# Quantum groupoids and dynamical categories*

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**Abstract**

In this paper we realize the dynamical categories introduced in our previous paper as categories of modules over bialgebroids; we study the bialgebroids arising in this way. We define quasitriangular structure on bialgebroids and present examples of quasitriangular bialgebroids related to the dynamical categories. We show that dynamical twists over an arbitrary base give rise to bialgebroid twists.

We prove that the classical dynamical r-matrices over an arbitrary base manifold are in one-to-one correspondence with a special class of coboundary Lie bialgebroids.

Key words: Dynamical category, dynamical twist, dynamical Yang-Baxter equation, bialgebroid, quantum groupoid, Lie bialgebroid.

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# Contents



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# 1 Introduction

In our recent paper [DM1], we introduced a procedure of dynamization of monoidal categories. The categorical approach naturally led to a definition of the dynamical Yang-Baxter equation (DYBE), both classical and quantum, over an arbitrary base[1]. In this way, the constructions of twists from [Xu2] and [EE1] acquired a categorical meaning. The dynamical

[1]For an introduction to the theory of dynamical Yang-Baxter equation and the bibliography see [ESch1]

categories of [DM1] generalize the dynamical categories which were introduced by Etingof and Varchenko [EV2] for commutative cocommutative Hopf algebras.

In the framework of the categorical approach, we developed a fusion procedure which led to a construction of dynamical twists. Those dynamical twists were used for equivariant star product quantization of vector bundles on the coadjoint orbits of reductive Lie groups, including the algebras of functions (see also [AL] and [KMST]). In a recent paper of Etingof and Enriquez [EE2], this fusion procedure was extended further, including a class of infinite dimensional Lie algebras.

The goal of the present paper is to realize the dynamical categories as representations of certain $\mathcal{L}$-bialgebroids, were $\mathcal{L}$ is a base algebra over a Hopf algebra $\mathcal{H}$ in the sense of Definition 2.1. The simplest bialgebroid of this kind, namely the smash product $\mathcal{L} \rtimes \mathcal{H}$, was introduced in [Lu]. It is interesting to note that bialgebroids of [Lu] were considered over exactly the same class of base algebras that was used for the definition of dynamical categories in [DM1]. In the present paper we link the theory of dynamical Yang-Baxter equations over a non-abelian base with the bialgebroids of [Lu]. We pursue a further study of those bialgebroids and their descenders. In particular, we show that their certain quotients have a quasitriangular structure.

The infinitesimal analogs of bialgebroids are Lie bialgebroids. We consider Lie bialgebroids which are quasi-classical limits of the bialgebroids related to the dynamical categories. In this way we come to the most general definition of dynamical r-matrix over an arbitrary base manifold as the space of dynamical parameters. We show that the classical dynamical r-matrices are in one-to-one correspondence with a special class of coboundary Lie bialgebroids.

In the present paper we obtain the following results.

We define quasitriangular structure and the notion of universal R-matrix on bialgebroids.

We study quasitriangular bialgebroids (quantum groupoids) related to the dynamical categories.

We give an interpretation to the antipode of [Lu] as a an isomorphism between two different bialgebroids over different bases.

We prove that a dynamical twist over an arbitrary base gives rise to a twist of bialgebroids. This is a generalization of the results of [Xu1].

We present an example of a "dual" bialgebroid over an non-abelian base.

We define a classical dynamical r-matrix over a Poisson base algebra $\mathcal{L}_0$ as a coboundary Lie bialgebroid of a special type over $\mathcal{L}_0$.

The paper is organized as follows.

Section 2 recalls the construction of dynamical categories over an $\mathcal{H}$-base algebra $\mathcal{L}$ for some Hopf algebra, $\mathcal{H}$.

Section 3 contains basic definitions from the theory of bialgebroids.

Section 4 introduces a bialgebroid extension of a quasitriangular Hopf algebra $\mathcal{H}$ by its quasi-commutative module algebra $\mathcal{L}$. Therein we show that a certain quotient bialgebroid $\mathcal{H}_{\mathcal{L}}$ has a quasitriangular structure.

In Section 5 we give an interpretation of Lu's antipode on the smash product bialgebroid as an isomorphism between a pair of bialgebroids. We prove that the antipode is carried over to the quotient quantum groupoid $\mathcal{H}_{\mathcal{L}}$.

Section 6 establishes a relation between dynamical cocycles and bialgebroid twists. We start from the trivial extension of the bialgebroid $\mathfrak{D}\mathcal{H}_{\mathcal{L}}$, where $\mathfrak{D}\mathcal{H}$ is the double of $\mathcal{H}$, by a Hopf algebra $\mathcal{U}$ containing $\mathcal{H}$. We show that the element $\Psi = \mathcal{F}\Theta$ built out of a dynamical cocycle $\mathcal{F} \in \mathcal{U}\otimes\mathcal{U}\otimes\mathcal{L}$ and a universal R-matrix $\Theta$ of the double, is a twist of the bialgebroid $\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}$.

Section 7 realizes dynamical categories as representations of bialgebroids.

In Section 8 we present a "bimodule" algebra over the tensor product bialgebroid $\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}$ twisted by a dynamical twist. It is, in fact, a bialgebroid and may be considered as a dynamical FRT algebra, in case $\mathcal{U}$ is quasitriangular.

In Section 9 we give the most general, to our knowledge, definition of the classical dynamical r-matrix over arbitrary base. We prove that a classical dynamical r-matrix in the sense of that definition is the same as a special coboundary Lie bialgebroid structure on the base manifold.

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# 2 Dynamical categories

## 2.1 Hopf algebras and the double

In this subsection we fix some notation and set up general conventions concerning Hopf algebras[2] that will be used in the paper.

Let $k$ denote a field of zero characteristic or a topological algebra of formal power series in one variable with coefficients in the field. By an algebra we mean an associative unital algebra over $k$; all the homomorphisms of algebras are unital. Unless otherwise specified, ideals are assumed to be two-sided ideals. The symbol $\otimes$ stands for the (completed) tensor product in the category of (complete) $k$-modules.

Let $\mathcal{H}$ be a Hopf algebra over $k$ with invertible antipode $\gamma$. We use the symbolic Sweedler notation for the coproduct $\Delta(h) = h^{(1)} \otimes h^{(2)} \in \mathcal{H} \otimes \mathcal{H}$ and mark the tensor components in the standard way, e.g., $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 \in \mathcal{H} \otimes \mathcal{H}$. We use analogous notation for an $\mathcal{H}$-coaction $\delta$ on a (left) comodule $A$, namely, $\delta(a) = a^{(1)} \otimes a^{[2]}$, where the square brackets label the $A$-component and the parentheses mark the component belonging to $\mathcal{H}$. The Hopf algebra with the opposite multiplication will be denoted by $\mathcal{H}_{op}$ whereas the Hopf algebra with the opposite comultiplication will be denoted by $\mathcal{H}^{op}$.

All $\mathcal{H}$-modules are assumed to be left. Recall that an associative algebra and $\mathcal{H}$-module $\mathcal{A}$ is called an $\mathcal{H}$-module algebra, or simply $\mathcal{H}$-algebra, if the action is non-degenerate (the unit acts as the identity operator), the multiplication in $\mathcal{A}$ is $\mathcal{H}$-equivariant, and the unit in $\mathcal{A}$ generates the trivial submodule. Recall also that a (left) $\mathcal{H}$-comodule algebra $\mathcal{A}$ is an algebra and $\mathcal{H}$-comodule such that the coaction $\mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ is an algebra homomorphism.

Assuming $\mathcal{H}$ is quasitriangular with the universal R-matrix $\mathcal{R}$, we will use the standard notation

$$\mathcal{R}^+ = \mathcal{R}, \quad \mathcal{R}^- = \mathcal{R}_{21}^{-1}. \tag{1}$$

The matrix $\mathcal{R}^-$ is an alternative quasitriangular structure on $\mathcal{H}$. We will use the following well known equalities relating the R-matrix and the antipode

$$(\gamma \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1} = (\mathrm{id} \otimes \gamma^{-1})(\mathcal{R}), \quad (\gamma \otimes \gamma)(\mathcal{R}) = \mathcal{R}. \tag{2}$$

If an $\mathcal{H}$-algebra $\mathcal{A}$ satisfies the condition

$$\lambda\mu = \big(\mathcal{R}_2 \triangleright \mu\big)\,\big(\mathcal{R}_1 \triangleright \lambda\big), \tag{3}$$

[2]For a guide in the Hopf algebras and quantum groups the reader is referred to Drinfeld's report [Dr1] or to one of the textbooks, e.g. [K] or [Mj]

for all $\lambda, \mu \in \mathcal{A}$, then $\mathcal{A}$ is called $\mathcal{R}$-commutative or $\mathcal{H}$-commutative (or simply quasi-commutative if $\mathcal{H}$ and $\mathcal{R}$ are clear from the context). Note that this definition is independent on the choice of the matrices $\mathcal{R}^{\pm}$.

By the dual $\mathcal{H}^*$ to Hopf algebra $\mathcal{H}$ we understand a Hopf algebra equipped with the non-degenerate Hopf pairing $\langle .,. \rangle \colon \mathcal{H} \otimes \mathcal{H}^* \to k$.

Twist by a cocycle $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ of a Hopf algebra $\mathcal{H}$ with the coproduct $\Delta$ is a Hopf algebra with the same multiplication and with the coproduct $h \mapsto \mathcal{F}^{-1}\Delta(h)\mathcal{F}$. Given two Hopf algebras $\mathcal{A}$ and $\mathcal{B}$, a bicharacter $\mathcal{F}$ is a non-zero element from $\mathcal{B} \otimes \mathcal{A}$ obeying

$$(\Delta_{\mathcal{B}} \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23} \in \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{A}, \quad (\mathrm{id} \otimes \Delta_{\mathcal{A}})(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{12} \in \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A}.$$

A bicharacter defines a cocycle in the Hopf algebra $\mathcal{A} \otimes \mathcal{B}$, see [RS]; the corresponding twisted Hopf algebra $\mathcal{A} \overset{\mathcal{F}}{\otimes} \mathcal{B}$ is called a twisted tensor product of $\mathcal{A}$ and $\mathcal{B}$. The comultiplication in $\mathcal{A} \overset{\mathcal{F}}{\otimes} \mathcal{B}$ has the form

$$\Delta(a \otimes b) = (a^{(1)} \otimes \mathcal{F}_1^{-1} b^{(1)} \mathcal{F}_1) \otimes (\mathcal{F}_2^{-1} a^{(2)} \mathcal{F}_2 \otimes b^{(2)}) \tag{4}$$

It is convenient for our exposition to define the double $\mathfrak{D}\mathcal{H}$ of the Hopf algebra $\mathcal{H}$, [Dr1], as a double cross product $\mathcal{H} \bowtie \mathcal{H}^*_{op}$, [Mj]. This is equivalent to the standard definition of the double as $\mathcal{H} \bowtie \mathcal{H}^{*op}$, having in mind the isomorphism between $\mathcal{H}^*_{op}$ and $\mathcal{H}^{*op}$ realized via the antipode. Algebraically, $\mathfrak{D}\mathcal{H}$ is dual to the tensor product $\mathcal{H}^* \otimes \mathcal{H}^{op}$ twisted by the canonical element $\sum_i e_i \otimes e^i \in \mathcal{H}^{op} \otimes \mathcal{H}^*$ of the pairing $\langle .,. \rangle$, where $\{e_i\}$ is the basis in $\mathcal{H}$ and $\{e^i\}$ its dual in $\mathcal{H}^*_{op}$. Explicitly, the cross relations between elements of $\mathcal{H}$ and $\mathcal{H}^*_{op}$ are given by

$$\eta^{(1)} * h^{(1)} \langle \eta^{(2)}, h^{(2)} \rangle = \langle \eta^{(1)}, h^{(1)} \rangle h^{(2)} * \eta^{(2)},$$

$h \in \mathcal{H}$, $\eta \in \mathcal{H}^*_{op}$. Then $\Theta = \sum_i e^i \otimes e_i$ is naturally considered as an element from the tensor square of $\mathfrak{D}\mathcal{H}$ is a universal R-matrix of $\mathfrak{D}\mathcal{H}$.

We will also deal with the situation when $\mathcal{H}$ is a Hopf subalgebra in another Hopf algebra, $\mathcal{U}$. Then we can define a generalized double $\mathcal{H} \bowtie \mathcal{U}^*_{op}$ as the dual to the twisted tensor product of $\mathcal{H}^*$ and $\mathcal{U}^{op}$ (this twist is induced from the subalgebra $\mathcal{H}^* \otimes \mathcal{H}^{op} \subset \mathcal{H}^* \otimes \mathcal{U}^{op}$). Clearly the projection $\mathcal{U}^* \to \mathcal{H}^*$ extends to a Hopf algebra map $\mathcal{H} \bowtie \mathcal{U}^*_{op} \to \mathfrak{D}\mathcal{H}$.

A quasitriangular structure $\mathcal{R}$ on $\mathcal{H}$ defines two Hopf algebra maps $\mathcal{R}^{\pm} \colon \mathcal{H}^*_{op} \to \mathcal{H}$ given by

$$\mathcal{R}^{\pm}(\eta) = \langle \mathcal{R}^{\pm}_2, \eta \rangle \mathcal{R}^{\pm}_1, \quad \eta \in \mathcal{H}^*_{op}. \tag{5}$$

These maps extend to Hopf algebra epimorphisms $\mathfrak{D}\mathcal{H} \to \mathcal{H}$,

$$x \otimes \eta \mapsto x\mathcal{R}^+(\eta), \quad x \otimes \eta \mapsto x\mathcal{R}^-(\eta), \quad x \otimes \eta \in \mathcal{H} \bowtie \mathcal{H}^*_{op}. \tag{6}$$

The universal R-matrix $\Theta$ of the double goes over into $\mathcal{R}^\pm$ under (6).

## 2.2 Base algebras

Recall that one can assign to any monoidal category $\mathcal{C}$ a braided category $Z(\mathcal{C})$ called the center of $\mathcal{C}$. Its objects are the pairs $(X, \sigma)$, were $X$ is an object of $\mathcal{C}$ equipped with a family of natural isomorphisms $\sigma = \{\sigma_A\}$, $X \otimes A \xrightarrow{\sigma_A} A \otimes X$ for all objects of $\mathcal{C}$ (these permutations should satisfy certain functorial conditions, see [K]). When $\mathcal{C}$ is a category of $\mathcal{H}$-modules, $Z(\mathcal{C})$ is equivalent to the category of modules over the double $\mathfrak{D}\mathcal{H}$.

**Definition 2.1.** [DM1] Let $\mathcal{C}$ be a monoidal category and $Z(\mathcal{C})$ its center. A commutative algebra in $Z(\mathcal{C})$ is called a $\mathcal{C}$-***base algebra***.

When $\mathcal{C}$ is a category of $\mathcal{H}$-modules, we use the term $\mathcal{H}$-***base algebra***. An $\mathcal{H}$-base algebra can be alternatively defined as a $\mathfrak{D}\mathcal{H}$-commutative algebra.

Equivalently, an $\mathcal{H}$-base algebra can be defined as an $\mathcal{H}$-module algebra and simultaneously a left $\mathcal{H}$-comodule algebra satisfying the conditions

$$\delta(h \triangleright \lambda) = h^{(1)}\lambda^{(1)}\gamma(h^{(3)}) \otimes h^{(2)} \triangleright \lambda^{[2]}, \tag{7}$$

$$\lambda\mu = (\lambda^{(1)} \triangleright \mu)\, \lambda^{[2]}, \tag{8}$$

for all $\lambda, \mu \in \mathcal{L}$ and $h \in \mathcal{H}$. This definition is equivalent to the definition of base algebra given in [DM1]. Remark that an $\mathcal{H}$-module and $\mathcal{H}$-comodule fulfilling condition (7) is called a Yetter-Drinfeld module.

Let $\Theta$ denote the standard quasitriangular structure on $\mathfrak{D}\mathcal{H}$. For simplicity, we think of $\Theta$ as an element of $\mathfrak{D}\mathcal{H} \otimes \mathfrak{D}\mathcal{H}$. For $\mathcal{H}$ finite dimensional, it is a canonical element of the Hopf pairing between $\mathcal{H}$ and $\mathcal{H}^*$, $\Theta \in \mathcal{H}^*_{op} \otimes \mathcal{H}$. Such an interpretation is valid for infinite dimensional Hopf algebras close to universal enveloping algebras and their quantizations, if $\mathcal{H}^*$ is understood as a restricted dual, and the tensor product is completed in some topology. In terms of the R-matrix $\Theta = \Theta_1 \otimes \Theta_2 \in (\mathfrak{D}\mathcal{H})^{\otimes 2}$ of the double, the coaction $\delta$ reads

$$\lambda \mapsto \lambda^{(1)} \otimes \lambda^{[2]} = \Theta_2 \otimes \Theta_1 \triangleright \lambda. \tag{9}$$

Actually, in our constructions we may understand by $\mathfrak{D}\mathcal{H}$ ***any*** quasitriangular Hopf algebra that contains $\mathcal{H}$ and whose universal R-matrix belongs to $\mathfrak{D}\mathcal{H} \otimes \mathcal{H}$. Then any $\mathfrak{D}\mathcal{H}$-commutative algebra belongs to the center of the category of $\mathcal{H}$-modules and therefore is an $\mathcal{H}$-base algebra. The $\mathcal{H}$-coaction is expressed by the formula (9) with $\Theta$ replaced by $\mathcal{R}$.

**Remarks 2.2.** Let $\mathcal{L}$ be an $\mathcal{H}$-base algebra. Then we can state the following.

1. $\mathcal{L}$ is also an $\mathcal{H}^*_{op}$-base algebra, as readily follows from the definition. The corresponding $\mathcal{H}^*_{op}$-coaction is given by $\lambda \mapsto \Theta^-_2 \otimes \Theta^-_1 \triangleright \lambda$, see notation (1).

2. If $\mathcal{H}$ is quasitriangular and $\mathcal{L}$ is $\mathcal{H}$-commutative, then $\mathcal{L}$ has two $\mathcal{H}$-base algebra structures defined by $\mathcal{R}^\pm$, where $\mathcal{R}$ is the R-matrix of $\mathcal{H}$. Namely, the double $\mathfrak{D}\mathcal{H}$ acts on $\mathcal{L}$ through the projections (6) to $\mathcal{H}$. The Hopf algebra homomorphisms (6) sends $\Theta^\pm$ to $\mathcal{R}^\pm$, hence the algebra $\mathcal{L}$ is $\mathfrak{D}\mathcal{H}$-commutative. In terms of the R-matrix, the $\mathcal{H}$-coactions are given by

$$\delta^+(\lambda) = \mathcal{R}^-_2 \otimes \mathcal{R}^-_1 \triangleright \lambda = \mathcal{R}^{-1}_1 \otimes \mathcal{R}^{-1}_2 \triangleright \lambda, \quad \delta^-(\lambda) = \mathcal{R}^+_2 \otimes \mathcal{R}^+_1 \triangleright \lambda = \mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright \lambda. \quad (10)$$

   We denote by $\mathcal{L}_\pm$ the two $\mathcal{H}$-base algebra structures on $\mathcal{L}$ that correspond to the coactions $\delta^\pm$.

3. Combining two previous remarks, we state that an $\mathcal{H}$-base algebra has two *different* $\mathfrak{D}\mathcal{H}$-base algebra structures. The $\mathcal{H}$- and $\mathcal{H}^*_{op}$-coactions expressed through $\Theta^\pm$ may be considered as $\mathfrak{D}\mathcal{H}$-coactions via the embeddings of $\mathcal{H}$ and $\mathcal{H}^*_{op}$ into $\mathfrak{D}\mathcal{H}$. The $\mathfrak{D}\mathcal{H}$-coactions are given by $\lambda \mapsto \Theta^\pm_2 \otimes \Theta^\pm_1 \triangleright \lambda$.

4. Let us fix that $\mathfrak{D}\mathcal{H}$-base algebra structure on $\mathcal{L}$ which corresponds to the $\mathcal{H}$-coaction, cf. the previous remark. Assume that $\mathcal{H}$ is a Hopf subalgebra in a Hopf algebra $\mathcal{U}$, thus there is a natural projection $\mathcal{U}^* \to \mathcal{H}^*$ inducing an $\mathcal{U}^*_{op}$-action on $\mathcal{L}$. Then $\mathcal{L}$ is a base algebra over the generalized double $\mathcal{H} \bowtie \mathcal{U}^*_{op}$.

**Lemma 2.3.** *Let $\mathcal{L}$ be an $\mathcal{H}$-base algebra. Then any $\mathcal{H}$-invariant element in $\mathcal{L}$ belongs to the center $Z(\mathcal{L})$.*

*Proof.* Let $\mu \in \mathcal{L}$ be $\mathcal{H}$-invariant. Then for any $\lambda \in \mathcal{L}$ one has $\lambda\mu = (\lambda^{(1)} \triangleright \mu)\, \lambda^{[2]} = \varepsilon(\lambda^{(1)})\, \mu\, \lambda^{[2]} = \mu\lambda$. The first equality follows from the $\mathfrak{D}\mathcal{H}$-commutativity of $\mathcal{L}$. ☐

**Definition 2.4.** An $\mathcal{H}$-base algebra $\mathcal{L}$ is called quasi-transitive if $\mathcal{L}^{\mathfrak{D}\mathcal{H}}$, the set of $\mathfrak{D}\mathcal{H}$-invariant elements in $\mathcal{L}$, coincides with $k$.

