Exact triangles in monopole homology and the Casson-Walker invariant

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Contents

1 Introduction

In the series of papers [2] [5] [6] [7], we studied how the Seiberg-Witten-Floer homology for an integral homology 3-sphere behaves under surgery: there exists a surgery triangle for the Seiberg-Witten-Floer theory. If K is a knot in a homology 3-sphere Y, the Seiberg-Witten-Floer homologies of Y and of the manifolds Y_1 and Y_0 obtained by Dehn surgery on K with framing 1 and 0, respectively, are related by an exact triangle

$$\begin{tikzcd}
HF_*^{SW}(Y_1, g_1) \arrow[r, "w_*^1"] & HF_*^{SW}(Y, g_0, \mu) \arrow[dl, "w_*^0"] \\
\bigoplus_{k\in\mathbb{Z}} HF_{(*)}^{SW}(Y_0, \mathfrak{s}_k) \arrow[u, "w_{(*)}"'] &
\end{tikzcd}$$

In this triangle the maps w_*^1, w_*^0 and $w_{(*)}$ are induced by the surgery cobordisms connecting Y_1 and Y, Y and Y_0, Y_0 and Y_1, respectively, and μ is the surgery perturbation that simulates the effect of Dehn surgery. In this paper, we extend this exact triangle to any closed oriented 3-manifold with a smooth embedded knot.

Before we state the main theorem of this paper, we introduce some notation. Let Y be a closed oriented 3-manifold, with a smooth embedded knot K, let $\nu(K)$ be the tubular neighbourhood of K in Y. Choose an identification of $\nu(K)$ with $D^2 \times S^1$:

$$\nu(K) \cong D^2 \times S^1, \tag{1}$$

where K is mapped to the core of the solid torus $D^2 \times S^1$. Under the identification (1), on the boundary T^2, we fix a basis m, l of $H_1(T^2, \mathbb{Z})$ such that l is the longitude (parallel to K under the identification (1)) and m is the right-handed meridian (intersecting l once), the orientation determined by $m \wedge l$ coincides with the orientation induced from Y. The corresponding longitude and meridian in the knot complement

$$V = Y \backslash \nu(K)$$

are denoted by m', l' respectively. Similarly, let m'' and l'' be the meridian and longitude in the tubular neighbourhood of the knot $\nu(K)$. The meridian m'' bounds a disk D^2 in $\nu(K)$, and l'' generates $H_1(\nu(K), \mathbb{Z})$ and parallels to K. Let p and q be two relatively prime integers, the Dehn surgery with coefficient $p/q \in \mathbb{Q} \cup \{\infty\}$ on K is the operation of removing $\nu(K)$ and gluing in $D^2 \times S^1$ by an orientation reversing diffeomorphism $f_{p/q}$ of T^2 that satisfies

$$f_{p/q}(m'') = pm' - ql'.$$

The resulting manifold is denoted by $Y_{p/q}$. Note that in general $Y_{p/q}$ depends on the choice of the identification (1) if K is not a null-homologous knot.

Let $\mathfrak{s}$ be a Spin^c structure on Y. We shall see that, with a suitable choice of metrics and perturbation, $(Y, \mathfrak{s})$ has non-empty monopole moduli space only if $\mathfrak{s}|_{\nu(K)}$ has trivial determinant, hence we shall always assume that the Spin^c structure $\mathfrak{s}$ is trivial around K. If K represents a trivial homology class in $H_1(Y, \mathbb{Z})$, then there is only one Spin^c structure on Y which agrees with $\mathfrak{s}$ over $Y - \nu(K)$ and $\nu(K)$. Suppose that K represents a torsion element of order n in $H_1(Y, \mathbb{Z})$, which means,

$$\frac{|\text{Torsion}(H_1(Y, \mathbb{Z}))|}{|\text{Torsion}(H_1(Y - \nu(K), \mathbb{Z}))|} = n. \tag{2}$$

In other words, n is the minimal positive integer such that $n[K]$ is homologous to zero in $H_1(Y, \mathbb{Z})$. Then there exists a $\mathbb{Z}_n$-family of Spin^c structures

$$\mathfrak{s} \otimes L_k \qquad (k = 1, \cdots, n)$$

which agree with $\mathfrak{s}$ over $Y - \nu(K)$ and $\nu(K)$, where L_k is a complex line bundle whose Euler class is given by $kPD([K])$. If $[K] \neq 0$ in $H_1(Y, \mathbb{Q})$, then there exists a $\mathbb{Z}$-family of Spin^c structures

$$\mathfrak{s} \otimes L_k \qquad (k \in \mathbb{Z})$$

which agree with $\mathfrak{s}$ over $Y - \nu(K)$ and $\nu(K)$, where L_k is a complex line bundle whose Euler class is given by $kPD([K])$.

Let Y_1 be the $(+1)$-surgery on K, obtained via an orientation reversing diffeomorphism f_1 of T^2 that satisfies

$$(f_1)_*(m'' = [pt \times \partial D^2]) = m' - l' \qquad (f_1)_*(l'') = l'.$$

Let Y_0 be the 0-surgery on K, obtained via an orientation reversing diffeomorphism f_0 of T^2 that satisfies

$$(f_0)_*(m'') = l' \qquad (f_0)_*(l'') = m'.$$

We shall consider separately the following cases.

• First we assume that Y is a rational homology 3-sphere with a smoothly embedded knot K representing a torsion element of order n in $H_1(Y, \mathbb{Z})$ in the sense of (2). Then Y_1 is a rational homology 3-sphere, and Y_0 is a rational homology $S^1 \times S^2$. Let $\mathfrak{s}$ be a Spin^c structure on Y, which is trivial on $\nu(K)$. Gluing the Spin^c structures $s|_{Y-\nu(K)}$ and $\mathfrak{s}|_{\nu(K)}$ along T^2 via different gauge transformations on T^2 results in a $\mathbb{Z}_n$-family of Spin^c structures on Y and Y_1, and a $\mathbb{Z}$-family of Spin^c structures on Y_0. Without confusion, thinking K as the core of the attaching solid torus $\nu(K)$, we denote these structures on Y, Y_1 and Y_0 all by

$$\mathfrak{s} \otimes L_k \qquad (k \in Z),$$

where L_k is a complex line bundle of Euler class $kPD([K])$. With this notation, it is understood that for Y and Y_1, and with $n[K]=0$, there is only a $\mathbb{Z}_n$-family of Spin^c structures. Then, with a careful choice of metrics and perturbations on Y, Y_1 and Y_0 as in Part I [2], we will obtain the following exact triangles for the Seiberg-Witten-Floer homologies:

$$\begin{tikzcd} HF_*^{SW}(Y_1, \mathfrak{s}\otimes L_m, g_1) \arrow[r, "w_*^1"] & HF_*^{SW}(Y, \mathfrak{s}\otimes L_m, g, \mu) \arrow[dl, "w_*^0"'] \\ \bigoplus_{k\in\mathbb{Z}} HF_{(*)}^{SW}(Y_0, \mathfrak{s}\otimes L_{nk+p}) \arrow[u, "w_*"] & \end{tikzcd}$$

which holds for any fixed choice of m and p in $\{0, \dots, n-1\}$, and for a corresponding choice of perturbation, and

$$\begin{tikzcd} \bigoplus_{k=1}^n HF_*^{SW}(Y_1, \mathfrak{s}\otimes L_k, g_1) \arrow[r, "w_*^1"] & \bigoplus_{k=1}^n HF_*^{SW}(Y, \mathfrak{s}\otimes L_k, g, \mu) \arrow[dl, "w_*^0"'] \\ \bigoplus_{k\in\mathbb{Z}} HF_{(*)}^{SW}(Y_0, \mathfrak{s}\otimes L_k) \arrow[u, "w_*"] & \end{tikzcd}$$

In both cases the homomorphisms w_*^1, w_*^0 and w_* are induced by the surgery cobordisms. These exact triangles generalize the results of Part I-IV [2] [5][6] [7], in the sense that the above exact triangle reduces to the exact triangle for an integral homology 3-sphere when $n=1$.

