Big(\frac{z}{a}\Delta_{k,a}\Big), \quad \text{for } \operatorname{Re} z \geq 0. \tag{2.3}$$

We note that $\mathscr{I}_{0,2}(z)$ is the Hermite semigroup $I(z)$, and $\mathscr{I}_{0,1}(z)$ is the Laguerre semigroup (see [5], [7], respectively).

Theorem B. *Suppose $a > 0$ and $a + 2\langle k \rangle + N - 2 > 0$.*
1) *$\mathscr{I}_{k,a}(z)$ is a holomorphic semigroup in the complex right-half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ in the sense that $\mathscr{I}_{k,a}(z)$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ satisfying*

$$\mathscr{I}_{k,a}(z_1) \circ \mathscr{I}_{k,a}(z_2) = \mathscr{I}_{k,a}(z_1 + z_2), \quad (\operatorname{Re} z_1, \operatorname{Re} z_2 > 0),$$

and that the scalar product $(\mathscr{I}_{k,a}(z)f, g)$ is a holomorphic function of z for $\operatorname{Re} z > 0$, for any $f, g \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.
2) *$\mathscr{I}_{k,a}(z)$ is a one-parameter group of unitary operators on the imaginary axis $\operatorname{Re} z = 0$.*

3. (k, a)-generalized Fourier transforms $\mathscr{F}_{k,a}$

As we mentioned in Theorem B (2), the 'boundary value' of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ on the imaginary axis gives a one-parameter family of unitary operators. The underlying idea may be interpreted as a descendent of Sato's hyperfunction theory [12] and also that of the Gelfand–Gindikin program (see [4] and references therein) for unitary representation of real reductive groups.

The case $z = 0$ gives the identity operator, namely, $\mathscr{I}_{k,a}(0) = \mathrm{id}$. The particularly interesting case is when $z = \frac{\pi i}{2}$, and we set

$$\mathscr{F}_{k,a} := c\, \mathscr{I}_{k,a}\Big(\frac{\pi i}{2}\Big) = c \exp\Big(\frac{\pi i}{2a}(\|x\|^{2-a}\Delta_k - \|x\|^a)\Big)$$

by multiplying the phase factor $c = \exp(i\pi\frac{N+2\langle k\rangle+a-2}{2a})$. Then, the unitary operator $\mathscr{F}_{k,a}$ for general a and k satisfies the following significant properties:

Theorem C. *Suppose $a > 0$ and $a + 2\langle k\rangle + N - 2 > 0$.*

1) *$\mathscr{F}_{k,a}$ is a unitary operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*
2) $\mathscr{F}_{k,a} \circ E = -(E + N + 2\langle k\rangle + a - 2) \circ \mathscr{F}_{k,a}$. *Here,* $E = \sum_{j=1}^{N} x_j\partial_j$.
3) $\mathscr{F}_{k,a} \circ \|x\|^a = -\|x\|^{2-a}\Delta_k \circ \mathscr{F}_{k,a}$, $\mathscr{F}_{k,a} \circ (\|x\|^{2-a}\Delta_k) = -\|x\|^a \circ \mathscr{F}_{k,a}$.
4) *$\mathscr{F}_{k,a}$ is of finite order if and only if $a \in \mathbb{Q}$. Its order is $2p$ if a is of the form $a = \frac{p}{q}$, where p and q are positive integers that are prime to each other.*

We call $\mathscr{F}_{k,a}$ a (k,a)*-generalized Fourier transform*. As indicated in Diagram 1, $\mathscr{F}_{k,a}$ reduces to the Euclidean Fourier transform $\mathscr{F}$ on $\mathbb{R}^N$ if $k \equiv 0$ and $a = 2$; to the Dunkl transform $\mathscr{D}_k$ introduced by C. Dunkl himself in [2] (E_k in his notation) if $k > 0$ and $a = 2$.