It follows from Lemma 2.3 that $\mathcal{L}^{\mathfrak{D}\mathcal{H}}$ is a commutative algebra belonging to the center of $\mathcal{L}$. Let $\chi$ be a character of $\mathcal{L}^{\mathfrak{D}\mathcal{H}}$, i.e. a one dimensional representation. Consider the ideal $J_\chi$ in $\mathcal{L}$ generated by the kernel of $\chi$.

**Proposition 2.5.** *The quotient $\mathcal{L}/J_\chi$ is a quasi-transitive $\mathcal{H}$-base algebra.*

*Proof.* The ideal $J_\chi$ is obviously $\mathfrak{D}\mathcal{H}$-invariant, hence the quotient $\mathcal{L}/J_\chi$ is an $\mathfrak{D}\mathcal{H}$-algebra. It is quasi-commutative, being a quotient of a quasi-commutative algebra. By construction, the subalgebra of invariants in $\mathcal{L}/J_\chi$ coincides with $k$. $\square$

**Examples 2.6.** Let us give some examples of base algebras. A detailed consideration to some of them is given in [DM1].

1. $\mathcal{H}$ itself is an (quasi-transitive) $\mathcal{H}$-base algebra, being equipped with the adjoint action and the coproduct coaction.

2. $\mathcal{H}^*_{op}$ is a (quasi-transitive) $\mathcal{H}$-base algebra due to the symmetry $\mathcal{H} \leftrightarrow \mathcal{H}^*_{op}$ in the definition of base algebras.

3. Consider an FRT algebra associated with a finite dimensional representation of a quasitriangular Hopf algebra $\mathcal{H}$, [FRT]. It is a commutative algebra in the category of $\mathcal{H}$-bimodules, whence it is an $\mathcal{H} \otimes \mathcal{H}_{op}$-base algebra (cf. Remark 2.2.2).

4. Assume again that $\mathcal{H}$ is quasitriangular. A reflection equation algebra (studied in [KSkl]) is, in fact, a commutative algebra in the category of modules over the twisted tensor product $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$, [DM2]. Therefore it is $\mathfrak{D}\mathcal{H}$-commutative and thus an $\mathcal{H}$-base algebra.

5. Let $\mathcal{L}$ and $\mathcal{L}_1$ be two $\mathcal{H}$-base algebras. On the linear space $\mathcal{L} \otimes \mathcal{L}_1$ define an associative algebra structure by the multiplication

$$(\lambda \otimes \mu)(\alpha \otimes \beta) := \lambda(\Theta_2 \triangleright \alpha) \otimes (\Theta_1 \triangleright \mu)\beta.$$

   This algebra is a braided tensor product of two $\mathfrak{D}\mathcal{H}$-commutative algebras, hence it is $\mathfrak{D}\mathcal{H} \overset{\Theta}{\otimes} \mathfrak{D}\mathcal{H}$-commutative and has two structures of $\mathfrak{D}\mathcal{H}$-base algebra. In case $\mathcal{L}_1 = \mathcal{H}$, it coincides with the smash product $\mathcal{L} \rtimes \mathcal{H}$ as an associative algebra.

### 2.3 Dynamical categories

The notion of dynamical extension (dynamization) of a monoidal category admits various formulations, [DM1], which become equivalent under certain circumstances. We will work with a category $\mathcal{M}_{\mathcal{H}}$ of $\mathcal{H}$-modules and its extension $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ over an $\mathcal{H}$-base algebra $\mathcal{L}$ in the sense of the following definition.

**Definition 2.7.** [DM1] ***Dynamization*** of the category $\mathcal{M}_{\mathcal{H}}$ over the $\mathcal{H}$-base algebra $\mathcal{L}$ is a strict monoidal category $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ defined by the following conditions

1. objects of $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ are the objects of $\mathcal{M}_{\mathcal{H}}$,

2. $\mathrm{Hom}_{\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}}(X,Y)$ is the set of $\mathcal{H}$-equivariant linear maps from $X$ to $Y\otimes\mathcal{L}$. The composition $\phi\circ\psi$ of two morphisms $\phi\in\mathrm{Hom}_{\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}}(X,Y)$ and $\psi\in\mathrm{Hom}_{\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}}(Y,Z)$ is the composition map
$$X\stackrel{\phi}{\longrightarrow}Y\otimes\mathcal{L}\stackrel{\psi\otimes\mathrm{id}_{\mathcal{L}}}{\longrightarrow}Z\otimes\mathcal{L}\otimes\mathcal{L}\stackrel{\mathrm{id}_Z\otimes\mathrm{m}_{\mathcal{L}}}{\longrightarrow}Z\otimes\mathcal{L},$$
where $\mathrm{m}_{\mathcal{L}}$ is the multiplication in $\mathcal{L}$,

3. tensor product of objects from $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ is the same as in $\mathcal{M}_{\mathcal{H}}$,

4. tensor product of morphisms $\phi\in\mathrm{Hom}_{\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}}(X,X')$ and $\psi\in\mathrm{Hom}_{\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}}(Y,Y')$ is given by the composition
$$X\otimes Y\stackrel{\phi\otimes\psi}{\longrightarrow}X'\otimes\mathcal{L}\otimes Y'\otimes\mathcal{L}\stackrel{\tau_{Y'}}{\longrightarrow}X'\otimes Y'\otimes\mathcal{L}\otimes\mathcal{L}\stackrel{\mathrm{m}_{\mathcal{L}}}{\longrightarrow}X'\otimes Y'\otimes\mathcal{L}.$$
where $\tau_{Y'}$ is the permutation $\mathcal{L}\otimes Y'\to Y'\otimes\mathcal{L}$ expressed via the $\mathcal{H}$-coaction on $\mathcal{L}$ by the formula $\lambda\otimes y\mapsto\lambda^{(1)}\triangleright y\otimes\lambda^{[2]}$.

The category $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ generalizes the category of Etingof and Varchenko that was constructed in [EV2] for commutative cocommutative $\mathcal{H}$ and $\mathcal{L}$ being a certain extension of $\mathcal{H}$. The category $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ was introduced in [DM1] in order to formulate the classical and quantum dynamical Yang-Baxter equations for an arbitrary Lie bialgebras and their quantizations. The purpose of the present paper is to realize $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ and its important subcategories via representations of bialgebroids. The notion of bialgebroid is a generalization of the notion of Hopf algebra, [Lu]. The next section is a brief introduction to this theory.

# 3 Some basics on bialgebroids

## 3.1 General definition and examples

The reconstruction theorem states that a fiber functor from a monoidal category $\mathcal{C}$ to the monoidal category of vector spaces gives rise to a bialgebra whose category of representations is equivalent to $\mathcal{C}$, see e.g. [Mj]. Not all monoidal categories admit such a fiber functor, thus not all of them are related to bialgebras, [GK]. A more general concept of functor to the monoidal category of bimodules over some associative algebra leads to the notion of bialgebroid [Lu]. Similarly to the bialgebra case, representations of a bialgebroid also form a monoidal category.

**Definition 3.1.** Let $\mathcal{L}$ be an associative unital algebra over $k$. An associative unital algebra $\mathfrak{B}$ over $k$ is called a ***bialgebroid over base*** $\mathcal{L}$ or $\mathcal{L}$-bialgebroid if there exist

1. an algebra homomorphism $s\colon \mathcal{L} \to \mathfrak{B}$ (source map) and an algebra anti-homomorphism $t\colon \mathcal{L} \to \mathfrak{B}$ (target map) making $\mathfrak{B}$ an $\mathcal{L}$-bimodule by $\lambda \llcorner a := s(\lambda)a$, $a \lrcorner \lambda := t(\lambda)a$, $\lambda \in \mathcal{L}$, $a \in \mathfrak{B}$,

2. a coassociative bimodule map (comultiplication) $\Delta\colon \mathfrak{B} \to \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ which is a homomorphism into the unital associative algebra specified by the condition

$$\{z \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B} |\; z\big(t(\lambda) \otimes 1\big) = z\big(1 \otimes s(\lambda)\big), \forall \lambda \in \mathcal{L}\}, \tag{11}$$

3. a bimodule map (counit) $\varepsilon\colon \mathfrak{B} \to \mathcal{L}$ such that $\varepsilon(1_{\mathfrak{B}}) = 1_{\mathcal{L}}$,

$$\varepsilon\big(a\ (s \circ \varepsilon)(b)\big) = \varepsilon(ab) = \varepsilon\big(a\ (t \circ \varepsilon)(b)\big), \quad a, b \in \mathfrak{B}, \quad \text{and} \tag{12}$$

$$(\varepsilon \otimes_{\mathcal{L}} \mathrm{id}_{\mathfrak{B}}) \circ \Delta = \mathrm{id}_{\mathfrak{B}} = (\mathrm{id}_{\mathfrak{B}} \otimes_{\mathcal{L}} \varepsilon) \circ \Delta \tag{13}$$

under the identification $\mathcal{L} \otimes_{\mathcal{L}} \mathfrak{B} \simeq \mathfrak{B} \simeq \mathfrak{B} \otimes_{\mathcal{L}} \mathcal{L}$.

**Remarks 3.2.** 1. The images of the source and target maps in $\mathfrak{B}$ commute, by virtue of condition 1.

2. In general, the tensor product $\mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ has no natural structure of associative algebra. However, the element $z(a \otimes b) := z_1 a \otimes_{\mathcal{L}} z_2 b \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ is well defined for any $z \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ and $a \otimes b \in \mathfrak{B} \otimes \mathfrak{B}$. The condition 2 selects a natural algebra in $\mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$.

3. Since $\Delta$ is a bimodule map, one has $\Delta \circ s = s \otimes_{\mathcal{L}} 1$ and $\Delta \circ t = 1 \otimes_{\mathcal{L}} t$.

4. Condition 3 implies the identities $\varepsilon \circ s = \varepsilon \circ t = \mathrm{id}_{\mathcal{L}}$ and makes $\mathcal{L}$ a left $\mathfrak{B}$-module by
$$a \vdash \lambda := \varepsilon\big(as(\lambda)\big) = \varepsilon\big(at(\lambda)\big), \quad a \in \mathfrak{B}, \lambda \in \mathcal{L}, \tag{14}$$
where the right equality is a consequence of (12). The $\mathcal{L}$-bimodule structure on $\mathcal{L}$ induced by this action coincides with the standard one. The action (14) is called ***anchor***. One can check that
$$a\, s(\lambda) = s(a^{(1)} \vdash \lambda)\, a^{(2)}, \quad a\, t(\lambda) = t(a^{(2)} \vdash \lambda)\, a^{(1)}. \tag{15}$$
Sometimes the anchor is introduced separately; then the condition (12) is dropped from definition of bialgebroid, see [Lu]. In our definition we follow [Szl].

Any left $\mathfrak{B}$-module $V$ is a natural $\mathcal{L}$-bimodule. We call this correspondence the forgetful functor. Given two $\mathfrak{B}$-modules $V$ and $W$, the tensor product $V \otimes_{\mathcal{L}} W$ acquires a left $\mathfrak{B}$-module structure via the coproduct, due to condition 2 of Definition 3.1. The whole set of axioms from Definition 3.1 ensures that the left $\mathfrak{B}$-modules form a monoidal category, with $\mathcal{L}$ being the unit object. The forgetful functor to the category of $\mathcal{L}$-bimodules is strong monoidal, i.e. preserves tensor products. Conversely, suppose a pair of algebras $(\mathfrak{B}, \mathcal{L})$ satisfies condition 1 of Definition 3.1 and there is a monoidal structure on the category of left $\mathfrak{B}$-modules. Suppose the forgetful functor to the category of $\mathcal{L}$-bimodules is strong monoidal. Then $\mathfrak{B}$ is an $\mathcal{L}$-bialgebroid, see [Szl].

**Remark 3.3.** The bialgebroid $\mathfrak{B}$ from Definition 3.1 is a ***left*** one. This means that the $\mathcal{L}$-bimodule structure on $\mathfrak{B}$ is defined by the source and target maps and multiplication from the left. Alternatively, one can consider $\mathfrak{B}$ as a $\mathcal{L}$-bimodule using multiplication from the right and require that the right $\mathfrak{B}$-modules form a monoidal category with the forgetful functor to a $\mathcal{L}$-bimodules. Such bialgebroids are called ***right*** ones; one can readily recover their definition by the apparent modification of Definition 3.1. Although right modules over $\mathfrak{B}$ are the same as left modules over $\mathfrak{B}_{op}$, sometimes the notion of right bialgebroid proves to be convenient to work with.

**Example 3.4 (Bialgebroid $\mathrm{End}(\mathcal{L})$).** Let $\mathcal{L}$ be a finite dimensional associative unital algebra over the field $k$. Denote by $\mathfrak{E}$ the algebra of endomorphisms of $\mathcal{L}$ over $k$. For $a \in \mathcal{L}$ let $L_a$ and $R_a$ be the linear operators acting on $\mathcal{L}$ via the left and right multiplication by $a$; they define an algebra and anti-algebra maps from $\mathcal{L}$ to $\mathfrak{E}$, respectively. Thus $\mathfrak{E}$ is a natural $\mathcal{L}$-bimodule: the element $a \otimes b \in \mathcal{L} \otimes_k \mathcal{L}_{op}$ acts on $\mathfrak{E}$ by multiplication by $L_a R_b$ from the left. The algebra $\mathfrak{E}$ is in fact an $\mathcal{L}$-bialgebroid with the coproduct defined by $\Delta(f)(a \otimes b) := f(ab)$ and the counit $\varepsilon(f) := f(e)$, see [Lu].

**Example 3.5 (Bialgebroid structure on $\mathcal{L} \otimes \mathcal{L}_{op} \otimes \mathcal{H}$).** Suppose $\mathcal{L}$ is a left $\mathcal{H}$-module algebra for some Hopf algebra $\mathcal{H}$. The action of $\mathcal{H}$ on $\mathcal{L}$ is denoted by $\triangleright$. Consider the associative algebra $\mathfrak{B}$ built on $\mathcal{L} \otimes \mathcal{L}_{op} \otimes \mathcal{H}$ and equipped with the multiplication

$$(\lambda \otimes \mu \otimes f)(\zeta \otimes \eta \otimes g) := \lambda(f^{(1)} \triangleright \zeta) \otimes \mu(f^{(3)} \triangleright \eta) \otimes f^{(2)} g.$$

Let $\iota$ denote the (anti-algebra) identity map from $\mathcal{L}$ to $\mathcal{L}_{op}$. It is not difficult to show that $\mathfrak{B}$ is a bialgebroid with the source map $s\colon \lambda \mapsto \lambda \otimes 1 \otimes 1$ the target map $t\colon \lambda \mapsto 1 \otimes \iota(\lambda) \otimes 1$, the coproduct $\Delta(\lambda \otimes \mu \otimes h) := \big(\lambda \otimes 1 \otimes h^{(1)}\big) \otimes_{\mathcal{L}} \big(1 \otimes \mu \otimes h^{(2)}\big)$ and the counit $\varepsilon(\lambda \otimes \mu \otimes h) := \lambda\, \iota^{-1}(\mu)\varepsilon(h)$. The anchor action (14) is given explicitly by

$$(\lambda \otimes \mu \otimes h) \llcorner \zeta = \lambda(h \triangleright \zeta)\iota^{-1}(\mu),$$

for $(\lambda \otimes \mu \otimes h) \in \mathfrak{B}$ and $\zeta \in \mathcal{L}$.

**Example 3.6 (Bialgebras).** A bialgebra over the field $k$ is a bialgebroid whose base is $k$.

**Example 3.7 (Tensor product of bialgebroids).** Let $(\mathfrak{B}_i, \mathcal{L}_i, s_i, t_i, \Delta_i, \varepsilon_i)$, $i = 1, 2$, be a pair of bialgebroids. Then one can build their tensor product bialgebroid over the base $\mathcal{L}_1 \otimes \mathcal{L}_2$. As an associative algebra, this is the standard tensor product $\mathfrak{B}_1 \otimes \mathfrak{B}_2$. The source, target, and counit maps are respectively $s_1 \otimes s_2$, $t_1 \otimes t_2$, and $\varepsilon_1 \otimes \varepsilon_2$. The coproduct is given by

$$\Delta(x \otimes y) := (x^{(1)} \otimes y^{(1)}) \otimes_{(\mathcal{L}_1 \otimes \mathcal{L}_2)} (x^{(2)} \otimes y^{(2)}).$$

In particular, if one of the bialgebroids, say $\mathfrak{B}_1$ is a Hopf algebra, then the tensor product bialgebroid will be over the base $\mathcal{L}_2$.

**Definition 3.8.** Let $(\mathfrak{B}_i, \mathcal{L}, s_i, t_i, \Delta_i, \varepsilon_i)$, $i = 1, 2$, be two $\mathcal{L}$-bialgebroids. An algebra map $\varphi\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ is called a homomorphism of bialgebroids if it is an $\mathcal{L}$-bimodule map and

$$\varepsilon_2 \circ \varphi = \varepsilon_1, \quad (\varphi \otimes_{\mathcal{L}} \varphi) \circ \Delta_1 = \Delta_2 \circ \varphi. \tag{16}$$

**Example 3.9.** For an arbitrary bialgebroid $\mathfrak{B}$ over a finite dimensional base $\mathcal{L}$ the anchor map $\mathcal{L} \to \mathrm{End}_k(\mathcal{L})$ is a homomorphism of Lie bialgebroids.

Of particular interest for us will be the notion of quotient bialgebroid.

**Definition 3.10.** Let $\mathfrak{B}$ be an $\mathcal{L}$-bialgebroid. A two-sided ideal $J$ in the algebra $\mathfrak{B}$ is called a ***biideal*** if $\Delta(J) \subset J \otimes_{\mathcal{L}} \mathfrak{B} + \mathfrak{B} \otimes_{\mathcal{L}} J$ and $\varepsilon(J) = 0$.

Given a biideal $J \subset \mathfrak{B}$ the quotient $\mathfrak{B}/J$ is naturally endowed with an $\mathcal{L}$-bialgebroid structure such that the projection $j\colon \mathfrak{B} \to \mathfrak{B}/J$ is a bialgebroid homomorphism.

**Remark 3.11.** Note that any biideal lies in the kernel of the anchor map since the latter is expressed through the counit by formula (14).

## 3.2 Quasitriangular structure and twist

In the Hopf algebra theory, a quasitriangular structure on a Hopf algebra is essentially the same as a braiding in the monoidal category of its modules. Analogously to Hopf algebras one can define quasitriangular bialgebroids, with inevitable complications caused by non-commutativity of the base. A quasitriangular structure on a bialgebroid gives rise to a braiding in the category of its modules.

Let $\mathfrak{B}$ be an $\mathcal{L}$-bialgebroid. Then every $\mathfrak{B}$-module, and $\mathfrak{B}$ in particular, is also a natural $\mathcal{L}_{op}$-bimodule with respect to the left and right $\mathcal{L}_{op}$-actions defined through the target and source maps, correspondingly. Given two $\mathfrak{B}$-modules $M_1$ and $M_2$, the flip $M_1 \otimes M_2 \to M_2 \otimes M_1$ induces an invertible map $\sigma_{M_1,M_2}\colon M_1 \otimes_{\mathcal{L}} M_2 \to M_2 \otimes_{\mathcal{L}_{op}} M_1$. Let us define a structure of an $\mathcal{L}_{op}$-bialgebroid, $\mathfrak{B}^{op}$, on the algebra $\mathfrak{B}$. The target and source maps from $\mathcal{L}$ to $\mathfrak{B}$ viewed as algebra and anti-algebra maps from $\mathcal{L}_{op}$ to $\mathfrak{B}$ give, respectively, the source and target maps of the $\mathcal{L}_{op}$-bialgebroid $\mathfrak{B}^{op}$. To define $\mathfrak{B}^{op}$, it is enough to specify the corresponding monoidal structure on the left $\mathfrak{B}$-modules. Let us define a new tensor product of two $\mathfrak{B}$-modules $M_1$ and $M_2$ as the $\mathcal{L}_{op}$-bimodule $M_1 \otimes_{\mathcal{L}_{op}} M_2$ equipped with the following $\mathfrak{B}$-action:

$$M_1 \otimes_{\mathcal{L}_{op}} M_2 \xrightarrow{\sigma^{-1}_{M_2,M_1}} M_2 \otimes_{\mathcal{L}} M_1 \xrightarrow{\Delta(a)} M_2 \otimes_{\mathcal{L}} M_1 \xrightarrow{\sigma_{M_2,M_1}} M_1 \otimes_{\mathcal{L}_{op}} M_2, \quad a \in \mathfrak{B} \tag{17}$$

This tensor product is associative, as follows from the coassociativity of $\Delta$ and the "hexagon" identity obeyed by $\sigma$. One can check that the corresponding coproduct $\mathfrak{B}^{op}$ is given by $\Delta^{op} = \sigma_{\mathfrak{B},\mathfrak{B}} \circ \Delta$, and the counit is $\iota \circ \varepsilon_{\mathfrak{B}}$, where $\iota$ is the anti-isomorphism $\mathcal{L} \to \mathcal{L}_{op}$ implemented by the identity map.

