THE FORMAL DEGREE OF DISCRETE SERIES REPRESENTATIONS OF CENTRAL SIMPLE ALGEBRAS OVER p-ADIC FIELDS

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1. INTRODUCTION AND RESULTS

Let F be a p-adic field and $A|F$ a central simple algebra of reduced degree N. This means that

$$A = M_m(D_d)\,, \quad md = N$$

is a matrix algebra of order m over a central division algebra $D_d|F$ of index d.

We consider irreducible square integrable representations of the multiplicative group A^*. The Abstract Matching Theorem of Deligne, Kazhdan and Vigneras [BDKV] gives character preserving (up to sign) correspondences between these sets of representations for the various algebras $A|F$ with $N = md$ fixed. Moreover the correspondences – which are a special instance of Langlands' functoriality principle – preserve the formal degree of the representations. On the other hand there are the explicit constructions of the square integrable representations in the "extreme" cases $A = M_N(F)$ ([BK], [C]) and $A = D_N$ a division algebra ([C], [Zi$_1$]). From these constructions a certain system $\mathcal{T}_N^-$ of parameters has emerged which is a noncanonical substitute for the indecomposable degree N representations of the complex Weil-Deligne group W_F'. Therefore it is reasonable to consider the following problems:

1) Explicit construction of the discrete series representations of A^* for all $A|F$ with N fixed by using the same system $\mathcal{T}_N^-$ of parameters.

2) Verification of how the explicit constructions fit with the Abstract Matching Theorem.

Concerning Problem 1) there are precise predictions how to construct the representations but the verification of all details has not been finished so far. This paper can be considered as a test of the ruling principles. Using the two central conjectures concerning the structure of Hecke algebras (section 3) and concerning the geometry of conjugacy classes (section 7) we examine the procedure

$$t \in \mathcal{T}_N^- \mapsto \Pi_t^A\,,$$

and we derive an explicit formula for the formal degree $\deg(\Pi_t^A, d\bar{g})$ of the irreducible square integrable representation Π_t^A of A^* which does not depend on the algebra A but only on the parameter t.

Given an admissible irreducible representation Π of the multiplicative group A^* let $V|\mathbb{C}$ be the representation space of Π, $\tilde{V}$ its admissible dual and let $\omega_\Pi : F^* \to \mathbb{C}^*$ be the central character of Π.

Π is called square integrable if ω_Π is unitary and if all matrix coefficients of Π are square integrable modulo the center:

$$\int_{\bar{g}\in A^*/F^*} |\langle \Pi(g)\circ v, \tilde{v}\rangle|^2_{\mathbb{C}}\, d\bar{g} < \infty \qquad \text{for all } v \in V,\ \tilde{v} \in \tilde{V}.$$

This implies that Π is a preunitary representation. Namely fix some $\tilde{v} \neq 0 \in \tilde{V}$. Then:

$$(v_1|v_2) = \int_{\bar{g}\in A^*/F^*} \langle \Pi(g)\circ v_1, \tilde{v}\rangle \cdot \overline{\langle \Pi(g)\circ v_2, \tilde{v}\rangle}\, d\bar{g} < \infty$$

is an A^*-invariant Hermitian product on V. Because V is irreducible, the product $(v_1|v_2)$ is unique up to a positive real factor, and with respect to the Haar measure $d\bar{g}$ on A^*/F^* the formal degree $\deg(\Pi, d\bar{g})$ is defined by:

$$(1) \qquad \int_{A^*/F^*} |(v_1|\Pi(g)\circ v_2)|^2_{\mathbb{C}}\, d\bar{g} = \frac{1}{\deg(\Pi, d\bar{g})} \cdot (v_1|v_1)(v_2|v_2)\,,$$

which is independent of the choice of the product $(\cdot\,|\,\cdot)$.

Remarks. 1. Let $\mathfrak{K}$ be a fixed compact mod center subgroup of A^*. Dividing (1) by $\mathrm{vol}(\mathfrak{K}/F^*, d\bar{g})$ one concludes that the product $\mathrm{vol}(\mathfrak{K}/F^*, d\bar{g}) \cdot \deg(\Pi, d\bar{g})$ does not depend on the Haar measure $d\bar{g}$.
2. We say that $\Pi \in (A^*)^\wedge$ is essentially square integrable or a discrete series representation, if there exists a character $\chi : F^* \to \mathbb{C}^*$, such that $\tilde{\chi}\otimes\Pi$ is a square integrable representation of A^*. ($\tilde{\chi}(g) := \chi(\mathrm{Nrd}_{A|F}(g))$ for $g \in A^*$). Then it is easily seen that an unramified character χ can be found such that $\tilde{\chi}\otimes\Pi$ is square integrable. This is due to the fact that a discrete series representation Π is square integrable iff its central character ω_Π is unitary. For a discrete series representation Π of A^* the formal degree $\deg(\Pi, d\bar{g})$ is defined via a square integrable representation which is obtained as a character twist of Π.

We consider $A|F$ central simple of reduced degree N fixed. Then we know a parameter system

$$(2) \qquad \mathcal{T}_N^- = \{t = [\phi, \beta]\,;\, \deg t | N\}$$

for the discrete series representations of A^*. It is obtained by fixing an approximation procedure, hence a set of *minus polynomials* in $F[T]_{\mathrm{irr}}$ the set of all irreducible monic polynomials provided with the distinguished exponential distance w_F. (See section 2 for more details.) $t = [\phi, \beta]$ denotes a Galois orbit of pairs (ϕ, β) where $\beta \in \bar{F}$ is the root of a minus polynomial and ϕ is a tame character of the multiplicative group of an unramified extension field $K|F(\beta)$ which is regular over $F(\beta)$. The degree of t is defined as $\deg t := [K : F]$. For all $A|F$ of reduced degree N we may define maps

$$t = [\phi, \beta] \in \mathcal{T}_N^- \mapsto [\phi, \beta, \lambda_\beta^A] \mapsto \Pi_t^A \in (A^*)^\wedge$$

by fixing a character $\lambda_\beta^A : F(\beta)^* \to \mathbb{C}^*$ for each β (=root of a minus polynomial). We write λ_β^A because so far it is not clear that the compatibility relations of the system $\{\lambda_\beta^A\}_\beta$ can be formulated without using A. $t_0 = [\phi_0, 0]$, where ϕ_0 is the trivial character of F^*, is mapped to the Steinberg representation St^A of A^*. In the case when $A = D$ is a division algebra, St^A is the trivial representation.

Our aim is to give an A-independent formula for the formal degree $\deg(\Pi_t^A, d\bar{g})$ if the Haar measure $d\bar{g}$ on A^*/F^* is normalized in such a way that $\deg(\mathrm{St}^A, d\bar{g}) = 1$.

1.1 Theorem. *Let $A|F$ be of reduced degree N, and let $t = [\phi, \beta] \in \mathcal{T}_N^-$ be a parameter. Then:*

$$\deg(\Pi_t^A, d\bar{g}) = f \cdot \frac{q^N - 1}{q^{N/e} - 1} \cdot q^{\frac{1}{2}[N^2 \cdot d_{f(T)} - N(1-1/e)]}$$

where e, f are the ramification exponent and the inertial degree of the parameter t and where $d_{f(T)} \geq 0 \in \mathbb{Q}$ is a nonnegative rational number depending on the minimal polynomial $f(T)$ of β over F. $q = |k_F|$ denotes the order of the residue field of F.

Remarks. 1. To the pairs $(\phi, \beta) \in t$ the towers of fields $K \supset F(\beta) \supset F$ have been assigned and by definition $e_t = e_{K|F}$, $f_t = f_{K|F}$ are the ramification exponent and the inertial degree of t.
2. If $f(T)$ is the minimal polynomial of β over F then $d_{f(T)} = 0$ iff $\deg f(T) = 1$, i. e. $\beta \in F$. In such a case one has $e = e_t = 1$, hence $\deg(\Pi_t^A, d\bar{g}) = f$. This especially applies to level 0 representations where the parameters are such that $\beta = 0$.

We are going to explain the polynomial invariant $d_{f(T)} \geq 0 \in \mathbb{Q}$ for irreducible polynomials $f(T)$. It depends on the distinguished exponential distance $w_F(f(T), g(T)) \in \mathbb{Q} \cup \{\infty\}$ on $F[T]_{\mathrm{irr}}$. Namely let be

$$\deg_j(f(T)) := \min\{\deg g(T)\,;\, w_F(f(T), g(T)) \geq j\}$$

the minimal degree of all irreducible polynomials in a "j-neighbourhood" of $f(T)$. Then we have $\deg_j(f(T)) = 1$ for $j << 0$, and $\deg_0(f(T)) = \deg(f(T))$ if $f(T)$ is a minus polynomial. Especially this applies if $f(T)$ is assigned to a parameter $t = [\phi, \beta] \in \mathcal{T}_N^-$ as the minimal polynomial of β over F.

1.2 Definition. If $f(T) \in F[T]_{\mathrm{irr}}$ is a minus polynomial and e is a natural number which is divisible by the ramification exponent of $f(T)$ (which refers to the corresponding isomorphism class of extension fields of F), then the invariant $d_{f(T)} \geq 0 \in \mathbb{Q}$ is given as:

$$d_{f(T)} := \frac{1}{e} \cdot \sum_{\substack{v \geq 0 \\ v \in \frac{1}{e}\mathbb{Z}}} \left[1 - 1/\deg_{-v}(f(T))\right] \tag{3}$$

Concerning the definition we make the following

Remarks. (see section 3)
1. $\deg_v(f(T)) \mid \deg f(T)$ for all $v \in \mathbb{Q}$.
2. v is a *degree-jump* of $f(T)$ if $\deg_v(f(T))$ properly divides $\deg_{v+\varepsilon}(f(T))$ for all $\varepsilon > 0$.
3. denominator$(v)|$ramification exponent of $f(T)$, if v is a degree jump of $f(T)$. Especially we conclude denominator$(v)|e$ such that (3) is independent from the choice of e. The minimal choice would be e = ramification exponent of $f(T)$, and this implies $N^2 \cdot d_{f(T)} \in \mathbb{Z}$ because $e|N$ and $\deg_v(f(T)) \mid \deg f(T) \mid N$ for all v.

4. Because of $\deg_v\big(f(T)\big) \geq 1$ for all v we see $d_{f(T)} = 0$ iff $\deg_0\big(f(T)\big) = 1$ which means $\deg\big(f(T)\big) = 1$ since we only consider minus polynomials.

1.3 Reformulation in the tame case.

If $p \nmid N$, the parameter system $\mathcal{T}_N^-$ can be replaced by the Galois orbits of Howe's admissible pairs $(K|F,\chi)$ (see 2.8 below) such that $[K:F]|N$. If t denotes the Galois orbit of $(K|F,\chi)$, then:

$$\deg(\Pi_t^A, d\bar{g}) = f \cdot \frac{q^N - 1}{q^{N/e} - 1} \cdot q^{\frac{1}{2}[N^2 \cdot d_t - N(1-1/e)]}$$

where $e = e_{K|F}$, $f = f_{K|F}$,

$$d_t = \frac{1}{e} \sum_{\substack{v \geq 0 \\ v \in \frac{1}{e}\mathbf{Z}}} \big(1 - 1/\deg_{-v}(t)\big)$$

$\deg_{-v}(t) = [K_{-v} : F]$ where $K_{-v}|F$ is the minimal extension such that $\chi|_{U_K^{ev+1}} = \chi_v \circ N_{K|K_{-v}}$ factorizes via the corresponding norm map (see also 2.8.3 below).