Thus, in these classical setting, our approach uses the following expressions of $\mathscr{F}_{k,a}$:

$$\mathscr{F} = \exp(\frac{i\pi N}{4}) \exp \frac{\pi i}{4}(\Delta - \|x\|^2) \qquad \text{(Fourier transform)},$$

$$\mathscr{D}_k = \exp(\frac{i\pi(2\langle k\rangle + N)}{4}) \exp \frac{\pi i}{4}(\Delta_k - \|x\|^2) \qquad \text{(Dunkl transform)}.$$

For $a = 1$ and $k \equiv 0$, the unitary operator $\mathscr{F}_{0,1} = \exp(\frac{i\pi(N-1)}{2}) \exp\Big(\frac{\pi i}{2}\|x\|(\Delta - 1)\Big)$ arises as the *unitary inversion operator* of the Schrödinger model of the minimal representation of the conformal group $O(N + 1, 2)$ (see [7]). Its Dunkl analogue

$$\mathscr{H}_k \ (= \mathscr{F}_{k,1}) := \exp\left(\frac{i\pi(2\langle k\rangle + N - 1)}{2}\right) \exp\left(\frac{\pi i}{2}\|x\|(\Delta_k - 1)\right)$$

is an involutive unitary operator on $L^2(\mathbb{R}^N, \vartheta_{k,1}(x)dx)$ whose kernel is explicitly given by using the formula (4.4) below.

Our study also contributes to the theory of special functions, in particular orthogonal polynomials; indeed we derive several new identities, for example, the (k,a)-deformation of the classical Hecke identity where the Gaussian function and harmonic polynomials in the classical setting are replaced respectively with $\exp(-\frac{1}{a}\|x\|^a)$ and polynomials annihilated by the Dunkl Laplacian. We also have:

Theorem D (Heisenberg type inequality)**.** *Let $\| \ \|_k$ denote by the norm on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. Then,*

$$\Big\| \|x\|^{\frac{a}{2}} f(x)\Big\|_k \Big\| \|\xi\|^{\frac{a}{2}} \mathscr{F}_{k,a} f(\xi)\Big\|_k \geq \frac{2\langle k\rangle + N + a - 2}{2} \|f(x)\|_k^2$$

for any $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. The equality holds if and only if f is a scalar multiple of $\exp(-c\|x\|^a)$ *for some $c > 0$.*

This inequality was previously known by Rösler [11] for the $a = 2$ case (i.e. the Dunkl transform $\mathscr{D}_k$). On the other hand, for $a = 1$, we may think of the function where the equality holds in Theorem D as a ground state in physics terms; indeed when $a = c = 1$ it is exactly the wave function for the Hydrogen atom with the lowest energy.

4. Integral representation of $\mathscr{I}_{k,a}(z)$ and $\mathscr{F}_{k,a}$

The Euclidean Fourier transform is given by the integral against the kernel $(2\pi)^{-\frac{N}{2}} e^{-i\langle x,\xi\rangle}$, and the Hermite semigroup is given by the Mehler kernel. In this section, we discuss the integral expression of the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ and a holomorphic semigroup $\mathscr{I}_{k,a}(z)$, as a generalization of the above mentioned case ($k \equiv 0$, $a = 2$).

In this subsection, we assume $k \geq 0$. *Dunkl's intertwining operator* V_k is a topological linear isomorphism of the space of continuous functions on $\mathbb{R}^N$, which intertwines the Dunkl operators and the directional derivatives [2]. For a continuous function $h(t)$ of one variable, we set $h_y(\cdot) := h(\langle \cdot, y\rangle)$ ($y \in \mathbb{R}^N$), and define

$$(\widetilde{V_k}h)(x, y) := (V_k h_y)(x).$$

Then $(\widetilde{V_k}h)(x, y)$ is a continuous function on $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Suppose more generally that h is a continuous function on the closed interval $[-1, 1]$. By using the Laplace type representation of V_k (see Rösler [11]), we see that $(\widetilde{V_k}h)(x, y)$ is well-defined as a continuous function on $B \times B$, where B denotes the unit ball in $\mathbb{R}^N$. Further, $(\widetilde{V_k}h)(x, y) = (\widetilde{V_k}h)(y, x)$. We note that $(\widetilde{V_k}h)(x, y) = h(\langle x, y\rangle)$ if $k \equiv 0$.