• Now we assume that $(Y, \mathfrak{s})$ is a closed oriented 3-manifold of $b_1(Y)>0$ and with a Spin^c structure $\mathfrak{s}$. Let K be a knot representing a torsion homology class in $H_1(Y,\mathbb{Z})$ of order n in the sense of (2). If $\mathfrak{s}$ has non-trivial determinant in the sense that $c_1(\mathfrak{s})\neq 0$ in $H^2(Y,\mathbb{Q})$, then the Seiberg-Witten-Floer homology for $(Y,\mathfrak{s})$ is $\mathbb{Z}_{2\ell}$-graded, where 2ℓ is the multiplicity of $c_1(\mathfrak{s})$ in $H^2(Y,\mathbb{Z})/\text{Torsion}$, i.e, $c_1(\mathfrak{s})(H_2(Y,\mathbb{Z})/\text{Torsion}) = 2\ell$. The gluing of the Spin^c structures corresponding to these three different surgeries gives rise to a $\mathbb{Z}_n$-family of Spin^c structures on Y, Y_1 and a $\mathbb{Z}$-family of Spin^c structures on Y_0, which we denote by

$$\mathfrak{s}\otimes L_k \qquad (k\in\mathbb{Z}),$$

with the convention as before. Then, for any Spin^c structure $\mathfrak{s}\otimes L_k$ $(k=1,\cdots,n)$ on Y and Y_1, the corresponding Seiberg-Witten-Floer homologies are all $\mathbb{Z}_{2\ell}$-graded, while for any Spin^c structure $\mathfrak{s}\otimes L_k$ $(k\in\mathbb{Z})$ on Y_0, the Seiberg-Witten-Floer homology $HF_*^{SW}(Y_0,\mathfrak{s}\otimes L_k)$ is $\mathbb{Z}_{\ell_{[k]}}$-graded, where $\ell_{[k]}$ is the greatest common factor in 2ℓ and $2k$. Similar to Part IV [7], the $\mathbb{Z}_{\ell_{[k]}}$-graded homology $HF_*^{SW}(Y_0,\mathfrak{s}\otimes L_k)$ can be lifted to a $\mathbb{Z}_\ell$-graded homology $HF^{SW}_{(*)}(Y_0, \mathfrak{s}\otimes L_k)$. For this case, we derive in this paper the following exact triangles:

$$\begin{tikzcd} HF^{SW}_*(Y_1, \mathfrak{s}\otimes L_m, g_1) \arrow[r, "w^1_*"] & HF^{SW}_*(Y, \mathfrak{s}\otimes L_m, g, \mu) \arrow[dl, "w^0_*"] \\ \bigoplus_{k\in\mathbb{Z}} HF^{SW}_{(*)}(Y_0, \mathfrak{s}\otimes L_{nk+p}). \arrow[u, "w_{(*)}"] & \end{tikzcd} \tag{3}$$

which holds for any choice of m and p in $\{0, \ldots, n-1\}$, and for a corresponding choice of perturbation, and

$$\begin{tikzcd} \bigoplus_{k=1}^n HF^{SW}_*(Y_1, \mathfrak{s}\otimes L_k, g_1) \arrow[r, "w^1_*"] & \bigoplus_{k=1}^n HF^{SW}_*(Y, \mathfrak{s}\otimes L_k, g, \mu) \arrow[dl, "w^0_*"] \\ \bigoplus_{k\in\mathbb{Z}} HF^{SW}_{(*)}(Y_0, \mathfrak{s}\otimes L_k). \arrow[u, "w_{(*)}"] & \end{tikzcd} \tag{4}$$

• If $\mathfrak{s}$ is a torsion Spin^c structure on Y, then for any $k = 1, \cdots, n$, $HF^{SW}_*(Y, \mathfrak{s}\otimes L_k, \mathbb{Z}[[t]])$ and $HF^{SW}_*(Y_1, \mathfrak{s}\otimes L_k, \mathbb{Z}[[t]])$ are $\mathbb{Z}$-graded with $\mathbb{Z}[[t]]$-coefficient. The reduced versions $HF^{SW}_*(Y, \mathfrak{s}\otimes L_k)$ and $HF^{SW}_*(Y_1, \mathfrak{s}\otimes L_k)$ are obtained by setting $t = 0$. The Seiberg-Witten-Floer homology $HF^{SW}_*(Y_0, \mathfrak{s}\otimes L_k)$ is $\mathbb{Z}_{2k}$-graded, and can be lifted to a $\mathbb{Z}$-graded version, denoted by $HF^{SW}_{(*)}(Y_0, \mathfrak{s}\otimes L_k)$. Then the exact triangle takes the same form as (4).

• The remaining case is when a smoothly embedded knot K represents a non-trivial homology class in $H_1(Y, \mathbb{Q})$. Let n be the minimal positive integer such that there is a 2-cycle in $H_2(Y, \mathbb{Z})$ intersecting K at n points. Then the 0-surgery on Y along K yields Y_0 satisfying $b_1(Y_0) = b_1(Y) - 1$. More precisely, $H_1(Y_0, \mathbb{Z})$ is obtained by replacing the $\mathbb{Z}$-component $\mathbb{Z}\langle [K]\rangle$ of $H_1(Y, \mathbb{Z})$ by $\mathbb{Z}_n\langle [m''] = [l']\rangle$. Notice that Y can be thought of as the manifold obtained by 0-surgery on $m' \subset Y - \nu(K) \subset Y_0$, and Y_1 can be thought as the result of $(+1)$-surgery on $m' \subset Y - \nu(K) \subset Y_0$. Since m' represents a torsion element of order n in $H_1(Y_0, \mathbb{Z})$ in the sense of (2), we have the exact triangle for $(Y, \mathfrak{s}, K)$ obtained from the corresponding exact triangle for $(Y_0, \mathfrak{s}, m')$ in the form of (4). Thus, it is enough to establish the exact triangle for a general closed oriented 3-manifold Y with a smoothly embedded knot representing a torsion element of order n in $H_1(Y, \mathbb{Z})$.

We now summarize the main theorem of this paper.

Theorem 1.1. *Let $(Y, \mathfrak{s})$ be a closed oriented 3-manifold with a* Spin^c *structure which is trivial around a smoothly embedded knot K. Assume that K*

represents a torsion element of order n in $H_1(Y,\mathbb{Z})$ in the sense of (2). Assume that the canonical framing of K is given by an identification of $D^2\times S^1$ with the tubular neighbourhood $\nu(K)$ such that K is given by $\{0\}\times S^1$. Here the parallel simple curve on T^2 provides the longitude of K and the right handed meridian is given by $\partial(D^2)\times\{pt\}$. The orientation determined by $m\wedge l$ coincides with the orientation induced from Y. Let Y_1 and Y_0 be the manifolds obtained, respectively, by $(+1)$ and 0 surgery along K in Y. With a careful choice of metrics and perturbations, we obtain the following exact triangle induced by the surgery cobordisms after possible grading shifts:

$$\begin{tikzcd}
\bigoplus_{k=1}^n HF_*^{SW}(Y_1,\mathfrak{s}\otimes L_k) \arrow[r, "w_*^1"] & \bigoplus_{k=1}^n HF_*^{SW}(Y,\mathfrak{s}\otimes L_k) \arrow[dl, "w_*^0"'] \\
\bigoplus_{k\in\mathbb{Z}} HF_{(*)}^{SW}(Y_0,\mathfrak{s}\otimes L_k) \arrow[u, "w_{(*)}"'] &
\end{tikzcd}$$

Here $\mathfrak{s}\otimes L_k$ is the Spin^c *structure obtained by tensoring a* Spin^c *structure $\mathfrak{s}$ with a complex line bundle L_k of Euler class $kPD([K])$. Moreover, for any fixed $m,p\in\{0,\cdots,n-1\}$, we have the following more refined version of the exact triangle:*

$$\begin{tikzcd}
HF_*^{SW}(Y_1,\mathfrak{s}\otimes L_m) \arrow[r, "w_*^1"] & HF_*^{SW}(Y,\mathfrak{s}\otimes L_m) \arrow[dl, "w_*^0"'] \\
\bigoplus_{k\in\mathbb{Z}} HF_{(*)}^{SW}(Y_0,\mathfrak{s}\otimes L_{nk+p}) \arrow[u, "w_{(*)}"'] &
\end{tikzcd}$$

Here again the maps are induced by the surgery cobordisms, possibly after a shift in the grading.

The major technical analysis required in the proof has already been developed in [2][5][6][7]. Thus, in this paper, we shall address only those issues that are relevant to this general exact triangle, while we refer to the previous papers for the general setting and the basic results.

In the last section of this paper, we will show that a modified version of the Seiberg-Witten invariant agrees with the Casson-Walker invariants, for any closed and oriented rational homology 3-sphere. For any rational homology 3-sphere $(Y,\mathfrak{s},g)$ with a Spin^c structure $\mathfrak{s}$ and a Riemannian metric, the counting of the irreducible Seiberg-Witten monopoles defines the Seiberg-Witten invariant

$$SW_Y(\mathfrak{s},g)=\#\big(\mathcal{M}_Y^*(\mathfrak{s},g)\big),$$

where each irreducible monopole in $\mathcal{M}_Y^*(\mathfrak{s},g)$ has a natural orientation from the linearization of the Seiberg-Witten equations. As studied in [4],

$SW_Y(\mathfrak{s}, g)$ depends on the metric and perturbation used in the definition, in order to obtain a topological invariant, we can modify $SW_Y(\mathfrak{s}, g)$ by a metric and perturbation dependent correction term as follows. Choose any four manifold X with boundary Y, such that X is endowed with a cylindrical-end metric modeled on (Y, g_Y). Choose a Spin^c structure $\mathfrak{s}_X$ on X which agrees with $\mathfrak{s}$ on Y over the end, and choose a connection A on $(X, \mathfrak{s}_X)$ which extends the unique reducible on $(Y, \mathfrak{s})$. Then we set

$$\xi_Y(\mathfrak{s}, g) = Ind_{\mathbb{C}}(\not{D}_A^X) - \frac{1}{8}\big(c_1(\mathfrak{s}_X^2) - \sigma(X)\big), \tag{5}$$

where $Ind_{\mathbb{C}}(\not{D}_A^X)$ is the complex index of the Dirac operator on $(X, \mathfrak{s}_X)$ twisted with the extending Spin^c connection A and $\sigma(X)$ is the signature of X. By the Atiyah-Patodi-Singer index theorem, $\xi_Y(\mathfrak{s}, g)$ is independent of the choice of $(X, \mathfrak{s}_X)$ and A. Moreover, $\xi_Y(\mathfrak{s}, g)$ can be expressed as a combination of the Atiyah-Patodi-Singer eta invariants for the Dirac operator and signature operator on $(Y, \mathfrak{s})$. The modified version of the Seiberg-Witten invariant is defined as

$$S\hat{W}_Y(\mathfrak{s}) = SW_Y(\mathfrak{s}, g) - \xi_Y(\mathfrak{s}, g). \tag{6}$$

Then we prove the following equivalence between $S\hat{W}_Y$ and the Casson-Walker invariant.

