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by

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Preprint Series 2015 (18)

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MPIM 15-18

On the existence of a torsor structure for Galois covers over a complete discrete valuation ring

Mohamed Saïdi & Nicholas Williams

Abstract

In this note we investigate the problem of existence of a torsor structure for Galois covers of (formal) schemes over a complete discrete valuation ring of residue characteristic $p>0$ in the case of abelian Galois p-groups.

§0. Introduction

In this paper R denotes a *complete discrete valuation ring*, with uniformiser π, residue field k of characteristic $p>0$, and fraction field $K:=\text{Fr}R$. For an R-(formal)scheme Z we write $Z_K:=Z\times_{\text{Spec}R}\text{Spec}K$ and $Z_k:=Z\times_{\text{Spec}R}\text{Spec}k$ for the generic and special fibre, respectively, of Z. (In the case where Z is a formal R-scheme by its generic fibre Z_K we mean the associated rigid analytic space.) Let X be a (*formal*) R-scheme of finite type which is *normal*, geometrically connected, and flat over R. We further assume that the special fibre X_k of X is *integral*. Let $f_K:Y_K\to X_K$ be an *étale torsor* under a finite étale K-group scheme $\widetilde{G}$ of rank p^t ($t\geq 1$), with Y_K *geometrically connected*, and $f:Y\to X$ the corresponding morphism of *normalisation*. (Thus, Y is the normalisation of X in Y_K.) We are interested in the following question.

Question 1. *When is $f:Y\to X$ a* torsor *under a finite and flat R-group scheme G which extends $\widetilde{G}$, i.e., with $G_K=\widetilde{G}$?*

The following is well known.

Theorem A. (Proposition 2.4 in [Saïdi]; Theorem 5.1 in [Tossici]) *If* $\text{char}(K)=0$ *we assume that R contains a primitive p-th root of* 1, *and X is* locally factorial. *Let η be the generic point of X_k and $\mathcal{O}_\eta$ the local ring of X at η, which is a discrete valuation ring with fraction field $K(X)$: the function field of X. Let $f_K:Y_K\to X_K$ be an étale torsor under a finite étale K-group scheme $\widetilde{G}$ of* **rank p**, *with Y_K connected, and let $K(X)\to L$ be the corresponding extension of function fields. Assume that the ramification index above $\mathcal{O}_\eta$ in the field extension $K(X)\to L$ equals 1. Then $f:Y\to X$ is a torsor under a finite and flat R-group scheme G of rank p which extends $\widetilde{G}$ (i.e., with $G_K=\widetilde{G}$).*

Strictly speaking the above references treat the case where $\text{char}(K)=0$. For the equal characteristic $p>0$ case see [Saïdi1], Theorem 2.2.1. Theorem A also holds when

X is the formal spectrum of a complete discrete valuation ring (cf. [Saïdi2], Proposition 2.3, and the references therein in the unequal characteristic case, as well as Proposition 2.3.1 in [Saïdi3] in the equal characteristic $p > 0$ case). It is well known that the analog of Theorem A is *false* in general. There are counterexamples to the statement in Theorem A where $\widetilde{G}$ is cyclic of rank p^2, see [Tossici], Example 6.2.12, for instance.

Next, we describe the setting in this paper. Let $n \geq 1$, and for $i \in \{1, \cdots, n\}$ let