In terms of "Galois" representations the parameter $(K|F,\chi)$ corresponds to $\sigma = \mathrm{Ind}_{K\uparrow F}(\chi)$ which is an irreducible representation of the Weil group $W = W_F$ of $\bar{F}|F$, and for $t = \sigma$ we obtain:

$$\begin{aligned} e &= \text{dimension of the irreducible constituents of } \sigma_I \\ f &= \text{number of irreducible constituents of } \sigma_I \\ \deg_{-v}(t) &= \min\{\dim\tau\,;\ \tau \in \hat{W},\ \mathrm{Hom}_{W^{v+}}(\tau,\sigma) \neq 0\} \end{aligned}$$

where I is the inertia group of W, W^j are the higherramification groups in upper notation, and W^{v+} is the closure of $\bigcup_{j>0} W^j$. Note that $\mathrm{Hom}_{W^{v+}}(\tau,\sigma) \neq 0$ is equivalent to saying that the exponential distance is $w_F(\tau,\sigma) \geq -v$, where $w_F(\tau,\sigma) := -\max\{v\,;\ \mathrm{Hom}_{W^v}(\tau,\sigma) = 0\} = -\min\{v\,;\ \mathrm{Hom}_{W^{v+}}(\tau,\sigma) \neq 0\}$.

We remark that in the split case $A = M_N(F)$ and in the division algebra case $A = D_N$ this includes the results of [CMS].

2. Parameters for discrete series representations

Our aim is to express $\deg(\Pi, d\bar{g})$ of a discrete series representation Π of A^* in terms of a certain set of parameters for those representations. We briefly recall what the parameters look like:

Consider $F[T]_{\mathrm{irr}}$ the set of irreducible polynomials of degree ≥ 1 where the highest coefficient is 1, and let $F \hookrightarrow F[T]_{\mathrm{irr}}$, $a \mapsto T - a$ be the natural embedding. Then the exponential distance $\nu_F(a-b) \in \mathbb{Z}$ on F has a certain extension to an exponential distance $w_F\big(f(T), g(T)\big) \in \mathbb{Q}$ on$F[T]_{\mathrm{irr}}$, i. e.

$$\begin{aligned} w_F\big(f(T), g(T)\big) &\geq \min\{w_F\big(f(T), h(T)\big), w_F\big(h(T), g(T)\big)\} \\ w_F(T-a, T-b) &= \nu_F(a-b) \qquad \text{for } a, b \in F\,. \end{aligned}$$

(see [Zi3] 1.8). Moreover there exist approximation procedures on $F[T]_{\mathrm{irr}}$ with respect to the exponential distance w_F.

2.1. *An approximation procedure is a map*

$$F[T]_{\text{irr}} \times \mathbb{Q} \to F[T]_{\text{irr}}\,, \quad \big(f(T), j\big) \mapsto f^j(T) \tag{1}$$

such that:

(i) $f^j(T) = T$ *for all* j *if* $f(T) = T$

(ii) $w_F(f, f^j) \geq j$ *and* $f^j(T) = f^{f+\varepsilon}(T)$ *if* $w_F(f, f^j) \geq j + \varepsilon$ *for some* $\varepsilon > 0$

(iii) $\deg f^j(T) | \deg f(T)$ *and the same divisibility holds for the ramification exponent and inertial degree of the polynomials.*

(iv) $w_F(f, g) \geq j$ *implies* $f^j(T) = g^j(T)$.

The existence of approximation procedures was proved by H. Koch [Ko$_1$].

2.2. *Note that for* $f(T) \in F[T]_{\text{irr}}$, $\nu_F(\alpha) \in \mathbb{Q}$ *is the same for all roots* α *of* $f(T)$ *in a fixed algebraic closure* $\bar{F}|F$, *and* $f^j(T) = T$ *for* $j \leq \nu_F(\alpha)$ *i. e. the approximation of* $f(T)$ *starts from the polynomial* $T \in F[T]_{\text{irr}}$ *(which is the "zero element") and it ends up with* $f^\infty(T) = f(T)$.

There is no p-adic expansion of irreducible polynomials but it is suggestive to think of $f^j(T)$ as of the partial sum of a p-adic expansion. Just as for p-adic numbers there are many approximation procedures and we have to fix one of them.

We define

2.3 A polynomial $f(T) \in F[T]_{\text{irr}}$ is called a *minus polynomial* with respect to the fixed approximation procedure if already $f^0(T) = f(T)$. The set of minus polynomials is denoted $F[T]^-_{\text{irr}}$. (To make this clear see the example 2.8.)

2.4 Consider pairs (ϕ, β) where $\beta \in \bar{F}$ is the root of a minus polynomial and $\phi : K^*/1 + \mathfrak{p}_K \to \mathbb{C}^*$ is a tame character of a field K such that:

(i) $K|F(\beta)$ is an unramified extension of fields,

(ii) ϕ is regular over $F(\beta)$, i. e. all conjugate characters are different.

The Galois group $\mathfrak{G}_F = \mathrm{Gal}(\bar{F}|F)$ acts as follows:

$\sigma \circ (\phi, \beta) := \big(\phi \circ \sigma^{-1}, \sigma(\beta)\big)$ for $\sigma \in \mathfrak{G}_F$, and by $t = [\phi, \beta]$ the Galois orbit of the pair (ϕ, β) is denoted. The degree of such a parameter is defined as $\deg t = [K : F]$, and a twist with tamely ramified characters $\chi : F^*/1 + \mathfrak{p}_F \to \mathbb{C}^*$ is given as: $\chi \otimes t := \big[(\chi \circ N_{K|F})\phi, \beta\big]$.

2.5 *If* $A|F$ *is a central simple algebra of reduced degree* N *then* $\mathcal{T}_N^- = \{t = [\phi, \beta]\,;\, \deg t | N\}$ *may serve as a system of parameters for the irreducible discrete series representation of* A^*.

(The minus sign in $\mathcal{T}_N^-$ reminds to the fact that the numbers β are roots of minus polynomials over F.) If $A|F$ is a division algebra, 2.5 has been proved in [Zi$_1$], and if $A|F$ is split it has been proved in [Zi$_3$], [Zi$_4$] using the work of [BK] and the Abstract Matching Theorem[BDKV].

The parameter set $\mathcal{T}_N^-$ is not canonical because it is necessary to make choices when constructing a discrete series representation Π_t out of a parameter t. In order to obtain a well defined Π_t one has to fix a character $\lambda_\beta : F(\beta)^* \to \mathbb{C}^*$ for all β such that the following compatibility relations are fulfilled.

2.6. (i) $\lambda_\beta \circ \sigma^{-1} = \lambda_{\sigma(\beta)}$ *for all* $\sigma \in \mathfrak{G}_F$.

(ii) $\lambda_\beta \equiv 1$ *the unit character of* F^* *if* $\beta = 0$.

(iii) $\big(\lambda_\beta[\lambda_b \circ N_{F(\beta)|F}]^{-1}\big)(1+x) = \psi \circ Tr_{F(\beta)|F}\big((\beta - b)x\big)$ *for* $x \in \mathfrak{p}_{F(\beta)}^{[j/2]+1}$ *and* $j = -\nu_{F(\beta)}(\beta - b)$ *if* $b \in F$. *(Note that* $\nu_{F(\beta)}(\beta - b) = e_{F(\beta)|F} \cdot w_F(f_\beta(T), T - b)$ *is a negative integer because* β, b *are roots of minus polynomials and* $b \in F$*).*

(iv) $\lambda_\beta(\beta) = 1$.

Conditions (iii), (iv) are compatible because $\nu_{F(\beta)}(\beta) < 0$ *implies that the cyclic group* $\langle\beta\rangle$ *and the principal units of* $F(\beta)^*$ *have trivial intersection.*

Unfortunately the compatibility relations of 2.6 are not complete because what we need in (iii) is compatibility between λ_β and λ_γ for arbitrary γ whereas we have assumed $\gamma = b \in F$. So far the general compatibility between λ_β and λ_γ can be expressed only in terms of the algebra A at hand such that fixing a compatible system of characters $\{\lambda_\beta\}_\beta$ might depend on A.

2.7 Let $A|F$ be central simple of reduced degree N and let $A^{*\wedge}_{\text{discrete}}$ be the set of equivalence classes of irreducible discrete series representations of A^*. Fixing a map $\mathcal{T}_N^- \to A^{*\wedge}_{\text{discrete}}$, $t = [\phi, \beta] \mapsto \Pi_t^A$ means to fix a compatible system $\{\lambda_\beta^A\}_\beta$ of characters $\lambda_\beta^A : F(\beta)^* \to \mathbb{C}^*$, which gives a well defined map

$$t = [\phi, \beta] \mapsto [\phi, \beta, \lambda_\beta^A] \mapsto \Pi_t^A \,. \tag{4}$$

We remark that the construction of Π_t uses all characters λ_γ^A where the minimal polynomial of γ is an approximation polynomial of the minimal polynomial of β.

2.8 In the tame case our formula for the formal degree can be expressed in terms of Howe's admissible characters. First our metric w_F on $F[T]_{\text{irr}}$ can be described very easily:

2.8.1 Proposition [Zi$_2$]. *The metric is given as*

$$w_F\big(f(T), g(T)\big) = \max\{v_F(\alpha - \beta)\,;\, f(\alpha) = g(\beta) = 0\}$$

if the irreducible polynomials $f(T)$, $g(T)$ *correspond to tame extensions of* F. □

Let $\tilde{F}|F$ be the maximal tame extension in $\bar{F}$ and let $\Gamma = \mathrm{Gal}(\tilde{F}|F)$ be the corresponding Galois group. We get a natural bijection:

$$\Gamma\backslash\tilde{F} \leftrightarrow F[T]_{\text{irr, tame}}\,, \tag{5}$$

and from the Proposition an approximation procedure on $F[T]_{\text{irr, tame}}$ is obtained as follows:

In F^* we fix a complementary group C_F with respect to the principal units, $F^* = C_F \times U_F^1$. Then there is a uniquely determined complementary group $\tilde{C}$ such that $\tilde{F}^* = \tilde{C} \times U_{\tilde{F}}^1$ and $\tilde{C} \supset C_F$, hence $\tilde{C}$ is a Γ-module. Moreover every $x \in \tilde{F}$ has a unique $\tilde{C}$-expansion

$$x = \sum_{v \in \mathbb{Q}} x_v$$

where $x_v \in \tilde{C} \cup \{0\}$ and $\nu_F(x_v) = v$ if $x_v \neq 0$. For $\sigma \in \Gamma$ the $\tilde{C}$-expansion of $\sigma(x)$ is $\sigma(x) = \sum_{v \in \mathbb{Q}} \sigma(x_v)$, because Γ preserves $\tilde{C}$. Hence the approximation procedure $\tilde{F} \times \mathbb{Q} \to \tilde{F}$, $(x,j) \mapsto x(j) = \sum_{v<j} x_v$ induces

$$\Gamma \backslash \tilde{F} \times \mathbb{Q} \to \Gamma \backslash \tilde{F}, \qquad [x](j) := [x(j)],$$

where $[x]$ denotes the Γ-orbit of $x \in \tilde{F}$. Via (5) this is an approximation procedure of $F[T]_{\text{irr, tame}}$, and denoting $\tilde{F}^- := \{x \in \tilde{F}\,;\, x = x(0) = \sum_{v<0} x_v\}$ we obtain the natural bijection

$$(5)^- \qquad \Gamma \backslash \tilde{F}^- \leftrightarrow F[T]^-_{\text{irr, tame}}.$$

Our system $\mathcal{T}^-$ of parameters now comes down to $\mathcal{T}^-_{\text{tame}}$ consisting of Γ-orbits $t = [\phi, \beta]$ such that $\beta \in \tilde{F}^-$. And as a more precise variant of 2.6 we can prove:

2.8.2 Proposition. *There exist maps $\beta \in \tilde{F}^- \mapsto \{\lambda_\beta : F(\beta)^* \to \mathbb{C}^*\}$ with the following properties:*

(i) $\lambda_\beta \circ \sigma^{-1} = \lambda_{\sigma(\beta)}$ *for all* $\sigma \in \Gamma$,

(ii) $\lambda_\beta \equiv 1$ *if* $\beta = 0 \in \tilde{F}^-$,

(iii) $\left[\lambda_{\beta_1} \circ N_{K|F(\beta_1)} \cdot (\lambda_{\beta_2} \circ N_{K|F(\beta_2)})^{-1}\right](1+x) = \psi \circ Tr_{K|F}\big((\beta_1 - \beta_2)x\big)$ *for* $x \in \mathfrak{p}_K^{[j/2]+1}$, *where* $j = -e_{K|F} \cdot \nu_F(\beta_1 - \beta_2)$, $K = F(\beta_1, \beta_2)$ *and* ψ *is a fixed additive character of* F *of conductor* $\mathfrak{p}_F$,

(iv) $\lambda_\beta(\beta) = 1$.