Theorem 1.2. *Let Y be a rational homology 3-sphere. Then,*

$$\sum_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} S\hat{W}_Y(\mathfrak{s}) = \frac{1}{2}|H_1(Y, \mathbb{Z})|\lambda(Y),$$

where $\lambda(Y)$ is the Casson-Walker invariant of Y (cf. [11]).

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2 The geometric triangle

In this section, we identify the monopoles on Y with monopoles on Y_1 and Y_0. Suppose given a smoothly embedded knot K in $(Y, \mathfrak{s})$, which represents a torsion element of order n in $H_1(Y,\mathbb{Z})$. We can split Y along a torus as in [2],

$$Y = V \cup_{T^2} \nu(K).$$

We choose a metric on Y with a long cylinder $[-r,r]\times T^2$, and denote the resulting manifold as

$$Y(r) = V \cup_{T^2} ([-r,r]\times T^2) \cup_{T^2} \nu(K).$$

We additionally require that the chosen metric on Y satisfies the condition of Lemma 3.18 in [2] in the neighbourhood $\nu(K)$ of the knot: it has non-negative scalar curvature, strictly positive away from the boundary. This induces a natural metric on $Y_0(r)$. On $Y_1(r)$, we need to choose a metric which agrees with the original metric on $Y(r)$ when restricted to the knot complement V. The induced metric from $Y_1(r)$ in the torus neighbourhood of $\nu(K)$ is the metric described in Lemma 3.21 [2].

With this choice of the metric, the moduli space of monopoles on $(Y,\mathfrak{s})$ is non-empty only if $\mathfrak{s}|_{\nu(K)}$ is trivial. Gluing the two Spin^c structures $\mathfrak{s}|_V$ and $\mathfrak{s}|_{\nu(K)}$ along T^2 by a gauge transformation on T^2 gives rise to a $\mathbb{Z}_n$-family of Spin^c structures on Y and Y_1, and to a $\mathbb{Z}$-family of Spin^c structures on Y_0. The resulting Spin^c structures can be classified as the result of tensoring the original Spin^c structure $\mathfrak{s}$ with complex line bundles $L_k (k\in\mathbb{Z})$ whose Euler class is given by $kPD([K])$. The gluing theorem for 3-dimensional monopoles and the perturbation μ on $\nu(K)$, "simulating the effect of surgery", provide the decomposition of the moduli space for

$$\cup_{k\in\{1,\cdots,n\}}\mathcal{M}_Y(\mathfrak{s}\otimes L_k).$$

Theorem 2.1. *With the choice of perturbations and metrics on Y, Y_1 and Y_0 described above, we have the following relation between the critical sets of the Chern-Simons-Dirac functional on the manifolds Y, Y_1 and Y_0:*

$$\begin{aligned} & \bigcup_{k\in\{1,\cdots,n\}} \mathcal{M}_Y(\mathfrak{s}\otimes L_k) \\ = \ & \bigcup_{k\in\{1,\cdots,n\}} \mathcal{M}_{Y_1}(\mathfrak{s}\otimes L_k) \cup \bigcup_{k\in\mathbb{Z}} \mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_k). \end{aligned}$$

Proof. First we assume that Y is a rational homology 3-sphere. When stretching the neck in $Y(r)$, as $r\to\infty$, we get two manifolds, each endowed with an infinite cylindrical end,

$$V(\infty) = V \cup_{T^2} ([0,\infty)\times T^2)$$

$$\nu(K)(\infty) = \nu(K) \cup_{T^2} ((-\infty, 0] \times T^2).$$

The Seiberg-Witten monopole moduli space of $Y(r)$, for sufficiently large r, can be described in terms of the moduli spaces of $V(\infty)$ and $\nu(K)(\infty)$ as analyzed in [2].

Notice that the moduli spaces of flat connections on T^2, modulo the subgroups of the gauge transformations on T^2 which can be extended to the whole manifolds V or $\nu(K)$, define the following character varieties:

$$\begin{aligned} & \chi_0(T^2, V) \\ = \ & H^1(T^2, \mathbb{R})/2Im(H^1(V, \mathbb{Z}) \to H^1(T^2, \mathbb{Z}) \\ = \ & H^1(T^2, \mathbb{R})/2n\mathbb{Z}\langle PD([l])\rangle, \end{aligned}$$

$$\begin{aligned} & \chi_0(T^2, \nu(K)) \\ = \ & H^1(T^2, \mathbb{R})/2Im(H^1(\nu(K), \mathbb{Z}) \to H^1(T^2, \mathbb{Z}) \\ = \ & H^1(T^2, \mathbb{R})/2\mathbb{Z}\langle PD([m])\rangle. \end{aligned}$$

Thus the character variety $\chi_0(T^2, Y)$ is

$$\chi_0(T^2, Y) = H^1(T^2, \mathbb{R}).$$

The covering maps between these character varieties are illustrated as follows

$$\begin{tikzcd} 2\mathbb{Z}\langle PD([m])\rangle \arrow[dr] & & 2n\mathbb{Z}\langle PD([l])\rangle \arrow[dl] \\ & \chi_0(T^2, Y) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \chi_0(T^2, V) \arrow[dr] & & \chi_0(T^2, \nu(K)) \arrow[dl] \\ & \chi(T^2) & \end{tikzcd}$$

where the maps π_i are the covering maps with fibers as indicated.

Based on the analysis of the space $\mathcal{M}_V^*(\mathfrak{s}|_V)$ of irreducible monopoles on a 3-manifold with a cylindrical end modelled on T^2, as in [2], we see that the asymptotic limit map defines a continuous map:

$$\mathcal{M}_V^*(\mathfrak{s}|_V) \overset{\partial_\infty}{\to} \chi_0(T^2, V).$$

The reducibles on V embed in the character variety $\chi_0(T^2, V)$

$$\mathcal{M}_V^{red} \to \chi_0(T^2, V)$$

as the connections whose holonomy around the meridian is trivial. Then, for a sufficiently large r, the gluing theorem gives a diffeomorphism:

$$\#_Y : \mathcal{M}_V^* \backslash \partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2,Y)} \chi(\nu(K)) \to \bigcup_{k=1}^{n} \mathcal{M}_{Y(r)}^*(\mathfrak{s} \otimes L_k),$$

here U_θ is a small neighbourhood of the "bad points" θ in $\chi_0(T^2, Y)$, and $\chi(\nu(K))$ are the reducible lines in $\chi_0(T^2, Y)$. The above fibred product is obtained (cf.[2]) by taking the pullbacks of the images of the boundary value maps under the projections

$$\begin{tikzcd} & \chi_0(T^2, Y) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \chi_0(T^2, V) & & \chi_0(T^2, \nu(K)). \end{tikzcd}$$

Let (u, v) be the coordinates on $\chi_0(T^2, Y) \cong H^1(T^2, \mathbb{R})$ corresponding to the holonomy around the longitude l and the meridian m respectively. In the gluing map $\#_Y$ above, $\chi(\nu(K))$ corresponds to the lines $\{v = 2k, k = 1, \cdots, n\}$. For each line $\{v = 2k\}$, the image of the gluing map gives a diffeomorphism onto $\mathcal{M}_{Y(r)}^*(\mathfrak{s} \otimes L_k)$.

For each Spin^c structure $\mathfrak{s} \otimes L_k$, there is a unique reducible monopole on $Y(r)$, which is given by the intersection of $\chi(V, \mathfrak{s})$, the flat connections on $(V, \mathfrak{s})$, with the line $\{v = 2k\}$ in $\chi_0(T^2, Y)$:

$$\theta_Y(k) = \{u = u(\mathfrak{s})\} \times_{\chi_0(T^2,Y)} \{v = 2k\}.$$

Here $u(\mathfrak{s})$ is the holonomy of the flat connections in $\chi(V, \mathfrak{s})$ around the longitude l'.

Similarly, we have gluing maps for monopoles on $Y_1(r)$ and $Y_0(r)$, respectively. In the gluing map $\#_{Y_1}$ for $\mathcal{M}_{Y_1(r)}^*(\mathfrak{s} \otimes L_k)$,

$$\#_{Y_1} : \mathcal{M}_V^* \backslash \partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2,Y_1)} \chi(\nu(K)) \to \bigcup_{k=1}^{n} \mathcal{M}_{Y_1(r)}^*(\mathfrak{s} \otimes L_k),$$

$\chi(\nu(K))$ is identified with $v - u = 2k + 1$. Similarly, we have the gluing map $\#_{Y_0}$ for $\mathcal{M}_{Y_0(r)}^*(\mathfrak{s} \otimes L_k)$,

$$\#_{Y_0} : \mathcal{M}_V^* \backslash \partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2,Y_0)} \chi(\nu(K)) \to \bigcup_{k \in \mathbb{Z}} \mathcal{M}_{Y_0(r)}^*(\mathfrak{s} \otimes L_k),$$

where $\chi(\nu(K))$ is given by $u = 2k$. For each $k \in \{1, \cdots, n\}$, the reducible monopole for $(Y_1, \mathfrak{s} \otimes L_k)$, consists of the unique point

$$\theta_{Y_1}(k) = \{u = u(\mathfrak{s})\} \times_{\chi_0(T^2, Y)} \{v = u + 2k + 1\}.$$

For Y_0 with a non-trivial Spin^c structure $\mathfrak{s} \otimes L_k$ $(k \in \mathbb{Z}, k \neq 0)$, the set of reducibles is empty for any generic perturbation, and for $\mathfrak{s} \otimes L_0 = \mathfrak{s}$, it consists of one circle of reducibles $u = 0$ in the cylinder $\chi_0(T^2, Y_0) = \chi_0(T^2, V)$, which can be perturbed away by introducing a small perturbation as in Theorem 6.13 [2].