$$f_{i,K} : X_{i,K} \to X_K$$

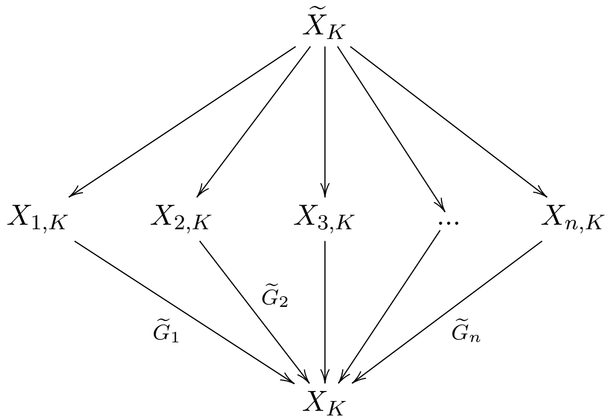
be an étale torsor under an étale finite *commutative* K-group scheme $\widetilde{G}_i$, with $X_{i,K}$ *geometrically connected*, such that the $\{f_{i,K}\}_{i=1}^n$ are *generically pairwise disjoint*. Assume that $f_{i,K} : X_{i,K} \to X_K$ extends to a torsor

$$f_i : X_i \to X$$

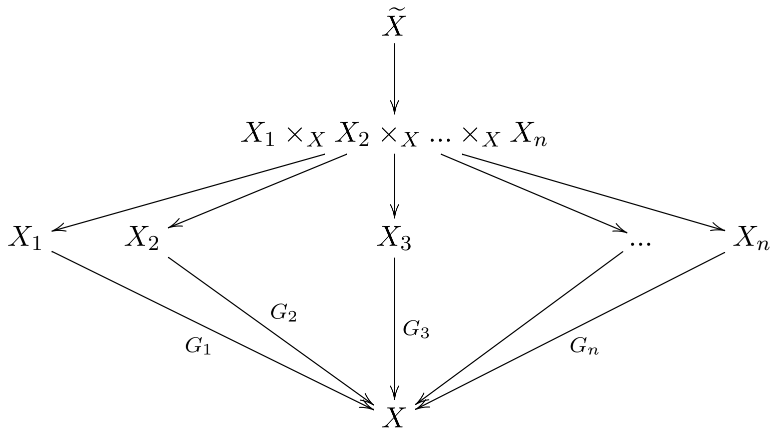
under a finite and flat (necessarily commutative) R-group scheme G_i with $(G_i)_K = \widetilde{G}_i$, and with X_i *normal*, $\forall i \in \{1, \cdots, n\}$. (Thus, X_i is the normalisation of X in $X_{i,K}$.) Let

$$\widetilde{X}_K := X_{1,K} \times_{X_K} X_{2,K} \times_{X_K} \cdots \times_{X_K} X_{n,K},$$

and $\widetilde{X}$ the *normalisation* of X in $\widetilde{X}_K$. Thus, $\widetilde{X}_K$ is the generic fibre of $\widetilde{X}$ and we have the following commutative diagrams



and



where $X_1 \times_X X_2 \times_X \cdots \times_X X_n$ denotes the fibre product of the $\{X_i\}_{i=1}^n$ over X, the morphism $\widetilde{X} \to X_1 \times_X X_2 \times_X \cdots \times_X X_n$ is birational and is induced by the natural *finite* morphisms $\widetilde{X} \to X_i$, $\forall i \in \{1, \cdots, n\}$. Note that $f_K : \widetilde{X}_K \to X_K$ (resp. $\tilde{f} : X_1 \times_X X_2 \times_X \cdots \times_X X_n \to X$) is a torsor under the étale finite commutative K-group scheme $\widetilde{G} := \widetilde{G}_1 \times_{\mathrm{Spec}K} \widetilde{G}_2 \times_{\mathrm{Spec}K} \cdots \times_{\mathrm{Spec}K} \widetilde{G}_n$ (resp. a torsor under the finite and flat commutative R-group scheme $G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$), as follows easily from the various definitions. Note that $(G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n)_K = \widetilde{G}$.

In this setup Question 1 reads as follows.

Question 2. *When is $f : \widetilde{X} \to X$ a torsor under a finite and flat (necessarily commutative) R-group scheme G which extends $\widetilde{G}$, i.e., with $G_K = \widetilde{G}$?*

Our main result in this paper is the following.