**Definition 3.12.** A bialgebroid $\mathfrak{B}$ is called ***quasitriangular*** if there is a monoidal isomorphism $\mathrm{Mod}\,\mathfrak{B} \to \mathrm{Mod}\,\mathfrak{B}^{op}$ identical on objects, and the transformation of tensor products is defined by an element $\mathcal{R} = \mathcal{R}_1 \otimes_{\mathcal{L}_{op}} \mathcal{R}_2 \in \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$ (universal R-matrix):

$$M_1 \otimes_{\mathcal{L}} M_2 \xrightarrow{\mathcal{R}} M_1 \otimes_{\mathcal{L}_{op}} M_2, \quad x_1 \otimes_{\mathcal{L}} x_2 \mapsto \mathcal{R}_1 x_1 \otimes_{\mathcal{L}_{op}} \mathcal{R}_2 x_2,$$

for any pair of modules $M_1, M_2$.

It follows that there exists an element $\bar{\mathcal{R}} \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ such that $\mathcal{R}\bar{\mathcal{R}} = 1 \otimes_{\mathcal{L}_{op}} 1$ and $\bar{\mathcal{R}}\mathcal{R} = 1 \otimes_{\mathcal{L}} 1$. The element $\bar{\mathcal{R}}$ implements the inverse isomorphism $\mathrm{Mod}\,\mathfrak{B}^{op} \to \mathrm{Mod}\,\mathfrak{B}$ and it is a quasi-triangular structure on the coopposite bialgebroid $\mathfrak{B}^{op}$.

We will use the term ***quantum groupoid*** for a quasitriangular bialgebroid.

**Proposition 3.13.** *An element $\mathcal{R} \in \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$ defines a quasitriangular structure on $\mathfrak{B}$ if and only if*

1. $\mathcal{R}\big(t(\lambda) \otimes 1\big) = \mathcal{R}\big(1 \otimes s(\lambda)\big)$, *for all* $\lambda \in \mathcal{L}$,

2. *for all* $a \in \mathfrak{B}$ *equation* $\mathcal{R}\Delta(a) = \Delta^{op}(a)\mathcal{R}$ *holds in* $\mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$,

3. *equations*

$$
\begin{aligned}
(\Delta^{op} \otimes_{\mathcal{L}_{op}} \mathrm{id})(\mathcal{R}) &= \mathcal{R}_{23}\mathcal{R}_{13} := \mathcal{R}_1 \otimes_{\mathcal{L}_{op}} \mathcal{R}(1 \otimes \mathcal{R}_2), \quad (18)\\
(\mathrm{id} \otimes_{\mathcal{L}_{op}} \Delta^{op})(\mathcal{R}) &= \mathcal{R}_{12}\mathcal{R}_{13} := \mathcal{R}(\mathcal{R}_1 \otimes 1) \otimes_{\mathcal{L}_{op}} \mathcal{R}_2) \quad (19)
\end{aligned}
$$

   *hold in* $\mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$,

4. *there exists an element* $\bar{\mathcal{R}} \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ *such that* $\bar{\mathcal{R}}\big(s(\lambda) \otimes 1\big) = \bar{\mathcal{R}}\big(1 \otimes t(\lambda)\big)$, *for all* $\lambda \in \mathcal{L}$, *and* $\mathcal{R}\bar{\mathcal{R}} = 1 \otimes_{\mathcal{L}_{op}} 1$ *and* $\bar{\mathcal{R}}\mathcal{R} = 1 \otimes_{\mathcal{L}} 1$.

*Proof.* A direct computation. □

**Remarks 3.14.** Let us make a few comments on the conditions of Proposition 3.13.

1. By condition 1, one has $\mathcal{R}(a \otimes_{\mathcal{L}} b) \in \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$ for all $a, b \in \mathfrak{B}$. Analogously, condition 4 implies $\bar{\mathcal{R}}(a \otimes_{\mathcal{L}_{op}} b) \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ for all $a, b \in \mathfrak{B}$. The R-matrix $\mathcal{R}$ lives, in fact, in $\mathfrak{B}^{op} \otimes_{\mathcal{L}_{op}} \mathfrak{B}^{op}$. This explains appearance of the opposite coproduct in (18-19). On the contrary, the inverse $\bar{\mathcal{R}}$ is supported in $\mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$.

2. Both sides of the equation from condition 2 are well defined, cf. remark 1.

3. The right-hand side expressions in (18-19) are correctly defined, i.e. are independent on the representative $\mathcal{R}_1 \otimes \mathcal{R}_2$ of $\mathcal{R} \in \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$ in $\mathfrak{B} \otimes \mathfrak{B}$. Indeed, from condition 2 one deduces

$$\mathcal{R}\big(1 \otimes t(\lambda)\big) = t(\lambda)\mathcal{R}_1 \otimes_{\mathcal{L}_{op}} \mathcal{R}_2, \quad \mathcal{R}\big(s(\lambda) \otimes 1\big) = \mathcal{R}_1 \otimes_{\mathcal{L}_{op}} s(\lambda)\mathcal{R}_2 \quad (20)$$

   for all $\lambda \in \mathcal{L}$. Equations (20) imply that the two maps $\mathfrak{B} \otimes \mathfrak{B} \to \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$ defined by

$$\hat{\jmath}_{23} \colon x \otimes y \mapsto x \otimes_{\mathcal{L}_{op}} \mathcal{R}(1 \otimes y), \quad \hat{\jmath}_{12} \colon x \otimes y \mapsto \mathcal{R}(x \otimes 1) \otimes_{\mathcal{L}_{op}} y, \quad (21)$$

   are factored through maps $\jmath_{12}, \jmath_{23} \colon \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B} \to \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$. Let us check this, say, for $\hat{\jmath}_{23}$. In view of (20), we have

$$x \otimes_{\mathcal{L}_{op}} \mathcal{R}\big(1 \otimes t(\lambda)y\big) = x \otimes_{\mathcal{L}_{op}} t(\lambda)\mathcal{R}_1 \otimes_{\mathcal{L}_{op}} \mathcal{R}_2 y = s(\lambda)x \otimes_{\mathcal{L}_{op}} \mathcal{R}(1 \otimes y)$$

for all $x, y \in \mathfrak{B}$, $\lambda \in \mathcal{L}$. This shows that $\hat{\jmath}_{23}\big(x \otimes t(\lambda)y\big) = \hat{\jmath}_{23}\big(s(\lambda)x \otimes y\big)$, i.e. the value of $\hat{\jmath}_{23}$ depends only on the class of a representative of $\mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$ in $\mathfrak{B} \otimes \mathfrak{B}$.

Now notice that the right-hand sides of equations (18-19) are equal to $\jmath_{23}(\mathcal{R})$ and $\jmath_{12}(\mathcal{R})$, respectively.

**Theorem 3.15.** *Suppose a bialgebroid $\mathfrak{B}$ is quasitriangular. Then the collection of morphisms $\sigma^{-1}_{M_2,M_1} \circ (\rho_1 \otimes_{\mathcal{L}_{op}} \rho_2)(\mathcal{R}) \in \mathrm{Hom}_{\mathfrak{B}}(M_1 \otimes_{\mathcal{L}} M_2, M_2 \otimes_{\mathcal{L}} M_1)$, where $(M_i, \rho_i)$, $i = 1, 2$, are $\mathfrak{B}$-modules, is a braiding in the monoidal category* $\mathrm{Mod}\,\mathfrak{B}$.

*Proof.* Follows from the definition of $\mathcal{R}$. □

Analogously to Hopf algebras, one can consider twists of bialgebroids.

**Definition 3.16 ([Xu1]).** An element $\Psi = \Psi_1 \otimes_{\mathcal{L}} \Psi_2 \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$, where $\mathfrak{B}$ is an $\mathcal{L}$-bialgebroid, is called a twisting cocycle if

$$\Delta(\Psi_1)\Psi \otimes_{\mathcal{L}} \Psi_2 = \Psi_1 \otimes_{\mathcal{L}} \Delta(\Psi_2)\Psi. \tag{22}$$

and $(\varepsilon \otimes_{\mathcal{L}} \mathrm{id})(\Psi) = (\mathrm{id} \otimes_{\mathcal{L}} \varepsilon)(\Psi) = 1 \otimes_{\mathcal{L}} 1$.

Given a twisting cocycle, the space $\mathcal{L}$ is equipped with a new multiplication

$$\lambda * \mu := (\Psi_1 \vdash \lambda)(\Psi_2 \vdash \lambda),$$

making it an associative algebra, $\tilde{\mathcal{L}}$. Applying equation (22) to $\mathfrak{B} \otimes \mathcal{L} \otimes \mathcal{L}$, $\mathcal{L} \otimes \mathfrak{B} \otimes \mathcal{L}$, and $\mathcal{L} \otimes \mathcal{L} \otimes \mathfrak{B}$, one obtains that

$$\tilde{s}(\lambda) := s(\Psi_1 \vdash \lambda)\Psi_2, \quad \tilde{t}(\lambda) := t(\Psi_2 \vdash \lambda)\Psi_1, \quad \lambda \in \mathfrak{B}, \tag{23}$$

are, respectively, an algebra and anti-algebra maps from $\tilde{\mathcal{L}}$ to $\mathfrak{B}$ and their images commute in $\mathfrak{B}$. Thus $\mathfrak{B}$ becomes an $\tilde{\mathcal{L}}$-bimodule by means of the new source and target maps, $\tilde{s}$ and $\tilde{t}$. Applying formulas (15), one can check that $\Psi\big(\tilde{t}(\lambda) \otimes 1\big) = \Psi\big(1 \otimes \tilde{s}(\lambda)\big)$.

Thus twisting cocycle defines an operator acting from $\mathfrak{B} \otimes_{\tilde{\mathcal{L}}} \mathfrak{B}$ to $\mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ by the mapping $a \otimes_{\tilde{\mathcal{L}}} b \mapsto \Psi_1 a \otimes_{\mathcal{L}} \Psi_2 b$. It is called invertible if there is an element $\Psi^{-1} \in \mathfrak{B} \otimes_{\tilde{\mathcal{L}}} \mathfrak{B}$ such that $\Psi\Psi^{-1} \in 1 \otimes_{\mathcal{L}} 1$ and $\Psi^{-1}\Psi \in 1 \otimes_{\tilde{\mathcal{L}}} 1$.

**Proposition 3.17 ([Xu1]).** *Let $\mathfrak{B}$ be an $\mathcal{L}$-bialgebroid and $\Psi \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ be an invertible twisting cocycle. Let $\tilde{\Delta}$ denote the map*

$$a \mapsto \Psi^{-1}\Delta(a)\Psi \tag{24}$$

*from $\mathfrak{B}$ to $\mathfrak{B} \otimes_{\tilde{\mathcal{L}}} \mathfrak{B}$. Then $(\mathfrak{B}, \tilde{\mathcal{L}}, \tilde{s}, \tilde{t}, \tilde{\Delta}, \varepsilon)$ is an $\tilde{\mathcal{L}}$-bialgebroid called the twist of $\mathfrak{B}$ by $\Psi$.*

**Remark 3.18.** Given two $\mathcal{L}$-bialgebroids $\mathfrak{B}_i$, $i = 1, 2$, a twist $\Psi \in \mathfrak{B}_1 \otimes_{\mathcal{L}} \mathfrak{B}_1$, and a homomorphism $\varphi \colon \mathfrak{B}_1 \to \mathfrak{B}_2$, the element $(\varphi \otimes_{\mathcal{L}} \varphi)(\Psi) \in \mathfrak{B}_2 \otimes_{\mathcal{L}} \mathfrak{B}_2$ is a twist in $\mathfrak{B}_2$. Then $\varphi$ becomes a homomorphism of twisted bialgebroids, $\varphi \colon \tilde{\mathcal{B}}_1 \to \tilde{\mathcal{B}}_2$.

A bialgebroid twist induces a transformation of monoidal categories. For any pair $M_1$, $M_2$ of $\mathfrak{B}$-modules, the twist $\Psi$ gives a map $M_1 \otimes_{\tilde{\mathcal{L}}} M_2 \to M_1 \otimes_{\mathcal{L}} M_2$ intertwining the actions of $\tilde{\mathfrak{B}}$ and $\mathfrak{B}$. If $\mathfrak{B}$ is quasitriangular, then the braiding in $\mathrm{Mod}\, \mathfrak{B}$ defines a braiding in $\mathrm{Mod}\, \tilde{\mathfrak{B}}$. This follows from the following fact.

**Proposition 3.19.** *Let $\mathfrak{B}$ be a quasitriangular $\mathcal{L}$-bialgebroid with the universal R-matrix $\mathcal{R}$ and let $\Psi \in \mathfrak{B} \otimes_{\mathcal{L}} \mathfrak{B}$ be a twisting cocycle. Then the twisted bialgebroid $\tilde{\mathfrak{B}}$ is quasitriangular, with the universal R-matrix $\tilde{\mathcal{R}} := (\Psi_{21})^{-1}\mathcal{R}\Psi$, where $\Psi_{21} = \sigma_{\mathfrak{B},\mathfrak{B}}(\Psi) \in \mathfrak{B} \otimes_{\mathcal{L}_{op}} \mathfrak{B}$.*

*Proof.* First of all notice that $\tilde{\mathcal{R}} = (\Psi_{21})^{-1}\mathcal{R}\Psi$ is a well defined element of $\tilde{\mathfrak{B}} \otimes_{\tilde{\mathcal{L}}_{op}} \tilde{\mathfrak{B}}$. The proof is carried out by a direct computation. □

**Remark 3.20.** The R-matrix $\mathcal{R}$ is a special twist of the coopposite bialgebroid $\mathfrak{B}^{op}$, analogously to the Hopf algebra case.

# 4 Bialgebroids over a quasi-commutative base

## 4.1 Bialgebroid $\mathcal{L} \rtimes \mathcal{H}$

In this subsection we assume that the Hopf algebra $\mathcal{H}$ is quasitriangular, with the universal R-matrix $\mathcal{R}$. The module algebra $\mathcal{L}$ is assumed to be $\mathcal{H}$-commutative, i.e.

$$(\mathcal{R}_2 \triangleright \mu)(\mathcal{R}_1 \triangleright \lambda) = \lambda\mu \tag{25}$$

for all $\lambda, \mu \in \mathcal{H}$. We use the standard notation $\mathcal{R}^+ = \mathcal{R}$ and $\mathcal{R}^- = \mathcal{R}_{21}^{-1}$. Recall that $\mathcal{R}^-$ gives an alternative quasitriangular structure on $\mathcal{H}$, and for $\mathcal{L}$ to be quasi-commutative does not depend on the choice of $\mathcal{R} = \mathcal{R}^\pm$.

Recall that $\mathcal{L}$ is equipped with two structures $\mathcal{L}_\pm$ of $\mathcal{H}$-base algebra corresponding to the two coactions $\delta^\pm$, cf. Remark 2.2. Consider the associative algebra $\mathcal{L} \rtimes \mathcal{H}$ endowed with the smash product multiplication

$$(\lambda \otimes f)(\mu \otimes g) \quad := \quad \lambda(f^{(1)} \triangleright \mu) \otimes f^{(2)} g. \tag{26}$$

Introduce linear maps $s$ and $t^\pm$ from $\mathcal{L}$ to $\mathcal{L} \rtimes \mathcal{H}$ by

$$s(\lambda) = \lambda \otimes 1, \quad t^\pm(\lambda) = \mathcal{R}_2^\pm \triangleright \lambda \otimes \mathcal{R}_1^\pm, \tag{27}$$

for $\lambda \in \mathcal{L}$. By the construction of smash product, $s$ is an algebra embedding. The maps $t^{\pm}$ are expressed through the $\mathcal{H}$-coactions by the formula

$$t^{\pm}(\lambda) = \lambda^{[2]} \otimes \gamma^{-1}(\lambda^{(1)}), \tag{28}$$

where $\delta^{\pm} = \lambda^{(1)} \otimes \lambda^{[2]}$ and $\gamma$ is the antipode in $\mathcal{H}$, cf. formulas (2).

**Lemma 4.1.** *The maps $t^{\pm}$ are algebra anti-homomorphisms. For every pair $\lambda, \mu \in \mathcal{L}$ one has $s(\lambda)\, t^{\pm}(\mu) = t^{\pm}(\mu) s(\lambda)$.*

*Proof.* Let us check the statement for $t = t^{+}$. Using the bicharacter properties

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$

we find

$$t(\lambda\mu) \;=\; \mathcal{R}_2 \triangleright (\lambda\mu) \otimes \mathcal{R}_1 = (\mathcal{R}_2^{(1)} \triangleright \lambda)(\mathcal{R}_2^{(2)} \triangleright \mu) \otimes \mathcal{R}_1 = (\mathcal{R}_2 \triangleright \lambda)(\mathcal{R}_{2'} \triangleright \mu) \otimes \mathcal{R}_{1'}\mathcal{R}_1.$$

On the other hand,

$$\begin{aligned} t(\mu)t(\lambda) &= (\mathcal{R}_{2'} \triangleright \mu \otimes \mathcal{R}_{1'})(\mathcal{R}_2 \triangleright \lambda \otimes \mathcal{R}_1) = (\mathcal{R}_{2'} \triangleright \mu)\big(\mathcal{R}_{1'}^{(1)}\mathcal{R}_2 \triangleright \lambda\big) \otimes \mathcal{R}_{1'}^{(2)}\mathcal{R}_1 \\ &= (\mathcal{R}_{2'} \triangleright \mu)\big(\mathcal{R}_{1'}^{(1)}\mathcal{R}_2 \triangleright \lambda\big) \otimes \mathcal{R}_{1'}^{(2)}\mathcal{R}_1 = (\mathcal{R}_{2''}\mathcal{R}_{2'} \triangleright \mu)\big(\mathcal{R}_{1''}\mathcal{R}_2 \triangleright \lambda\big) \otimes \mathcal{R}_{1'}\mathcal{R}_1 \\ &= (\mathcal{R}_2 \triangleright \lambda)(\mathcal{R}_{2'} \triangleright \mu) \otimes \mathcal{R}_{1'}\mathcal{R}_1. \end{aligned}$$

In the last transformation, we have used the quasi-commutativity of the algebra $\mathcal{L}$.

Employing the same arguments, we find

$$\begin{aligned} t(\mu)s(\lambda) &= (\mathcal{R}_{2'} \triangleright \mu \otimes \mathcal{R}_{1'})(\lambda \otimes 1) = (\mathcal{R}_{2'} \triangleright \mu)(\mathcal{R}_{1'}^{(1)} \triangleright \lambda) \otimes \mathcal{R}_{1'}^{(2)} \\ &= (\mathcal{R}_{2''}\mathcal{R}_{2'} \triangleright \mu)(\mathcal{R}_{1''} \triangleright \lambda) \otimes \mathcal{R}_{1'} = \lambda(\mathcal{R}_{2'} \triangleright \mu) \otimes \mathcal{R}_{1'} = s(\lambda)t(\mu). \end{aligned} \tag{29}$$

We have proven the statement regarding the map $t^{+}$. The case of $t^{-}$ is treated similarly, with $\mathcal{R}$ replaced by $\mathcal{R}^{-}$. ∎

**Proposition 4.2 ([Lu]).** *The associative algebra $\mathcal{L} \rtimes \mathcal{H}$ is equipped with two $\mathcal{L}$-bialgebroid structures, $\mathcal{L}_{\pm} \rtimes \mathcal{H}$, with the source map $s$ and the target map $t^{\pm}$ from Lemma 4.1, coproduct $\Delta(\lambda \otimes h) := (\lambda \otimes h^{(1)}) \otimes_{\mathcal{L}} (1 \otimes h^{(2)})$, and the counit $\varepsilon(\lambda \otimes h) := \lambda\, \varepsilon(h)$, $\lambda \otimes h \in \mathcal{L} \rtimes \mathcal{H}$. The anchor action of $\mathcal{L} \rtimes \mathcal{H}$ on $\mathcal{L}$ is given by $(\mu \otimes h) \vdash \lambda = \mu(h \triangleright \lambda)$, for $\lambda \in \mathcal{L}$ and $\mu \otimes h \in \mathcal{L} \rtimes \mathcal{H}$.*

Remark that the bialgebroid structures $\mathcal{L}_{\pm} \rtimes \mathcal{H}$ on the same associative algebra $\mathcal{L} \rtimes \mathcal{H}$ are determined solely by the structures of the base algebra $\mathcal{L}_{\pm}$ on $\mathcal{L}$ (in other words, by the $\mathcal{H}$-coactions). By default, we understand by $\mathcal{L} \rtimes \mathcal{H}$ the bialgebroid $\mathcal{L}_{+} \rtimes \mathcal{H}$.

Each bialgebroid $\mathcal{L}_\pm \rtimes \mathcal{H}$ has a natural sub-bialgebroid. To describe them, let us recall that the R-matrices $\mathcal{R}^\pm$ define two Hopf algebra maps from $\mathcal{H}^*_{op}$ to $\mathcal{H}$, by formulas (5). Then $\mathcal{R}^\pm \in \mathcal{H}^{\{\pm\}} \otimes \mathcal{H}^{\{\mp\}} \subset \mathcal{H} \otimes \mathcal{H}$, where $\mathcal{H}^{\{\pm\}}$ are Hopf subalgebras in $\mathcal{H}$ that are the images of $\mathcal{R}^\pm$. Note that $\mathcal{L}_\pm$ is a base algebra over the Hopf algebra $\mathcal{H}^{\{\pm\}}$, since the coaction $\delta^\pm$ actually takes its values in $\mathcal{H}^{\{\pm\}} \otimes \mathcal{L}$, see (10). The algebra $\mathcal{L} \rtimes \mathcal{H}$ contains $\mathcal{L} \rtimes \mathcal{H}^{\{\pm\}}$ as subalgebras.