Sketch of proof. We choose a fundamental domain Δ^- for $\Gamma \backslash \tilde{F}^-$ including one representative of each Γ-orbit in such a way that $\beta \in \Delta^-$ implies $\beta(j) \in \Delta^-$ for all approximations. Δ^- is constructed by induction on the number of nonvanishing terms in the $\tilde{C}$-expansion of β, which is finite because of $\beta(0) = \beta$. Then we make choices

$$\beta \in \Delta^- \mapsto \{\lambda_\beta : F(\beta)^* \to \mathbb{C}^*\}$$

such that the conditions (ii)–(iv) are fulfilled. Once again this is done by induction on the length of the $\tilde{C}$ expansion of β. The induction begins with (ii) which is taken as a definition. In the last step we extend the construction from Δ^- to $\tilde{F}$ putting

$$\lambda_{\sigma(\beta)} := \lambda_\beta \circ \sigma^{-1} \qquad \text{for } \beta \in \Delta^-,\ \sigma \in \Gamma.$$

Note that $\sigma(\beta) = \sigma'(\beta')$ implies $\beta = \sigma^{-1}\sigma'(\beta')$, hence $\beta = \beta'$ because these are elements from Δ^-. Therefore denoting $K = F\big(\sigma(\beta)\big)$ we see that $\sigma, \sigma' : F(\beta) \to K$ are identical isomorphisms, hence $\lambda_{\sigma(\beta)}$ is well defined. $\square$

We fix once and for all a map $\beta \in \tilde{F}^- \mapsto \{\lambda_\beta : F(\beta)^* \to \mathbb{C}^*\}$ with properties (i)–(iv) of the Proposition. It induces a bijection

$$(6) \qquad (\phi, \beta) \mapsto \phi \cdot (\lambda_\beta \circ N_{K|F(\beta)})$$

between pairs (ϕ, β) such that $\beta \in \tilde{F}^-$ and ϕ is a regular tame character of K^* for an unramified extension $K|F(\beta)$ on one hand and admissible characters (in the sense of Howe) of tame extension fields $K|F$ on the other hand. The injectivity of (6) is obvious, and the surjectivity follows from [Rei], Lemma 2.3. Moreover (6) is compatible with the action of the Galois group Γ on both sides, where $\sigma \circ (\phi, \beta)$ is given by 2.4 and $\sigma \circ (K|F, \chi) = (\sigma(K)|F, \chi\sigma^{-1})$. Hence the map (6) induces a bijection between $\mathcal{T}^{-}_{\mathrm{tame}}$ and the set of conjugacy classes of admissible pairs $(K|F, \chi)$. Fixing a bijection (6) implies to fix a Howe decomposition for any admissible pair $(K|F, \chi)$. Namely for a given χ we have a well defined pair (ϕ, β) such that $\chi = \phi \cdot (\lambda_\beta \circ N_{K|F(\beta)})$, and the Howe decomposition of χ is:

$$\begin{aligned} \chi &= (\lambda_\beta \circ N_{K|F(\beta)}) \cdot \phi \\ &= \left\{ \prod_{j=\nu_F(\beta)}^{-\varepsilon} \left[\lambda_{\beta(j+\varepsilon)}(\lambda^{-1}_{\beta(j)} \circ N_{F(\beta(j+\varepsilon))|F(\beta(j))})\right] \circ N_{K|F(\beta(j+\varepsilon))} \right\} \cdot \phi \end{aligned} \tag{7}$$

where the product is over all $j \in \frac{1}{e_{K|F}}\mathbb{Z}$, $\nu_F(\beta) \leq j \leq -\varepsilon$, $\varepsilon = \frac{1}{e_{K|F}} = \frac{1}{e_{F(\beta)|F}}$, and $\beta(j) = \sum_{v<j} \beta_v$ are the partial sums of the $\tilde{C}$-expansion of $\beta \in \tilde{F}^{-}$. Note that $\beta(j) = 0$ for $j \leq \nu_F(\beta)$ and $\beta(0) = \beta$. 2.8.2(iii) implies $\lambda_{\beta_1} \circ N_{K|F(\beta_1)} = \lambda_{\beta_2} \circ N_{K|F(\beta_2)}$ on $1 + \mathfrak{p}_K^{j+1}$, where $j = -e_{K|F}\nu_F(\beta_1 - \beta_2)$. If $L|K$ is a tame extension we obtain $N_{L|K}(1 + \mathfrak{p}_L^{e_{L|K}j+1}) = (1 + \mathfrak{p}_L^{e_{L|K}j+1}) \cap K = 1 + \mathfrak{p}_L^{j+1}$, such that in 2.8.2(iii) we may replace K by any other tame extension of F containing β_1 and β_2.

Applying this to (7) we conclude:

$$\chi|_{U_K^{-e_{K|F}j+1}} = \lambda_{\beta(j)} \circ N_{K|F(\beta(j))}|_{U_K^{-e_{K|F}j+1}} \qquad \text{for all } j \leq 0. \tag{8}$$

From the proof of [Rei], Lemma 2.3 we see that $\lambda_\beta : F(\beta)^* \to F^*$ does not factorize via any intermediate norm map of $F(\beta)|F$, for all $\beta \in \tilde{F}^{-}$. Hence:

2.8.3 Proposition. *If under* (6) *the admissible pair* $(K|F, \chi)$ *is given as* $\chi = \phi \cdot (\lambda_\beta \circ N_{K|F(\beta)})$, *then for all* $j \leq 0$ $F(\beta(j))|F$ *is the minimal subextension in* $K|F$ *such that* $\chi|_{U_K^{-e_{K|F}j+1}}$ *factorizes via the norm map* $N_{K|F(\beta(j))}$.

3. Hecke algebras and formal degree

The aim of this section is to relate the formal degree $\deg(\Pi_t^A, d\bar{g})$ to the dimension $\dim(\pi_t^\#)$, where $\pi_t^\#$ is a distinguished representation of a compact modulo center subgroup of A^* which is contained in Π_t^A. Relying on a conjectural Hecke algebra isomorphism (see [BK] and [SZ] for special instances), the quotient $\deg(\Pi_t^A, d\bar{g})/\dim\pi_t^\#$ can be expressed in terms of the formal degree of a Steinberg representation which is explicitely known by a formula of Macdonald's. We begin with some general remarks on Hecke algebras which are applied to express the formal degree in terms of an idempotent of the Hecke algebra (see 3.5). Then we make use of the conjectural Hecke algebra isomorphism.

Let G denote a totally disconnected, separable unimodular group, Z the center of G, and $\mathfrak{K} \supset Z$ an open compact mod Z subgroup of G. Let dx denote a Haar measure on G/Z. For any Z–invariant measurable subset $X \subset G$ write

$$\mathrm{vol}(X/Z) = \mathrm{vol}(X/Z, dx) = \int_{X/Z} dx\,.$$

Let χ be a unitary character of Z and let (ρ, W) be an irreducible unitary representation of $\mathfrak{K}$ with central character χ which acts in a complex vector space W, $\dim(W) = \dim(\rho)$. Let $[\rho_{ij}(k)]$ $(k \in \mathfrak{K}; 1 \leq i, j \leq \dim(\rho))$ denote the matrix of ρ with respect to an orthonormal basis of W.

Let $\mathcal{H}_\chi = C_c^\infty(G, \chi)$ denote the convolution algebra on G/Z consisting of all mod Z compactly supported functions ψ on G such that

$$\psi(gz) = \chi(z)\psi(g)\,.$$

Let $\mathcal{H}_\rho = \mathcal{H}(G, \rho)$ denote the convolution algebra on G/Z of all mod Z compactly supported $\mathrm{End}(W)$ valued functions h on G which satisfy

$$h(k_1 g k_2) = \rho(k_1) h(g) \rho(k_2) \tag{1}$$

for $k_1, k_2 \in \mathfrak{K}$ and $g \in G$. With respect to an orthonormal basis for W the space $\mathcal{H}_\rho$ may be identified with the space of matrix-valued functions $[h_{ij}(g)]$ such that $h_{ij} \in \mathcal{H}_\chi$ which satisfy the matrix product relation

$$[h_{ij}(k_1 g k_2)] = [\rho_{ij}(k_1)][h_{ij}(g)][\rho_{ij}(k_2)]\,;$$

explicitly

$$h_{ij}(k_1 g k_2) = \sum_{1 \leq i', j' \leq \dim(\rho)} \rho_{ii'}(k_1) h_{i'j'}(g) \rho_{j'j}(k_2)\,. \tag{2}$$

Under the identification of endomorphisms with matrices the convolution product of elements of $\mathcal{H}_\rho$ becomes identified with integration of the matrix product; thus we have

$$(h^{(1)} * h^{(2)})_{ij}(g) = \sum_{1 \leq \lambda \leq \dim(\rho)} \int_{\mathfrak{K}/Z} h^{(1)}_{i\lambda}(x) h^{(2)}_{\lambda j}(x^{-1}g)dx \qquad (1 \leq i, j \leq \dim(\rho))$$

for $h^{(1)}, h^{(2)} \in \mathcal{H}_\rho$.

Let θ_ρ denote the character of ρ,

$$\deg(\rho) = \deg(\rho, dx) := \frac{\dim(\rho)}{\mathrm{vol}(\mathfrak{K}/Z)}\,,$$

and define

$$e(g) = e_\rho(g) = \begin{cases} \deg(\rho)\theta_\rho(g), & \text{if } g \in \mathfrak{K}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $e_\rho(g)$ is a locally constant function which is compactly supported mod Z. Schur's orthogonality relations ([W], p.73) imply that $e_\rho(g)$ is an idempotent function on G/Z.

More generally, for any fixed i, $1 \leq i \leq \dim(\rho)$, let

$$e_i(g) = \begin{cases} \deg(\rho)\rho_{ii}(g), & \text{if } g \in \mathfrak{K}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $e_i(g)$ is also an idempotent function on G/Z. For $i \neq j$ the functions $e_i(g)$ and $e_j(g)$ are orthogonal with respect to convolution on G/Z. Moreover,

$$e_\rho(g) = \sum_{i=1}^{\dim(\rho)} e_i(g)\,.$$

Define $\mathcal{H}_e = \mathcal{H}_e(G)$ to be the subalgebra of $\mathcal{H}_\chi$ consisting of all ψ such that

$$e_\rho * \psi = \psi * e_\rho = \psi\,.$$

Clearly, e_ρ is the identity element of $\mathcal{H}_e$. For any $h = [h_{ij}] \in \mathcal{H}_\rho$ we have $h_{ij} \in \mathcal{H}_e$ for $1 \leq i, j \leq \dim(\rho)$. More precisely, let

$$\mathcal{H}_{ij} = e_i * \mathcal{H}_e * e_j \qquad (1 \leq i, j \leq \dim(\rho))\,.$$

Then, as a vector space

$$\mathcal{H}_e = \oplus_{1\leq i,j\leq \dim(\rho)} \mathcal{H}_{ij}\,,$$

the decomposition being an orthogonal direct sum. For each i the subspace $\mathcal{H}_{ii}$ is a subalgebra of $\mathcal{H}_e$ with e_i as identity element.