Theorem B. *We use the same notations as above. Assume that $\widetilde{X}_k$ is reduced. Then the following three statements are equivalent.*

1. *$f : \widetilde{X} \to X$ is a torsor under a finite and flat commutative R-group scheme G, in which case $G = G_1 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$ necessarily.*

2. *$\widetilde{X} = X_1 \times_X X_2 \times_X \cdots \times_X X_n$, in other words $X_1 \times_X X_2 \times_X \cdots \times_X X_n$ is normal.*

3. *$(X_1 \times_X X_2 \times_X \cdots \times_X X_n)_k$ is reduced.*

Note that the above condition in Theorem B that $\widetilde{X}_k$ is reduced is always satisfied after possibly passing to a finite extension R'/R of R (cf. [Epp]). It implies that the $(X_i)_k$ are reduced, $\forall i \in \{1, \cdots, n\}$. Moreover, Theorem A and Theorem B provide a "complete" answer to Question 1 in the case of Galois covers of type $(p, \cdots, p)$, i.e., the case where $\mathrm{rank}(G_i) = p$, $\forall i \in \{1, \cdots, n\}$.

In the case of (relative) *smooth curves* one can prove the following more precise result.

Theorem C. *We use the same notations and assumptions as in Theorem B. Assume further that X is a (relative)* ***smooth R-curve****, $n \geq 2$, and R is* ***strictly henselian****. If* $\text{char}(K) = 0$ *we assume that K contains a primitive p-th root of* 1*. Then the three (equivalent) conditions in Theorem B are equivalent to the following.*

4. ***At least n-1*** *of the finite flat R-group schemes G_i acting on $f_i : X_i \to X$ are* ***étale****, for $i \in \{1, \cdots, n\}$.*

Remarks D. *1. Theorem B holds true if X is the formal spectrum of a complete discrete valution ring (cf. the details of the proof of Theorem B in* §1 *which applies as it is in this case).*

2. In §3 *we provide examples showing that Theorem C* doesn't hold *in relative dimension* > 1.

§1. Proof of Theorem B

In this section we prove Theorem B. We start by the following.

Proposition 1.1 *Let G be a finite and flat commutative R-group scheme whose generic fibre is a product of group schemes of the form*

$$G_K = \widetilde{G}_1 \times_{\text{Spec}K} \widetilde{G}_2 \cdots \times_{\text{Spec}K} \widetilde{G}_n,$$

where the $\{\widetilde{G}_i\}_{i=1}^n$ are finite and flat commutative K-group schemes. Then G is a product of finite and flat commutative R-group schemes $\{G_i\}_{i=1}^n$, i.e.,

$$G = G_1 \times_{\text{Spec}R} G_2 \times_{\text{Spec}R} \cdots \times_{\text{Spec}R} G_n,$$

with $(G_i)_K = \widetilde{G}_i$.

Proof. First, we treat the case $n = 2$. Thus, we have $G_K = \widetilde{G}_1 \times_{\text{Spec}K} \widetilde{G}_2$ and need to show $G = G_1 \times_{\text{Spec}R} G_2$ where $(G_i)_K = \widetilde{G}_i$, for $i = 1, 2$. Let G_i be the *schematic closure* of $\widetilde{G}_i$ in G, for $i = 1, 2$ (cf. [Raynaud], 2.1). Therefore, G_1 and G_2 are closed subgroup schemes of G which are finite and flat over Spec$R$ (cf. loc. cit.). We have a short exact sequence:

$$1 \to G_1 \to G \to G/G_1 \to 1,$$

and likewise

$$1 \to G_2 \to G \to G/G_2 \to 1,$$

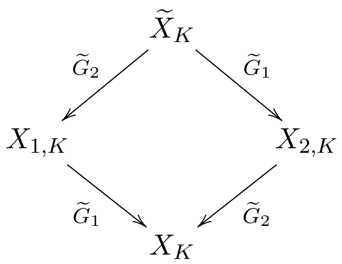
of finite and flat commutative R-group schemes (cf. loc. cit.). It remains for the proof to show that the composite homomorphism $G_2 \to G \to G/G_1$ is an isomorphism. The morphism $G \to G/G_1$ is finite. The morphism $G_2 \to G$ is a closed immersion, hence finite. The composite $G_2 \to G/G_1$ of the above morphisms is then finite. We will show it is an isomorphism. The morphism $G_2 \to G/G_1$ is a closed immersion since its kernel is trivial. Indeed, on the generic fibre the kernel is trivial: $(G_1 \cap G_2)_K = \widetilde{G_1} \cap \widetilde{G_2} = \{1\}$. The map $G_2 \to G/G_1$ is then an isomorphism as both group schemes have the same rank.

Similarly, the morphism $G_1 \to G/G_2$ is an isomorphism. Therefore, $G = G_1 \times_{\mathrm{Spec} R} G_2$ as required. Now an easy devissage argument along the above lines of thought, using induction on n, reduces immediately to the above case $n = 2$. $\square$

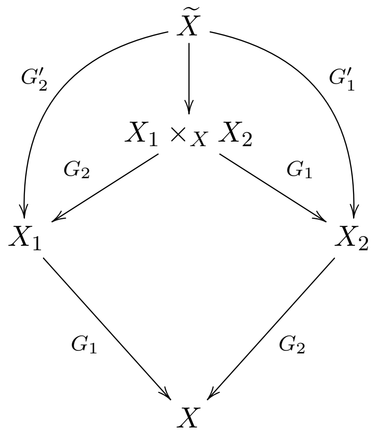
Proof of Theorem B

Proof. ($1 \Rightarrow 2$) Assume that $f : \widetilde{X} \to X$ is a torsor under a finite and flat R-group scheme G. In particular, $G_K = \widetilde{G}$ and G is necessarily commutative. We will show that $\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$, i.e., show that $X_1 \times_X X_2 \times_X ... \times_X X_n$ is normal (this will imply that $G = G_1 \times_{\mathrm{Spec} R} \cdots \times_{\mathrm{Spec} R} G_n$ necessarily, as $G_1 \times_{\mathrm{Spec} R} ... \times_{\mathrm{Spec} R} G_n$ is the group scheme of the torsor $\tilde{f} : X_1 \times_X X_2 \times_X \cdots \times_X X_n \to X$). One reduces easily by a devissage argument to the case $n = 2$ which we will treat below.

Assume $n = 2$. We have the following commutative diagrams of torsors



and



where $\widetilde{X} \to X_i$ is a torsor under a finite and flat R-group scheme G'_i, for $i = 1, 2$. Moreover, $G'_1 = \left(\widetilde{G}_1\right)^{\text{schematic closure}}$, and $G'_2 = \left(\widetilde{G}_2\right)^{\text{schematic closure}}$ (where the schematic closure is taken inside G) holds necessarily, so that $G = G'_1 \times_{\mathrm{Spec} R} G'_2$ (cf. Proposition 1.1). Note that $\widetilde{X}/G'_1 = X_2$ must hold as the quotient $\widetilde{X}/G'_1$ is normal: since $\left(\widetilde{X}/G'_1\right)_k$ is reduced (as $\widetilde{X}_k$ is reduced and $\widetilde{X}$ dominates $\widetilde{X}/G'_1$), and $\left(\widetilde{X}/G'_1\right)_K = X_{2,K}$ is normal (cf. [Liu], 4.1.18)). Similarly $\widetilde{X}/G'_2 = X_1$ holds. We want to show that $\widetilde{X} = X_1 \times_X X_2$, and we claim that this reduces to showing that the natural morphism $G \to G_1 \times_{\mathrm{Spec}R} G_2$ (cf. the map ϕ below) is an isomorphism. Indeed, if one has two torsors, in this case $\widetilde{X} \to X$ and $X_1 \times_X X_2 \to X$ above the same X, under isomorphic group schemes, which are isomorphic on the generic fibres, and if we have a morphism $\widetilde{X} \to X_1 \times_X X_2$ which is compatible with the torsor structure and the given identification of group schemes (cf. above diagrams), then this morphism must be an isomorphism. (This is a consequence of Lemma 4.1.2 in [Tossici]. In [Tossici] $\mathrm{char}(K) = 0$ is assumed, the same proof however applies if $\mathrm{char}(K) = p$.) We have two short exact sequences of finite and flat commutative R-group schemes (cf. above diagrams and discussion for the equalities $G_1 = G/G_2'$ and $G_2 = G/G_1'$)