**Proposition 4.3.** *$\mathcal{L} \rtimes \mathcal{H}^{\{\pm\}}$ are sub-bialgebroids in $\mathcal{L}_\pm \rtimes \mathcal{H}$.*

*Proof.* The formula (28) shows that the maps $t^\pm$ take values in $\mathcal{L} \rtimes \mathcal{H}^{\{\pm\}}$. Therefore $\mathcal{L} \rtimes \mathcal{H}^{\{\pm\}}$ are $\mathcal{L}$-sub-bimodules in $\mathcal{L}_\pm \rtimes \mathcal{H}$. The coproduct in $\mathcal{L}_\pm \rtimes \mathcal{H}$ restricts to $\mathcal{L} \rtimes \mathcal{H}^{\{\pm\}}$, thus we conclude that $\mathcal{L} \rtimes \mathcal{H}^{\{\pm\}}$ are sub-bialgebroids. □

**Remark 4.4.** An arbitrary Hopf algebra $\mathcal{H}$ is identified with $(\mathfrak{D}\mathcal{H})^{\{-\}}$, if the double $\mathfrak{D}\mathcal{H}$ is equipped with the quasitriangular structure $\Theta \in \mathcal{H}^*_{op} \otimes \mathcal{H} \subset (\mathfrak{D}\mathcal{H})^{\otimes 2}$. Given an $\mathcal{H}$-base algebra $\mathcal{L}$, one can build an $\mathcal{L}$-bialgebroid $\mathcal{L} \rtimes \mathcal{H} \simeq \mathcal{L} \rtimes (\mathfrak{D}\mathcal{H})^{\{-\}} \subset \mathcal{L}_- \rtimes \mathfrak{D}\mathcal{H}$, according to the Proposition 4.3. The target map in $\mathcal{L} \rtimes \mathcal{H}$ is expressed through the coaction by formula

$$t(\lambda) = \lambda^{[2]} \otimes \gamma^{-1}(\lambda^{(1)}), \tag{30}$$

as a specialization of (28).

## 4.2 Quantum groupoid $\mathcal{H}_\mathcal{L}$

In this subsection we build a quasitriangular bialgebroid $\mathcal{H}_\mathcal{L}$ as a quotient of $\mathcal{L}_\pm \rtimes \mathcal{H}$ by a certain biideal. This quotient eliminates the distinctions between the two bialgebroids $\mathcal{L}_+ \rtimes \mathcal{H}$ and $\mathcal{L}_- \rtimes \mathcal{H}$.

To proceed with the study of the bialgebroid $\mathcal{L}_\pm \rtimes \mathcal{H}$, we need an algebraic construction to be described next.

**Lemma 4.5.** *Let $\phi$ be an endomorphism of an associative algebra $\mathfrak{B}$. The left ideal $J_\phi$ generated by the image of the endomorphism $\phi - \mathrm{id}$ is a $\phi$-invariant two-sided ideal. It is the minimal among $\phi$-invariant two-sided ideals such that the endomorphism of $\mathfrak{B}/J_\phi$ induced by $\phi$ is identical.*

*Proof.* The identity $\big(\phi(a) - a\big)b = \big(\phi(ab) - ab\big) - \phi(a)\big(\phi(b) - b\big)$ being valid for any pair $a, b \in \mathfrak{B}$ shows that $J_\phi$ is a two-sided ideal. The minimality property is obvious. □

**Lemma 4.6.** *a) The linear endomorphism $\phi\colon \mathcal{L} \rtimes \mathcal{H} \to \mathcal{L} \rtimes \mathcal{H}$ given by*

$$\phi(\lambda \otimes h) := (\mathcal{R}_2 \mathcal{R}_{1'}) \triangleright \lambda \otimes \mathcal{R}_1 \mathcal{R}_{2'} h. \tag{31}$$

*is an algebra automorphism. b) The ideal $J_\phi$ can be presented in the form $s(\mathcal{L})(\phi - \mathrm{id})(\mathcal{L} \rtimes \mathcal{H})$. c) As a two-sided ideal, $J_\phi$ is generated by the image of the map $t^+ - t^-$, i.e. by the set $(t^+ - t^-)(\mathcal{L})$.*

*Proof.* Denote by $v$ the Drinfeld element $\mathcal{R}_1 \gamma(\mathcal{R}_2) \in \mathcal{H}$, [Dr2]. It satisfies the identities

$$\mathcal{R}_{21} \mathcal{R} = \Delta(v^{-1})(v \otimes v), \quad v h v^{-1} = \gamma^{-2}(h), \quad h \in \mathcal{H}. \tag{32}$$

It is easy to check, using (32), that the map $\phi_0\colon \lambda \otimes h \mapsto v \triangleright \lambda \otimes v h v^{-1}$ is an automorphism of the algebra $\mathcal{L} \rtimes \mathcal{H}$. Then $\phi$ from (31) coincides with the composition of two automorphisms $\mathrm{Ad}^{-1}(1 \otimes v) \circ \phi_0$; this proves a). Since $\phi$ is identical on $1 \otimes \mathcal{H}$, the image of $\phi - \mathrm{id}$ is invariant under the left regular $\mathcal{H}$-action on $\mathcal{L} \rtimes \mathcal{H}$. Therefore $J_\phi$ can be presented as a left $\mathcal{L}$-submodule generated by the image of $\phi - \mathrm{id}$; this proves b). Remark that as a two-sided ideal, $J_\phi$ is generated by the image of the map $(\phi - \mathrm{id}) \circ s$ or, in terms of the Drinfeld element, by the relations

$$(v \triangleright \lambda) \otimes 1 = (1 \otimes v)(\lambda \otimes 1)(1 \otimes v)^{-1}, \quad \lambda \in \mathcal{L}. \tag{33}$$

Notice that $\phi \circ t^- = t^+$; this implies the equality $t^- \equiv t^+ \mod J_\phi$ or, explicitly,

$$\mathcal{R}_2^+ \lambda \otimes \mathcal{R}_1^+ \equiv \mathcal{R}_2^- \lambda \otimes \mathcal{R}_1^- \mod J_\phi, \quad \lambda \in \mathcal{L}. \tag{34}$$

On the other hand, $(\phi - \mathrm{id})(\lambda \otimes 1) = \big(t^+ - t^-\big)(\mathcal{R}_2^- \lambda)(1 \otimes \mathcal{R}_1^-)$, hence $J_\phi$ lies in the ideal generated by $\big(t^+ - t^-\big)(\mathcal{L})$; this proves c). $\square$

Remark that the ideal $J_\phi$ is zero if $\mathcal{H}$ is triangular, i.e. $\mathcal{R}^+ = \mathcal{R}^-$.

**Proposition 4.7.** *The ideal $J_\phi$ is a biideal in both bialgebroids $\mathcal{L}_\pm \rtimes \mathcal{H}$. The quotient bialgebroid $\mathcal{H}_\mathcal{L} := \big(\mathcal{L}_+ \rtimes \mathcal{H}\big)/J_\phi = \big(\mathcal{L}_- \rtimes \mathcal{H}\big)/J_\phi$ is quasitriangular, with the universal R-matrix being the image of*

$$(1 \otimes \mathcal{R}_1) \otimes_{\mathcal{L}_{op}} (1 \otimes \mathcal{R}_2) \tag{35}$$

*under the projection along $J_\phi$.*

*Proof.* By Lemma 4.6.b, $J_\phi$ is a left $\mathcal{L}$-module generated by the image of $\phi - \mathrm{id}$. Applying the counit $\varepsilon$ from Proposition 4.2 to the formula (31), we find $\varepsilon \circ (\phi - \mathrm{id}) = 0$. Thus $J_\phi$ lies in the kernel of $\varepsilon$.

Let us prove that $J_\phi$ is a biideal in $\mathcal{L}_+ \rtimes \mathcal{H}$. By Lemma 4.6.c, $J_\phi$ is generated by the set $(t^- - t^+)(\mathcal{L})$. Therefore, it is sufficient to check that $(\Delta \circ t^-)(\lambda) \equiv (\Delta \circ t^+)(\lambda)$, where the symbol $\equiv$ means equality modulo $J_\phi \otimes_{\mathcal{L}} \big(\mathcal{L} \rtimes \mathcal{H}\big) + \big(\mathcal{L} \rtimes \mathcal{H}\big) \otimes_{\mathcal{L}} J_\phi$ for all $\lambda \in \mathcal{L}$. We have for $(\Delta \circ t^-)(\lambda)$

$$\begin{aligned}
\Delta(\mathcal{R}_2^- \triangleright \lambda \otimes \mathcal{R}_1^-) &= \big(\mathcal{R}_2^- \mathcal{R}_{2'}^- \triangleright \lambda \otimes \mathcal{R}_1^-\big) \otimes_{\mathcal{L}} (1 \otimes \mathcal{R}_{1'}^-) \equiv \big(\mathcal{R}_2^+ \mathcal{R}_{2'}^- \triangleright \lambda \otimes \mathcal{R}_1^+\big) \otimes_{\mathcal{L}} (1 \otimes \mathcal{R}_{1'}^-) \\
&= t^+(\mathcal{R}_{2'}^- \triangleright \lambda) \otimes_{\mathcal{L}} (1 \otimes \mathcal{R}_{1'}^-) = 1 \otimes_{\mathcal{L}} s(\mathcal{R}_{2'}^- \triangleright \lambda)(1 \otimes \mathcal{R}_{1'}^-) \\
&= 1 \otimes_{\mathcal{L}} (\mathcal{R}_2^- \triangleright \lambda \otimes \mathcal{R}_1^-) \equiv 1 \otimes_{\mathcal{L}} (\mathcal{R}_2^+ \triangleright \lambda \otimes \mathcal{R}_1^+).
\end{aligned}$$

But the last expression is equal to $1 \otimes_{\mathcal{L}} t^+(\lambda) = (\Delta \circ t^+)(\lambda)$ since $\Delta$ is an $\mathcal{L}$-bimodule map. This proves that $J_\phi$ is a biideal in $\mathcal{L}_+ \rtimes \mathcal{H}$. This also implies that $J_\phi$ is a biideal in $\mathcal{L}_- \rtimes \mathcal{H}$, in view of the symmetry $+ \leftrightarrow -$. The quotient $\big(\mathcal{L}_+ \rtimes \mathcal{H}\big)/J_\phi$ is canonically isomorphic to $\big(\mathcal{L}_- \rtimes \mathcal{H}\big)/J_\phi$ as a bialgebroid, since $t^+ \equiv t^- \mod J_\phi$.

Let us show that (35) is a universal $R$-matrix in $\mathcal{H}_{\mathcal{L}}$. Only condition 1 of Definition 3.12 requires verification. The other conditions follow from the properties of $\mathcal{R}$ as a universal R-matrix of the Hopf algebra $\mathcal{H}$.

Computing the element $(1 \otimes \mathcal{R}_1)t^+(\lambda) \otimes_{\mathcal{L}_{op}} (1 \otimes \mathcal{R}_2)$ modulo the ideal $J_\phi$ we find

$$\begin{aligned}
(1 \otimes \mathcal{R}_1)(\mathcal{R}_2 \triangleright \lambda \otimes \mathcal{R}_1) \otimes_{\mathcal{L}_{op}} (1 \otimes \mathcal{R}_2) &= (\mathcal{R}_1^{(1)} \mathcal{R}_{2'} \triangleright \lambda \otimes \mathcal{R}_1^{(2)} \mathcal{R}_{1'}) \otimes_{\mathcal{L}_{op}} (1 \otimes \mathcal{R}_2) \\
&= s(\mathcal{R}_1^{(1)} \mathcal{R}_{2'} \triangleright \lambda)(1 \otimes \mathcal{R}_1^{(2)} \mathcal{R}_{1'}) \otimes_{\mathcal{L}_{op}} (1 \otimes \mathcal{R}_2) \\
&= (1 \otimes \mathcal{R}_1^{(2)} \mathcal{R}_{1'}) \otimes_{\mathcal{L}_{op}} t^+(\mathcal{R}_1^{(1)} \mathcal{R}_{2'} \triangleright \lambda)(1 \otimes \mathcal{R}_2).
\end{aligned}$$

Since $t^+ \equiv t^- \mod J_\phi$, this expression becomes equal to

$$(1 \otimes \mathcal{R}_{1''} \mathcal{R}_{1'}) \otimes_{\mathcal{L}_{op}} (\mathcal{R}_{2'} \triangleright \lambda \otimes \mathcal{R}_{2''}) \quad = \quad (1 \otimes \mathcal{R}_{1'} \mathcal{R}_1) \otimes_{\mathcal{L}_{op}} (\mathcal{R}_2 \triangleright \lambda \otimes \mathcal{R}_{2'})$$

and, finally, to $(1 \otimes \mathcal{R}_1) \otimes_{\mathcal{L}_{op}} (1 \otimes \mathcal{R}_2)s(\lambda)$, as required. Let us comment that one can also deduce this fact directly from Remark 3.20, noticing that the inverse R-matrix of $\mathcal{H}$ is a twisting cocycle of $\mathfrak{B}$. □

**Remark 4.8.** In what follows we will abuse notation suppressing the projection $\mathcal{L}_\pm \rtimes \mathcal{H} \to \mathcal{H}_{\mathcal{L}}$ when writing elements of $\mathcal{H}_{\mathcal{L}}$. In other words, the reader can perceive calculations in $\mathcal{H}_{\mathcal{L}}$ as those in $\mathcal{L}_\pm \rtimes \mathcal{H}$ done modulo $J_\phi$. The most important feature for us is the identity $\mathcal{R}_2^+ \lambda \otimes \mathcal{R}_1^+ = \mathcal{R}_2^- \lambda \otimes \mathcal{R}_1^-$, which is valid in $\mathcal{H}_{\mathcal{L}}$ for all $\lambda \in \mathcal{L}$.

# 5 On the antipode

In this subsection we study antipodes in bialgebroids $\mathcal{L} \rtimes \mathcal{H}$ and $\mathcal{H}_{\mathcal{L}}$. It turns out that they can be defined as isomorphisms between opposite and coopposite bialgebroids, analogously to Hopf algebras. However, contrary to the Hopf algebra case, there is no canonical way to define the opposite bialgebroid. Even the coopposite bialgebroid, although defined canonically in Subsection 3.2, is in fact over the opposite base. Nevertheless, using the specific form of the bialgebroids under consideration, the opposite bialgebroids can be introduced.

## 5.1 Bialgebroid $(\mathcal{L} \rtimes \mathcal{H})_{op}$

In this and the next subsections we consider the $\mathcal{L}$-bialgebroid $\mathcal{L} \rtimes \mathcal{H}$, where $\mathcal{L}$ is a base algebra for a general (not necessarily quasitriangular) Hopf algebra $\mathcal{H}$, cf. Remark 4.4.

**Lemma 5.1.** *Let $\mathcal{H}$ be a Hopf algebra. Let $\mathcal{L}$ be an $\mathcal{H}$-base algebra with the $\mathcal{H}$-action $\triangleright$ and the coaction $\delta$. Then $\mathcal{L}_{op}$ is a base algebra over $\mathcal{H}_{op}$ with respect to the $\mathcal{H}_{op}$-action $x \blacktriangleright \ell := \gamma^{-1}(x) \triangleright \ell$ and the $\mathcal{H}_{op}$-coaction $\delta^\bullet = \delta$.*

*Proof.* Obviously $\mathcal{L}_{op}$ is a left $\mathcal{H}_{op}$-module algebra with respect to $\blacktriangleright$ and a left $\mathcal{H}_{op}$-comodule algebra with respect to $\delta^\bullet$. Let us show that it is a Yetter-Drinfeld module with respect to $\mathcal{H}_{op}$. This is equivalent to the condition

$$\delta(x \blacktriangleright \ell) = x^{(1)} \cdot \ell^{(1)} \cdot \gamma^{-1}(x^{(3)}) \otimes x^{(2)} \blacktriangleright \ell^{[2]},$$

which is the formula (7) translated to the case of $\mathcal{L}_{op}$ and $\mathcal{H}_{op}$ instead of $\mathcal{L}$ and $\mathcal{H}$ (note that $\gamma^{-1}$ the antipode for $\mathcal{H}_{op}$). The dots mean the opposite multiplication.

Finally, the $\mathcal{H}$-commutativity condition (8) in $\mathcal{L}$ transforms into the $\mathcal{H}_{op}$-commutativity condition $\lambda \cdot \mu = (\lambda^{(1)} \blacktriangleright \mu) \cdot \lambda^{[2]}$ in $\mathcal{L}_{op}$. □

Consider the opposite associative algebra $(\mathcal{L} \rtimes \mathcal{H})_{op}$. This algebra contains $\mathcal{L}_{op}$ and $\mathcal{H}_{op}$ as subalgebras and, in fact, has the form of smash product $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$, where the action of $\mathcal{H}_{op}$ on $\mathcal{L}_{op}$ is specified in Lemma 5.1. But $\mathcal{L}_{op}$ is a base algebra over $\mathcal{H}_{op}$, by Lemma 5.1, thus $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$ is equipped by the structure of $\mathcal{L}_{op}$-bialgebroid in the standard way. Thus there is a canonical bialgebroid structure on the opposite algebra $(\mathcal{L} \rtimes \mathcal{H})_{op}$. We denote by $s^\bullet$, $t^\bullet$, $\Delta^\bullet$, and $\varepsilon^\bullet$ respectively, the source, target, coproduct, and counit maps of the opposite bialgebroid $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$.

**Definition 5.2.** The opposite bialgebroid $(\mathcal{L} \rtimes \mathcal{H})_{op}$ to the bialgebroid $\mathcal{L} \rtimes \mathcal{H}$ is an $\mathcal{L}_{op}$-bialgebroid $(\mathcal{L}_{op} \rtimes \mathcal{H}_{op}, \mathcal{L}_{op}, s^\bullet, t^\bullet, \Delta^\bullet, \varepsilon^\bullet)$

## 5.2 The antipode in $\mathcal{L} \rtimes \mathcal{H}$

Recall that $\Theta \in \mathcal{H}^*_{op} \otimes \mathcal{H} \subset (\mathfrak{D}\mathcal{H})^{\otimes 2}$ denotes the standard quasitriangular structure on $\mathfrak{D}\mathcal{H}$. Let us compute the target map of the bialgebroid $(\mathcal{L} \rtimes \mathcal{H})_{op}$ defined in the previous subsection, in terms of $\mathcal{L}$, $\mathcal{H}$, and $\Theta$.

**Lemma 5.3.** *The map $\lambda \mapsto \Theta_1 \triangleright \lambda \otimes \gamma(\Theta_2)$ from $\mathcal{L}$ to $\mathcal{L} \otimes \mathcal{H}$ yields the target map $t^\bullet$ of the $\mathcal{L}_{op}$-bialgebroid $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$.*

*Proof.* Let us apply the formula (30) to the bialgebroid $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$. The coaction $\delta^\bullet$ coincides with $\delta$, which has the form $\lambda \mapsto \Theta_2 \otimes \Theta_1 \triangleright \lambda$. Now the lemma follows from the fact that the antipode in $\mathcal{H}_{op}$ is the inverse antipode in $\mathcal{H}$. $\square$

Consider the map $\zeta\colon \mathcal{L} \rtimes \mathcal{H} \to \mathcal{L}_{op} \rtimes \mathcal{H}_{op}$ defined by

$$\zeta\colon \lambda \otimes h \mapsto \Theta_1 \triangleright \lambda \otimes \gamma(\Theta_2) \cdot \gamma(h), \tag{36}$$

where $\gamma$ is the antipode of $\mathcal{H}$ and $\Theta$ is the R-matrix of $\mathfrak{D}\mathcal{H}$ (here we suppressed the anti-isomorphism $\iota\colon \mathcal{L} \to \mathcal{L}_{op}$).