Now we apply the perturbation to simulate the effect of Dehn surgery. This amounts to a careful choice of perturbation as in Section 6 [2], which we now briefly describe.

Choose a compactly supported 2-form μ representing the generator of $H^2_{cpt}(D^2 \times S^1)$, defined as in Lemma 3.18 [2], such that we have

$$\int_{D^2 \times \{pt\}} \mu = 1 \tag{7}$$

for any point on S^1. Under the isomorphism $H^2_{cpt}(\nu(K)) \cong H_1(\nu(K))$, given by Poincaré duality, this form corresponds to the generator $[\mu] = PD_{\nu(K)}(l)$. The class of μ in $H^2(D^2 \times S^1)$ is trivial, and we can write $\mu = d\nu$, where ν is a 1-form satisfying $\int_{S^1 \times \{pt\}} \nu = 1$, i.e. $\nu = PD_{T^2}(l)$. Choose on $\nu(K)$ a metric as in Lemma 3.18 [2].

Fix a $U(1)$-connection A_0 representing the trivial connection on T^2, For any $U(1)$-connection A, define T_A to be

$$T_A(z) = -i \int_{\{z \in D^2\} \times S^1} (A - A_0).$$

For any given $\epsilon > 0$, we can choose a function $\hat{f} : \mathbb{R} \to \mathbb{R}$ with the following properties.

(a) $\hat{f}$ is continuously differentiable on $(-1, 1)$ and satisfies the periodicity $\hat{f}(t + 2) = \hat{f}(t)$

(b) the derivative $\hat{f}'$ has range $\hat{f}'(t) \in [-1, 1]$ for all $t \in [-1, 1]$, and satisfies $\hat{f}'(1 - t) = \hat{f}'(1 + t)$ for $t \in \mathbb{R}$.

(c) the following estimate holds:

$$\sup_{t \in [-1+\epsilon, 1-\epsilon]} |\hat{f}'(t) - t| < \epsilon.$$

Now, for the Spin^c structure $\mathfrak{s}\otimes L_k$ $(k=1,\cdots,n)$, consider the function $f'_k(t) = \hat{f}'(t+1)+2k$ and define a perturbation of the Seiberg-Witten equations on $(Y,\mathfrak{s}\otimes L_k)$ in the following way:

$$\begin{cases} F_A = *\sigma(\psi,\psi) + f'_k(T_A)\mu \\ \partial\!\!\!/_A(\psi) = 0 \end{cases} . \tag{8}$$

With respect to the chosen metric on $\nu(K)$, with sufficiently large positive scalar curvature on the support of μ as specified in Lemma 3.18, the only solutions of the perturbed monopole equations are reducibles $(A,0)$, that satisfy

$$F_A = f'_k(T_A)\mu. \tag{9}$$

In addition to this surgery perturbation, we consider another perturbation of the Seiberg–Witten equations on the tubular neighbourhood $\nu(K)$ in Y, Y_1, and Y_0. This perturbation has the effect of producing a global shift in the character variety to avoid the bad points on $H^1(T^2,\mathbb{R})$ when we deform the unperturbed geometric triangles in $\chi(T^2)$ to the perturbed geometric triangles in $\chi(T^2)$.

Let μ be a compactly supported 2-form in $D^2\times S^1$ satisfying (7). Let $\eta>0$ be some small real parameter. Consider an additional perturbation

$$F_A = *\sigma(\psi,\psi) \pm \eta\mu \tag{10}$$

of the curvature equation on $\nu(K)$ inside Y and inside Y_0.

This perturbation has the effect of shifting the asymptotic values by an amount η. We choose the sign so that the line of reducibles on $\nu(K)\subset Y$ for $\mathfrak{s}\otimes L_k$ becomes $\{(u,v)|v=2k+\eta\}$, the line of reducibles on $\nu(K)\subset Y_1$ for $\mathfrak{s}\otimes L_k$ remains the same $\{(u,v)|v-u=2k+1\}$, and the line of reducibles on $\nu(K)\subset Y_0$ for $\mathfrak{s}\otimes L_k$ becomes $\{(u,v)|u=2k+\eta\}$.

On $\nu(K)$ inside Y for $\mathfrak{s}\otimes L_k$ we shall consider the perturbed curvature equation

$$F_A = *\sigma(\psi,\psi) + (f'_k(T_A-\eta)+\eta)\mu. \tag{11}$$

Therefore, we can partition the moduli spaces for $(Y,\mathfrak{s}\otimes L_k)$ $(k=1,\cdots,n)$ into the union of the moduli spaces for $(Y_1,\mathfrak{s}\otimes L_k)$ $(k=1,\cdots,n)$ and $(Y,\mathfrak{s}\otimes L_k)$ $(k\in\mathbb{Z})$, as in Theorem 6.3 [2]. This completes the proof of the theorem for the case of rational homology 3-sphere Y with a knot K representing a torsion element of order n in $H_1(Y,\mathbb{Z})$.

For a general 3-manifold Y with a smoothly embedded knot K representing a torsion element of order n in $H_1(Y, \mathbb{Z})$, the proof is essentially the same as the case of rational homology spheres discussed above, and we omit the details here.

$\square$

The perturbation can be illustrated as in Figure 1 where $n = 4$. Here we used the notation

$$L_Y(\mathfrak{s}_k) = \{(u, v)|\, v = f_k'(u - \eta) + \eta\}$$
$$L_{Y_1}(\mathfrak{s}_k) = \{(u, v)|\, v = u + 2k + 1\}$$
$$L_{Y_0}(\mathfrak{s}_k) = \{(u, v)|\, v = 2k + \eta\}.$$

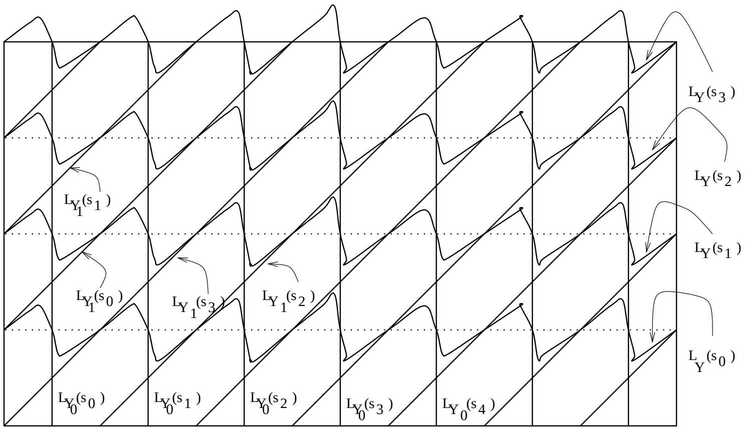

Figure 1: The perturbed geometric triangle as in Theorem 2.1

With a more careful study of the perturbed geometric triangles, we have the following decomposition of 3-dimensional monopoles on Y under the Dehn surgery.

Theorem 2.2. *With the perturbations and metrics on Y, Y_1 and Y_0 as in Theorem 2.1, and for any fixed $m, p \in \{0, \cdots, n-1\}$, there exists a further perturbation on Y, such that we have the following relation between the critical sets of the Chern-Simons-Dirac functional on the manifolds Y, Y_1 and Y_0:*

$$\mathcal{M}_Y(\mathfrak{s} \otimes L_m) = \mathcal{M}_{Y_1}(\mathfrak{s} \otimes L_m) \cup \bigcup_{k \in \mathbb{Z}} \mathcal{M}_{Y_0}(\mathfrak{s} \otimes L_{nk+p}). \tag{12}$$

Proof. In the proof of Theorem 2.1, we know that the perturbed Seiberg-Witten monopoles on Y, Y_1 and Y_0 are given by the following gluing models (here we assume that Y is a rational homology 3-sphere):

$$\mathcal{M}^*_{Y(r)}(\mathfrak{s}\otimes L_k) \cong \mathcal{M}^*_V\backslash\partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2,Y)} \{v = 2k\},$$

$$\mathcal{M}^*_{Y_1(r)}(\mathfrak{s}\otimes L_k) \cong \mathcal{M}^*_V\backslash\partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2,Y_1)} \{v = u + 2k + 1\},$$

$$\mathcal{M}^*_{Y_0(r)}(\mathfrak{s}\otimes L_k) \cong \mathcal{M}^*_V\backslash\partial_\infty^{-1}(U_\theta) \times_{\chi_0(T^2,Y_0)} \{u = 2k\}.$$

Note that the additional perturbation (10) of the curvature equation on $\nu(K)$ inside Y and inside Y_0 introduce a shift of coordinates (u, v) to $(u+\eta, v+\eta)$. We can introduce these new coordinates, still denoted by (u, v). Then the reducible line for V is given by $u = -\eta$ in $H^1(T^2, \mathbb{R})$, and the set of "bad points" is given by $\{(-\eta, -\eta + 2k) : k \in \mathbb{Z}\}$.