Let $\mathcal{A}_\rho$ denote the tensor product algebra $\mathcal{H}_\rho \otimes_{\mathbb{C}} \mathrm{End}(W)$, the multiplication in $\mathcal{A}_\rho$ being

$$(h^{(1)} \otimes T_1) \cdot (h^{(2)} \otimes T_2) = h^{(1)} * h^{(2)} \otimes T_1 T_2\,.$$

3.1 Theorem (Bushnell/Kutzko). *Let (ρ, W) be, as above, an irreducible unitary representation of $\mathfrak{K}$ in the complex vector space W of dimension $\dim(\rho)$. For $h \otimes T \in \mathcal{A}_\rho$ define*

$$\mu(h \otimes T)(x) = \deg(\rho)\,trace(h(x)T^t)\,.$$

Then μ is an algebra isomorphism of $\mathcal{A}_\rho$ to $\mathcal{H}_e$.

Proof. Fix an orthonormal basis for W and identify $\mathcal{H}_\rho$ with an algebra of matrix-valued functions as above. The elements of $\mathrm{End}(W)$ may also be identified with matrices. Let E_{ij} denote the matrix with 1 in the i,j–th place and elsewhere all entries 0, $1 \leq i, j \leq \dim(\rho)$. For any $h \in \mathcal{H}_\rho$ and i, j set

$$\mu_{ij}(h) = \mu(h \otimes E_{ij})\,.$$

Then

$$\mu_{ij}(h) = \deg(\rho) h_{ij}\,.$$

Thus μ_{ij} is a linear mapping of $\mathcal{H}_\rho$ to the subspace $\mathcal{H}_{ij} \subset \mathcal{H}_e$. Since, by Schur orthogonality and (2),

$$\begin{aligned}\deg(\rho)\rho_{i'i} * h_{ij} * \deg(\rho)\rho_{jj'} &= (\deg(\rho)\rho_{i'i} * \sum_{\lambda=1}^{\dim(\rho)} \rho_{i\lambda})(I) h_{\lambda j} * \deg(\rho)\rho_{jj'} \\ &= h_{i'j'}\,,\end{aligned}$$

it is easily seen that μ_{ij} is bijective. This implies that μ is bijective too. A similar calculation, again applying Schur orthogonality, also shows that

$$\deg(\rho) h^{(1)}_{i\lambda} * \deg(\rho) h^{(2)}_{\lambda' j} = \deg(\rho)(h^{(1)} * h^{(2)})_{ij} \delta_{\lambda\lambda'}\,,$$

which implies that μ is an algebra isomorphism. $\square$

3.2 Corollary. *Let $E \neq 0$ be an idempotent in End(W). Then the mapping*

$$h \mapsto \mu(h \otimes E)$$

defines an imbedding (i. e., an algebra monomorphism) of $\mathcal{H}_\rho$ into $\mathcal{H}_e$.

Choosing $E = E_{ii}$ $(1 \leq i \leq \dim(\rho))$, we see that Corollary 3.2 implies that $\mathcal{H}_\rho$ is isomorphic to $\mathcal{H}_{ii}$:

3.3 Corollary. *The mapping*

$$\mu_{ii} : h \mapsto \deg(\rho) h_{ii}$$

defines an isomorphism of $\mathcal{H}_\rho$ to the algebra of scalar-valued functions $\mathcal{H}_{ii}$.

Mautner used the imbedding of Corollary 3.3 in his pioneering study of "spherical function algebras" on the group $PGL_2(F)$ ([Mt], Lemma 2.1). We shall use this imbedding to deduce Proposition 3.5.

Now let G be the group of F points of a connected reductive F–group. Then G is a separable totally disconnected unimodular group. Let $\mathfrak{K}$ be, as before, an open subgroup containing the center Z of G such that $\mathfrak{K}/Z$ is compact. We consider the Schwartz space $\mathcal{C}_*(G)$ of Harish-Chandra ([Si], 4.4, p. 174); in particular we recall that $\mathcal{C}_*(G)$ is a convolution algebra on G/Z.

Let $\Pi(g) = [\Pi_{ij}(g)]$ $(g \in G; 1 \leq i,j < \infty)$ be a unitary discrete series matrix representation of G and let dx be a Haar measure on G/Z. Then the matrix coefficients $\Pi_{ij}(g)$ of Π are elements of $\mathcal{C}_*(G)$ ([Si], Corollary 4.4.5) and the formal degree $\deg(\Pi, dx)$ is defined such that the Schur orthogonality relations hold for the matrix coefficients of Π:

$$\int_{G/Z} \Pi_{ij}(x) \bar{\Pi}_{i'j'}(x) dx = \delta_{ii'} \delta_{jj'} \deg(\Pi, dx)^{-1}$$

or

$$(\Pi_{ij} * \Pi_{j'k})(g) = \delta_{jj'} \Pi_{ik}(g) \deg(\Pi, dx)^{-1}.$$

In particular, the constant $\deg(\Pi, dx)$ is uniquely specified by the condition that, for any diagonal matrix coefficient,

$$\deg(\Pi, dx)\Pi_{ii} * \deg(\Pi, dx)\Pi_{ii} = deg(\Pi, dx)\Pi_{ii}. \tag{3}$$

We also have

$$\deg(\Pi, dx) \int_{G/Z} |\Pi_{ij}(x)|^2 dx = 1$$

and

$$\deg(\Pi, dx) \mathrm{vol}(\mathfrak{K}/Z, dx) = \text{constant}, \tag{4}$$

a constant which is independent of the choice of dx.

Let $\mathcal{S}_\rho \supset \mathcal{H}_\rho$ [respectively, $\mathcal{S}_e \supset \mathcal{H}_e$] denote the closure of $\mathcal{H}_\rho$ [respectively, $\mathcal{H}_e$] with respect to the topology of $\mathcal{C}_*(G)$. Then both $\mathcal{S}_\rho$ and $\mathcal{S}_e$ are convolution algebras on G. It is easy to see that the isomorphism $\mathcal{H}_\rho \to \mathcal{H}_{ii}$ given by Corollary 3.3 extends to an isomorphism $\mathcal{S}_\rho \to \mathcal{S}_{ii}$ and is bicontinuous in the Schwartz topology.

3.4 Lemma. *Assume that ρ occurs simply in the restriction $\Pi_{\mathfrak{K}}$ of the discrete series representation Π. Then W_ρ may be identified with a unique subspace of the representation space $\mathfrak{H}_\Pi$. Let $\phi = [\phi_{ij}]$ denote the $dim(\rho) \times dim(\rho)$ submatrix of the infinite unitary matrix $[\Pi_{ij}]$ consisting of the matrix coefficients of Π with respect to an orthonormal basis of the subspace $W_\rho \subset \mathfrak{H}_\Pi$. Then $\phi \in \mathcal{S}_\rho$ and the linear mapping $h \mapsto h * \phi$ of $\mathcal{S}_\rho \to \mathcal{S}_\rho$ stabilizes the one-dimensional subspace of $\mathcal{S}_\rho$ which is spanned by ϕ. Define*

$$\lambda : \mathcal{S}_\rho \to \mathbb{C}$$

by setting

$$h * \phi = \lambda(h)\phi .$$

Then λ is an algebra homomorphism.

Proof. That $\phi \in \mathcal{S}_\rho$ follows from the fact that matrix coefficients of discrete series representations belong to $\mathcal{C}_*(G)$; it is obvious that ϕ satisfies (1). It is well known ([Si], Lemma 1.10.9) that for any irreducible admissible representation of G the space spanned by the matrix coefficients is a module for $\mathcal{H}_\chi$. Since ρ occurs in $\Pi_{\mathfrak{K}}$ with multiplicity one, for any i there is a one-dimensional subspace of the space of matrix coefficients of Π which belongs to $\mathcal{S}_{ii}$. Since $\mathcal{S}_{ii}$ is an algebra, this one-dimensional space must be stabilized under convolution, from both sides, by the algebra $\mathcal{S}_{ii}$. Thus, for $\psi \in \mathcal{S}_{ii}$,

$$\psi * \phi_{ii} = \lambda(\psi)\phi_{ii} ,$$

where $\lambda(\psi)$ is a scalar. That $\psi \mapsto \lambda(\psi)$ is an algebra homomorphism follows from the associativity of the convolution product in $\mathcal{S}_{ii}$. Using μ_{ii}^{-1} of Corollary 3.3 to pull back λ to $\mathcal{S}_\rho$, we obtain the conclusions. □

The consequence of Corollary 3.3 which we shall use is the following:

3.5 Proposition. *Let*

$$\Phi = \frac{\deg(\Pi, dx)}{\deg(\rho)}\phi .$$

Then the function Φ is an idempotent in $\mathcal{S}_\rho$. Moreover,

$$\mathit{trace}(\Phi(I)) = \deg(\Pi, dx) .$$

Proof. By Corollary 3.3 the mapping μ_{ii} is an isomorphism of the algebra $\mathcal{S}_\rho$ to $\mathcal{S}_{ii}$. Thus,

$$\mu_{ii}(\Phi) = \deg(\rho)\Phi_{ii} = \deg(\Pi, dx)\phi_{ii} ,$$

which, by (3), implies that $\mu_{ii}(\Phi)$ is an idempotent in $\mathcal{S}_{ii}$. Therefore, since μ_{ii} is an isomorphism, Φ is an idempotent in $\mathcal{S}_\rho$. Clearly,

$$\mathrm{trace}(\Phi(I)) = \deg(\rho)\Phi_{ii}(I) = \deg(\Pi, dx)\phi_{ii}(I) = \deg(\Pi, dx) ,$$

since ϕ_{ii}, being a diagonal matrix coefficient of a representation, has the value 1 at I. □

Now we recall some details of the construction

$$t = [\phi, \beta] \in \mathcal{T}_N^- \mapsto [\phi, \beta, \lambda_\beta^A] \mapsto \Pi_t^A \in (A^*)^\wedge ,$$

where $A = M_m(D_d)$ is a central simple algebra over the p-adic field F. Namely let

$$K \supset E = F(\beta) \supset F$$

be the tower of fields associated to $(\phi, \beta) \in t$. Let A_E be the centralizer of E in A. Then $A_E \cong M_{m_0}(D_{d_0})$ is a central simple algebra over E of index $N/[E:F]$, with $d_0 = d/(d, [E:F])$ and $m_0 = (m, N/[E:F])$ (see [Zi$_5$], 1.). Let $L' \subset A_E$ be a maximal subfield such that

$$f_{L'|E} = [K:E]/([K:E], d_0) .$$

Let $\mathfrak{A} = \mathfrak{A}_{L'|F}$ be the principal order in A which is normalized by $(L')^*$. (See the Theorem of Benz and Fröhlich [F], [Zi$_5$].)

Assuming that the character ϕ is unitary, we obtain from t a unitary representation $\pi_t^\#$ of $E^*\mathfrak{A}^*$ to which are associated $N/[K:F]$ discrete series representations Π_t^A in a single unramified twist class with the following additional properties: (1) $\Pi_t^A|_{E^*\mathfrak{A}^*}$ contains $\pi_t^\#$; (2) $\pi_t^\#$ occurs with multiplicity one in $\Pi_t^A|_{E^*\mathfrak{A}^*}$; and (3) these $N/[K:F]$ discrete series representations are the only discrete series representations of A^* which contain $\pi_t^\#$ in their restrictions to $E^*\mathfrak{A}^*$.

We depend upon the following isomorphism of Hecke algebras:

Conjecture.

$$\mathcal{H}(A_K^*, K^*\mathfrak{A}_{L|K}^*, \mathbf{1}) \cong \mathcal{H}(A^*, E^*\mathfrak{A}^*, \pi_t^\#) ,$$

where $L|K$ is maximal and fully ramified; thus the principal order unit group $\mathfrak{A}_{L|K}^*$ is minimal parahoric in A_K^*.