**Proposition 5.4.** *The map (36) defines an isomorphism of $\mathcal{L}_{op}$-bialgebroids*

$$\zeta\colon (\mathcal{L} \rtimes \mathcal{H})^{op} \to \mathcal{L}_{op} \rtimes \mathcal{H}_{op}. \tag{37}$$

*Proof.* The map (36) is an algebra homomorphism when restricted to the subalgebras $\mathcal{L} \otimes 1$ and $1 \otimes \mathcal{H}$. By construction, it respects the product $(\lambda \otimes 1)(1 \otimes h)$. To complete the proof, one must check that $\zeta$ respects the product $(1 \otimes h)(\lambda \otimes 1)$:

$$\begin{aligned}
\zeta(1 \otimes h)\zeta(\lambda \otimes 1) &= \big(1 \otimes \gamma(h)\big)(\Theta_1 \triangleright \lambda \otimes \Theta_2) = \gamma(h^{(2)}) \blacktriangleright \Theta_1 \triangleright \lambda \otimes \gamma(h^{(1)}) \cdot \gamma(\Theta_2) \\
&= h^{(2)}\Theta_1 \triangleright \lambda \otimes \gamma(h^{(1)}) \cdot \gamma(\Theta_2) = h^{(2)}\Theta_1 \triangleright \lambda \otimes \gamma(h^{(1)}\Theta_2).
\end{aligned}$$

Since $\Theta$ is an R-matrix of the double $\mathfrak{D}\mathcal{H}$, the above expression can be rewritten as

$$(\Theta_1 h^{(1)} \triangleright)\lambda \otimes \gamma(\Theta_2) \cdot \gamma(h^{(2)}) = \zeta(h^{(1)} \triangleright \lambda \otimes h^{(2)}) = \zeta\big((1 \otimes h)(\lambda \otimes 1)\big).$$

The target map $t^{op}$ of the $\mathcal{L}_{op}$-bialgebroid $(\mathcal{L} \rtimes \mathcal{H})^{op}$ comes from the source map of the $\mathcal{L}$-bialgebroid $\mathcal{L} \rtimes \mathcal{H}$ and it is equal to $s \circ \iota^{-1}$. It follows from Lemma 5.3 that $\zeta \circ t^{op} = t^\bullet$. Let us prove that $\zeta \circ s^{op}$ equals the source map $s^\bullet$ of $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$. Suppressing the notation of the map $\iota$, we have

$$(\zeta \circ s^{op})(\lambda) = (\zeta \circ t)(\lambda) = \zeta\big(\Theta_1 \triangleright \lambda \otimes \gamma^{-1}(\Theta_2)\big) = (\Theta_{1'}\Theta_1) \triangleright \lambda \otimes \gamma(\Theta_{2'}) \cdot \Theta_2 = \lambda \otimes 1.$$

Here we have used the fact $\Theta_{1'}\Theta_1 \otimes \Theta_2\gamma(\Theta_{2'}) = 1 \otimes 1$, which is equivalent to the standard identity $\Theta_1 \otimes \gamma^{-1}(\Theta_2) = \Theta^{-1}$ for the universal R-matrix. Thus we have shown that $\zeta$ is a morphism of $\mathcal{L}_{op}$-bimodules.

Now let us show that $\zeta$ respects the coproducts. Indeed, we have

$$\big((\zeta \otimes_{\mathcal{L}_{op}} \zeta) \circ \Delta^{op}\big)(\lambda \otimes h) = \zeta(1 \otimes h^{(2)}) \otimes_{\mathcal{L}_{op}} \zeta(\lambda \otimes h^{(1)}) = \big(1 \otimes \gamma(h^{(2)})\big) \otimes_{\mathcal{L}_{op}} t^\bullet(\lambda)\gamma(h^{(1)}).$$

The rightmost expression is equal to $\Delta^\bullet\big(t^\bullet(\lambda)\gamma(h)\big) = \Delta^\bullet\big(\zeta(\lambda \otimes h)\big)$. Thus we have checked the right equation from (16). The left one, concerning the counits, readily follows from the definition of $\zeta$. $\square$

Replacing $\mathcal{L}$ and $\mathcal{H}$ by $\mathcal{L}_{op}$ and $\mathcal{H}_{op}$ and taking the inverse map in (37), we obtain a bialgebroid isomorphism

$$\mathcal{L} \rtimes \mathcal{H} \to (\mathcal{L}_{op} \rtimes \mathcal{H}_{op})^{op}. \tag{38}$$

Using the argument after the proof of Lemma 5.1, we can consider the map (38) as an anti-isomorphism of the associative algebra $\mathcal{L} \rtimes \mathcal{H}$, which reads

$$\lambda \otimes h \mapsto \big(1 \otimes \gamma(h)\big)t(v^{-1} \triangleright \lambda), \quad \lambda \otimes h \in \mathcal{L} \rtimes \mathcal{H}, \tag{39}$$

where $v$ is the Drinfeld element, cf. (32). We denote the map (38) by $\gamma$ regarding it as an extension of the antipode of $\mathcal{H} \subset \mathcal{L} \rtimes \mathcal{H}$.

**Remark 5.5.** The map (39) coincides with the antipode of [Lu]. However it was considered there just as an operator on $\mathcal{L} \rtimes \mathcal{H}$ possessing a certain set properties. We would like to emphasize the bialgebroid meaning of the map (39). Namely, it implements an isomorphism (38) between *different* bialgebroids over *different* bases. This gives rise to the categorical interpretation of the antipode of [Lu]. It gives rise to an isomorphism of the corresponding monoidal categories of modules.

## 5.3 The antipode in the quantum groupoid $\mathcal{H}_\mathcal{L}$

In this subsection we will investigate the behavior of the antipode (38) under the projection $\mathcal{L} \rtimes \mathcal{H} \to \mathcal{H}_\mathcal{L}$ assuming $\mathcal{H}$ quasitriangular with R-matrix $\mathcal{R}$ and $\mathcal{L}$ quasi-commutative; $\mathcal{L}$ is equipped with the $\mathcal{H}$-base algebra structure $\mathcal{L}_+$, cf. Remark 4.4. Denote by $v^\bullet$ the Drinfeld element of $\mathcal{H}_{op}$ and by $\phi^\bullet$ the automorphism (31) specialized to the case of the bialgebroid $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$. According to Proposition 4.7, the ideal $J_{\phi^\bullet}$ is a biideal in $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$.

Thus we have two two-sided ideals in the algebra $\mathcal{L} \rtimes \mathcal{H}$, namely $J_\phi$ and $J_{\phi^\bullet}$, corresponding to the bialgebroids $\mathcal{L} \rtimes \mathcal{H}$ and $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$ (two-sided ideals are the same in opposite algebras).

**Lemma 5.6.** *The ideals $J_\phi$ and $J_{\phi^\bullet}$ coincide.*

*Proof.* In the course of the proof of Proposition 4.7 we have shown that the ideal $J_\phi$ is generated by the relations (33). Specializing (33) for the bialgebroid $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$, we find that the ideal $J_{\phi^\bullet}$ is generated by the relations

$$v^\bullet \blacktriangleright \lambda \otimes 1 = (1 \otimes v^\bullet) \cdot (\lambda \otimes 1) \cdot (1 \otimes v^\bullet)^{-1}, \tag{40}$$

where the dots stand for the multiplication in $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$.

Observe that the antipode in $\mathcal{H}_{op}$ is the inverse antipode in $\mathcal{H}$. Therefore $v^\bullet = v^{-1}$ and $\gamma^{-1}(v^{-1}) = v$, since $v^{-1}$ implements the squared antipode by conjugation. Taking into account this argument, equation (40) translates into equation (33) in $\mathcal{L} \rtimes \mathcal{H}$. □

Following Proposition 4.7 we can introduce a quantum groupoid that is opposite to $\mathcal{H}_\mathcal{L}$.

**Definition 5.7.** The opposite quantum groupoid $(\mathcal{H}_\mathcal{L})_{op}$ is a quasitriangular $\mathcal{L}_{op}$-bialgebroid that is the quotient of $\mathcal{L}_{op} \rtimes \mathcal{H}_{op}$ by the biideal $J_{\phi^\bullet}$.

The coopposite $\mathcal{L}$-bialgebroid $(\mathcal{H}_\mathcal{L})_{op}^{op}$ to the $\mathcal{L}_{op}$-bialgebroid $(\mathcal{H}_\mathcal{L})_{op}$ is defined canonically, see Subsection 3.2.

**Proposition 5.8.** *The antipode (38) descends to an isomorphism of quantum groupoids $\gamma : \mathcal{H}_\mathcal{L} \to (\mathcal{H}_\mathcal{L})_{op}^{op}$.*

*Proof.* By Lemma 5.6, the ideal $J_\phi$ defining $\mathcal{H}_\mathcal{L}$ coincides with the ideal defining $(\mathcal{H}_\mathcal{L})_{op}$ and thus $(\mathcal{H}_\mathcal{L})_{op}^{op}$. Therefore, it suffices to check that $J_\phi$ is invariant with respect to the antipode (38). Identifying $\mathcal{L}$ and $\mathcal{H}$ with the corresponding subalgebras in $\mathcal{L} \rtimes \mathcal{H}$, we can write

$$\gamma(v\lambda v^{-1}) \;=\; v\gamma(\lambda)v^{-1} = vt^+(v^{-1} \triangleright \lambda)v^{-1} = \big((v^{(1)}\mathcal{R}_2 v^{-1}) \triangleright \lambda\big)(v^{(2)}\mathcal{R}_1 v^{-1}). \tag{41}$$

for any $\lambda \in \mathcal{L}$. Using (32) and the identity $(\gamma \otimes \gamma)(\mathcal{R}) = \mathcal{R}$, we find the last expression to be equal to $(\mathcal{R}_1^{-1} \triangleright \lambda)\mathcal{R}_2^{-1} = t^-(\lambda) \equiv t^+(\lambda) \mod J_\phi$ and therefore to $\gamma(v \triangleright \lambda)$ modulo the ideal $J_\phi$, by formula (39) for $h = 1$. Thus the relations (33) are preserved by $\gamma$, modulo $J_\phi$.

The induced homomorphism $\gamma \colon \mathcal{H}_\mathcal{L} \to (\mathcal{H}_\mathcal{L})_{op}^{op}$ of bialgebroids relates the quasitriangular structures of $\mathcal{H}_\mathcal{L}$ and $(\mathcal{H}_\mathcal{L})_{op}^{op}$, i.e. $\gamma \otimes \gamma$ leaves the R-matrix invariant. Thus $\gamma$ is an isomorphism of quantum groupoids. □

# 6 Dynamical cocycles and twisting bialgebroids

## 6.1 Twisting by dynamical cocycles

The present section establishes a relation between bialgebroid twists and dynamical cocycles over a non-abelian base from [DM1]. The case of abelian base was treated in [Xu1].

A categorical definition of dynamical twist is given in [DM1]. Here we will work with the equivalent definition in terms of universal dynamical twisting cocycle. A ***universal dynamical cocycle*** over an $\mathcal{H}$-base algebra $\mathcal{L}$ is an invertible element $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3 \in \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$, where $\mathcal{U}$ is a Hopf algebra containing $\mathcal{H}$, satisfying the invariance condition

$$h^{(1)}\mathcal{F}_1 \otimes h^{(2)}\mathcal{F}_2 \otimes h^{(3)} \triangleright \mathcal{F}_3 \quad = \quad \mathcal{F}_1 h^{(1)} \otimes \mathcal{F}_2 h^{(2)} \otimes \mathcal{F}_3, \quad \forall h \in \mathcal{H}, \tag{42}$$

the shifted cocycle condition

$$(\Delta \otimes \mathrm{id})(\mathcal{F}) \left(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3^{(1)} \otimes \mathcal{F}_3^{[2]}\right) \quad = \quad (\mathrm{id} \otimes \Delta)(\mathcal{F})(\mathcal{F}_{23}), \tag{43}$$

and the normalization condition

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\mathcal{F}) = \quad 1 \otimes 1 \otimes 1 \quad = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\mathcal{F}). \tag{44}$$

Note that equation (43) holds in $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$.

Assume that $\mathcal{H}$ is quasitriangular with the R-matrix $\mathcal{R}$ and $\mathcal{L}$ is $\mathcal{H}$-commutative. Recall that $\mathcal{L}$ can be equipped with two $\mathcal{H}$-base algebra structures $\mathcal{L}_\pm$ by the coactions (10). Consider the tensor product bialgebroid $\mathcal{U} \otimes (\mathcal{L}_+ \rtimes \mathcal{H})$, as in Example 3.7.

**Proposition 6.1.** *Let $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3 \in \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}_-$ be a dynamical twist. Then the element $\Psi \in (\mathcal{U} \otimes \mathcal{L}_+ \rtimes \mathcal{H}) \otimes_{\mathcal{L}} (\mathcal{U} \otimes \mathcal{L}_+ \rtimes \mathcal{H})$,*

$$\Psi \quad := \quad (\mathcal{F}_1 \otimes \mathcal{F}_3 \otimes \mathcal{R}_1) \otimes_{\mathcal{L}} (\mathcal{F}_2 \mathcal{R}_2 \otimes 1 \otimes 1), \tag{45}$$

*is a bialgebroid twist.*

*Proof.* Explicitly, the comultiplication in $\mathcal{U} \otimes (\mathcal{L}_+ \rtimes \mathcal{H})$ is written as

$$\Delta(u \otimes \lambda \otimes h) \quad := \quad (u^{(1)} \otimes \lambda \otimes h^{(1)}) \otimes_{\mathcal{L}} (u^{(2)} \otimes 1 \otimes h^{(2)}) \tag{46}$$

for any $u \otimes \lambda \otimes h \in \mathcal{U} \otimes (\mathcal{L} \rtimes \mathcal{H})$. Then the right-hand side of (22) is equal to

$$(\mathcal{F}_1 \otimes \mathcal{F}_3 \otimes \mathcal{R}_1) \otimes_{\mathcal{L}} (\mathcal{F}_2^{(1)} \mathcal{R}_2^{(1)} \mathcal{F}_{1'} \otimes \mathcal{F}_{3'} \otimes \mathcal{R}_{1'''}) \otimes_{\mathcal{L}} (\mathcal{F}_2^{(2)} \mathcal{R}_2^{(2)} \mathcal{F}_{2'} \mathcal{R}_{2'''} \otimes 1 \otimes 1). \tag{47}$$

By the standard Hopf algebra technique, the identity (42) implies

$$h^{(1)}\mathcal{F}_1 \otimes h^{(2)}\mathcal{F}_2 \otimes \mathcal{F}_3 = \mathcal{F}_1 h^{(1)} \otimes \mathcal{F}_2 h^{(2)} \otimes \gamma^{-1}(h^{(3)}) \triangleright \mathcal{F}_3, \quad \forall h \in \mathcal{H},$$

where $\gamma$ is the antipode in $\mathcal{H}$. Using this, we transform (47) to

$$(\mathcal{F}_1 \otimes \mathcal{F}_3 \otimes \mathcal{R}_1) \otimes_{\mathcal{L}} (\mathcal{F}_2^{(1)}\mathcal{F}_{1'}\mathcal{R}_2^{(1)} \otimes \gamma^{-1}\mathcal{R}_2^{(3)} \triangleright \mathcal{F}_{3'} \otimes \mathcal{R}_{1'''}) \otimes_{\mathcal{L}} (\mathcal{F}_2^{(2)}\mathcal{F}_{2'}\mathcal{R}_2^{(2)}\mathcal{R}_{2'''} \otimes 1 \otimes 1) =$$
$$(\mathcal{F}_1 \otimes \mathcal{F}_3 \otimes \bar{\mathcal{R}}_{1''}\mathcal{R}_1) \otimes_{\mathcal{L}} (\mathcal{F}_2^{(1)}\mathcal{F}_{1'}\mathcal{R}_2^{(1)} \otimes \bar{\mathcal{R}}_{2''} \triangleright \mathcal{F}_{3'} \otimes \mathcal{R}_{1'''}) \otimes_{\mathcal{L}} (\mathcal{F}_2^{(2)}\mathcal{F}_{2'}\mathcal{R}_2^{(2)}\mathcal{R}_{2'''} \otimes 1 \otimes 1).$$

The element $\bar{\mathcal{R}}$ denotes $\mathcal{R}_1 \otimes \gamma^{-1}\mathcal{R}_2$, which is the inverse to $\mathcal{R}$. The term $\bar{\mathcal{R}}_{2''} \triangleright \mathcal{F}_{3'}$ in the middle tensor factor can be pulled to the left as the factor $s(\bar{\mathcal{R}}_{2''} \triangleright \mathcal{F}_{3'})$. Using the definition of tensor product over $\mathcal{L}$ we transform this expression to

$$t^+(\bar{\mathcal{R}}_{2''} \triangleright \mathcal{F}_{3'})\Big(\mathcal{F}_1 \otimes \mathcal{F}_3 \otimes \bar{\mathcal{R}}_{1''}\mathcal{R}_1\Big) \otimes_{\mathcal{L}} \Big(\mathcal{F}_2^{(1)}\mathcal{F}_{1'}\mathcal{R}_2^{(1)} \otimes 1 \otimes \mathcal{R}_{1'''}\Big) \otimes_{\mathcal{L}} \ldots$$
$$= \Big(1 \otimes (\mathcal{R}_{2''}\bar{\mathcal{R}}_2) \triangleright \mathcal{F}_{3'} \otimes \mathcal{R}_{1''}\Big)\Big(\mathcal{F}_1 \otimes \mathcal{F}_3 \otimes \bar{\mathcal{R}}_1\mathcal{R}_1\Big) \otimes_{\mathcal{L}} \Big(\mathcal{F}_2^{(1)}\mathcal{F}_{1'}\mathcal{R}_2^{(1)} \otimes 1 \otimes \mathcal{R}_{1'''}\Big) \otimes_{\mathcal{L}} \ldots$$
$$= \Big(\mathcal{F}_1 \otimes \mathcal{F}_3\big((\mathcal{R}_{2''}\bar{\mathcal{R}}_2) \triangleright \mathcal{F}_{3'}\big) \otimes \mathcal{R}_{1''}\bar{\mathcal{R}}_1\mathcal{R}_1\Big) \otimes_{\mathcal{L}} \Big(\mathcal{F}_2^{(1)}\mathcal{F}_{1'}\mathcal{R}_2^{(1)} \otimes 1 \otimes \mathcal{R}_{1'''}\Big) \otimes_{\mathcal{L}} \ldots$$
$$= (\mathcal{F}_1 \otimes \mathcal{F}_3\mathcal{F}_{3'} \otimes \mathcal{R}_{1''}\mathcal{R}_{1'}) \otimes_{\mathcal{L}} (\mathcal{F}_2^{(1)}\mathcal{F}_{1'}\mathcal{R}_{2'} \otimes 1 \otimes \mathcal{R}_{1'''}) \otimes_{\mathcal{L}} (\mathcal{F}_2^{(2)}\mathcal{F}_{2'}\mathcal{R}_{2''}\mathcal{R}_{2'''} \otimes 1 \otimes 1).$$

Here we employed the fact that the image of the map $t^+$ commutes with all the elements $(x \otimes \mu \otimes 1) \in \mathcal{U} \otimes \mathcal{L}_+ \rtimes \mathcal{H}$.

On the other hand, the left-hand side of (22) turns to

$$\Big(\mathcal{F}_1^{(1)}\mathcal{F}_{1'} \otimes \mathcal{F}_3(\mathcal{R}_1 \triangleright \mathcal{F}_{3'}) \otimes \mathcal{R}_{1''}\mathcal{R}_{1'}\Big) \otimes_{\mathcal{L}}$$
$$\otimes_{\mathcal{L}}\Big(\mathcal{F}_1^{(2)}\mathcal{F}_{2'}\mathcal{R}_{2'} \otimes 1 \otimes \mathcal{R}_{1'''}\Big) \otimes_{\mathcal{L}} \Big(\mathcal{F}_2\mathcal{R}_2\mathcal{R}_{2''}\mathcal{R}_{2'''} \otimes 1 \otimes 1\Big).$$

Thus $\Psi$ satisfies equation (22) if $\mathcal{F}$ is a dynamical cocycle over the base algebra $\mathcal{L}_-$, with the coaction $\delta^-(\lambda) = \mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright \lambda$. □

**Corollary 6.2.** *A dynamical cocycle $\mathcal{F} \in \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}_-$ defines a new $\mathcal{L}$-bialgebroid structure $\widetilde{\mathcal{U} \otimes (\mathcal{L}_+} \rtimes \mathcal{H})$ on the algebra $\mathcal{U} \otimes (\mathcal{L}_+ \rtimes \mathcal{H})$, with the same counit and target map $\tilde{t} := t$ but the new source map $\tilde{s}(\lambda) := \mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright \lambda \otimes 1$ and comultiplication (24) with $\Psi$ given by (45)*

*Proof.* This is a corollary of Proposition 3.17 and Proposition 6.1. The source and target maps are readily calculated. □

In fact, the twist $\Psi$ is supported in the sub-bialgebroid $\mathcal{U} \otimes (\mathcal{L} \rtimes \mathcal{H}^{\{+\}})$. Therefore the sequence of bialgebroid homomorphisms

$$\mathcal{U} \otimes \mathcal{L} \rtimes \mathcal{H}^{\{+\}} \to \mathcal{U} \otimes \mathcal{L}_+ \rtimes \mathcal{H} \to \mathcal{U} \otimes \mathcal{H}_{\mathcal{L}},$$

where the left arrow is embedding and the right one is projection, gives rise to the sequence of homomorphisms of twisted bialgebroids, cf. Remark 3.18:

$$\mathcal{U}\otimes\widetilde{\mathcal{L}\rtimes\mathcal{H}}^{\{+\}}\to\mathcal{U}\otimes\widetilde{\mathcal{L}_+\rtimes\mathcal{H}}\to\widetilde{\mathcal{U}\otimes\mathcal{H}_{\mathcal{L}}}.$$

**Corollary 6.3.** *Suppose $\mathcal{U}$ is quasitriangular with the universal R-matrix $\Omega$. Then the bialgebroid $\widetilde{\mathcal{U}\otimes\mathcal{H}_{\mathcal{L}}}$ is quasitriangular, with the universal R-matrix*

$$(\mathcal{R}_{2'}^{-1}\mathcal{R}_{2'''}^{-1}\tilde{\Omega}_1\otimes 1\otimes\mathcal{R}_1\mathcal{R}_{1''})\otimes_{\mathcal{L}_{op}}(\tilde{\Omega}_2\mathcal{R}_{2''}\otimes\mathcal{R}_{1'''}^{-1}\triangleright\tilde{\Omega}_3\otimes\mathcal{R}_{1'}^{-1}\mathcal{R}_2),\tag{48}$$

*where $\tilde{\Omega}:=\mathcal{F}_{21}^{-1}\Omega\mathcal{F}=\tilde{\Omega}_1\otimes\tilde{\Omega}_2\otimes\tilde{\Omega}_3\in\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{L}$.*

*Proof.* If $\mathcal{U}$ is quasitriangular with the universal R-matrix $\Omega$, then the tensor product bialgebroid $\mathcal{U}\otimes\mathcal{H}_{\mathcal{L}}$ is also quasitriangular with the universal R-matrix

$$(\Omega_1\otimes 1\otimes\mathcal{R}_1)\otimes_{\mathcal{L}_{op}}(\Omega_2\otimes 1\otimes\mathcal{R}_2).\tag{49}$$

The quasitriangular structure on $\widetilde{\mathcal{U}\otimes\mathcal{H}_{\mathcal{L}}}$ is obtained from (49) by twisting, following Proposition 3.19. It can be expressed through the element $\tilde{\Omega}$ by the formula (48). Let us remark that the element $\tilde{\Omega}$ is a solution to the DYBE over the base algebra $\mathcal{L}$, see [DM1]. ☐

## 6.2 The twisted tensor product $\mathcal{U}\overset{\mathcal{R}}{\otimes}\mathcal{H}_{\mathcal{L}}$

The element $\mathcal{F}=1\otimes 1\otimes 1$ is a particular case of dynamical cocycle. So we can always build a twist by (45) $\Psi_{\mathcal{R}}:=\Psi|_{\mathcal{F}=1}$. A slight modification of the proof of Proposition 6.1 shows that the bialgebroid $\mathcal{U}\otimes(\mathcal{L}_+\rtimes\mathcal{H})$ has one more twist, namely if $\mathcal{R}$ in $\Psi_{\mathcal{R}}$ is replaced by $\mathcal{R}^-$. The same holds true for the bialgebroid $\mathcal{U}\otimes(\mathcal{L}_-\rtimes\mathcal{H})$, which differs from $\mathcal{U}\otimes(\mathcal{L}_+\rtimes\mathcal{H})$ by the alternative choice of the quasitriangular structure on $\mathcal{H}$. These twists are analogous to the twisted tensor products of Hopf algebras, cf. Subsection 2.1 and Example 3.7. Following this analogy, we reserve the special notation, $\mathcal{U}\overset{\mathcal{R}}{\otimes}(\mathcal{L}\rtimes\mathcal{H})$ and $\mathcal{U}\overset{\mathcal{R}}{\otimes}\mathcal{H}_{\mathcal{L}}$ of the bialgebroids $\mathcal{U}\otimes(\mathcal{L}\rtimes\mathcal{H})$ and $\mathcal{U}\otimes\mathcal{H}_{\mathcal{L}}$ twisted with $\Psi_{\mathcal{R}}$.