We will show that there exists a further surgery perturbation on Y that suits the purpose of identifying monopoles on Y, Y_1 and Y_0 as stated in the Theorem. Recall that, in the case of rational homology 3-spheres, the Seiberg-Witten-Floer homology is metric and perturbation dependent (cf. [4]). A change in the Floer homology occurs whenever the restriction to $\chi_0(T^2, Y)$ of the perturbed reducible on Y passes through one of the "bad points" $\{(-\eta, -\eta + 2k) : k \in \mathbb{Z}\}$ along the line $u = -\eta$. We will see later that no wall-crossing takes place, as long as we take the same m $\mathfrak{s}\otimes L_m$ on Y and Y_1. Without loss of the generality, after possible coordinates change, we can assume that $m = 0$. Fix $p \in \{0, \cdots, n-1\}$. We will construct a function $f_0' : \mathbb{R} \to \mathbb{R}$, which depends on a small $\epsilon > 0$, such that, as $\epsilon \to 0$, the curve $v = f_0'(u)$ approaches the union of lines

$$L_{Y_1}(\mathfrak{s}\otimes L_0) \cup \bigcup_{k\in\mathbb{Z}} L_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$$

with

$$L_{Y_1}(\mathfrak{s}\otimes L_0) = \{v = u + 1\}$$

and

$$L_{Y_0}(\mathfrak{s}\otimes L_{nk+p}) = \{u = 2nk + 2p\}.$$

We identify $\chi_0(T^2, V)$ with the fundamental domain

$$\{u \in \mathbb{R}\} \times \{0 \leq v < 2n\},$$

in $H^1(T^2, \mathbb{R})$. The asymptotic values $\partial_\infty(\mathcal{M}_V^*) \subset \chi_0(T^2, V)$ can be lifted to $H^1(T^2, \mathbb{R})$ periodically. Using this $2n$–periodicity, we only need to construct a function $f_0 : [-1, 2n-1] \to [0, 2n]$, which depends on ϵ, such that, for any given $\epsilon \le \epsilon_0$, we have

$$\sup_{t \in [-1, 2n-1] \setminus [-\epsilon, \epsilon]} |f_0'(t) - t| < \epsilon.$$

Such function can be easily constructed as in the proof of Theorem 2.1. Then over $[2nk-1, 2nk+2n-1]$, $f_0'(t)$ is defined to be $f_0'(t) + 2nk$. See Figure 2 where the perturbation is illustrated in the cases with $n = 4, m = p = 0$ and with $n = 4, m = 0, p = 2$, respectively. The general case can be proved by a similar method. $\square$

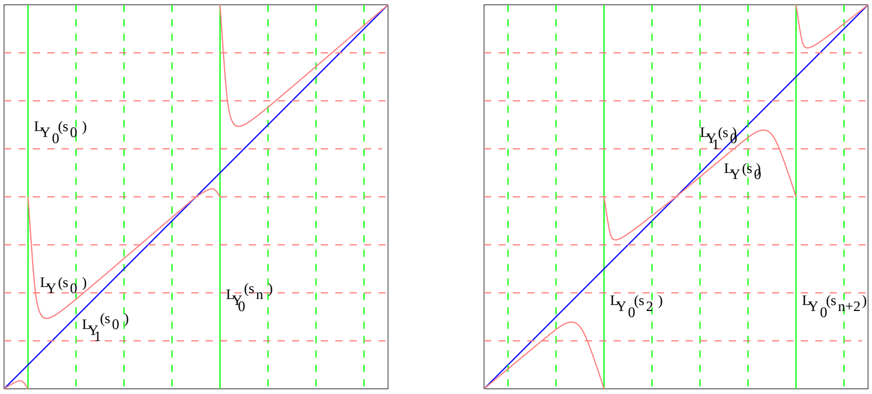

Figure 2: The perturbed geometric triangle as in Theorem 2.2

3 The relative gradings

In the previous section, we have the following decomposition:

$$\mathcal{M}_{Y,\mu}(\mathfrak{s} \otimes L_m) = \mathcal{M}_{Y_1}(\mathfrak{s} \otimes L_m) \cup \bigcup_{k \in \mathbb{Z}} \mathcal{M}_{Y_0}(\mathfrak{s} \otimes L_{nk+p})$$

for any fixed $m, p \in \{0, \cdots, n-1\}$. Assume that Y is a rational homology 3-sphere. First we fix a grading on $\mathcal{M}_{Y,\mu}(\mathfrak{s} \otimes L_0)$ defined in terms of the spectral flow of the linearization operator for the 3-dimensional Seiberg-Witten equations of a path connecting an irreducible monopole in $\mathcal{M}_{Y,\mu}(\mathfrak{s} \otimes L_0)$ to the unique reducible $\theta_Y(0)$ in the configuration space for $(Y, \mathfrak{s} \otimes L_0)$. Then the analysis of the relative grading in Part I section 7 [2] can be applied to induce a compatible grading on $\mathcal{M}_{Y_1}(\mathfrak{s}\otimes L_0)\cup\bigcup_{k\in\mathbb{Z}}\mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$ as follows, cf. Proposition 7.3 – Corollary 7.7 in [2].

Proposition 3.1. *Let Y be a rational homology 3-sphere. For any fixed $p\in\{0,\cdots,n-1\}$, the Floer complexes*

$$C_*(Y,\mathfrak{s}\otimes L_0,\mu)=\oplus_{a\in\mathcal{M}_{Y,\mu}(\mathfrak{s}\otimes L_0)}\mathbb{Z}\langle a\rangle,$$

$$C_*(Y_1,\mathfrak{s}\otimes L_0)=\oplus_{a\in\mathcal{M}_{Y_1}(\mathfrak{s}\otimes L_0)}\mathbb{Z}\langle a\rangle,$$

$$C_*(Y_0,\mathfrak{s}\otimes L_{nk+p})=\oplus_{a\in\mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p})}\mathbb{Z}\langle a\rangle,$$

have a compatible relative grading of generators in the following sense.

1. *Suppose given two irreducible critical points a,b in $\mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0)$, and the corresponding elements a^ϵ,b^ϵ in $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$ under the above decomposition (12). Then*

$$\deg_{Y,\mu}(a^\epsilon)-\deg_{Y,\mu}(b^\epsilon)=\deg_{Y_1}(a)-\deg_{Y_1}(b).$$

2. *Suppose given two monopoles a,b in $\mathcal{M}^*_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$, and the corresponding elements a^ϵ,b^ϵ in $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$ under the above decomposition (12). Then*

$$\deg_{Y_0,\mathfrak{s}_k}(a)-\deg_{Y_0,\mathfrak{s}_k}(b)=\deg_{Y,\mu}(a^\epsilon)-\deg_{Y,\mu}(b^\epsilon)\quad mod\ (2nk+2p).$$

   Therefore, the grading $\deg_{Y,\mu}$ defines a $\mathbb{Z}$-valued lift of the $\mathbb{Z}_{2nk+2p}$-valued relative index on $\mathcal{M}^*_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$ under the decomposition of $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$.

For a general 3-manifold $(Y,\mathfrak{s})$ with $b_1(Y)>0$ and a knot K representing a torsion element of order n, if $\mathfrak{s}$ is a torsion Spin^c structure, we know that, for any $k\in\{0,\cdots,n-1\}$, the Spin^c structure $\mathfrak{s}\otimes L_k$ also has a torsion class $c_1(\mathfrak{s}\otimes L_k)$. Thus, after a small perturbation to get rid of the $(S^1)^{b_1(Y)}$-family of reducibles, we have a $\mathbb{Z}$-graded

$$\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_k)\cong\mathcal{M}_{Y,\mu}(\mathfrak{s}\otimes L_k).$$

Then it is easy to see that the results in Part I section 7 [2] hold in this case as well without any substantial change.

Now assume that $c_1(\mathfrak{s})$ is a non-torsion element, with multiplicity 2ℓ in $H^2(Y,\mathbb{Z})/\text{Torsion}$. Then, for any $k\in\{0,\cdots,n-1\}$, the set of generators

$$\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_k)\cong\mathcal{M}_{Y,\mu}(\mathfrak{s}\otimes L_k)$$

is 2ℓ-graded, and so is

$$\mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_k)\cong\mathcal{M}_{Y_1}(\mathfrak{s}\otimes L_k).$$

Then for any $k\in\mathbb{Z}$, any non-empty moduli space

$$\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_k)\cong\mathcal{M}_{Y,\mu}(\mathfrak{s}\otimes L_k)$$

is $\mathbb{Z}_{2\ell_{[k]}}$-graded, where $2\ell_{[k]}$ is the maximum common factor of 2ℓ and $2k$. We can choose a relative grading on $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$ by the spectral flow of the linearization operator along a path connecting any irreducible monopole in $\mathcal{M}^*_Y(\mathfrak{s}\otimes L_0)$ to a fixed monopole a_0 in $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$. Note that this grading is 2ℓ-graded. Then the analysis in Part I section 7 can also be applied to obtain the following proposition.