Special instances of such an isomorphism can be found in [BK](5.6.6.) and in [SZ]. Since the Hecke algebras $\mathcal{H}(A_K^*, K^*\mathfrak{A}_{L|K}^*, \mathbf{1})$ and $\mathcal{H}(A^*, E^*\mathfrak{A}^*, \pi_t^\#)$ are isomorphic, the Schwartz algebras $\mathcal{S}(A_K^*, K^*\mathfrak{A}_{L|K}^*, \mathbf{1})$ and $\mathcal{S}(A^*, E^*\mathfrak{A}^*, \pi_t^\#)$ are isomorphic too.

Each discrete series representation Π_t^A which restricted to $E^*\mathfrak{A}^*$ contains $\pi_t^\#$ is represented by an idempotent $\Phi_t^A \in \mathcal{S}(A^*, E^*\mathfrak{A}^*, \pi_t^\#)$. Similarly, each discrete series representation of A_K^* which contains a $K^*\mathfrak{A}_{L|K}^*$-fixed vector is represented by an idempotent $\Phi^{A_K} \in \mathcal{S}(A_K^*, K^*\mathfrak{A}_{L|K}^*, \mathbf{1})$. Note that in both cases all the possible discrete series representations only differ by an unramified twist, hence they have the same formal degree. In the second case this is the formal degree of the Steinberg representation St^{A_K} because from [Bo] we conclude that only unramified twists of St^{A_K} can occur.

3.6 Proposition. *The formal degrees of the discrete series representations associated to the algebras $\mathcal{S}(A_K^*, K^*\mathfrak{A}_{L|K}^*, \mathbf{1})$ and $\mathcal{S}(A^*, E^*\mathfrak{A}^*, \pi_t^\#)$ are related by the formula*

$$vol(K^*\mathfrak{A}_{L|K}^*/K^*, dg_{A_K}) \cdot \frac{\deg(St^{A_K}, dg_{A_K})}{dim(\mathbf{1})} = vol(E^*\mathfrak{A}^*/F^*, dg_A) \cdot \frac{\deg(\Pi_t^A, dg_A)}{dim(\pi_t^\#)} .$$

Remark. Of course $\dim(\mathbf{1}) = 1$. This quantity is included in the denominator on the left to emphasize the symmetry of the formula.

Proof. Assume first that

$$(5) \qquad \mathrm{vol}(K^*\mathfrak{A}^*_{L|K}/K^*, dg_{A_K}) = \mathrm{vol}(E^*\mathfrak{A}^*/F^*, dg_A) = 1\,.$$

In this case, for each of the algebras $\mathcal{S}(A_K^*, K^*\mathfrak{A}^*_{L|K}, \mathbf{1})$ and $\mathcal{S}(A^*, E^*\mathfrak{A}^*, \pi_t^\#)$, the identity element is the function h_I which has support in the identity double coset and the respective values 1 and $I_{\pi_t^\#}$ at the respective identity elements I of A_K^* and A^*. Therefore, because the algebras are isomorphic, the coefficients of h_I in the expressions for the idempotents $\Phi_{St}^{A_K}$ and Φ_t^A are equal. Thus, with (5) holding, Proposition 3.5 implies that

$$\frac{\deg(St^{A_K}, dg_{A_K})}{\dim(\mathbf{1})} = \frac{\deg(\Pi_t^A, dg_A)}{\dim(\pi_t^\#)}\,.$$

This appears to be a special case of the Proposition. But the general case follows from (4). $\square$

Now we use:

3.7 Macdonald's formula [Mc]. *Let $A = M_m(D_d)$ be a central simple algebra over F, and $\mathfrak{A} = \mathfrak{A}_r$ a principal order of period r in A. r is a divisor of m and we denote $s = s(\mathfrak{A}) := m/r$. Then for any Haar measure $d\bar{g}$ on A^*/F^*:*

$$vol(F^*\mathfrak{A}^*/F^*, d\bar{g}) \cdot \deg(St^A, d\bar{g}) = \frac{1}{N} \cdot \prod_{i=1}^{s} (q^{di} - 1)^r / (q^N - 1)$$

where $q = |k_F|$ is the order of the residue field of F.

Remark. We consider the special case where $A = D$ is a division algebra hence $r = s = m = 1$, $d = N$, and $\mathfrak{A} = \mathfrak{O}_D$ is the unique principal order. Moreover St^D is the trivial representation of D^*. Then we obtain

$$\mathrm{vol}(F^*\mathfrak{O}_D^*/F^*, d\bar{g}) \cdot \deg(\mathrm{St}^D, d\bar{g}) = \frac{1}{N}$$

reflecting the fact that $\deg(\mathrm{St}^D, d\bar{g}) = 1$ iff $\mathrm{vol}(D^*/F^*, d\bar{g}) = 1$.

Now we consider our case $\mathfrak{A} = \mathfrak{A}_{L'|F}$ and we put $r' = r(\mathfrak{A})$, $s' = s(\mathfrak{A})$ in this case. Normalizing $d\bar{g}$ in such a way that $\deg(\mathrm{St}^A, d\bar{g}) = 1$ we conclude:

$$(6) \qquad \begin{aligned} \mathrm{vol}(E^*\mathfrak{A}^*/F^*, d\bar{g}) &= e_{E|F} \cdot \mathrm{vol}(F^*\mathfrak{A}^*/F^*, d\bar{g}) \\ &= \frac{e_{E|F}}{N} \cdot \prod_{i=1}^{s'} (q^{di} - 1)^{r'} / (q^N - 1)\,. \end{aligned}$$

We want to apply Macdonald's formula in order to compute the left hand side of 3.6. We have to replace

N by $N_K = N/[K:F]$,
A by $A_K = M_{m_K}(D_{d_K})$,
q by $q^f = |k_K|$, where $f = f_{K|F} = f_t$.
Because $L|K$ is fully ramified we have $r(\mathfrak{A}_{L|K}) = m_K$, $s(\mathfrak{A}_{L|K}) = 1$ such that:

$$(7) \quad \mathrm{vol}(K^*\mathfrak{A}^*_{L|K}/K^*, d\bar{g}) \cdot \deg(\mathrm{St}^{A_K}, d\bar{g}) = \frac{1}{N_K}(q^{fd_K} - 1)^{m_K}/(q^{fN_K} - 1).$$

Using 3.6, (6), (7) we have reduced the computation of $\deg(\Pi_t^A, d\bar{g})$ to that of $\dim(\pi_t^{\#})$, namely:

$$\deg(\Pi_t^A, d\bar{g}) = \dim\pi_t^{\#} \cdot \frac{(q^{fd_K} - 1)^{m_K} \cdot N(q^N - 1)}{N_K(q^{fN_K} - 1) \cdot e_{E|F} \prod_{i=1}^{s'}(q^{di} - 1)^{r'}}$$

4. Computing the dimension of $\pi_t^{\#}$

Consider $t = [\phi, \beta] \in \mathcal{T}_N^-$. The embedding of $K \supset E = F(\beta) \supset F$ in A takes β into a pure element of type $(e_{L'|F}, f_{L'|F}, \mathfrak{A}_{L'|F})$ because $\beta \in L'$ and $(L')^*$ normalizes $\mathfrak{A}_{L'|F} =: \mathfrak{A}$. Since β is the root of a minus polynomial it generates a simple stratum $\beta + \mathfrak{A}$. Let $\mathfrak{P} = \mathrm{Jac}\mathfrak{A}$ be the Jacobson radical and $U^1(\mathfrak{A}) = 1 + \mathfrak{P}$ be the principal units in $\mathfrak{A}^*$. We recall:

4.1 Proposition. (i) *There is a construction*

$$\beta + \mathfrak{A} \mapsto \pi_\beta^1 \in U^1(\mathfrak{A})^\wedge$$

of an irreducible representation π_β^1 of $U^1(\mathfrak{A})$ which makes use of the characters λ_γ^A for all approximations γ of β (i. e. the minimal polynomial of γ is an approximation polynomial of the minimal polynomial of β with respect to the fixed approximation procedure for polynomials). The selfintertwining of π_β^1 is:

$$(1) \qquad I_{A^*}(\pi_\beta^1) = U^1(\mathfrak{A}) \cdot A_E^* \cdot U^1(\mathfrak{A})$$

(ii) *$\dim\pi_\beta^1 = q^{\frac{1}{2}[N^2 \cdot d_{f(T)} - N(s' - f_0'/e_{E|F})]}$ where $d_{f(T)} \geq 0 \in \mathbb{Q}$ is the numerical invariant 2.2 of the minimal polynomial $f(T)$ of β over F and where $s' = s(\mathfrak{A})$, $f_0' = s(\mathfrak{A}_{L'|E})$.* □

(Note that $\mathfrak{A} = \mathfrak{A}_{L'|F}$ and $\mathfrak{A}_{L'|E} = \mathfrak{A} \cap A_E$.)

Remark. The construction of 4.1(i) is essentially the same as in [BK] and [Zi$_1$] respectively and the dimension formula is an immediate consequence of the construction procedure (see section 7 below).

We are going to explain how the dimensions of π_β^1 and of $\pi_t^{\#}$ are related. Let $\mathfrak{K} = \mathfrak{K}_{L'|F}$ be the normalizer of $\mathfrak{A}$ and let $\mathfrak{K}_\beta := \mathfrak{K} \cap A_E$. Because of $U^1(\mathfrak{A}) \subset \mathfrak{K}$ from (1) we deduce $I_{A^*}(\pi_\beta^1) \cap \mathfrak{K} = \mathfrak{K}_\beta \cdot U^1(\mathfrak{A})$, and since $U^1(\mathfrak{A})$ is a normal subgroup in $\mathfrak{K}$ we can say:

4.2. $\mathfrak{K}_\beta \cdot U^1(\mathfrak{A})$ *is the normalizer of* π_β^1 *in* $\mathfrak{K}$.

Now we use $\pi_t^{\#} = \mathrm{Ind}(\tau_\phi \otimes \tilde{\pi}_\beta)$ where the induction is from $E^*\mathfrak{A}^*_{L'|E}J^1_\beta$ onto $E^*\mathfrak{A}^*$. Let us do the induction in two steps namely $E^*\mathfrak{A}^*_{L'|E}J^1_\beta \uparrow E^*\mathfrak{A}^*_{L'|E}U^1(\mathfrak{A})$ and $E^*\mathfrak{A}^*_{L'|E}U^1(\mathfrak{A}) \uparrow E^*\mathfrak{A}^*$. Because τ_ϕ is a supercuspidal representation of $E^*\mathfrak{A}^*_{L'|E}/U^1(\mathfrak{A}_{L'|E})$, the first induction yields

$$\mathrm{Ind}_1(\tau_\phi \otimes \tilde{\pi}_\beta) = \tau_\phi \otimes \mathrm{Ind}_1(\tilde{\pi}_\beta)\,, \tag{2}$$

and $\mathrm{Ind}_1(\tilde{\pi}_\beta)$ is an extension of π^1_β. From 4.2 we see that $E^*\mathfrak{A}^* \cap \mathfrak{K}_\beta \cdot U^1(\mathfrak{A}) = E^*\mathfrak{A}^*_{L'|E}U^1(\mathfrak{A})$ is the normalizer of π^1_β in $E^*\mathfrak{A}^*$. Therefore

$$\pi_t^{\#} = \mathrm{Ind}_2 \circ \mathrm{Ind}_1(\tau_\phi \otimes \tilde{\pi}_\beta) \tag{3}$$

is an irreducible representation of $E^*\mathfrak{A}^*$. From (2), (3) we conclude:

$$\mathrm{dim}\pi_t^{\#} = \left(\mathfrak{A}^* : \mathfrak{A}^*_{L'|E}U^1(\mathfrak{A})\right) \cdot \mathrm{dim}\tau_\phi \cdot \mathrm{dim}\pi^1_\beta \tag{4}$$

because $\mathrm{Ind}_1(\tilde{\pi}_\beta)$ and π^1_β are equidimensional. So we are left with the computation of the first two factors on the right hand side of (4).