$$1 \to G_2' \to G \to G_1 = G/G_2' \to 1,$$

and

$$1 \to G_1' \to G \to G_2 = G/G_1' \to 1.$$

The morphisms $G \to G_1$, and $G \to G_2$, are finite. Consider the following exact sequence

$$1 \to \mathrm{Ker}(\phi) \to G \to G_1 \times_{\mathrm{Spec}R} G_2,$$

where $\phi : G \to G_1 \times_{\mathrm{Spec}R} G_2$ is the morphism induced by the above morphisms. We want to show that the map $\phi : G \to G_1 \times_{\mathrm{Spec}R} G_2$ is an isomorphism. We have $\mathrm{Ker}(\phi) = G_1' \cap G_2'$ by construction. However, $G_1' \cap G_2' = \{1\}$ since $G = G_1' \times_{\mathrm{Spec}R} G_2'$ by Proposition 1.1, and therefore $\mathrm{Ker}(\phi) = \{1\}$ which means $\phi : G \to G_1 \times_{\mathrm{Spec}R} G_2$ is a closed immersion. Finally, G and $G_1 \times_{\mathrm{Spec}R} G_2$ have the same rank as group schemes which implies ϕ is an isomorphism, as required.

($2 \Rightarrow 3$) Clear.

($3 \Rightarrow 1$) By assumption $(X_1 \times_X X_2 \times_X \dots \times_X X_n)_k$ is reduced. Moreover, we have $(X_1 \times_X X_2 \times_X \dots \times_X X_n)_K = \widetilde{X}_K$ is normal. Hence $X_1 \times_X X_2 \times_X \dots \times_X X_n$ is normal (cf. [Liu], 4.1.18), and $\widetilde{X} = X_1 \times_X X_2 \times_X \dots \times_X X_n$. We know that $\tilde{f} : X_1 \times_X X_2 \times_X \dots \times_X X_n \to X$ is a torsor under the group scheme $G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \dots \times_{\mathrm{Spec}R} G_n$, so $f : \widetilde{X} \to X$ is a torsor under the group scheme $G = G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \dots \times_{\mathrm{Spec}R} G_n$. □

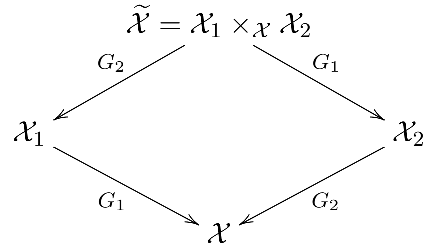
§2. Proof of Theorem C

In this section we prove Theorem C.

Proof. ($1 \Rightarrow 4$) Suppose that $\tilde{f} : \widetilde{X} \to X$ is a torsor under a finite and flat R-group scheme G; in which case $\widetilde{X} = X_1 \times_X X_2 \times_X \dots \times_X X_n$ and $G = G_1 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$ (cf. Theorem B). We will show that *at least* $n-1$ of the finite flat R-group schemes G_i (acting on $f_i : X_i \to X$) are étale, for $i \in \{1, \cdots, n\}$. We argue by induction on the rank of G.

Base case: The base case pertains to $\mathrm{rank}(G) = p^2$ and $n = 2$. Thus, $\mathrm{rank}(G_1) = \mathrm{rank}(G_2) = p$. We assume $\widetilde{X} = X_1 \times_X X_2$ and prove that at least one of the two group schemes G_1 or G_2 is étale. We assume that X is a scheme, and not a formal scheme, in which case the argument of proof is the same.