The goal of the present subsection is to establish the following commutative diagram of bialgebroid homomorphisms:

$$\begin{array}{ccccc}
\mathcal{L}_-\rtimes\mathcal{H} & \longrightarrow & \mathcal{H}\overset{\mathcal{R}^+}{\otimes}(\mathcal{L}_-\rtimes\mathcal{H}) & \longrightarrow & \mathcal{U}\overset{\mathcal{R}^+}{\otimes}(\mathcal{L}_-\rtimes\mathcal{H})\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{H}_{\mathcal{L}} & & \mathcal{H}\overset{\mathcal{R}^+}{\otimes}\mathcal{H}_{\mathcal{L}} & \longrightarrow & \mathcal{U}\overset{\mathcal{R}^+}{\otimes}\mathcal{H}_{\mathcal{L}}
\end{array}\tag{50}$$

The similar diagram takes place upon interchanging $+ \leftrightarrow -$. The horizontal arrows on the right are obvious: the twist of the rightmost bialgebroids is transferred from the middle ones via bialgebroid homomorphisms, cf. Remark 3.18. The horizontal arrow on the left descends from the coproduct $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. Thus it can be viewed as a generalization of the Hopf algebra embedding $\mathcal{H} \xrightarrow{\Delta} \mathcal{U} \overset{\mathcal{R}^\pm}{\otimes} \mathcal{H}$. The blank space on the left of the bottom line means that there is no homomorphisms from $\mathcal{H}_\mathcal{L}$ to $\mathcal{H} \overset{\mathcal{R}^+}{\otimes} \mathcal{H}_\mathcal{L}$ in general. One can construct a homomorphism from $\mathcal{H}_\mathcal{L}$ into the quotient of $\mathcal{H} \overset{\mathcal{R}^+}{\otimes} \mathcal{H}_\mathcal{L}$ by the ideal generated by the image of $J_\phi \subset \mathcal{L}_- \rtimes \mathcal{H}$. Note that the two-sided ideal generated by the image of a biideal under a bialgebroid homomorphism is always a biideal. We do not focus on this issue here.

**Proposition 6.4.** *Let $\mathcal{H}$ be a quasitriangular Hopf algebra with the R-matrix $\mathcal{R}$ and $\mathcal{L}$ is an $\mathcal{H}$-commutative algebra considered as a base algebra $\mathcal{L}_-$. The map*

$$\eta\colon \lambda \otimes h \mapsto \mathcal{R}_2 h^{(1)} \otimes \mathcal{R}_1 \triangleright \lambda \otimes h^{(2)} \tag{51}$$

*from $\mathcal{L}_- \rtimes \mathcal{H}$ to $\mathcal{H} \overset{\mathcal{R}^+}{\otimes} (\mathcal{L}_- \rtimes \mathcal{H})$ is a bialgebroid embedding.*

*Proof.* Let us prove that (51) is an algebra homomorphism. When restricted to $\mathcal{L}$, $\eta$ coincides with the source map $\tilde{s}$, whereas the restriction to $\mathcal{H}$ descends from the coproduct of $\mathcal{H}$. Thus $\eta$ is a homomorphism on the subalgebras $\mathcal{L}$ and $\mathcal{H}$ in $\mathcal{L} \rtimes \mathcal{H}$. We have $\eta(\lambda \otimes 1)\eta(1 \otimes h) = \eta\big((\lambda \otimes 1)(1 \otimes h)\big) = \eta(\lambda \otimes h)$ by construction. Further,

$$\eta\big((1 \otimes h)(\lambda \otimes 1)\big) = \mathcal{R}_2 h^{(2)} \otimes \mathcal{R}_1 h^{(1)} \triangleright \lambda \otimes h^{(3)} = h^{(1)}\mathcal{R}_2 \otimes h^{(2)}\mathcal{R}_1 \triangleright \lambda \otimes h^{(3)} = \eta(1 \otimes h)\eta(\lambda \otimes 1).$$

This proves that $\eta$ is an algebra homomorphism. It is an embedding, since there is a projection $\varepsilon_\mathcal{H} \otimes \mathrm{id}\colon \mathcal{H} \overset{\mathcal{R}}{\otimes} (\mathcal{L}_- \rtimes \mathcal{H}) \to \mathcal{L}_- \rtimes \mathcal{H}$ (in fact, this is a bialgebroid map) and the composition of $\eta$ with this projection is identical on $\mathcal{L}_- \rtimes \mathcal{H}$.

Let us show that $\eta$ is an $\mathcal{L}$-bimodule map. The equality $\tilde{s} = \eta \circ s$ is obvious, so let us consider the target maps. We have, for $\lambda \in \mathcal{L}$,

$$(\eta \circ t^-)(\lambda) = \eta(\mathcal{R}_2^- \triangleright \lambda \otimes \mathcal{R}_1^-) = \mathcal{R}_{2'}\mathcal{R}_{1''}^- \otimes \mathcal{R}_{1'}\mathcal{R}_{2''}^-\mathcal{R}_2^- \triangleright \lambda \otimes \mathcal{R}_1^- = 1 \otimes \mathcal{R}_2^- \triangleright \lambda \otimes \mathcal{R}_1^- = \tilde{t}^-(\lambda).$$

Here we used $\mathcal{R}^- = \mathcal{R}_{21}^{-1}$.

We must show that $\eta$ respects the coproducts. It is obvious for $\eta$ restricted to $\mathcal{L} \subset \mathcal{L}_- \rtimes \mathcal{H}$, so it suffices to check this on the elements of $\mathcal{H} \subset \mathcal{L}_- \rtimes \mathcal{H}$. When restricted to $\mathcal{H}$, the map $\eta$ coincides with $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. The bialgebroid coproducts in $\mathcal{L}_- \rtimes \mathcal{H}$ and $\mathcal{H} \overset{\mathcal{R}^+}{\otimes} (\mathcal{L}_- \rtimes \mathcal{H})$, when restricted to $\mathcal{H}$ and respectively to $\mathcal{H} \otimes \mathcal{H}$, are obtained from the Hopf algebra coproducts in $\mathcal{H}$ and $\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$ by projecting the tensor products over $k$ to those over $\mathcal{L}$. Recall that the coproduct of $\mathcal{H}$ defines a Hopf algebra map from $\mathcal{H}$ to $\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$. It follows from here that $\eta$ respects the bialgebroid coproducts when restricted to $\mathcal{H}$, since $\eta$ is an $\mathcal{L}$-bimodule map.

Finally, it is obvious that $\eta$ respects the counits. $\square$

**Remark 6.5.** If $\mathcal{F}$ is a dynamical twisting cocycle as in Proposition 6.1, then the twisted quantum groupoid $\widetilde{\mathcal{U}\otimes\mathcal{H}_{\mathcal{L}}}$ can be considered as a result of two consecutive twists

$$\mathcal{U}\otimes\mathcal{H}_{\mathcal{L}} \xrightarrow{\Psi_{\mathcal{R}}} \mathcal{U}\overset{\mathcal{R}}{\otimes}\mathcal{H}_{\mathcal{L}} \xrightarrow{\Psi_{\mathcal{R}^{-1}\mathcal{F}\mathcal{R}}} \widetilde{\mathcal{U}\otimes\mathcal{H}_{\mathcal{L}}}$$

**Corollary 6.6.** *Suppose that $\mathcal{H}$ is an arbitrary Hopf algebra and $\mathcal{L}$ is an $\mathcal{H}$-base algebra. Let $\mathcal{H}$ be a Hopf subalgebra in a Hopf algebra $\mathcal{U}$. Then there exists a bialgebroid homomorphism from $\mathcal{L}\rtimes\mathcal{H}$ to $\mathcal{U}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}}$, where $\Theta\in\mathcal{H}^*_{op}\otimes\mathcal{H}$ is the standard quasitriangular structure on $\mathfrak{D}\mathcal{H}$.*

*Proof.* Recall from Remark 4.4 that $\mathcal{H}=(\mathfrak{D}\mathcal{H})^{\{-\}}$ with respect to the standard quasitriangular structure $\Theta\in\mathcal{H}^*_{op}\otimes\mathcal{H}\subset(\mathfrak{D}\mathcal{H})^{\otimes 2}$. Applying Proposition 6.4 to this case, we obtain the sequence of bialgebroid homomorphisms

$$\mathcal{L}\rtimes\mathcal{H}\to\mathcal{L}_-\rtimes\mathfrak{D}\mathcal{H}\to\mathfrak{D}\mathcal{H}\overset{\Theta}{\otimes}(\mathcal{L}_-\rtimes\mathfrak{D}\mathcal{H})\to\ \mathfrak{D}\mathcal{H}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}},$$

where the left arrow is embedding and the right one is projection along the ideal $\mathfrak{D}\mathcal{H}\otimes J_\phi$. The middle arrow is the map (51) where $\mathcal{H}$ is replaced by $\mathfrak{D}\mathcal{H}$ and $\mathcal{R}$ by $\Theta$. This map is constructed out of the coaction $\delta\colon\mathcal{L}\to\mathcal{H}\otimes\mathcal{L}$ and the coproduct of $\mathfrak{D}\mathcal{H}$. It remains to notice that, as a coalgebra, the double $\mathfrak{D}\mathcal{H}$ is a trivial tensor product of coalgebras $\mathcal{H}$ and $\mathcal{H}^*_{op}$, whence the composite map takes the values in $\mathcal{H}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}}\subset\mathcal{U}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}}$. $\square$

**Remark 6.7.** Replacing $\mathcal{H}$ by $\mathcal{H}_{op}$, $\mathcal{L}$ by $\mathcal{L}_{op}$, and $\Theta$ by $\bar\Theta=\Theta^{-1}$ in Corollary 6.6, one can construct the twisted opposite bialgebroids $\mathcal{U}\overset{\bar\Theta}{\otimes}(\mathcal{L}\rtimes\mathcal{H})_{op}$ and $\mathcal{U}\overset{\bar\Theta}{\otimes}(\mathcal{H}_{\mathcal{L}})_{op}$, cf. Definitions 5.2 and 5.7.

# 7 Dynamical categories and representations of bialgebroids

In this section we establish relations between dynamical categories from Definition 2.7 and representations of bialgebroids over quasi-commutative base. In this section $\mathcal{H}$ is an arbitrary Hopf algebra and $\mathcal{L}$ is an $\mathcal{H}$-base algebra.

## 7.1 Category $\mathrm{Mod}\,\mathcal{L} \rtimes \mathcal{H}$

A central role in our further consideration belongs to Lemma 7.1 below. Let $X$ be a left $\mathcal{H}$-module. Denote by $\tilde{X}_{\mathcal{L}}$ the $\mathcal{L}$-bimodule $X \otimes \mathcal{L}$ with respect to the following left and right action:

$$\lambda \llcorner (x \otimes \mu) = \Theta_2 \triangleright x \otimes (\Theta_1 \triangleright \lambda)\mu, \quad (x \otimes \mu) \lrcorner \lambda = x \otimes \mu\lambda \tag{52}$$

where $x \otimes \mu \in \tilde{X}_{\mathcal{L}}$ and $\lambda \in \mathcal{L}$. For an $\mathcal{H}$-equivariant map $\psi \colon X \to Y \otimes \mathcal{L}$ let $\tilde{\psi}_{\mathcal{L}}$ denote the composition map

$$X \otimes \mathcal{L} \stackrel{\psi \otimes \mathrm{id}_{\mathcal{L}}}{\longrightarrow} Y \otimes \mathcal{L} \otimes \mathcal{L} \stackrel{\mathrm{id}_Y \otimes \mathrm{m}_{\mathcal{L}}}{\longrightarrow} Y \otimes \mathcal{L},$$

where $\mathrm{m}_{\mathcal{L}}$ is the multiplication in $\mathcal{L}$. It is a morphism of $\mathcal{L}$-bimodules, due to the $\mathcal{H}$-invariance of $\psi$ and the quasi-commutativity of $\mathcal{L}$.

**Lemma 7.1.** *The correspondence $X \to \tilde{X}_{\mathcal{L}}$, $\psi \to \tilde{\psi}_{\mathcal{L}}$ commutes with taking tensor products and defines a strong monoidal functor, $\Xi$, from $\overline{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$ to the category $\mathcal{B}i(\mathcal{L})$ of $\mathcal{L}$-bimodules.*

*Proof.* Straightforward. □

Denote by $\mathrm{Mod}_0\,\mathcal{L} \rtimes \mathcal{H}$ the full subcategory of $\mathcal{L} \rtimes \mathcal{H}$-modules of the form $X \otimes \mathcal{L}$, where $X$ is an $\mathcal{H}$-module.

**Theorem 7.2.** *The functor $\Xi$ establishes an isomorphism from the dynamical category $\overline{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$ to* $\mathrm{Mod}_0\,\mathcal{L} \rtimes \mathcal{H}$.

*Proof.* Let $X \in \mathrm{Ob}\,\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ be an $\mathcal{H}$-module, and $\tilde{X}_{\mathcal{L}} = \Xi(X)$ its image in $\mathcal{B}i(\mathcal{L})$. Consider $\tilde{X}_{\mathcal{L}}$ as an $\mathcal{H}$-module being the tensor product of $\mathcal{H}$-modules $X$ and $\mathcal{L}$. One can check that this action together with the left action of $\mathcal{L}$ gives rise to an action of $\mathcal{L} \rtimes \mathcal{H}$. Further, the tensor product of two $\mathcal{L} \rtimes \mathcal{H}$-modules $\tilde{X}_{\mathcal{L}}$ and $\tilde{Y}_{\mathcal{L}}$ is $\Xi(X \otimes Y)$, due to Lemma 7.1.

For any morphism $\psi \in \mathrm{Hom}_{\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}}(X, Y)$ the map $\tilde{\psi}_{\mathcal{L}} \colon \tilde{X}_{\mathcal{L}} \to \tilde{Y}_{\mathcal{L}}$ commutes with the action of $\mathcal{L} \rtimes \mathcal{H}$. Conversely, let $\phi \colon \tilde{X}_{\mathcal{L}} \to \tilde{Y}_{\mathcal{L}}$ be an $\mathcal{L} \rtimes \mathcal{H}$-intertwiner. Then $\phi$ is an $\mathcal{L}$-bimodule map and must have the form $\phi(x \otimes \mu) = \phi(x \otimes 1)\mu$. The map $\psi \colon X \to \tilde{X}_{\mathcal{L}}$, $\psi(x) := \phi(x \otimes 1)$, is $\mathcal{H}$-equivariant, therefore it is a morphism in $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ and $\phi = \tilde{\psi}_{\mathcal{L}}$. Thus we have proved that the image of $\Xi$ is a full subcategory in $\mathrm{Mod}\,\mathcal{L} \rtimes \mathcal{H}$. □

Now suppose that $\mathcal{H}$ is a quasitriangular Hopf algebra with the R-matrix $\mathcal{R}$ and $\mathcal{L}$ is $\mathcal{H}$-commutative. Denote by $\mathcal{M}'_{\mathcal{H}}$ the full subcategory in $\mathcal{M}_{\mathcal{H}}$ consisting of such $\mathcal{H}$-modules $X$ that

$$\mathcal{R}_1^+ \triangleright x \otimes \mathcal{R}_2^+ \triangleright \lambda = \mathcal{R}_1^- \triangleright x \otimes \mathcal{R}_2^- \triangleright \lambda \tag{53}$$

for all $x\otimes\lambda\in X\otimes\mathcal{L}$. Let $\overline{\mathcal{M}}'_{\mathcal{H},\mathcal{L}}$ denote the dynamization of $\mathcal{M}'_{\mathcal{H}}$, i.e. the full subcategory in $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ whose objects belong to $\mathcal{M}'_{\mathcal{H}}$. Denote by $\mathrm{Mod}_0\,\mathcal{H}_{\mathcal{L}}$ the full subcategory of $\mathcal{H}_{\mathcal{L}}$-modules of the form $X\otimes\mathcal{L}$, where $X$ is an $\mathcal{H}$-module satisfying the condition (53).

**Proposition 7.3.** *The category $\overline{\mathcal{M}}'_{\mathcal{H},\mathcal{L}}$ is a braided monoidal category. It is naturally isomorphic to the category $\mathrm{Mod}_0\,\mathcal{H}_{\mathcal{L}}$, which itself is a full subcategory in $\mathrm{Mod}\,\mathcal{L}\rtimes\mathcal{H}$.*

*Proof.* The bialgebroid $\mathcal{H}_{\mathcal{L}}$ is the quotient of the bialgebroid $\mathcal{L}\rtimes\mathcal{H}$ by the relations $\Theta^+_2\triangleright\lambda\otimes\Theta^+_1=\Theta^-_2\triangleright\lambda\otimes\Theta^-_1$, for all $\lambda\in\mathcal{L}$. Therefore $\mathrm{Mod}\,\mathcal{H}_{\mathcal{L}}$ consists of those $\mathcal{L}\rtimes\mathcal{H}$-modules whose annihilator contains this ideal, thus $\mathrm{Mod}\,\mathcal{H}_{\mathcal{L}}$ is a full subcategory in $\mathrm{Mod}\,\mathcal{L}\rtimes\mathcal{H}$. An $\mathcal{L}\rtimes\mathcal{H}$-module $\Xi(X)$, where $X\in\mathrm{Ob}\,\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$, belongs to $\mathrm{Mod}\,\mathcal{H}_{\mathcal{L}}$ if and only if $X\in\mathrm{Ob}\,\overline{\mathcal{M}}'_{\mathcal{H},\mathcal{L}}$. Applying Theorem 7.2, we conclude that restriction of the functor $\Xi$ to $\overline{\mathcal{M}}'_{\mathcal{H},\mathcal{L}}$ gives an isomorphism of $\overline{\mathcal{M}}'_{\mathcal{H},\mathcal{L}}$ with a full subcategory in $\mathrm{Mod}\,\mathcal{H}_{\mathcal{L}}$. Since $\mathrm{Mod}\,\mathcal{H}_{\mathcal{L}}$ is braided, the category $\overline{\mathcal{M}}'_{\mathcal{H},\mathcal{L}}$ is braided as well. □

## 7.2 Category $\mathrm{Mod}\,\widetilde{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}}$

In this subsection we assume that $\mathcal{H}$ is a Hopf subalgebra (not necessarily quasitriangular) of a quasitriangular Hopf algebra $\mathcal{U}$. The category $\mathcal{M}_{\mathcal{U}}$ of $\mathcal{U}$-modules is viewed as a natural subcategory in $\mathcal{M}_{\mathcal{H}}$.