For $a > 0$ and a non-negative multiplicity function k, we introduce the following normalization constant

$$c_{k,a} := \left(\int_{\mathbb{R}^N} \exp\left(-\frac{1}{a}\|x\|^a\right) \vartheta_{k,a}(x) dx\right)^{-1}. \tag{4.1}$$

The constant $c_{k,a}$ can be explicitly found by means of the gamma function owing to the work by Selberg, Macdonald, Heckman, Opdam, and others (see [9]).

Let $\widetilde{I}_\lambda(w) = (\frac{w}{2})^{-\lambda} I_\lambda(w)$ be the normalized modified Bessel function of the first kind, and $C_m^\nu(t)$ is the Gegenbauer polynomial. We set

$$\mathscr{J}(b, \nu; w; t) = \frac{\Gamma(b\nu + 1)}{\nu} \sum_{m=0}^{\infty} (m + \nu) \left(\frac{w}{2}\right)^{bm} \widetilde{I}_{b(m+\nu)}(w) C_m^\nu(t). \tag{4.2}$$

Then, the summation (4.2) converges absolutely and uniformly on any compact subset of

$$U = \{(b, \nu, w, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{C} \times [-1, 1] \ : \ 1 + b\nu > 0\}.$$

In particular, $\mathscr{J}(b,\nu;w;t)$ is a continuous function on U. The special values at $b = 1,2$ are given by

$$\mathscr{J}(1,\nu;w;t) = e^{wt}, \tag{4.3}$$

$$\mathscr{J}(2,\nu;w;t) = \Gamma\left(\nu+\frac{1}{2}\right)\widetilde{I}_{\nu-\frac{1}{2}}\Big(\frac{w(1+t)^{1/2}}{\sqrt{2}}\Big). \tag{4.4}$$

We introduce the following continuous function of t on the interval $[-1,1]$ with parameters $r,s>0$ and $z \in \{z\in\mathbb{C} \mid \operatorname{Re} z \geq 0\} \setminus i\pi\mathbb{Z}$

$$h_{k,a}(r,s;z;t) = \frac{\exp\big(-\frac{1}{a}(r^a+s^a)\coth(z)\big)}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}}\mathscr{J}\left(\frac{2}{a},\frac{2\langle k\rangle+N-2}{2};\frac{2(rs)^{\frac{a}{2}}}{a\sinh(z)};t\right).$$

By using the polar coordinate $x = r\omega$, $y = s\eta$, we set

$$\Lambda_{k,a}(x,y;z) = \widetilde{V}_k(h_{k,a}(r,s;z;\cdot))(\omega,\eta).$$

Here is an integration formula of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$.

Theorem E. *Suppose $a>0$ and k is a non-negative multiplicity function. Suppose* $\operatorname{Re} z \geq 0$ *and $z \notin i\pi\mathbb{Z}$. Then, $\mathscr{I}_{k,a}(z) = \exp(\frac{z}{a}\Delta_{k,a})$ is given by*

$$\mathscr{I}_{k,a}(z)f(x) = c_{k,a}\int_{\mathbb{R}^N} f(y)\Lambda_{k,a}(x,y;z)\vartheta_{k,a}(y)dy, \tag{4.5}$$

where $c_{k,a}$ is as in (4.1).

5. Hidden symmetries in $L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$

There is an obvious action of the Coxeter group $\mathfrak{C}$ on $L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$. In addition, we see that there is a hidden action of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ commuting with $\mathfrak{C}$. This action is the key to the proof of Theorems A to E.

5.1. $\mathfrak{sl}_2$ action.

We introduce the following differential-difference operators on $\mathbb{R}^N\setminus\{0\}$ by

$$\mathbb{E}^+_{k,a} := \frac{i}{a}\|x\|^a, \qquad \mathbb{E}^-_{k,a} := \frac{i}{a}\|x\|^{2-a}\Delta_k, \qquad \mathbb{H}_{k,a} := \frac{2}{a}\sum_{i=1}^N x_i\partial_i + \frac{N+2\langle k\rangle+a-2}{a}.$$

Our operator $\Delta_{k,a} = \frac{i}{a}(\mathbb{E}^+_{k,a} - \mathbb{E}^-_{k,a})$ can be interpreted in the framework of the (infinite dimensional) representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$:

Lemma F. *The differential-difference operators $\{\mathbb{H}_{k,a},\mathbb{E}^+_{k,a},\mathbb{E}^-_{k,a}\}$ form an $\mathfrak{sl}_2$-triple for any multiplicity-function k and any non-zero complex number a.*

Special cases of Lemma F was previously known: the case $k \equiv 0$ and $a = 2$ is the classical harmonic $\mathfrak{sl}_2$-triple (e.g. Howe [5]), the case $k > 0$ and $a = 2$ by Heckman [3], and $k \equiv 0$ and $a = 1$ by Kobayashi and Mano [7].