Let x be a *closed* point of X and $\mathcal{X}$ the *boundary of the formal germ* of X at x, so $\mathcal{X}$ is isomorphic to Spec $\big(R[[T]]\{T^{-1}\}\big)$ (cf. [Saïdi2], §1). We have a natural morphism $\mathcal{X} \to X$ of schemes. Write $\mathcal{X}_1 := \mathcal{X} \times_X X_1$, $\mathcal{X}_2 := \mathcal{X} \times_X X_2$, and $\widetilde{\mathcal{X}} := \mathcal{X} \times_X \widetilde{X}$. Thus, by base change, $\widetilde{\mathcal{X}} \to \mathcal{X}$ (resp. $\mathcal{X}_1 \to \mathcal{X}$, and $\mathcal{X}_2 \to \mathcal{X}$) is a torsor under the group scheme G (resp. under G_1, and G_2) and we have the following commutative diagram



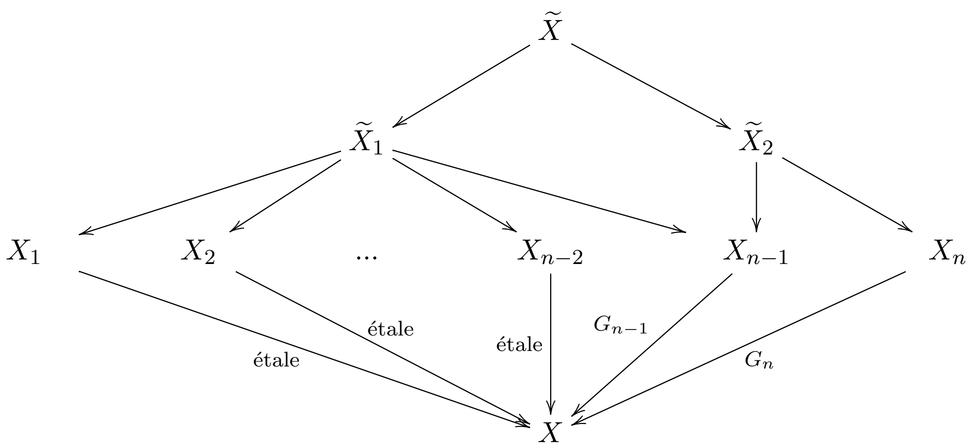
Note that $\widetilde{\mathcal{X}}$ is normal as $(\widetilde{\mathcal{X}})_k$ is reduced (recall $(\widetilde{X})_k$ is reduced) and $(\widetilde{\mathcal{X}})_K$ is normal (cf. [Liu], 4.1.18), hence $\widetilde{\mathcal{X}} = \mathcal{X}_2 \times_{\mathcal{X}} \mathcal{X}_2$ holds (cf. Theorem B and Remarks D, 1).

Assume now that G_1 and G_2 are both *non-étale* R-group schemes. Then we prove that $\widetilde{\mathcal{X}} \to \mathcal{X}$ can not have the structure of a torsor under a finite and flat R-group scheme which would then be a contradiction. More precisely, we will prove that $\mathcal{X}_2 \times_{\mathcal{X}} \mathcal{X}_2$ can not be normal in this case, hence the above conclusion (cf. Theorem B).