4.3 Proposition. *dim*$\tau_\phi = \prod_{i=1}^{f_0'-1}(q^{d_0 f_{E|F} i} - 1)^{e_0'}$
where $e_0' = r(\mathfrak{A}_{L'|E})$, $f_0' = s(\mathfrak{A}_{L'|E})$ *hence* $e_0' \cdot f_0' = m_0$.

Proof. We have $A_E = M_{m_0}(D_{d_0})$, hence $s(\mathfrak{A}_{L'|E}) = (m_0, f_{L'|E})$. Dividing $m_0 d_0 = N_E = [L:K]\cdot[K:E]$ by the product $([K:E], d_0)\cdot([L:K], m_0)$ we see that $\frac{m_0}{([L:K],m_0)} = \frac{[K:E]}{([K:E],d_0)}$ which is $f_{L'|E}$ by definition. Therefore $f_{L'|E}$ divides m_0 and

$$s(\mathfrak{A}_{L'|E}) = f_{L'|E}\,, \qquad r(\mathfrak{A}_{L'|E}) = m_0/s(\mathfrak{A}_{L'|E}) = ([L:K], m_0) \tag{5}$$

Let k_0' be the residue field of the central division algebra $D_{d_0}|E$. Then we get

$$(\mathfrak{A}_{L'|E}/\mathfrak{P}_{L'|E})^* \cong \left[\mathrm{GL}_{f_0'}(k_0')\right]^{e_0'}. \tag{6}$$

The tame character $\phi \in X(K^*)$ has reduction $\bar{\phi} \in X(k_K^*)$ which is regular over k_E. Let l be the composite field of k_K and k_0'. Because of $[k_0' : k_E] = d_0$ we conclude:

$$[l : k_0'] = [k_K : k_0' \cap k_K] = [K:E]/([K:E], d_0) = f_{L'|E}\,.$$

But we have already seen that $f_{L'|E} = f_0'$. Obviously $\bar{\phi}\circ N_{l|k_K} \in X(l^*)$ is a character which is regular over k_0', hence it gives rise to a supercuspidal representation σ of $\mathrm{GL}_{f_0'}(k_0')$.

To compute dimσ we note that $k_0' \supset k_E \supset k_F$ is of degree $d_0 f_{E|F}$, hence $|k_0'| = q^{d_0 f_{E|F}}$ and

$$\mathrm{dim}\sigma = \prod_{i=1}^{f_0'-1}(q^{d_0 f_{E|F} i} - 1)$$

Using (6) the tensor power $\sigma^{\otimes e_0'}$ inflates to a representation of $\mathfrak{A}^*_{L'|E}$ and extends to τ_ϕ of $E^*\mathfrak{A}^*_{L'|E}$. Therefore $\dim\tau_\phi = (\dim\sigma)^{e_0'}$ which proves 4.3. $\square$

The first factor on the right hand side of (4) is

$$(7) \qquad (\mathfrak{A}^* : \mathfrak{A}^*_{L'|E} U^1(\mathfrak{A})) = \frac{(\mathfrak{A}^* : U^1(\mathfrak{A}))}{(\mathfrak{A}^*_{L'|E} : U^1(\mathfrak{A}_{L'|E}))} .$$

With the notation of section 3 we get:

$$\mathfrak{A}^*/U^1(\mathfrak{A}) \cong \left[\mathrm{GL}_{s'}(k_D)\right]^{r'} ,$$

where k_D is the residue field of the central division algebra $D_d|F$. Hence

$$(7)_{\mathrm{num}} = \prod_{i=1}^{s'} (q^{di} - 1)^{r'} \cdot q^{\frac{1}{2} ds'(s'-1)r'}$$

is the numerator of (7). On the other hand from (6) and from the formula for $|k_0'|$ we see that

$$(7)_{\mathrm{den}} = \prod_{i=1}^{f_0'} (q^{d_0 f_{E|F} i} - 1)^{e_0'} \cdot q^{\frac{1}{2} d_0 f_{E|F} f_0'(f_0'-1)e_0'}$$

is the denominator of (7).

Now we take the results together in order to compute $\dim\pi_t^\#$. First we determine the q-power which is in $\dim\pi_t^\#$. From (4), 4.1(ii), $(7)_{\mathrm{num}}$, $(7)_{\mathrm{den}}$ we conclude:

$$q\text{-power} = \frac{1}{2}\left[N^2 \cdot d_{f(T)} - N(s' - f_0'/e_{E|F}) + ds'(s'-1)r' - d_0 f_{E|F} f_0'(f_0'-1)e_0'\right] .$$

We use $ds'r' = dm = N$ and $d_0 f_0' e_0' = d_0 m_0 = N_E$. Then:

$$(8) \qquad \begin{aligned} q\text{-power} &= \frac{1}{2}\left[N^2 \cdot d_{f(T)} + N f_0'/e_{E|F} - N - N_E \cdot f_{E|F}(f_0' - 1)\right] \\ &= \frac{1}{2}\left[N^2 \cdot d_{f(T)} - N + N_E f_{E|F}\right] = \frac{1}{2}\left[N^2 \cdot d_{f(T)} - N(1 - \frac{1}{e_{E|F}})\right] \end{aligned}$$

Let $(7)'_{\mathrm{num}}$, $(7)'_{\mathrm{den}}$, $(\dim\tau_\phi)'$ be the prime-to-p-part of these expressions. Then the prime-to-p-part of $\dim\pi_t^\#$ is:

$$(9) \qquad p'\text{-part} = \frac{(7)'_{\mathrm{num}} \cdot (\dim\tau_\phi)'}{(7)'_{\mathrm{den}}} = \frac{\prod_{i=1}^{s'} (q^{di} - 1)^{r'}}{(q^{d_0 f_{E|F} f_0'} - 1)^{e_0'}}$$

5. The formal degree

Let the Haar measure $d\bar{g}$ of A^*/F^* be normalized in such a way that $\deg(\mathrm{St}^A, d\bar{g}) = 1$. Then in section 3 we have obtained:

$$\deg(\Pi_t^A, d\bar{g}) = \dim\pi_t^{\#} \cdot \frac{(q^{fd_K}-1)^{m_K} \cdot N(q^N-1)}{N_K(q^{fN_K}-1)\cdot e_{E|F}\prod_{i=1}^{s'}(q^{di}-1)^{r'}}$$

We note that $N/N_K e_{E|F} = [K:F]/e_{E|F} = f_{K|F} = f$ is the inertial degree of our parameter t, hence $fN_K = N/e_{E|F}$, and $e_{E|F} = e$ is the ramification exponent of t. Therefore from 4.(8), (9) we obtain:

$$\deg(\Pi_t^A, d\bar{g}) = f \cdot \frac{(q^N-1)}{(q^{N/e}-1)} \cdot q^{\frac{1}{2}\left[N^2 d_{f(T)} - N(1-1/e)\right]} \cdot \frac{(q^{fd_K}-1)^{m_K}}{(q^{d_0 f_{E|F} f_0'}-1)^{e_0'}},$$

and we are left to show that the last factor is 1. For this we consider $A_K = M_{m_K}(D_{d_K})$ in $A_E = M_{m_0}(D_{d_0})$. Because A_K is the centralizer of K in A_E, the invariants of these algebras are related by

$$\mathrm{inv}[A_K] = \mathrm{inv}\big(\mathrm{res}_K[A_E]\big) = [K:E]\,\mathrm{inv}[A_E]$$

This yields $d_K = d_0/\big([K:E], d_0\big)$, hence $d_0 f_{E|F} f_0' =_{4.(5)} d_0 f_{E|F} \cdot \frac{[K:E]}{\big([K:E],d_0\big)} = f_{E|F}[K:E]\cdot d_K = fd_K$. Moreover $m_K = \big(m_0, N_E/[K:E]\big) = \big(m_0, [L:K]\big) =_{4.(5)} e_0'$.

6. Additional remarks in the supercuspidal case

We continue to consider the map

$$t = [\phi,\beta] \mapsto [\phi,\beta,\lambda_\beta^A] \mapsto \Pi_t^A \in (A^*)^\wedge,$$

where $A = M_m(D_d)$, $md = N$. As a conveniant notation we introduce $n_K := [K:F] = \deg t$, hence $n_K \cdot N_K = N$. Because under the Hecke algebra isomorphism (Conjecture of section 3) supercuspidal representations correspond, we see that $\pi_t^{\#}$ is contained in a supercuspidal representation iff A_K^* is a division algebra. Only in that case supercuspidal representations and representations with Iwahori fixed vector may agree. Moreover this is the case where $\mathcal{H}(A^*, E^*\mathfrak{A}^*, \pi_t^{\#})$ is of finite dimension $N_t = N_K$, and the N_t different supercuspidal representations of A^* which contain $\pi_t^{\#}$ are obtained by extending $\pi_t^{\#}$ and inducing. Therefore we get:

6.1 Proposition. *The representation Π_t^A is supercuspidal iff the following equivalent conditions are fulfilled:*

(i) *The centralizer A_K of K embedded into A is a division algebra.*
(ii) $lcm(n_K, d) = N$
(iii) $n_K = (n_K, d)\cdot m$
(iv) $m_K = (m, N_K) = 1$

. We only prove that (i)–(iv) are equivalent. Because $A_K = M_{m_K}(D_{d_K})$ where $m_K = (m, N_K)$ we see the equivalence of (i) and (iv). Moreover dividing $md = N = n_K \cdot N_K$ by (d, n_K) we obtain $m \cdot \frac{d}{(d,n_K)} = \frac{n_K}{(d,n_K)} \cdot N_K$, hence $(m, N_K) = 1$ iff $m = \frac{n_K}{(d,n_K)}$ which proves the equivalence of (iii) and (iv). Finally from $\frac{n_K}{(d,n_K)} = \frac{\operatorname{lcm}(n_k,d)}{d}$ we get the equivalence of (ii) and (iii). □

In the following we use the notation of 3.(1).

6.2 Proposition. *Assume that Π_t^A is a supercuspidal representation. Then up to conjugation holds:*

$$\begin{aligned} \mathfrak{A}_{L'|F} &= \mathfrak{A}_{L|F} \\ \mathfrak{A}_{L'|E} &= \mathfrak{A}_{L|E} = \mathfrak{A}_1(A_E)\,. \end{aligned}$$

Proof. For the first equation we have to show:

$$s(\mathfrak{A}_{L|F}) = s(\mathfrak{A}_{L'|F}), \qquad \text{i. e. } (m, f_{L|F}) = (m, f_{L'|F})\,.$$

By a result of Fröhlich (see [Zi$_5$], 3.(ii)) we know

$$s(\mathfrak{A}_{L|F}) = (s(\mathfrak{A}_{L|E})f_{E|F}, m)$$

for all maximal field extensions $L|E|F$. And the numerical invariant $s(\mathfrak{A})$ determines the principal order $\mathfrak{A}$ up to conjugacy. Therefore we are left to show

$$s(\mathfrak{A}_{L|E}) = s(\mathfrak{A}_{L'|E}), \quad \text{i.e.} \qquad (m_0, f_{L|E}) = (m_0, f'_{L'|E}).$$

In 4.3 we have introduced the notation $f'_0 := s(\mathfrak{A}_{L'|E})$ and we have seen that $f_{L'|E} | m_0$, hence $f'_0 = f_{L'|E}$. Now from 6.1(iv) and the last equation of section 5 we see

$$1 = m_K = e'_0 := r(\mathfrak{A}_{L'|E}) \quad \text{hence} \qquad f'_0 = m_0$$

because $e'_0 f'_0 = m_0$. On the other hand $f_{L'|E} | f_{L|E} = [K : E]$ such that altogether we obtain $s(\mathfrak{A}_{L|E}) = s(\mathfrak{A}_{L'|E}) = m_0$, and $s(\mathfrak{A}_{L|F}) = s(\mathfrak{A}_{L'|F}) = m_0(f_{E|F}, m/m_0)$.□

Remark. If Π_t^A is supercuspidal then of course we expect it to be induced from a (up to conjugation) unique maximal compact modulo center subgroup. The observation $\mathfrak{A}_{L|F} = \mathfrak{A}_{L'|F}$ i. e. $\mathfrak{K}_{L|F} = \mathfrak{K}_{L'|F}$ supports this.