Lemma F fits nicely into the theory of discretely decomposable representations of reductive groups [6], and we see that the above representation of $\mathfrak{sl}(2, \mathbb{R})$ on $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ lifts to the universal covering group $\widetilde{SL(2, \mathbb{R})}$:

Theorem G. *If $a > 0$ and $a + 2\langle k\rangle + N - 2 > 0$, then the representation of $\mathfrak{sl}(2, \mathbb{R})$ lifts to a unitary representation of $\widetilde{SL(2, \mathbb{R})}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*

The above unitary representation in the case $N = 1$ and $k \equiv 0$ is essentially the same with Kostant's realization [8] of highest weight representations of $\widetilde{SL(2, \mathbb{R})}$. For $N \geq 2$, this unitary representation contains countably many irreducible components of $\widetilde{SL(2, \mathbb{R})}$, which we can find explicitly by using Laguerre polynomials for general k and a.

5.2. Hidden symmetries for $a = 1$ and 2.

Theorem G asserts that the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ has a symmetry of the direct product group $\mathfrak{C} \times \widetilde{SL(2, \mathbb{R})}$ for all k and a. It turns out that this symmetry becomes larger for special values of k and a. In this subsection, we discuss these hidden symmetries.

First, in the case $k \equiv 0$, the Dunkl-Laplacian Δ_k becomes the Euclidean Laplacian Δ, and consequently, the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{0,a}(x)dx)$ has a larger symmetry $O(N) \times \widetilde{SL(2, \mathbb{R})}$.

Next, we observe that the Lie algebra of the direct product group $O(N) \times \widetilde{SL(2, \mathbb{R})}$ may be seen as a subalgebra of two different reductive Lie algebras $\mathfrak{sp}(N, \mathbb{R})$ and $\mathfrak{o}(N + 1, 2)$:

$$\mathfrak{o}(N) \oplus \mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{o}(N) \oplus \mathfrak{o}(1, 2) \subset \mathfrak{o}(N + 1, 2),$$
$$\mathfrak{o}(N) \oplus \mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{o}(N) \oplus \mathfrak{sp}(1, \mathbb{R}) \subset \mathfrak{sp}(N, \mathbb{R}).$$

Correspondingly, there are the following symmetries in the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{0,a}(x)dx)$ for $a = 1, 2$:

$$\begin{tikzcd}
 & & \boxed{O(N+1,2)^{\sim}} \\
\boxed{\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})}} \arrow[r, "k\to 0"] & \boxed{O(N) \times \widetilde{SL(2,\mathbb{R})}} \arrow[ur, "a \to 1"'] \arrow[dr, "a \to 2"] & \\
 & & \boxed{Mp(N,\mathbb{R})}
\end{tikzcd}$$

(k, a: general)

Diagram 2. Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

For $a = 2$, this unitary representation of $Mp(N, \mathbb{R})$ is nothing but the Weil representation, and its realization on $L^2(\mathbb{R}^N)$ is called the Schrödinger model. For $a = 1$, the unitary representation of $O(N+1, 2)\tilde{}$ (a double covering of the conformal group) on $L^2(\mathbb{R}^N, \|x\|^{-1}dx)$ is irreducible and has a similar nature to the Weil representation. Both of them are so-called the minimal representations and, in particular, they attain the minimum of their Gel'fand–Kirillov dimensions among the unitary dual.

In this sense, our continuous parameter $a > 0$ interpolates two minimal representations of different reductive groups by keeping smaller symmetries. The (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ gives a unitary inversion corresponding to the non-trivial Weyl group element in the L^2-model of minimal representations.

Detailed proof will appear elsewhere.

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