We will assume for simplicity that $\text{char}(K) = 0$ and K contains a primitive p-th root of 1. A similar argument used below holds in equal characteristic $p > 0$. First, $\widetilde{\mathcal{X}}$ is connected as $\widetilde{X}_k$ is *unibranche* (the finite morphism $\widetilde{X}_k \to X_k$ is radicial). As the group schemes G_1 and G_2 are non étale, their special fibres $(G_1)_k$ and $(G_2)_k$ are radicial isomorphic to either μ_p or α_p. We treat the case $(G_1)_k$ is isomorphic to $\mu_p := \mu_{p,k}$ and $(G_2)_k$ is isomorphic to $\alpha_p := \alpha_{p,k}$, the remaining cases are treated similarly. (Recall $\mathcal{X}$ is ismorphic to Spec $\big(R[[T]]\{T^{-1}\}\big)$.) For a suitable choice of the parameter T the torsor $\mathcal{X}_2 \to \mathcal{X}$ is given by an equation $Z_2^p = 1 + \pi^{np}T^m$ where n is a positive integer (satisfying a certain condition) and $m \in \mathbb{Z}$ (cf. [Saïdi2], Proposition 2.3 (b). Strictly speaking in loc. cit. this is shown to hold after a finite extension of R, however a close inspection of the proof in loc. cit. reveals that this finite extension can be chosen to be étale. Also see Proposition 2.3.1 in [Saïdi3] for the equal characteristic case), and the torsor $\mathcal{X}_1 \to \mathcal{X}$ is given by an equation $Z_1^p = f(T)$ where $f(T) \in R[[T]]\{T^{-1}\}$ is a unit whose reduction $\overline{f(T)}$ modulo π is not a p-power (cf. loc. cit.). We claim that $\widetilde{\mathcal{X}} = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$ can not hold. Indeed, by base change $\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2 \to \mathcal{X}_2$ is a G_1-torsor which is generically given by an equation $Z^p = f(T)$, where $f(T)$ is viewed as a function on $\mathcal{X}_2$. But in $\mathcal{X}_2$ the function T becomes a p-power modulo π as one easily deduces from the equation $Z_2^p = 1 + \pi^{np}T^m$ defining the torsor $\mathcal{X}_2 \to \mathcal{X}$. In particular, the reduction $\overline{f(T)}$ modulo π of $f(T)$, viewed as a function on $(\mathcal{X}_2)_k$, is a p-power. This means that $(\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2)_k$ is not reduced and $\widetilde{\mathcal{X}} \to \mathcal{X}_2$ can not be a $G_1 \simeq \mu_{p,R}$-torsor (cf. the proof of Proposition 2.3 in [Saïdi2]), and a fortiori $\widetilde{\mathcal{X}} \neq \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$.

Inductive hypothesis: Given G, we assume that the $(1 \Rightarrow 4)$ part in Theorem C holds true for $n \geq 2$ and cases where $\text{rank}(G_1) + \cdots + \text{rank}(G_n) < \text{rank}(G)$. Write $\widetilde{X}_1 := X_1 \times_X X_2 \times_X ... \times_X X_{n-1}$. Then $\widetilde{X}_1$ is normal (since its special fibre is reduced (as it is dominated by $\widetilde{X}$ whose special fibre is reduced) and its generic fibre is normal (cf. [Liu], 4.1.18)), hence at least $n-2$ of the corresponding G_i's, for $i \in \{1, \cdots, n-1\}$, are étale by the induction hypothesis. We will assume, without loss of generality, that G_i is étale for $1 \leq i \leq n-2$.

Inductive step: We have the following picture for our inductive step (the case for n):



We argue by contradiction. Suppose that neither G_{n-1} nor G_n is étale. This would mean that $\widetilde{X}_2 \to X$, where $\widetilde{X}_2$ is the normalisation of X in $(X_{n-1})_K \times_{X_K} (X_n)_K$, does not have the structure of a torsor (as this would contradict the induction hypothesis). This implies that $\widetilde{X} \to X$ does not have the structure of a torsor since it factorises $\widetilde{X} \to \widetilde{X}_2 \to X$, for otherwise $\widetilde{X}_2 \to X$ being a quotient of $\widetilde{X} \to X$ would be a torsor. Of course, $\widetilde{X} \to X$ is a torsor to start with by assumption and so this is a contradiction. Therefore, at least one of G_{n-1} and G_n is étale, as required.