Proposition 3.2. *For $p\in\{0,\cdots,n-1\}$ and $k\in\mathbb{Z}$, the Floer complexes $C_*(Y,\mathfrak{s}\otimes L_0), C_*(Y_1,\mathfrak{s}\otimes L_0)$, and $C_*(Y_0,\mathfrak{s}\otimes L_{nk+p})$ have a compatible relative grading of generators in the following sense.*

1. *Suppose given two irreducible critical points a,b in $\mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0)$, and the corresponding elements a^ϵ, b^ϵ in $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$, then*

   $$\deg_{Y,\mu}(a^\epsilon)-\deg_{Y,\mu}(b^\epsilon)=\deg_{Y_1}(a)-\deg_{Y_1}(b),$$

   as a $\mathbb{Z}_\ell$-valued function.

2. *Suppose given two monopoles a,b in $\mathcal{M}^*_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$, and the corresponding elements a^ϵ, b^ϵ in $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$, then*

   $$\deg_{Y_0,\mathfrak{s}_k}(a)-\deg_{Y_0,\mathfrak{s}_k}(b)=\deg_{Y,\mu}(a^\epsilon)-\deg_{Y,\mu}(b^\epsilon)\quad mod\ (2\ell_{[nk+p]}).$$

   Thus, the induced grading $\deg_{Y,\mu}$ on $\mathcal{M}^*_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$, under the decomposition (12), defines a choice of a $\mathbb{Z}_{2\ell}$-valued lifting of the $\mathbb{Z}_{2\ell_{[nk+p]}}$-valued relative index.

In the rest of this section we discuss the induced gradings on the various $\mathcal{M}^*_Y(\mathfrak{s}\otimes L_m)$, for $m\in\{1,\cdots,n-1\}$. Notice that, in the case of a rational homology 3-sphere Y, the perturbed line $L_Y(\mathfrak{s}\otimes L_m)$ is a deformation of the line $L_Y(\mathfrak{s}\otimes L_0)$. Along this deformation, the corresponding parameterized reducibles form a path connecting $\theta_Y(0)$ to $\theta_Y(m)$. As the following Lemma shows, this deformation can be realized as a perturbation of the monopole equations for $(Y,\mathfrak{s}\otimes L_0)$.

Lemma 3.3. *For any $m \in \{1, \cdots, n-1\}$, there exists a perturbation μ_m for the Seiberg-Witten monopole equations on $(Y, \mathfrak{s} \otimes L_0)$, such that there is a diffeomorphism:*

$$\mathcal{M}_{Y,\mu_m}(\mathfrak{s} \otimes L_0) \cong \mathcal{M}_Y(\mathfrak{s} \otimes L_m).$$

Moreover, if Y is a rational homology 3-sphere, the grading on $\mathcal{M}_Y^*(\mathfrak{s}\otimes L_m)$ induces a grading on $\mathcal{M}^*_{Y,\mu_m}(\mathfrak{s} \otimes L_0)$. Under the above identification, the resulting grading differs from the original grading on $\mathcal{M}_Y^*(\mathfrak{s} \otimes L_0)$ by the wall-crossing formulae studied in [4] for $(Y, \mathfrak{s} \otimes L_0)$. If $b_1(Y) > 0$, then the induced grading on $\mathcal{M}_Y(\mathfrak{s}\otimes L_m)$ from $\mathcal{M}_{Y,\mu_m}(\mathfrak{s}\otimes L_0)$ agrees with the relative grading on $\mathcal{M}_Y(\mathfrak{s} \otimes L_m)$.

Proof. The first claim follows from the gluing models of the monopoles in $\mathcal{M}_Y(\mathfrak{s} \otimes L_0)$ and $\mathcal{M}_Y(\mathfrak{s} \otimes L_m)$. Then, using the results in [4], we know that, in the case of the rational homology 3-sphere, the spectral flow of the twisted Dirac operator along the path of reducibles gives the index shift on $\mathcal{M}_Y^*(\mathfrak{s} \otimes L_0)$ and $\mathcal{M}^*_{Y,\mu_m}(\mathfrak{s} \otimes L_0)$ according to the wall-crossing formulae derived in [4]. Again by the results of [4], for Y with $b_1(Y) > 0$, the induced grading from the parametrized spectral flow is same as the original relative index on $\mathcal{M}_Y(\mathfrak{s} \otimes L_m)$. □

Thus, Lemma 3.3 provides a consistent way of assigning a choice of absolute grading on the various $\mathcal{M}_Y(\mathfrak{s} \otimes L_m)$. In fact, notice that, in the case of rational homology 3-sphere Y, for different values $p \in \{0, \cdots, n-1\}$, the additional perturbations on $(Y, \mathfrak{s} \otimes L_0)$ to achieve the decomposition

$$\mathcal{M}_Y^*(\mathfrak{s} \otimes L_0) \cong \mathcal{M}^*_{Y_1}(\mathfrak{s} \otimes L_0) \cup \bigcup \mathcal{M}^*_{Y_0}(\mathfrak{s} \otimes L_{nk+p}),$$

does not cross the wall where the grading would jump.

The degree shift of Lemma 3.3 can be described as follows. Let $\theta_Y(t)$ be the path of reducibles on $(Y, \mathfrak{s} \otimes L_0)$ for the family of perturbations connecting $\mathcal{M}_Y^*(\mathfrak{s} \otimes L_0)$ to $\mathcal{M}^*_{Y,\mu_m}(\mathfrak{s} \otimes L_0)$, then the wall crossing formulae in [4] tell us that the index shift is given by the spectral flow

$$SF(\not{\partial}_{\theta_Y(t)}). \tag{13}$$

4 Geometric limits and the holomorphic triangles

4.1 Surgery cobordisms

We first briefly describe the surgery cobordisms from Y_1 to Y, from Y to Y_0, and from Y_0 to Y_1, respectively, as in [6]. The cobordism W_1, from

Y_1 to Y, is obtained by removing from the trivial cobordism $Y_1 \times [0,1]$ an $S^1 \times D \cong \nu(K) \times \{1\}$, where D is a disk, and $\nu(K)$ is the tubular neighbourhood of the knot in Y_1, and then attaching a 2-handle with framing -1. We denote by D_1 the core disk of the 2-handle in W_1. Similarly, the cobordism W_0, form Y to Y_0, is obtained by removing from the trivial cobordism $Y_0 \times [0,1]$ an $S^1 \times D \cong \nu(K) \times \{0\}$ and attaching a 2-handle with framing zero. We denote by D_0 the core disk of the 2-handle in W_0. Attaching the two-handle has the effect of modifying the boundary component $Y_1 \times \{1\}$ in the trivial cobordism to the boundary component $Y \times \{1\}$ in the non-trivial cobordism W_1, or, respectively, the boundary component $Y_0 \times \{0\}$ in the trivial cobordism to the boundary component $Y \times \{0\}$ in W_0. The cobordism $\bar{W}_2$ connecting Y_0 and Y_1, satisfies the relation

$$\bar{W} = \bar{W}_2 \# \mathbb{CP}^2,$$

where $\bar{W}$ is the composite cobordism $\bar{W} = \bar{W}_0 \cup_Y \bar{W}_1$.

We assume that the 3-manifolds Y_1, Y, and Y_0 are endowed with metrics with a long cylinder $T^2 \times [-r,r]$, as specified in section 2 (see also [2]). We consider the manifolds W_1 and W_0 endowed with infinite cylindrical ends $Y_1 \times (-\infty, -T_0]$ and $Y \times [T_0, \infty)$, and $Y_0 \times [T_0, \infty)$ and $Y \times (-\infty, -T_0]$, respectively. As in [6], we can decompose the cobordisms W_i as

$$W_i = V \times \mathbb{R} \cup_{T^2 \times \mathbb{R}} T^2 \times [-r,r] \times \mathbb{R} \cup_{T^2 \times \mathbb{R}} W_i(\nu(K)). \tag{14}$$

The region $W_i(\nu(K))$ has the following property. There is a compact set $\mathcal{K}$ in W_i such that the intersection $\mathcal{K} \cap W_i(\nu(K))$ is obtained by attaching a 2-handle $D \times D$ to the product $\nu(K) \times [-T_0, T_0]$, and, outside of $\mathcal{K}$, the region $\mathcal{K}^c \cap W_i(\nu(K))$ consists of product regions $\nu(K) \times [T_0, \infty)$ and $\nu(K) \times (-\infty, -T_0]$, and $T^2 \times [r_0, r] \times [-T_0, T_0]$.

As in [6], consider an interior point x_i contained in the core disk of the 2-handle, $x_i \in D_i$, and we denote by $\hat{W}_i$ the punctured cobordism $\hat{W}_i = W_i \backslash \{x_i\}$. Similarly, we can consider the punctured manifold

$$\hat{W}_i(\nu(K)) = W_i(\nu(K)) \backslash \{x_i\}.$$

In the manifolds $\hat{W}_i(\nu(K))$, endowed with an extra asymptotic end of the form $S^3 \times [0, \infty)$ at the puncture, we can identify a product region

$$\mathcal{V} = \nu(K)_{r_0} \times \mathbb{R} \cong D \times (D_i \backslash \{x_i\}). \tag{15}$$

Thus, we identify the manifold W_i with a connected sum

$$W_i = \hat{W}_i \# Q_i,$$

with a long cylindrical neck $S^3 \times [-T(r), T(r)]$, and with Q_i a 4-ball, where S^3 is decomposed as the union of two solid tori in the standard way, $S^3 = \nu \cup \tilde{\nu}$, with $\nu \cong \tilde{\nu} \cong D \times S^1$. Then the product region $\mathcal{V}$ of (15) in W_i identifies the standard solid torus ν in S^3 with the neighbourhood $\nu(K)$ of the knot K in Y. Similarly, there is a product region $\tilde{\mathcal{V}}$ which identifies the other solid torus $\tilde{\nu}$ in S^3 with the tubular neighbourhood $\nu(K)$ in Y_i, after the surgery. The resulting punctured cobordism can be written as

$$\hat{W}_i(r) = (V_r \times \mathbb{R}) \cup \mathcal{V}(r) \cup \tilde{\mathcal{V}}(r). \tag{16}$$

We now impose a choice of metrics and perturbations for the Seiberg-Witten equations on the cobordisms as in subsection 2.2 of [6]. Then we can adopt the results of [5][6] to understand the asymptotic limits of finite energy monopoles, under the splitting of the punctured cobordisms as $r \to \infty$, as in (16).