Let $\mathcal{L}$ be an $\mathcal{H}$-base algebra $\mathcal{L}$ and $\mathcal{F}$ a dynamical twist. Consider the twisted bialgebroid $\widetilde{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}}$ built by means of twist $\Psi=\Psi_{\mathcal{F}\Theta}$ from Subsection 6.1.

**Proposition 7.4.** *Objects $V\otimes\mathcal{L}$, where $V$ is a $\mathcal{U}$-module, form a full monoidal subcategory, $\mathrm{Mod}_0\widetilde{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}}$, in $\mathrm{Mod}\widetilde{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}}$. It is isomorphic to $\mathcal{M}_{\mathcal{U}}$ if and only if $\mathcal{L}$ is quasi-transitive. In the particular case of the unit $\mathcal{F}$, the isomorphism from $\mathcal{M}$ to $\mathrm{Mod}_0\,\mathcal{U}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}}$ is enclosed in the commutative diagram*

$$
\begin{array}{ccc}
\mathcal{M}_{\mathcal{U}} & \longrightarrow & \overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}} \\
\wr\,\| & & \wr\,\|\;\Xi \\
\mathrm{Mod}_0\,\mathcal{U}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}} & \longrightarrow & \mathrm{Mod}_0\,\mathcal{L}\rtimes\mathcal{H}
\end{array}
\quad, \tag{54}
$$

*where the bottom line is induced by the bialgebroid homomorphism $\mathcal{U}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}}\longleftarrow\mathcal{L}\rtimes\mathcal{H}$.*

*Proof.* The $\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}$-modules of the form $V\otimes\mathcal{L}$, where $V$ is a $\mathcal{U}$ module, are closed under the tensor product induced by the twist. Thus they form a full monoidal subcategory in $\mathrm{Mod}\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}$. Obviously, $\mathrm{Hom}_{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}}(V\otimes\mathcal{L},W\otimes\mathcal{L})\simeq\mathrm{Hom}_{\mathcal{U}}(V,W)\otimes\mathrm{End}_{\mathfrak{D}\mathcal{H}}(\mathcal{L})$. There is a natural bijection between $\mathrm{End}_{\mathfrak{D}\mathcal{H}}(\mathcal{L})$ and $\mathcal{L}^{\mathfrak{D}\mathcal{H}}$, following from Lemma 2.3. Thus the category $\mathcal{M}_{\mathcal{U}}$ is isomorphic to $\mathrm{Mod}_0\,\widetilde{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}}$, provided $\mathcal{L}$ is quasi-transitive. In the particular case of the unit $\mathcal{F}$, the isomorphism from $\mathcal{M}_{\mathcal{U}}$ to $\mathrm{Mod}_0\,\mathcal{U}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}}$ is given by the functor $\Xi$. $\square$

Summarizing, we present a diagram of most important bialgebroids and their interrelations:

$$
\begin{array}{ccc}
\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}} & & \\
\wr\ \Psi_{\Theta} & & \\
\mathcal{U}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}} & \longleftarrow \mathcal{L}\rtimes\mathcal{H}\longrightarrow \mathfrak{D}\mathcal{H}_{\mathcal{L}}\ , & \\
\wr\ \Psi_{\Theta^{-1}\mathcal{F}\Theta} & & \\
\widetilde{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}} & &
\end{array}
$$

The columns represent the bialgebroid twist, cf. Remark 6.5. The horizontal arrows are bialgebroid homomorphisms.

Assuming the $\mathcal{H}$-base algebra $\mathcal{L}$ to be quasi-transitive and passing to the $\mathrm{Mod}_0$-modules, we obtain the following commutative diagram displaying the interrelations between the categories:

$$
\begin{array}{ccccc}
\mathrm{Mod}_0\,(\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}})\simeq\mathcal{M}_{\mathcal{U}} & \longrightarrow & \overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}} & \longleftarrow & \overline{\mathcal{M}}'_{\mathcal{H},\mathcal{L}} \\
\wr\,\|\ \Psi_{\Theta} & & \wr\,\|\ \Xi & & \wr\,\|\ \Xi \\
\mathrm{Mod}_0\,\mathcal{U}\overset{\Theta}{\otimes}\mathfrak{D}\mathcal{H}_{\mathcal{L}} & \longrightarrow & \mathrm{Mod}_0\,\mathcal{L}\rtimes\mathcal{H} & \longleftarrow & \mathrm{Mod}_0\,\mathfrak{D}\mathcal{H}_{\mathcal{L}} \\
\wr\,\|\ \Psi_{\Theta^{-1}\mathcal{F}\Theta} & & & & \\
\mathrm{Mod}_0\,\widetilde{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}}. & & & &
\end{array}
$$

# 8 Dual quantum groupoids (dynamical FRT algebras)

In this section we present an example of a module algebra over the bialgebroid $\widetilde{\mathcal{U}\otimes\mathfrak{D}\mathcal{H}_{\mathcal{L}}}$ constructed in Subsection 6.1. It turns out to be a bialgebroid and may be thought of as an analog of the dual Hopf algebra. This fact is not occasional and will be addressed in a separate publication.

## 8.1 Dynamical associative algebras

Let $\mathcal{H}$ be a Hopf algebra and $\mathcal{L}$ its base algebra. Recall that $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ denotes the dynamical extension over $\mathcal{L}$ of the category $\mathcal{M}_{\mathcal{H}}$ of left $\mathcal{H}$-modules.

**Definition 8.1 ([DM1]).** ***Dynamical associative algebra*** (or simply dynamical algebra) is an algebra in the monoidal category $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$.

A dynamical algebra $\mathcal{A}$ is an $\mathcal{H}$-module equipped with an $\mathcal{H}$-equivariant bilinear map $\circledast\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{L}$ such that the following diagram is commutative:

$$
\begin{array}{ccccccc}
\mathcal{A}\otimes\mathcal{L}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\tau_{\mathcal{A}}} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{L} & \xrightarrow{\circledast\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{L}\otimes\mathcal{L} & \xrightarrow{\mathrm{id}\otimes\mathrm{m}_{\mathcal{L}}} & \mathcal{A}\otimes\mathcal{L}\\
{\scriptstyle\circledast\otimes\mathrm{id}}\uparrow & & & & & & \|\\
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\circledast} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{L} & \xrightarrow{\circledast\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{L}\otimes\mathcal{L} & \xrightarrow{\mathrm{id}\otimes\mathrm{m}_{\mathcal{L}}} & \mathcal{A}\otimes\mathcal{L}
\end{array}
\tag{55}
$$

Here $\mathrm{m}_{\mathcal{L}}$ stands for the multiplication in $\mathcal{L}$ and $\tau_{\mathcal{A}}$ denotes the permutation $\mathcal{L}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{L}$, which is expressed through the coaction $\delta(\lambda)=\lambda^{(1)}\otimes\lambda^{[2]}$ or, equivalently, through the canonical R-matrix of the double $\mathfrak{D}\mathcal{H}$ by $\lambda\otimes a\mapsto\lambda^{(1)}\triangleright a\otimes\lambda^{[2]}=\Theta_2\triangleright a\otimes\Theta_1\triangleright\lambda$.

If the operation $\circledast$ takes values in $\mathcal{A}\otimes 1\subset\mathcal{A}\otimes\mathcal{L}$, the condition (55) reduces to the ordinary associativity. For example, suppose that $\mathcal{A}$ is a module algebra over a Hopf algebra $\mathcal{U}$ containing $\mathcal{H}$, then it is a dynamical algebra. Let $\mathcal{F}=\mathcal{F}_1\otimes\mathcal{F}_2\otimes\mathcal{F}_3\in\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{L}$ be a dynamical cocycle. Then the map $\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{L}$,

$$
a\circledast b:=(\mathcal{F}_1\triangleright a)(\mathcal{F}_2\triangleright b)\otimes\mathcal{F}_3,
\tag{56}
$$

defines a new structure of dynamical algebra on $\mathcal{A}$, [DM1].

**Proposition 8.2.** *A left $\mathcal{H}$-module $\mathcal{A}$ is a dynamical associative algebra if and only if $\mathcal{A}\otimes\mathcal{L}$ equipped with the $\mathcal{L}$-bimodule structure (52) is an algebra in the category of $\mathcal{L}$-bimodules.*

*Proof.* This follows from the isomorphism of categories $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}\simeq\mathrm{Mod}_0\,\mathcal{L}\rtimes\mathcal{H}$. ☐

**Remark 8.3.** Remark that an $\mathcal{L}$-bimodule and associative algebra $\mathcal{B}$ is an algebra in the category of $\mathcal{L}$-bimodules if and only if the multiplication and $\mathcal{L}$-actions are compatible: $(a\lrcorner\lambda)b=a(\lambda\llcorner b)$ for all $a,b\in\mathcal{B}$ and $\lambda\in\mathcal{L}$.

We will denote the associative algebra from proposition 8.2 by $\mathcal{A}\ltimes\mathcal{L}$

## 8.2 Bialgebroid $\mathcal{U}^*_{\mathcal{F}}\ltimes(\mathcal{L}\otimes\mathcal{L}_{op})$

In this subsection we construct a dynamical dual to a Hopf algebra $\mathcal{U}$. The dynamical dual can be regarded as a dynamical analog of the FRT algebra if $\mathcal{U}$ is quasitriangular.

We consider the dual Hopf algebra $\mathcal{U}^*$ as a $\mathcal{U}\otimes\mathcal{U}_{op}$-module, with respect to the coregular actions

$$
x\triangleright u:=u^{(1)}\langle x,u^{(2)}\rangle,\quad y\blacktriangleright u:=\langle y,u^{(1)}\rangle u^{(2)},
\tag{57}
$$

where $x \in \mathcal{U}$, $y \in \mathcal{U}_{op}$, $u \in \mathcal{U}^*$. By restriction, $\mathcal{U}^*$ is also an $\mathcal{H} \otimes \mathcal{H}_{op}$-module algebra. Recall from Lemma 5.1 that $\mathcal{L}_{op}$ is an $\mathcal{H}_{op}$-base algebra. By this reason, we can consider $\mathcal{U}^*$ as a dynamical associative algebra over the $\mathcal{H} \otimes \mathcal{H}_{op}$-base algebra $\mathcal{L} \otimes \mathcal{L}_{op}$. Applying Proposition 8.2, we construct an algebra, $\mathcal{U}^* \ltimes (\mathcal{L} \otimes \mathcal{L}_{op})$, in the category of $\mathcal{L} \otimes \mathcal{L}_{op}$-bimodules. Moreover, it is an algebra in the category of modules over $\big(\mathcal{U} \overset{\ominus}{\otimes} \mathcal{H}_{\mathcal{L}}\big) \otimes \big(\mathcal{U}_{op} \overset{\bar{\ominus}}{\otimes} (\mathcal{H}_{\mathcal{L}})_{op}\big)$, which is a tensor product of $\mathcal{L}$- and $\mathcal{L}_{op}$-bialgebroids, cf. Example 3.7 and Remark 6.7. By Remark 8.3, $\mathcal{U}^* \ltimes (\mathcal{L} \otimes \mathcal{L}_{op})$ is an associative algebra with the multiplication

$$(u \otimes \lambda \otimes \mu)(v \otimes \alpha \otimes \beta) := u\big(\lambda^{(1)} \triangleright \mu^{(1)} \blacktriangleright v\big) \otimes \lambda^{[2]}\alpha \otimes \mu^{[2]} \cdot \beta$$

for $u, v \in \mathcal{U}^*$, $\lambda, \alpha \in \mathcal{L}$ and $\mu, \beta \in \mathcal{L}_{op}$.

The following two propositions can be checked by direct but tedious calculations.

**Proposition 8.4.** *The algebra $\mathcal{U}^* \ltimes (\mathcal{L} \otimes \mathcal{L}_{op})$ is a right $\mathcal{L}$-bialgebroid (cf. Remark 3.3) over the base $\mathcal{L}$ with the target map $t\colon \lambda \mapsto 1 \otimes 1 \otimes \iota(\lambda)$, the source map $s\colon \lambda \mapsto 1 \otimes \lambda \otimes 1$, the coproduct $\Delta\big(u \otimes \lambda \otimes \mu\big) := \big(u^{(1)} \otimes 1 \otimes \mu\big) \otimes_{\mathcal{L}} (u^{(2)} \otimes \lambda \otimes 1)$, and the counit $\varepsilon\big(u \otimes \lambda \otimes \mu\big) := \varepsilon_{\mathcal{U}^*}(u)\, \iota^{-1}(\mu)\lambda$.*

Suppose that $\mathcal{F} \in \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$ is a universal dynamical cocycle over $\mathcal{L}$. Then one can check that $\bar{\mathcal{F}} := \mathcal{F}^{-1} \in \mathcal{U}_{op} \otimes \mathcal{U}_{op} \otimes \mathcal{L}_{op}$ is a universal dynamical cocycle over $\mathcal{L}_{op}$, which is an $\mathcal{H}_{op}$-base algebra. Thus $\mathcal{F} \otimes \bar{\mathcal{F}}$ is a universal twist in the dynamical extension of the category of $\mathcal{U}$-bimodules over the base algebra $\mathcal{L} \otimes \mathcal{L}_{op}$. Let $\mathcal{U}^*_{\mathcal{F}}$ denote the dynamical associative algebra over the base $\mathcal{L} \otimes \mathcal{L}_{op}$ obtained from $\mathcal{U}^*$ by the twist $\mathcal{F} \otimes \bar{\mathcal{F}}$, see (56).

**Proposition 8.5.** *The algebra $\mathcal{U}^*_{\mathcal{F}} \ltimes (\mathcal{L} \otimes \mathcal{L}_{op})$ is a right $\mathcal{L}$-bialgebroid with the same source and target maps, coproduct, and counit as $\mathcal{U}^* \ltimes (\mathcal{L} \otimes \mathcal{L}_{op})$. It is an algebra in the category of modules over the $\mathcal{L} \otimes \mathcal{L}_{op}$-bialgebroid $\mathcal{U} \,\widetilde{\otimes}\, \mathcal{H}_{\mathcal{L}} \otimes \mathcal{U}_{op} \,\widetilde{\otimes}\, (\mathcal{H}_{\mathcal{L}})_{op}$.*

**Remark 8.6.** The dynamical algebra $\mathcal{U}^*_{\mathcal{F}}$ is obtained from $\mathcal{U}^*$ by the dynamical twist $\mathcal{F}$ applied from the two sides. Applied to only one side, the dynamical twist gives a dynamical algebra in the category $\overline{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$, which participates in the equivariant star product quantization of vector bundles on coadjoint orbits of reductive Lie groups, [DM1].

# 9 On the quasi-classical limit and dynamical r-matrix

In this section we consider Lie bialgebroids that are relevant to quantum groupoids studied in this paper. For an exposition of the theory of Lie algebroids and Lie bialgebroids, the reader is referred to [K-Schw] and [MXu].

## 9.1 Lie bialgebroids

Let us recall that a Lie algebroid $\mathfrak{B}_0$ over a commutative algebra (sheaf) $\mathcal{L}_0$ is an $\mathcal{L}_0$-module equipped with a structure of Lie algebra together with a Lie algebra homomorphism (anchor) $\mathfrak{B}_0 \to Der(\mathcal{L}_0)$ such that $[\xi, f\eta] = f[\xi,\eta] + (\xi \triangleright f)\,\eta$ for all $\xi,\eta \in \mathfrak{B}_0$, $f \in \mathcal{L}_0$.

**Example 9.1.** Let $\mathfrak{h}$ be a Lie algebra acting on $\mathcal{L}_0$. Consider the trivial bundle $\mathcal{L}_0 \otimes \mathfrak{h}$ equipped with the following Lie algebra structure on section:

$$[f\otimes\xi, g\otimes\eta] := fg\otimes[\xi,\eta] + f(\xi\triangleright g)\otimes\eta - g(\eta\triangleright f)\otimes\xi$$

for $f,g \in \mathcal{L}_0$ and $\xi,\eta \in \mathfrak{h}$. The anchor map is determined by the action of $\mathfrak{h}$ on $\mathcal{L}_0$. We denote this Lie bialgebroid by $\mathcal{L}_0 \rtimes \mathfrak{h}$.

The Lie bracket on $\mathfrak{B}_0$ can be extended as the Schouten bracket $[.,.]$ to the exterior algebra $\wedge^\bullet\mathfrak{B}_0$ making it a Gerstenhaber algebra, see [K-Schw]. A Lie algebroid structure defines a "de Rham" differential $d$ of degree 1 with zero square on the graded exterior algebra $\wedge^\bullet\mathfrak{B}_0^*$ of the dual vector bundle $\mathfrak{B}^*$.

With every Lie algebroid $\mathfrak{B}_0$ one can associate a universal enveloping $\mathcal{L}_0$-bialgebroid, $\mathcal{U}(\mathfrak{B}_0)$, see [Xu1]. For the Lie algebroid from Example 9.1, it coincides with $\mathcal{L}_0 \rtimes \mathcal{U}(\mathfrak{h})$, where $\mathcal{U}(\mathfrak{h})$ is the universal enveloping algebra of $\mathfrak{h}$ (note that $\mathcal{L}_0$ is a base algebra for $\mathcal{U}(\mathfrak{h})$), in the sense of Definition 2.1.

Infinitesimal theory of quantized universal enveloping algebras leads to the notion of Lie bialgebras. Analogously, the problem of quantization of $\mathcal{U}(\mathfrak{B}_0)$ in the class of bialgebroids gives rise to the notion of Lie bialgebroid, [MXu]. By definition, a Lie bialgebroid is a pair $(\mathfrak{B}_0, \mathfrak{B}_0^*)$ of two Lie algebroids in duality satisfying the compatibility condition

$$d_*[\xi,\eta] = [d_*\xi,\eta] + [\xi, d_*\eta].$$

Below we give examples of Lie bialgebroids that are relevant to our study.

Let $(\mathfrak{h},\mathfrak{h}^*)$ be a Lie bialgebra and $\mathfrak{D}\mathfrak{h} := \mathfrak{h} \bowtie \mathfrak{h}_{op}$ its double Lie (bi)algebra.

**Definition 9.2 ([DM1]).** A Poisson $\mathfrak{h}$-base algebra $\mathcal{L}_0$ is a commutative algebra equipped with a left $\mathfrak{D}\mathfrak{h}$-action such that the canonical symmetric invariant tensor $\theta \in \mathfrak{D}\mathfrak{h}^{\otimes 2}$ vanishes on $\mathcal{L}_0$.

It follows from the definition that the classical r-matrix of $\mathfrak{D}\mathfrak{h}$ induces a Poisson bivector field on $\mathcal{L}_0$. This Poisson structure can be quantized to a $\mathcal{U}_q(\mathfrak{h})$-base algebra $\mathcal{L}$ for $\mathcal{U}_q(\mathfrak{h})$ being the quantization of $\mathcal{U}(\mathfrak{h})$ along $\mathfrak{h}^*$, [DM1].

One can check the following

**Proposition 9.3.** *Let $\mathfrak{h}$ be a Lie algebra and $\mathfrak{h}^*$ a Lie algebra structure on the dual space. The Lie algebroids $\mathcal{L}_0 \rtimes \mathfrak{h}$ and $\mathcal{L}_0 \rtimes \mathfrak{h}^*$ form a Lie bialgebroid* iff $(\mathfrak{h}, \mathfrak{h}^*)$ *is a Lie bialgebra and $\mathcal{L}_0$ is a Poisson $\mathfrak{h}$-base algebra. The differential $d_*$ is given by the Lie cobracket $\nu$ on $\mathfrak{h}$ considered as a constant section of $\wedge^2(\mathcal{L}_0 \rtimes \mathfrak{h})$.*

A Lie bialgebroid $\mathfrak{B}_0$ is called coboundary if the differential $d_*$ is generated by an element $\Lambda \in \wedge^2\mathfrak{B}_0$, namely, has the form $d_*(\zeta) := [\Lambda, \zeta]$, where $\Lambda$ satisfies the condition $[[\Lambda, \Lambda], \Lambda] = 0$. The Lie bialgebroid $\mathcal{L}_0 \rtimes \mathfrak{h}$ is not coboundary, in general, even in the case of coboundary Lie bialgebra $\mathfrak{h}$. However, if $\mathfrak{h}$ is quasitriangular, then there is an ideal in $\mathcal{L}_0 \rtimes \mathfrak{h}$ lying in the kernel of the anchor map. The quotient of $\mathcal{L}_0 \rtimes \mathfrak{h}$ by that ideal is a coboundary Lie bialgebroid. We will demonstrate this on the example of $\mathcal{L}_0 \rtimes \mathfrak{D}\mathfrak{h}$ (note that a Poisson $\mathfrak{h}$-base algebra is that for $\mathfrak{D}\mathfrak{h}$ as well).