We have based our computations on Proposition 3.6 which we have obtained as a consequence of the conjectural isomorphism of Hecke algebras. Finally we are going to explain that 3.6 simplifies considerably if Π_t^A is a supercuspidal representation. Namely we can use 6.1(i) and the remark following Mac Donald's formula 3.7. Then the left hand side of 3.6 is simply $1/N_K$, and we obtain:

$$\operatorname{vol}(E^*\mathfrak{A}^*/F^*, d\bar{g}) \cdot \deg(\Pi_t^A, d\bar{g}) = \frac{1}{N_K} \cdot \dim(\pi_t^{\#}) \tag{7}$$

if Π_t^A is a supercuspidal representation. Moreover due to 6.2 we need *not* to distinguish between $\mathfrak{A}_{L'|F}$ and $\mathfrak{A}_{L|F}$ in the supercuspidal case, hence $\mathfrak{A} = \mathfrak{A}_{L'|F} = \mathfrak{A}_{L|F}$. With $s' = s := (m, f)$ and $r' = r := m/(m, f)$ we obtain

$$\frac{\mathrm{vol}(\mathfrak{K}/F^*, d\bar{g})}{\mathrm{vol}(E^*\mathfrak{A}^*/F^*, d\bar{g})} = \frac{(\mathfrak{K} : F^*\mathfrak{A}^*)}{(E^*\mathfrak{A}^* : F^*\mathfrak{A}^*)} = \frac{rd}{e_{E|F}} = \frac{N}{s \cdot e_{E|F}},$$

where $\mathfrak{K}$ is the normalizer of $\mathfrak{A}$. Hence multiplying (7) by $N/s \cdot e_{E|F}$ yields

$$\mathrm{vol}(\mathfrak{K}/F^*, d\bar{g}) \cdot \deg(\Pi_t^A, d\bar{g}) = \frac{f}{s} \cdot \dim(\pi_t^\#) \tag{8}$$

We have $\mathfrak{A}_{L'|E} = \mathfrak{A}_1(A_E)$ (see 6.2) and we let $K'|E$ be of degree $([K : E], d_0)$ embedded into the central division algebra $D_{d_0}|E$, and we may assume that $D_{d_0}^*$ normalizes $\mathfrak{A}_1^*(A_E)$. Now we use the following well known facts:
a) $\tau_\phi \in (E^*\mathfrak{A}_1^*(A_E))^\wedge$ extends to $C^*\mathfrak{A}_1^*(A_E)$, where $C = \mathrm{Cent}(K', D_{d_0})$, and then irreducibly induces to a supercuspidal representation of A_E^*.
b) $\pi_t^\# = \mathrm{Ind}(\tau_\phi \otimes \tilde{\pi}_\beta) \in (E^*\mathfrak{A}^*)^\wedge$ extends to $C^*\mathfrak{A}^*$ and then irreducibly induces to a supercuspidal representation of A^*.
c) Π_t^A corresponds to an appropriate extension of $\pi_t^\#$.

We compute the index $(\mathfrak{K} : C^*\mathfrak{A}^*)$, where $\mathfrak{K}$ is the normalizer of $\mathfrak{A}$. Namely:

$$(\mathfrak{K} : C^*\mathfrak{A}^*) = (\mathfrak{K} : F^*\mathfrak{A}^*)/(C^*\mathfrak{A}^* : F^*\mathfrak{A}^*).$$

The numerator is rd and the denominator is:

$$(C^*\mathfrak{A}^* : K'\mathfrak{A}^*)(K'\mathfrak{A}^* : F^*\mathfrak{A}^*) = \frac{d_0}{([K : E], d_0)} \cdot e_{E|F}.$$

Hence:

$$(\mathfrak{K} : C^*\mathfrak{A}^*) = \frac{([K : E], d_0) rd}{d_0 e_{E|F}} = \frac{[K : E] rd}{m_0 d_0 e_{E|F}} = \frac{[K : F] rd}{N e_{E|F}} = \frac{f}{s}$$

because from (1), (2) we see $m_0 = f_0' = \frac{[K:E]}{([K:E], d_0)}$, and moreover $m_0 d_0 = N/[E : F]$.

Now remarks b), c) imply that Π_t^A is induced from a representation ϱ of $\mathfrak{K}$, such that

$$\dim\varrho = (\mathfrak{K} : C^*\mathfrak{A}^*) \dim\pi_t^\# = (f/s) \dim\pi_t^\#.$$

Putting this into (8) as a consequence of 3.(3) we obtain:

6.3. *If $t = [\phi, \beta]$ has the property lcm$(\deg t, d) = N$, then the supercuspidal representation Π_t^A is induced by an irreducible representation ϱ of $\mathfrak{K}_{L|F}$ (where $L|K$ is as in* 3.(1)) *and*

$$vol(\mathfrak{K}_{L|F}/F^*, d\bar{g}) \cdot \deg(\Pi_t^A, d\bar{g}) = dim\varrho.$$

We remark that the last equation can be proved independently from the Hecke algebra isomorphism 3.(1).

7. Proof of 4.1(ii)

Fix e, f such that $ef = N$ and a principal order $\mathfrak{A}$ in A. Let $\mathfrak{K}$ be the normalizer of $\mathfrak{A}$ in A^* and let $A(e, f, \mathfrak{A})$ be the set of $(e, f, \mathfrak{A})$-pure elements in A, i. e. $x \in A(e, f, \mathfrak{A})$ iff there exists a field extension $L|F$ in A such that $x \in L$, $e_{L|F} = e$, $f_{L|F} = f$, L^* normalizes $\mathfrak{A}$.

One has $A(e, f, \mathfrak{A}) \neq 0$ iff $r(\mathfrak{A}) = r(e, f) := m/(m, f) = e/(e, d)$, where the last equation is derived from $md = N = ef$. Moreover in this case one has a natural bijection (see [Zi$_5$], 6.)

$$\mathfrak{K}\backslash A(e, f, \mathfrak{A}) \leftrightarrow F[T]_{e,f} \tag{1}$$

between $\mathfrak{K}$-conjugacy classes of $(e, f, \mathfrak{A})$-pure elements and irreducible monic polynomials over F of ramification exponent dividing e and inertial degree dividing f. Under (1) the minimal polynomial of x over F is assigned to an $(e, f, \mathfrak{A})$-pure element x, and we have the

Isometry conjecture*. $\frac{1}{rd} \cdot \nu_{\mathfrak{P}}(\underline{x}, \underline{y}) = w_F\big(f(T), g(T)\big)$ *if the $\mathfrak{K}$-conjugacy classes $\underline{x}$, $\underline{y}$ correspond to the polynomials $f(T)$, $g(T)$ respectively.*

Here we have used the notation $r = r(\mathfrak{A})$, $\mathfrak{P} = \mathrm{Jac}(\mathfrak{A})$ the Jacobson radical of $\mathfrak{A}$, $\nu_{\mathfrak{P}}(\underline{x}, \underline{y}) = \max\{\, \nu_{\mathfrak{P}}(x' - y') \,;\, x' \in \underline{x}\,,\, y' \in \underline{y} \,\}$ and w_F is the distinguished exponential distance on the set of irreducible monic polynomials over F. From the isometry conjecture one deduces the existence of fundamental domains $\mathbf{D}^- \subset \mathbf{D} \subset A(e, f, \mathfrak{A})$ such that

$$\mathbf{D} \leftrightarrow \mathfrak{K}\backslash A(e, f, \mathfrak{A}) \leftrightarrow F[T]_{e,f} \tag{2}$$

$$\mathbf{D}^- \leftrightarrow \mathfrak{K}\backslash A(e, f, \mathfrak{A})/\mathfrak{A} \leftrightarrow F[T]^-_{e,f}\,, \tag{3}$$

i. e. each irreducible monic polynomial from $F[T]_{e,f}$ has precisely one root in $\mathbf{D}$ and each $\mathfrak{K}$-conjugacy class of $(e, f, \mathfrak{A})$-pure simple strata has precisely one representative in $\mathbf{D}^-$. This representative is the root of the corresponding "minus polynomial" in $\mathbf{D}$. To $\beta \in \mathbf{D}^-$ a $\mathfrak{K}$-conjugacy class $\{\pi^1_\beta\}$ of irreducible representations of $U^1(\mathfrak{A}) = 1 + \mathfrak{P}$ can be assigned which is not completely unique but depends on choices. Nevertheless the dimension $\dim(\pi^1_\beta)$ is well defined, and we are going to prove:

7.1 Proposition. *$\dim\pi^1_\beta = q^{\frac{1}{2}[N^2 \cdot d_{f(T)} - N(s - s_0/e_{F(\beta)|F})]}$ where $d_{f(T)} \geq 0 \in \mathbb{Q}$ is the numerical invariant 2.2 of the minimal polynomial $f(T)$ of β over F and where $s = s(\mathfrak{A})$, $s_0 = s(\mathfrak{A} \cap A_\beta)$.*

Proof. The argument is based on the equations (4), (5) and Lemma 7.2 below, which we take from [Zi$_5$]. To begin with we note:

$$\text{(4)}\quad \dim\pi^1_\beta = \sqrt{\mathfrak{P} : \mathfrak{P}^\perp_\beta}\,, \qquad \text{where } \mathfrak{P}^\perp_\beta := \{\, x \in \mathfrak{P} : \psi_A\big((xy - yx)\beta\big) \equiv 1\, \forall y \in \mathfrak{P} \,\}\,,$$

where $\psi : F^+ \to \mathbb{C}^*$ is our fixed additive character of conductor $\mathfrak{p}_F$ and $\psi_A = \psi \circ \mathrm{Trd}_{A|F}$.

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*This has been proved recently by P. Broussous.

Because each polynomial $f(T) \in F[T]_{e,f}$ has precisely one root $\beta \in \mathbf{D}$ and $\frac{1}{rd}\nu_{\mathfrak{P}}(\beta-\gamma) = w_F\big(f(T), g(T)\big)$ if $\gamma \in \mathbf{D}$ is the root of $g(T)$, the fixed approximation procedure $(f(T), j) \mapsto f^j(T)$ on $F[T]_{\mathrm{irr}}$ induces an approximation procedure on $\mathbf{D}$ where the approximation β_j of β is the root of $f^j(T)$ in $\mathbf{D}$. By definition $\nu_{\mathfrak{P}}(\beta-\beta_j) = rd\cdot w_F\big(f(T), f^j(T)\big) \geq rdj$ and for $f(T) \in F[T]_{e,f}$ all approximations $f^j(T)$ are covered for $j \in \frac{1}{e}\mathbb{Z}$ hence $rdj \in \mathbb{Z}$ because $r = e/(e,d)$ (see above), such that $e|rd$. For $\beta \in \mathbf{D}^-$ we have $\beta_0 = \beta$. Therefore it is enough to consider the approximations $\beta_{-v} \in \mathbf{D}^-$ for $v > 0$, $v \in \frac{1}{rd}\mathbb{Z}$. To compute (4) we make use of the formula

$$\mathfrak{P}_\beta^{\perp} = \sum_{v>0\,,\,v\in\frac{1}{rd}\mathbf{Z}} (\mathfrak{P}^{vrd} \cap A_{-v})\,, \tag{5}$$

where A_{-v} is the centralizer of β_{-v} in A.