4.2 Geometric limits and holomorphic triangles

We assume that Y and Y_1 are endowed with the Spin^c structure $\mathfrak{s} \otimes L_0$. The other Spin^c structures can be studied analogously.

On the surgery cobordism $W_1(r)$, there is a $\mathbb{Z}$-family of Spin^c structures whose restrictions to the two ends agree with $\mathfrak{s} \otimes L_0$ on Y and Y_1 respectively. We will study the moduli spaces of monopoles on $W_1(r)$ with asymptotic values in $\mathcal{M}_{Y_1}(\mathfrak{s} \otimes L_0)$ and $\mathcal{M}_{Y,\mu}(\mathfrak{s} \otimes L_0)$ at the two ends, where μ is the surgery perturbation defined by f_0' as in the proof of Theorem 2.2. In the case $b_1(Y) > 0$, we only consider the components of minimal energy, as defined in [6] [7], among all the possible moduli spaces with different Spin^c structures and with the given asymptotic values. For a rational homology 3-sphere Y, we only consider the moduli spaces of minimal dimension among the $\mathbb{Z}$-family of Spin^c structures.

With this convention understood, we denote by $\mathcal{M}^{W_1}(a_1, a)$ the moduli space with asymptotic values $a_1 \in \mathcal{M}_{Y_1}(\mathfrak{s} \otimes L_0)$ and $a \in \mathcal{M}_{Y,\mu}(\mathfrak{s} \otimes L_0)$. Similarly, we denote by $\mathcal{M}^{W_0}(a, a_0)$ the moduli space with asymptotic values $a \in \mathcal{M}_{Y,\mu}(\mathfrak{s} \otimes L_0)$ and $a_0 \in \mathcal{M}_{Y_0}(\otimes L_{nk+p})$ for $k \in \mathbb{Z}$, for a fixed $p \in \{0, \dots, n-1\}$. Under a generic choice of the perturbation, all these moduli spaces $\mathcal{M}^{W_1}(a_1, a)$ and $\mathcal{M}^{W_0}(a, a_0)$ are cut out transversely and of the expected dimension.

The convergence and gluing arguments developed in [4] can be applied to this case as well, to give the following compactifications of $\mathcal{M}^{W_1}(a_1, a)$ and $\mathcal{M}^{W_0}(a, a_0)$.

Proposition 4.1. *Suppose that $\mathcal{M}^{W_1}(a_1, a)$ is non-empty, then $\mathcal{M}^{W_1}(a_1, a)$ admits a compactification to a manifold with corners, where the codimension 1 boundary strata consist of*

$$\begin{aligned} &\bigcup_{c \in \mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)} \mathcal{M}^{W_1}(a_1, c) \times \hat{\mathcal{M}}_{Y,\mu}(c, a) \\ &\cup \bigcup_{c_1 \in \mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0)} \hat{\mathcal{M}}_{Y_1}(a_1, c_1) \times \mathcal{M}^{W_1}(c_1, a), \end{aligned} \tag{17}$$

and with extra components

$$\begin{aligned} &\hat{\mathcal{M}}_{Y_1}(a_1, \theta_1) \times U(1) \times \mathcal{M}^{W_1}(\theta_1, a) \\ &\mathcal{M}^{W_1}(a_1, \theta) \times U(1) \times \hat{\mathcal{M}}_{Y,\mu}(\theta, a), \end{aligned} \tag{18}$$

when splitting through the reducibles θ_1 and θ in $\mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0)$ and $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$ respectively . We also have the similar compactification for $\mathcal{M}^{W_0}(a, a_0)$.

Now we can describe the geometric limits of monopoles in $\mathcal{M}^{W_1}(a_1, a)$ and $\mathcal{M}^{W_0}(a, a_0)$ when stretching $r \to \infty$ inside

$$\hat{W}_i(r) = V_r \times \mathbb{R} \cup \mathcal{V}(r) \cup \tilde{\mathcal{V}}(r).$$

We only describe the case of $\hat{W}_1(r)$. With similar arguments we have the corresponding geometric limits for $\hat{W}_0(r)$. The proof of the results stated below on these geometric limits follows from the same arguments of [5][6].

Consider elements $a_i^{(1)} \in \mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0)$ and $a_j(\epsilon) \in \mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$, which we can write as

$$\begin{aligned} a_i^{(1)} &= [(A_i^-, \psi_i^-)\#(a^-_{\infty,i}, 0)] \\ a_j(\epsilon) &= [(A_j^+(\epsilon), \psi_j^+(\epsilon))\#(a^+_{\infty,j}(\epsilon), 0)], \end{aligned}$$

with

$$[A_i^-, \psi_i^-],\ [A_j^+(\epsilon), \psi_j^+(\epsilon)] \in\ \mathcal{M}^*_V(\mathfrak{s}\otimes L_0|_V)$$

and

$$\begin{aligned} a^-_{\infty,i} &\in\ L_{Y_1}(\mathfrak{s}\otimes L_0) \\ a^+_{\infty,j}(\epsilon) &\in\ L_{Y,\mu}(\mathfrak{s}\otimes L_0). \end{aligned}$$

Theorem 4.2. *(Proposition 3.1 and Remark 6.2 in [6]) Assume that $\mathcal{M}^{W_1(r)}\big(a_i^{(1)}, a_j(\epsilon)\big)$ is non-empty for all sufficiently large $r \geq r_0$. Then a family of solutions $[\mathcal{A}_1(r), \Psi_1(r)]$ in $\mathcal{M}^{W_1(r)}\big(a_i^{(1)}, a_j(\epsilon)\big)$ defines the following geometric limits on*

$$V_r \times \mathbb{R} \cup \mathcal{V}(r) \cup \tilde{\mathcal{V}}(r)$$

as $r \to \infty$.

(a). A finite energy solution $[\mathcal{A}', \Psi']^\epsilon$ *of the perturbed Seiberg-Witten equations on* $V \times \mathbb{R}$*, with a radial limit* $a_\infty(\epsilon)$ *in* $\partial_\infty(\mathcal{M}_V^*) \subset \chi_0(T^2, V)$*, and with temporal limits* $[A, \psi]_1^\epsilon$ *and* $[\tilde{A}, \tilde{\psi}]_1^\epsilon$ *in*

$$\partial_\infty^{-1}(a_\infty(\epsilon)) \subset \mathcal{M}_V^*.$$

(b). Two paths $[A(t), \psi(t)]_1^\epsilon$ *in* $\mathcal{M}_V^*$*, for* $t \in [-1, 0)$ *and* $t \in (0, 1]$*, with*

$$[A(-1), \psi(-1)]_1^\epsilon = [A_i^-, \psi_i^-]$$

$$\lim_{t \to 0-} [A(t), \psi(t)]_1^\epsilon = [A, \psi]_1^\epsilon$$

$$\lim_{t \to 0+} [A(t), \psi(t)]_1^\epsilon = [\tilde{A}, \tilde{\psi}]_1^\epsilon$$

$$[A(1), \psi(1)]_1^\epsilon = [A_j^+(\epsilon), \psi_j^+(\epsilon)].$$

These paths induce a continuous, piecewise smooth path

$$a_1^\epsilon(t) \subset \partial_\infty(\mathcal{M}_V^*) \subset \chi_0(T^2, V)$$

satisfying

$$a_1^\epsilon(t) = \partial_\infty[A(t), \psi(t)]_1^\epsilon,$$

with

$$a_1^\epsilon(-1) = a_{\infty,i}^- \quad a_1^\epsilon(0) = a_\infty(\epsilon) \quad a_1^\epsilon(1) = a_{\infty,j}^+(\epsilon).$$

As $\epsilon \to 0$*, these geometric limits define paths* $[A(t), \psi(t)]$ *and* $a(t)$ *with similar properties, and with*

$$a(-1) = a_{\infty,i}^- \quad a(0) = a_\infty \quad a(1) = a_{\infty,j}^+,$$

in $\partial_\infty(\mathcal{M}_V^*) \subset \chi_0(T^2, V)$*, with* $a_\infty = \lim_\epsilon a_\infty(\epsilon)$.