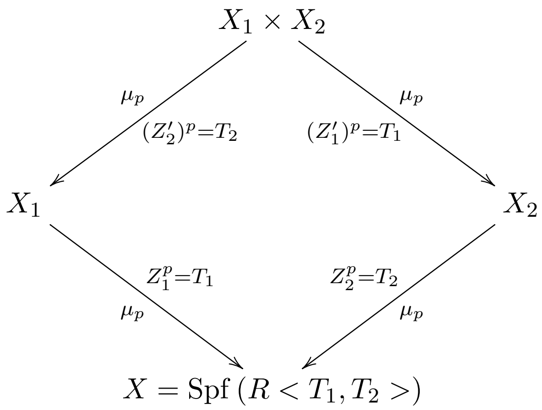
$(1 \Leftarrow 4)$ Suppose that at least $n-1$ of the G_i are étale, say: $G_1, G_2, \cdots, G_{n-1}$ are étale. Write $\widetilde{X}_1 := X_1 \times_X X_2 \times_X ... \times_X X_{n-1}$. Then $\widetilde{X}_1 \to X$ is a torsor under the finite étale R-group scheme $G'_1 := G_1 \times_{\text{Spec} R} G_2 \times_{\text{Spec} R} \cdots \times_{\text{Spec} R} G_{n-1}$. Moreover, $X_1 \times_X X_2 \times_X ... \times_X X_n = \widetilde{X}_1 \times_X X_n$, and $X_1 \times_X X_2 \times_X ... \times_X X_n \to X_n$ is an étale torsor under the group scheme G'_1 (by base change). In particular, $(X_1 \times_X X_2 \times_X ... \times_X X_n)_k$ is reduced as $(X_n)_k$ is reduced. (Indeed, $\widetilde{X}$ dominates X_n and $\widetilde{X}_k$ is reduced.) Hence $\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$ (cf. Theorem B) and $\widetilde{X} \to X$ is a torsor under the group scheme $G := G_1 \times_{\text{Spec} R} G_2 \times_{\text{Spec} R} \cdots \times_{\text{Spec} R} G_n$. □

§3. Counterexample to Theorem C in higher dimensions

Theorem C is not valid (under similar assumptions) for (formal) R-schemes of relative dimension ≥ 2. Here is a counterexample. Assume $\text{char}(K) = 0$ and K contains a primitive p-th root of 1. Let $X = \text{Spf}(A)$ where $A := R < T_1, T_2 >$ is the free R-Tate algebra in the two variables T_1 and T_2. Let $G_1 = G_2 = \mu_p := \mu_{p,R}$, neither being an étale R-group schemes. For $i = 1, 2$, consider the G_i-torsor $X_i \to X$ which is generically defined by the equation

$$Z_i^p = T_i.$$

We have the following commutative diagram



The torsor $X_1 \times_X X_2 \to X_2$ is a $G_1 = \mu_p$-torsor defined generically by the equation

$$(Z_1')^p = T_1$$

where T_1 is viewed as a function on X_2. This function is not a p-power modulo π as follows easily from the fact that the torsor $X_2 \to X$ is defined generically by the equation $Z_2^p = T_2$. In particular, $X_1 \times_X X_2 \to X_2$ is a non trivial μ_p-torsor, and $(X_1 \times_X X_2)_k \to (X_2)_k$ is a non trivial $\mu_{p,k}$-torsor. Hence $(X_1 \times_X X_2)_k$ is necessarily reduced (as $(X_2)_k$ is reduced since $(X_2)_k \to (X_1)_k$ is a non trivial $\mu_{p,k}$-torsor). Thus, $X_1 \times_X X_2$ is normal (cf. Theorem B) and $X_1 \times_X X_2 = \widetilde{X}$, where $\widetilde{X}$ is the normalisation of X in $(X_1 \times_X X_2)_K$, which contradicts the statement of Theorem C in this case.

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