Using the sequence $\mathfrak{P} = \mathfrak{P} + \mathfrak{P}_\beta^{\perp} \supset \mathfrak{P}^2 + \mathfrak{P}_\beta^{\perp} \supset \mathfrak{P}^3 + \mathfrak{P}_\beta^{\perp} \supset \ldots$ we deduce:

$$\begin{aligned}(\dim\pi_\beta^{1})^2 = (\mathfrak{P} : \mathfrak{P}_\beta^{\perp}) &= \prod_{i\geq 1}(\mathfrak{P}^i + \mathfrak{P}_\beta^{\perp} : \mathfrak{P}^{i+1} + \mathfrak{P}_\beta^{\perp}) \\ &= \prod_{i\geq 1} \frac{(\mathfrak{P}^i : \mathfrak{P}^{i+1})}{(\mathfrak{P}^i \cap \mathfrak{P}_\beta^{\perp} : \mathfrak{P}^{i+1} \cap \mathfrak{P}_\beta^{\perp})}\,.\end{aligned}$$

Moreover from (5) we obtain

$$\mathfrak{P}^i \cap \mathfrak{P}_\beta^{\perp} = \mathfrak{P}^i \cap A_{-i/rd} + \mathfrak{P}^{i+1} \cap \mathfrak{P}_\beta^{\perp}\,,$$

hence $(\mathfrak{P}^i \cap \mathfrak{P}_\beta^{\perp} : \mathfrak{P}^{i+1} \cap \mathfrak{P}_\beta^{\perp}) = (\mathfrak{P}^i \cap A_{-i/rd} : \mathfrak{P}^{i+1} \cap A_{-i/rd})$ which implies:

$$(\dim\pi_\beta^{1})^2 = \prod_{v>0\,,\,v\in\frac{1}{rd}\mathbf{Z}} \frac{(\mathfrak{P}^{vrd} : \mathfrak{P}^{vrd+1})}{(\mathfrak{P}^{vrd} \cap A_{-v} : \mathfrak{P}^{vrd+1} \cap A_{-v})}\,. \tag{6}$$

Now we make use of the following

7.2 Lemma.

(i) *If $L|F$ is a maximal field extension in A there is a uniquely determined principal order $\mathfrak{A}_{L|F}$ in A which is normalized by L^*.*

(ii) *If $\mathfrak{A} = \mathfrak{A}_{L|F}$ and K is an intermediate field $L \supset K \supset F$ then $\mathfrak{A} \cap A_K = \mathfrak{A}_{L|K}$.*

(iii) *Let $\mathfrak{P}$, $\mathfrak{P}_{L|K}$ denote the Jacobson radical of $\mathfrak{A}$ and $\mathfrak{A}_{L|K}$ respectively. Then $\mathfrak{P}^\nu \cap A_K$ is always a power of $\mathfrak{P}_{L|K}$, namely:*

$$\begin{aligned}\mathfrak{P}_{L|K}^{\nu} &= \mathfrak{P}^{e\cdot(\nu-1)+i} \cap A_K \qquad \textit{for } i = 1,\ldots,e \\ \nu_{\mathfrak{P}_{L|K}}(x) &= e\cdot \nu_{\mathfrak{P}}(x) \qquad \textit{for } x \in A_K\end{aligned}$$

where $e = e(\mathfrak{A}|\mathfrak{A}_{L|K}) = \big(f_{K|F}, f_{L|F}/s(\mathfrak{A})\big)$.

(i) is due to A. Fröhlich (1987) and (ii), (iii) are proved in [Zi_5], 2. Note the special cases $e = 1$ if $A = M_N(F)$ because $s(\mathfrak{A}_{L|F}) = f_{L|F}$, and $e = f_{K|F}$ if $A = D_N$ because $s(\mathfrak{A}_{L|F}) = 1$. (Every principal order $\mathfrak{A}$ in $A = M_m(D_d)$ has the invariants $r = r(\mathfrak{A})$, $s = s(\mathfrak{A})$ such that $rs = m$.)

We come back to our computation of $(\mathfrak{P} : \mathfrak{P}_\beta^{\frac{1}{2}})$ for $\beta \in \mathbf{D}^- \subset A(e, f, \mathfrak{A})$. Let f_{-v}, e_{-v} be the inertial degree and the ramification exponent of $F(\beta_{-v})|F$. From 7.2(iii) we conclude $e(\mathfrak{A}|\mathfrak{A} \cap A_{-v}) = (f_{-v}, f/s)$ where $s = s(\mathfrak{A}) = (m, f)$. Moreover $\mathfrak{A}/\mathfrak{P} \cong \left[M_s(k_D)\right]^r$ implies

$$(7) \qquad (\mathfrak{A} : \mathfrak{P}) = q^{ds^2r} = q^{Ns} \qquad \text{where } q = |k_F|.$$

We want to apply (7) to compute $(\mathfrak{A} \cap A_{-v} : \mathfrak{P} \cap A_{-v})$. The algebra A_{-v} is central over $F(\beta_{-v})$. Hence we have to replace q by $q^{f_{-v}} = |k_{F(\beta_{-v})}|$. Further N has to be replaced by $N_{-v} = N/[F(\beta_{-v}) : F]$ and s by $s_{-v} := s(\mathfrak{A} \cap A_{-v}) = (m, f/f_{-v})$. Therefore

$$(8) \qquad (\mathfrak{A} \cap A_{-v} : \mathfrak{P} \cap A_{-v}) = q^{f_{-v}N_{-v}s_{-v}} = q^{\frac{N}{e_{-v}} \cdot s_{-v}}$$

Because $(\mathfrak{P}^i : \mathfrak{P}^{i+1}) = (\mathfrak{A} : \mathfrak{P})$ for all $i \in \mathbb{Z}$ and

$$(\mathfrak{P}^i \cap A_{-v} : \mathfrak{P}^{i+1} \cap A_{-v}) = \begin{cases} (\mathfrak{A} \cap A_{-v} : \mathfrak{P} \cap A_{-v}) & \text{if } e(\mathfrak{A}|\mathfrak{A} \cap A_{-v}) \text{ divides } i \\ 1 & \text{otherwise} \end{cases}$$

we conclude:

$$\frac{(\mathfrak{P}^{vrd} : \mathfrak{P}^{vrd+1})}{(\mathfrak{P}^{vrd} \cap A_{-v} : \mathfrak{P}^{vrd+1} \cap A_{-v})} = \begin{cases} q^{Ns(1-s_{-v}/e_{-v}s)} & \text{if } e(\mathfrak{A}|\mathfrak{A} \cap A_{-v})|vrd \\ q^{Ns} & \text{otherwise} \end{cases}$$

$$(\dim \pi_\beta^{\frac{1}{2}})^2 = q^\mu, \quad \text{where}$$

$$(9) \qquad \mu = \sum_{v>0, v \in \frac{1}{rd}\mathbf{Z}} Ns(1 - \delta_v s_{-v}/e_{-v}s)$$

$$\delta_v = \begin{cases} 1 & \text{if } (f_{-v}, f/s)|vrd \\ 0 & \text{otherwise.} \end{cases}$$

7.3 Lemma. *Assume $f(T) \in F[T]_{e,f} \subset F[T]_{\mathrm{irr}}$ where $ef = N = md$ and let $v = j/rd$ (where $r = r(e, f) = e/(e, d)$) be a jump for the approximation of $f(T)$. Then $f|sj$ hence $(f/s)|j$.*

(Note that $s = m/r = (m, f)$ is a divisor of f).

Proof. We know that the jumps of the approximation of $f(T)$ are in $\frac{1}{e}\mathbb{Z}$, hence in $\frac{1}{rd}\mathbb{Z}$. Therefore if $v = j/rd$ is a jump then $j = j_0 \cdot \frac{rd}{e}$, where $j_0 \in \mathbb{Z}$. Hence $sj = j_0 \frac{N}{e} = j_0 f$ is divisible by f because $srd = N$. $\square$

Instead of μ (see(9)) we begin to compute

$$(10) \qquad \mu' = \sum_{v \geq 0, v \in \frac{1}{rd}\mathbf{Z}} Ns(1 - \delta_v s_{-v}/e_{-v}s)$$

where we have added a term for $v = 0$. Then

$$\mu' = \sum_{i=0}^{\infty} S_i \qquad \text{where}$$

$$S_i = \sum_{v=i(f/s)/rd}^{[i(f/s)+(f/s)-1]/rd} Ns(1 - \delta_v s_{-v}/e_{-v}s)\,. \tag{11}$$

We have $\beta = \beta_0$ because $\beta \in \mathbf{D}^-$, i. e. the minimal polynomial $f(T)$ of β over F is a minus polynomial, $f(t) = f^0(T)$. Therefore 7.3 implies that approximating $f(T)$, all jumps have the form $-v = -j/rd$ where $j > 0$ is a multiple of f/s. Consider the index set I_i for the sum S_i. We conclude that for $v \in I_i$ always $e_{-v} = e_{-i(f/s)/rd}$, $s_{-v} = s_{-i(f/s)/rd}$ because $e_{-v} \neq e_{-v+\varepsilon}$, $f_{-v} \neq f_{-v+\varepsilon}$ only if $v = i(f/s)/rd$. I_i consists of f/s numbers and we have $\delta_v = 1$ for $(f/s)/(f_{-v}, f/s)$ of these numbers namely if $(f_{-v}, f/s)|vrd$ whereas for the other $(f/s) - (f/s)/(f_{-v}, f/s)$ numbers we have $\delta_v = 0$. Hence:

$$S_i = Ns \cdot \frac{f/s}{(f_{-v}, f/s)} \cdot \left[1 - s_{-v}/e_{-v}s\right] + Ns\left[(f/s) - \frac{f/s}{(f_{-v}, f/s)}\right]$$

where $v = i(f/s)/rd = i/e$. We obtain:

$$S_i = Ns\left[(f/s) - \frac{(f/s)\cdot s_{-v}}{(f_{-v}, f/s)e_{-v}s}\right] = N\left[f - \frac{(f/s)\cdot s_{-v}}{(f_{-v}, f/s)e_{-v}}\right]$$

Now we remark $s_{-v} = (m, f/f_{-v}) = \big((m,f), f/f_{-v}\big) = (s, f/f_{-v})$, hence $f_{-v}s_{-v} = (f_{-v}s, f) = (f_{-v}, f/s)\cdot s$. Therefore:

$$\frac{(f/s)\cdot s_{-v}}{(f_{-v}, f/s)e_{-v}} = \frac{f\cdot s^{-1}s_{-v}f_{-v}}{(f_{-v}, f/s)[F(\beta_{-v}):F]} = f/[F(\beta_{-v}):F]$$

which finally implies:

$$S_i = Nf\big(1 - 1/[F(\beta_{-v}):F]\big) \qquad \text{for } v = i/e.$$

$$\mu' = \sum_{i=0}^{\infty} S_i = Nf \cdot \sum_{v\geq 0, v\in\frac{1}{e}\mathbf{Z}} \big(1 - 1/[F(\beta_{-v}):F]\big)$$

Because of $ef = N$ we obtain

$$\begin{aligned}\mu' &= N^2\cdot\frac{1}{e}\cdot\sum_{v\geq 0, v\in\frac{1}{e}\mathbf{Z}} \big(1 - 1/[F(\beta_{-v}):F]\big)\\ &= N^2\cdot d_{f(T)}\end{aligned}$$

where $f(T)$ is the minimal polynomial of β over F and $d_{f(T)}$ is as in 2.2. Namely $[F(\beta_{-v}) : F] = \deg f^{-v}(T) = \deg_{-v}\big(f(T)\big)$ because $\deg f^{-v}(T) = \gcd\{\deg g(T);\ w_F\big(f(T), g(T)\big) \geq -v\}$. (This is a property of an approximation procedure for irreducible polynomials.) To compute μ we recall:

$\mu = \mu' - Ns(1 - s_0/e_0s) = N^2 d_{f(T)} - N(s - s_0/e_0)$ where $e_0 = e_{F(\beta)|F}$, $s_0 = (m, f/f_0) = (m, f/f_{F(\beta)|F})$ because $\beta = \beta_0$. □

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