**Example 9.4.** Suppose that $\mathcal{L}_0$ is a Poisson $\mathfrak{h}$-base algebra. Let $\{h_i\}$ be a base in $\mathfrak{h}$ and $\{\eta^i\}$ its dual in $\mathfrak{h}^*_{op}$. Let $\theta := \frac{1}{2}\sum_i(h_i \otimes \eta^i + \eta^i \otimes h_i)$ denote the canonical symmetric invariant element of the double $\mathfrak{D}\mathfrak{h}$. Consider in $\mathcal{L}_0 \rtimes \mathfrak{D}\mathfrak{h}$ an $\mathcal{L}_0$-submodule generated by sections of the form $\theta_1 \triangleright f \otimes \theta_2$ for all $f \in \mathcal{L}_0$. It forms an ideal $J_0$ in the Lie algebra $\mathcal{L}_0 \rtimes \mathfrak{D}\mathfrak{h}$, and this ideal is $\mathfrak{D}\mathfrak{h}$-invariant. The quotient of $\mathcal{L}_0 \rtimes \mathfrak{D}\mathfrak{h}$ by $J_0$ is a coboundary Lie algebroid, $\mathfrak{D}\mathfrak{h}_{\mathcal{L}_0}$. Its dual Lie bialgebroid is the annihilator of $J_0$ in $\mathcal{L}_0 \rtimes \mathfrak{D}^*\mathfrak{h}$.

Suppose now that $\mathcal{L}_0$ is a function algebra on a Poisson $\mathfrak{h}$-base manifold $L$, which is assumed $\mathfrak{D}\mathfrak{h}$-homogeneous. Then $J_0$ can be considered as the space of sections of an $\mathfrak{h}$-vector bundle over $L$. Let us fix an origin in $L$ and let $\mathfrak{k} \in \mathfrak{h}$ be the Lie algebra of its stabilizer. Denote by $\mathfrak{k}_0$ the ideal in $\mathfrak{k}$ that is the kernel of the canonical invariant inner product in $\mathfrak{D}\mathfrak{h}$ restricted to $\mathfrak{k}$. Then the fiber of $J_0$ is isomorphic to $\mathfrak{k}_0$.

Given a Lie bialgebroid over $\mathcal{L}_0$, the latter is equipped with a Poisson structure $\{f, g\} := (df, d_*g)$. Quantization of a Lie bialgebroid $\mathfrak{B}_0$ over $\mathcal{L}_0$ means quantization, $\mathcal{L}_\hbar$, of $\mathcal{L}_0$ and construction of an $\mathcal{L}_\hbar$-bialgebroid whose a) classical limit is the universal enveloping $\mathcal{L}_0$-bialgebroid $\mathcal{U}(\mathfrak{B}_0)$ and b) the infinitesimal deformation is determined by the Lie bialgebroid $\mathfrak{B}_0$. Conversely, an $\mathcal{L}_\hbar$-bialgebroid $\mathfrak{B}_\hbar$ over $k = \mathbb{C}[[\hbar]]$ such that $\mathcal{L}_0 = \mathcal{L}_\hbar \mod \hbar$ and $\mathfrak{B}_\hbar = \mathcal{U}(\mathfrak{B}_0) \mod \hbar$ gives rise to a structure of Lie bialgebroid over $\mathcal{L}_0$ in the quasi-classical limit, [MXu].

Let $\mathfrak{h}$ be a Lie bialgebra, $\mathcal{L}_0$ a Poisson base algebra over $\mathfrak{h}$, $\mathcal{U}_\hbar(\mathfrak{h})$ and $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{h})$ the corresponding quantizations of the universal enveloping algebras, and $\mathcal{L}$ is the $\mathcal{U}_\hbar(\mathfrak{h})$-base algebra that is a quantization of $\mathcal{L}_0$. Then the $\mathcal{L}$-bialgebroid $\mathcal{L} \rtimes \mathcal{U}_\hbar(\mathfrak{h})$ is a quantization of the Lie bialgebroid $\mathcal{L}_0 \rtimes \mathfrak{h}$ from Proposition 9.3. The ideal $J_0$ from Example 9.4 is a classical limit of the biideal $J_\phi$ from Proposition 4.7, where the role of the double belongs to the arbitrary quasitriangular Hopf algebra. The quantum groupoid $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{h})_\mathcal{L}$ is a quantization of the coboundary Lie bialgebroid and $\mathfrak{D}\mathfrak{h}_{\mathcal{L}_0}$ from Example 9.4.

In the next subsection we describe more complicated coboundary Lie bialgebroids, which are related to dynamical r-matrices. The corresponding theory for commutative base was developed by Xu. Here we consider an arbitrary Lie bialgebra $\mathfrak{h}$ and its base manifold $L$. It turns out that the dynamical r-matrices are in one-to-one correspondence with a class of Lie bialgebroid structures on certain Lie algebroids.

## 9.2 Classical dynamical r-matrix and Lie bialgebroids

Let $\mathfrak{h}$ be a Lie bialgebra, with the Lie cobracket $\nu\colon \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$. Let $\mathcal{L}_0$ be a Poisson $\mathfrak{h}$-base algebra. The reader may think of $\mathcal{L}_0$ as a function algebra on a base manifold.

Suppose that $\mathfrak{h}$ is a subalgebra in a Lie algebra $\mathfrak{g}$. We emphasize that we do not assume *any* Lie bialgebra structure on $\mathfrak{g}$. The algebra $\mathcal{L}_0$ is equipped with an $(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})$-action assuming it trivial when restricted to $\mathfrak{g}$.

**Definition 9.5 (DM1).** An element $r(\lambda) \in \mathcal{L}_0 \otimes \wedge^2\mathfrak{g}$ is called a ***classical dynamical r-matrix*** over the Poisson base algebra $\mathcal{L}_0$ with values in $\wedge^2\mathfrak{g}$ if

1. for any $h \in \mathfrak{h}$
$$h \triangleright r(\lambda) + [h \otimes 1 + 1 \otimes h, r(\lambda)] = \nu(h), \tag{58}$$

2. $r$ satisfies the equation
$$\sum_i \mathrm{Alt}\big(h_i \otimes \eta^i \triangleright r(\lambda)\big) - \mathrm{CYB}\big(r(\lambda)\big) = \varphi(\lambda) \in \mathcal{L}_0^{\mathfrak{D}\mathfrak{h}} \otimes (\wedge^3\mathfrak{g})^{\mathfrak{g}} = (\mathcal{L}_0 \otimes \wedge^3\mathfrak{g})^{\mathfrak{g}\oplus\mathfrak{D}\mathfrak{h}}, \tag{59}$$
where $\mathrm{CYB}(\zeta) := [\zeta_{12}, \zeta_{13}] + [\zeta_{13}, \zeta_{23}] + [\zeta_{12}, \zeta_{23}]$, $\zeta \in \mathcal{L}_0 \otimes \wedge^2\mathfrak{g}$, is the Yang-Baxter operator, $\mathrm{Alt}(\xi_1 \otimes \xi_2 \otimes \xi_3) := \xi_1 \otimes \xi_2 \otimes \xi_3 - \xi_2 \otimes \xi_1 \otimes \xi_3 + \xi_2 \otimes \xi_3 \otimes \xi_1$, $\xi_i \in \mathcal{L}_0 \otimes \mathfrak{g}$, and $\varphi(\lambda)$ is some invariant element.

Consider the trivial Lie bialgebroid $\mathcal{L}_0 \otimes \mathfrak{g}$ with the zero anchor map. It is just a Lie algebra over $\mathcal{L}_0$. Denote by $(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$ the direct sum Lie algebroid $(\mathcal{L}_0 \otimes \mathfrak{g}) \oplus (\mathfrak{D}\mathfrak{h}_{\mathcal{L}_0})$. Let $\{h_i\}$ be a base in $\mathfrak{h}$ and $\{\eta^i\}$ its dual in $\mathfrak{h}^*_{op}$. Consider the sum $\Lambda_0 := \varpi + 2\varpi' \in \wedge^2(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})$, where $\varpi := \sum_i \eta^i \wedge h_i = \frac{1}{2}\sum_i(\eta^i \otimes h_i - h_i \otimes \eta^i) \in \wedge^2\mathfrak{D}\mathfrak{h}$ denotes the universal r-matrix of the double, and $\varpi' \in \mathfrak{g} \wedge \mathfrak{h}^*_{op}$ is obtained from $\varpi$ via the embedding $\mathfrak{h} \to \mathfrak{g}$. The element $\Lambda_0$ can be thought of as a constant section of the exterior square of the trivial vector bundle $(\mathcal{L}_0 \otimes \mathfrak{g}) \oplus (\mathcal{L}_0 \rtimes \mathfrak{D}\mathfrak{h})$. Let us denote the projection of $\Lambda_0$ to $\wedge^2(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$ by the same letter.

**Theorem 9.6.** *a) Let $r(\lambda) \in \mathcal{L}_0 \otimes \wedge^2\mathfrak{g}$ be a classical dynamical r-matrix, Then the element $\Lambda := r(\lambda) + \Lambda_0 \in \wedge^2(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$, generates a zero square differential on the graded Lie algebra $\wedge^\bullet(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$ and therefore defines a coboundary Lie bialgebroid on $(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$.*

*b) Suppose that $\mathfrak{h}^*_{op}$ acts effectively on $\mathcal{L}_0$ and the element $\Lambda = r(\lambda) + \Lambda_0 \in \wedge^2(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$ defines a coboundary Lie bialgebroid on $(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$. Then $r(\lambda) \in \mathcal{L}_0 \otimes \wedge^2\mathfrak{g}$ is a classical dynamical r-matrix.*

*Proof.* The element $\Lambda$ defines a coboundary Lie bialgebroid on $(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$ if and only if

$$[[\Lambda, \Lambda], f \otimes \xi] = 0 \mod J_0, \quad \forall f \in \mathcal{L}_0,\ \forall \xi \in \mathfrak{g} \oplus \mathfrak{D}\mathfrak{h}. \tag{60}$$

Explicitly, the ideal $J_0$ is generated by the relations

$$\theta_1 \triangleright f \otimes \theta_2 = \frac{1}{2}\sum_i \big((\eta^i \triangleright f) \otimes h_i + (h_i \triangleright f) \otimes \eta^i\big) = 0 \tag{61}$$

for all $f \in \mathcal{L}_0$, see Example 9.4.

Let us denote by Span(.) the $\mathcal{L}_0$-module generated by a given set. We will analyze the structure of the Schouten bracket $[\Lambda, \Lambda] \in \mathrm{Span}\wedge^3(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})$.

1) The contribution of $[\Lambda, \Lambda]$ to $\mathrm{Span}\big(\mathfrak{g} \wedge \mathfrak{D}\mathfrak{h} \wedge \mathfrak{D}\mathfrak{h}\big)$ is proportional to

$$-h_i \wedge [\eta^i, \eta^j] \wedge h_j - h_i \wedge \eta^j \wedge [\eta^i, h_j] + [h_i, h_j] \wedge \eta_i \wedge \eta_j,$$

where the summation over repeating indices is understood. This term is identically zero, which follows from definition of the double.

2) The contribution of $[\Lambda, \Lambda]$ to $\mathrm{Span}\big(\wedge^3\mathfrak{D}\mathfrak{h}\big)$ is equal to $[\varpi, \varpi]$ that is proportional to $[\theta_{12}, \theta_{23}]$. The latter is a $\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h}$-invariant, and $[[\varpi, \varpi], f]$ belongs to $J_0 \wedge \mathfrak{D}\mathfrak{h} \wedge \mathfrak{D}\mathfrak{h}$ for all $f \in \mathcal{L}_0$, see relations (61). It follows that the commutator of $[\varpi, \varpi]$ with all elements of $\wedge^\bullet(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$ vanishes.

3) The contribution of $[\Lambda, \Lambda]$ to $\mathrm{Span}\big(\wedge^3\mathfrak{g}\big)$ is equal to

$$\frac{4}{3}\varphi(\lambda) := [r(\lambda), r(\lambda)] - 4\sum_i h_i \wedge \big(\eta^i \triangleright r(\lambda)\big).$$

This definition of $\varphi(\lambda)$ is equivalent to (59). The element $\varphi$ commutes with all elements from $\mathcal{L}_0$. Its commutator with $\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h}$ cannot belong to $J_0 \wedge (\ldots)$ and hence vanishes if and only if $\varphi$ is $(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})$-invariant.

4) In fact, the contribution of $[\Lambda, \Lambda]$ to $\mathrm{Span}\big(\mathfrak{D}\mathfrak{h}\wedge\mathfrak{g}\wedge\mathfrak{g}\big)$ lies, modulo $J_0$, in $\mathrm{Span}\big(\mathfrak{h}^*_{op}\wedge\mathfrak{g}\wedge\mathfrak{g}\big)$. Using the identity (61) we find this contribution to be proportional, modulo $J_0$, to

$$\eta^i \wedge [h_i, r(\lambda)] + \eta^i \wedge (h_i \triangleright r(\lambda)) + [\eta^i, \eta^j] \wedge h_i \wedge h_j \tag{62}$$

(summation understood). If $r(\lambda)$ is a classical dynamical r-matrix, this term is identically zero. This follows from equation (58), since $\sum_i [\eta^i, \eta^j] \wedge h_i \wedge h_j = -\sum_i \eta^i \wedge \nu(h_i)$ (recall that the commutator is taken in the opposite Lie algebra $\mathfrak{h}^*_{op}$.

Thus we have proven that $\Lambda$ defines a structure of coboundary bialgebroid if $r(\lambda)$ is a dynamical r-matrix.

Conversely, let $\Lambda$ define a coboundary bialgebroid on $(\mathfrak{g} \oplus \mathfrak{D}\mathfrak{h})_{\mathcal{L}_0}$. The steps 1)-3) of the above proof imply that $r(\lambda)$ satisfies the equation (59). Assume now that $\mathfrak{h}^*_{op}$ acts effectively on $\mathcal{L}_0$. Taking commutator of (62) with an arbitrary element $f \in \mathcal{L}_0$, we find that the expression

$$(\eta^i \triangleright f) \otimes [h_i, r(\lambda)] + (\eta^i \triangleright f) \otimes (h_i \triangleright r(\lambda)) + ([\eta^i, \eta^j] \triangleright f) \otimes h_i \wedge h_j \tag{63}$$

(summation understood) is equal to zero if and only if the equation (58) is satisfied for all $h \in \mathfrak{h}$. This completes the proof. $\square$

**Remark 9.7.** The element $\varphi(\lambda)$ in the right-hand side of (59) is constant, i.e. belongs to $(\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ in case $\mathcal{L}_0$ is quasi-transitive, i.e. $\mathcal{L}_0^{\mathfrak{D}\mathfrak{h}} = k$.

Equation (59) with the zero right-hand side is called classical dynamical Yang-Baxter equation (CDYBE) over the Poisson base algebra $\mathcal{L}_0$. When the invariant element $\varphi$ is non-zero, it may be called modified CDYBE.

Suppose that the element $\varphi(\lambda)$ can be resolved by a symmetric element $\omega(\lambda) \in \mathcal{L}_0^{\mathfrak{D}\mathfrak{h}} \otimes \mathfrak{g}^{\otimes 2}$ in the sense of the equality $\varphi(\lambda) = -[\omega_{12}(\lambda), \omega_{23}(\lambda)]$. Then the element $r(\lambda) + \omega(\lambda)$ will satisfy equation (58) and equation (59) with zero $\phi$, although it will not be skew-symmetric. Conversely, if an element $r(\lambda) \in \mathcal{L}_0 \otimes \mathfrak{g} \otimes \mathfrak{g}$ with $\mathfrak{g}$-invariant symmetric part satisfies equations (58) and (59), then its skew-symmetric part is a dynamical r-matrix in the sense of Definition 9.5.

The classical dynamical r-matrices were conventionally defined on a "flat" base manifold, [BDFh, F, EV1, ESch2, Sch], namely the dual space $\mathfrak{h}^*$ with Lie algebra structure. This corresponds to the zero right-hand side of equation (58). A lot of progress in quantization of such r-matrices, including the Alekseev-Meinrenken solution [AM] and its generalizations, [ESch2], has been made in recent the recent papers of Enriquez and Etingof, [EE1, EE2].

Dynamical (non-skew) r-matrices over an arbitrary Lie bialgebra $\mathfrak{h}$ and an $\mathfrak{h}$-base manifold were introduced in [DM1]. The definition of dynamical r-matrix given in [DM1] was slightly less general then in the present paper. Namely, $\mathfrak{g}$ was assumed to be a Lie bialgebra containing $\mathfrak{h}$ as a sub-bialgebra. An example of dynamical r-matrix on a group manifold was given in [FhMrsh]. The existence of such r-matrices for a wide class of Lie bialgebras follows from the fusion procedure of [DM1] adopted to the quantum group case.

Thus there arises a problem of classification of dynamical r-matrices on non-flat base manifolds and the problem of their quantization. In view of Theorem 9.6, the second problem is closely related to the problem of quantization of Lie bialgebroids of a special class.

# References


[BDFh] J. Balog, L. Dabrowski, and L. Fehér: *Classical r-matrix and exchange algebra in WZNW and Toda field theories*, Phys. Lett. B, **244** # 2 (1990) 227–234.

[AL] A. Alekseev, A. Lachowska: *Invariant ∗-product on coadjoint orbits and the Shapovalov pairing*, arXiv: math.QA/0308100.

[AM] A. Alekseev, E. Meinrenken: *The non-commutative Weil algebra*, Invent. Math. **135** (2000) 135–172.

[Dr1] V. Drinfeld: *Quantum Groups*, in Proc. Int. Congress of Mathematicians, Berkeley, 1986, ed. A. V. Gleason, AMS, Providence (1987) 798–820.

[Dr2] V. Drinfeld: *Quasi-Hopf algebras*, Leningrad Math.J. **1** (1990) 1419–1457.

[DM1] J. Donin, A. Mudrov: *Dynamical Yang-Baxter equation and quantum vector bundles*, arXiv: math.QA/0306028.

[DM2] J. Donin, A. Mudrov: *Reflection Equation, Twist, and Equivariant Quantization*, Isr. J. Math., **136** (2003) 11–28.

[EE1] B. Enriquez, P. Etingof: *Quantization of Alekseev-Meinrenken dynamical r-matrices*, preprint math.QA/0302067.

[EE2] B. Enriquez, P. Etingof: *Quantization of classical dynamical r-matrices with non-abelian base*, preprint math.QA/0311224.

[ESch1] P. Etingof, O. Schiffmann: *Lectures on the dynamical Yang-Baxter equation. Quantum Groups and Lie theory*, London Math. Soc. Lecture Note **290** (2001) 89–129.

[ESch2] P. Etingof, O. Schiffmann: *On a moduli space of classical dynamical r-matrices*, Math. Res. Lett. **9** (2001) 157–170.

[EV1] P. Etingof, A. Varchenko: *Geometry and classification of solutions to the classical dynamical Yang-Baxter equation*, Commun. Math. Phys. **192** (1998) 77–120.

[EV2] P. Etingof, A. Varchenko: *Exchange dynamical quantum groups*, Commun. Math. Phys. **205** (1999) 19–52.

[F] G. Felder: *Conformal field theories and integrable models associated to elliptic curves*, Proc. ICM Zurich, (1994) 1247–1255.

[FhMrsh] L. Fehér and I. Marshall: *On a Poisson-Lie analogue of the classical dynamical Yang-Baxter equation for self dual Lie algebras*, arXiv:math.QA/0208159.

[FRT] L. Faddeev, N. Reshetikhin, and L. Takhtajan: *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990) 193–226.

[GK] S. Gelfand, D. Kazhdan: *Examples of tensor categories*, Invent. Math. **109** (1992) 595–617.

[K] Ch. Kassel: *Quantum groups*, Springer, NY 1995.

[K-Schw] Y. Kosmann-Schwarzbach: *Exact Gerstenhaber algebras and Lie bialgebroids*, Acta Appl. Math. **41** (1995) 153–165.

[KMST] E. Karolinsky, K. Muzykin, A. Stolin, V. Tarasov: *Dynamical Yang-Baxter equations, quasi-Poisson homogeneous spaces, and quantization*, arXiv:math.QA/0309203.

[KSkl] P. P. Kulish, E. K. Sklyanin: *Algebraic structure related to the reflection equation,* J. Phys. A **25** (1992) 5963–5976.

[Lu] J.-H. Lu: *Hopf algebroids and quantum groupoids*, Int. J. Math., **7** (1996) 47–70.

[Mj] S. Majid: *Foundations of quantum group theory*, Cambridge University Press, 1995.

[O] V. Ostrik: *Module categories, weak Hopf algebras, and modular invariants*, Transform. Group, **8** (2003) 177–206.

[MXu] K.C.H. Mackenzie, P. Xu: *Lie bialgebroids and Poisson groupoids* Duke Math. J. **73** (1994) 415–452.

[Sch] O. Schiffmann: *On classification of dynamical r-matrices*, Math. Res. Lett. **5** (1998) 13–30.

[RS] N. Reshetikhin, M. Semenov-Tian-Shansky: *Quantum R-matrices and factorization problem*, J. Geom. Phys. **5** (1988), 533–550.

[Szl] K. Szlachányi: *Finite quantum groupoids and inclusions of finite type*, Mathematical physics in mathematics and physics (Sienna, 2000) 393–407, in Fields Ins. Commun. **30**, Amer. Math. Soc., Providence, RI, 2001.

[Xu1] P. Xu: *Quantum groupoids*, Commun. Math. Phys., **216** (2001) 539–581.

[Xu2] P. Xu: *Quantum Dynamical Yang-Baxter Equation Over a Nonabelian basis*, Commun. Math. Phys. **226** (2002) 475–495.