(c) Moreover, we have a holomorphic triangle in $H^1(T^2, \mathbb{R})$ *with vertices*

$$\{a_{\infty,i}^-, \vartheta_1, a_{\infty,j}^+(\epsilon)\}$$

and sides given by parameterized arcs along the lines $\ell_1 = \{v - u = 1\}$*,* $\ell_\mu = \{v = f_0'(u)\}$ *and by*

$$\{a_1^\epsilon(t)\} \subset \ell = \partial_\infty(\mathcal{M}_V^*).$$

Here we have

$$\vartheta_1 = L_{Y_1}(\mathfrak{s} \otimes L_0) \cap L_{Y,\mu}(\mathfrak{s} \otimes L_0).$$

This Theorem shows that the moduli space $\mathcal{M}^{W_1(r)}\big(a_i^{(1)}, a_j(\epsilon)\big)$ is characterized by the geometric limits on $V \times \mathbb{R}$ from (a) and the holomorphic triangles in (c). Two typical holomorphic triangles for $\mathcal{M}^{W_1(r)}\big(a_i^{(1)}, a_j(\epsilon)\big)$ or $\mathcal{M}^{W_0(r)}\big(a_i(\epsilon), a_j^{(0)}\big)$ are illustrated in Figure 3 where $n = 4, m = k = 0$ and $p = 2$. Here the points ϑ_i are the intersection points

$$\vartheta_1 = L_{Y_1}(\mathfrak{s} \otimes L_0) \cap L_{Y,\mu}(\mathfrak{s} \otimes L_0)$$

for W_1 and

$$\vartheta_0 = L_{Y,\mu}(\mathfrak{s} \otimes L_0) \cap L_{Y_0}(\mathfrak{s} \otimes L_{nk+p})$$

in the case of W_0. In other words, ϑ_i is the restriction to $T^2 = \partial\nu = \partial\tilde{\nu}$ of the unique reducible point θ_{S^3} at the puncture in the cobordism.

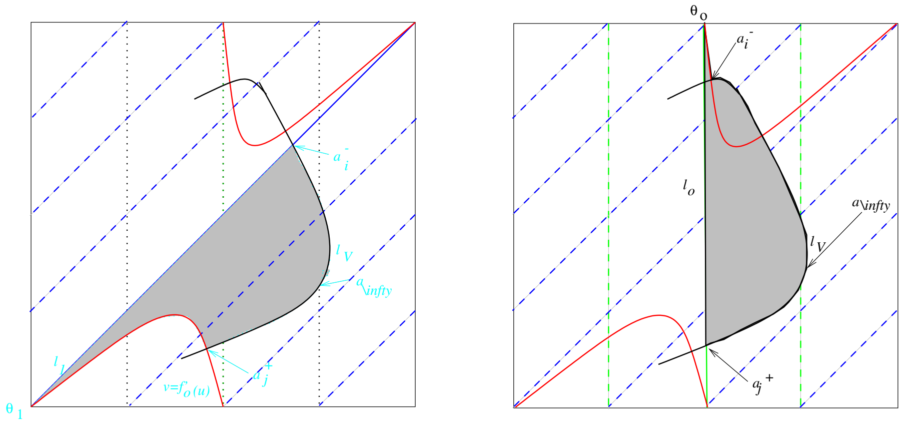

Figure 3: Holomorphic triangles for $W_1(r)$ and $W_0(r)$

In order to glue back the geometric limits and the holomorphic triangles, there are some admissible conditions for the geometric limits on $V \times \mathbb{R}$, which are studied in section 6.1, section 6.2 and section 6.3 [6]. As in section 6 [6], assuming that the element a_∞ is away from the bad points in the character variety of T^2, we denote by $\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty)$ the *balanced energy* moduli space of the Seiberg-Witten equations on $V \times \mathbb{R}$ with asymptotic value a_∞ in the radial direction. Then we have

$$\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty) = \bigcup_{[A,\psi],[\tilde{A},\tilde{\psi}]} \hat{\mathcal{M}}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty),$$

with

$$[A,\psi],[\tilde{A},\tilde{\psi}] \in \partial_\infty^{-1}(a_\infty) \subset \mathcal{M}_V^*.$$

Each moduli space

$$\mathcal{M}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty),$$

for a fixed choice of $[A,\psi]$ and $[\tilde{A},\tilde{\psi}]$ in $\partial_\infty^{-1}(a_\infty)$ in $\mathcal{M}_V^*$, is a smooth finite dimensional oriented manifold of the expected dimension, where the orientation is given by the corresponding determinant line bundle of the linearization operator for the monopole equations on $V\times\mathbb{R}$.

Remember that we are only considering the components with minimal energy or minimal dimension among all the moduli spaces of finite energy monopoles with fixed asymptotic values. Recall that there is a notion of admissible triples (cf. Definition 6.4 [6]), which singles out those elements $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ which arise as part of the geometric limits of solutions in $\mathcal{M}^{W_1(r)}(a_1,a)$ (or $\mathcal{M}^{W_0(r)}(a_1,a_0)$, or $\mathcal{M}_{Y(r)\times\mathbb{R}}(a,b)$ etc). For example, in the case of $\mathcal{M}^{W_1(r)}(a_1,a)$, with

$$a_1=[(A^-,\psi^-)\#(a^-,0)]$$

$$a=[(A^+,\psi^+)\#(a^+,0)],$$

an admissible triple $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ must satisfy the following conditions: there exists a smooth regular parameterization $a(t)$, for $t\in[-1,1]$ of the path in $\partial_\infty(\mathcal{M}_V^*)$ connecting a^- and a^+, such that $a(0)=a_\infty$, and corresponding smooth paths $[A(t),\psi(t)]$ in $\mathcal{M}_V^*$, for $t\in[-1,0)$ and $t\in(0,1]$, satisfying $\partial_\infty[A(t),\psi(t)]=a(t)$, and with

$$[A(-1),\psi(-1)]=[A^-,\psi^-] \quad\text{and}\quad \lim_{t\to 0_-}[A(t),\psi(t)]=[A,\psi]$$

$$\lim_{t\to 0_+}[A(t),\psi(t)]=[\tilde{A},\tilde{\psi}] \quad\text{and}\quad [A(1),\psi(1)]=[A^+,\psi^+].$$

By the results of [6], we know that the possible choices of a_∞ and of the admissible data in $\hat{\mathcal{M}}_{V\times\mathbb{R}}(a_\infty)$ are uniquely determined by the inequivalent holomorphic triangles Δ with vertices $\{a^-,\vartheta_1,a^+\}$ and sides along the union of Lagrangians $\ell\cup\ell_1\cup\ell_\mu$, with ℓ defined by the asymptotic values $\partial_\infty(\mathcal{M}_V^*)$. We denote by $\Xi_{W_1}(a_1,a)$ the set of such inequivalent holomorphic triangles. Moreover, under the identification of the choice of admissible triples $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ with the choice of inequivalent oriented holomorphic triangles in $\Xi_{W_1}(a_1,a)$, the gluing map gives an orientation preserving diffeomorphism

$$\#_{W_1}:\bigcup_{([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)\in\Xi_{W_1}(a_1,a)}\mathcal{M}_{V\times\mathbb{R}}([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)\to\mathcal{M}^{W_1}(a_1,a). \quad (19)$$

Similarly, there are orientation preserving diffeomorphisms given by the gluing maps for $W_0(r)$, $(Y(r)\times\mathbb{R},\mathfrak{s}\otimes L_0)$, $(Y_1(r)\times\mathbb{R},\mathfrak{s}\otimes L_0)$ and $(Y_0(r)\times\mathbb{R},\mathfrak{s}\otimes L_{nk+p})$ defined over the set of admissible triples, which are in turn determined by the corresponding inequivalent oriented holomorphic triangles or holomorphic discs. The corresponding sets are denoted by $\Xi_{W_0}(a,a_0)$, $\Xi_Y(a,b)$, $\Xi_{Y_1}(a_1,b_1)$ and $\Xi_{Y_0}(a_0,b_0)$, respectively. We summarize all the gluing theorems for the geometric limits and holomorphic triangles (discs) as follows:

Theorem 4.3. *Suppose given a pair of monopoles $a^{(1)},b^{(1)}$ in $\mathcal{M}_{Y_1}(\mathfrak{s}\otimes L_0)$, and a pair of monopoles $a(\epsilon),b(\epsilon)$ in $\mathcal{M}_{Y_1,\mu}(\mathfrak{s}\otimes L_0)$, where the surgery perturbation μ determined by f_0' depends on a small parameter ϵ. Suppose given a pair of monopoles $a^{(0)},b^{(0)}$ in $\mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$ for some $k\in\mathbb{Z}$ and $p\in\{0,\cdots,n-1\}$. Then, for sufficiently large r, the gluing maps give the following orientation preserving diffeomorphisms*

$$\begin{aligned}
&\textstyle\bigcup_{\Theta\in\Xi_Y(a(\epsilon),b(\epsilon))}\mathcal{M}_{V\times\mathbb{R}}(\Theta)\overset{\#_{Y(r)\times\mathbb{R}}}{\longrightarrow}\mathcal{M}_{Y\times\mathbb{R}}(a(\epsilon),b(\epsilon)),\\
&\textstyle\bigcup_{\Theta\in\Xi_{Y_1}(a^{(1)},b^{(1)})}\mathcal{M}_{V\times\mathbb{R}}(\Theta)\overset{\#_{Y_1(r)\times\mathbb{R}}}{\longrightarrow}\mathcal{M}_{Y_1\times\mathbb{R}}(a^{(1)},b^{(1)}),\\
&\textstyle\bigcup_{\Theta\in\Xi_{Y_0}(a^{(0)},b^{(0)})}\mathcal{M}_{V\times\mathbb{R}}(\Theta)\overset{\#_{Y_0(r)\times\mathbb{R}}}{\longrightarrow}\mathcal{M}_{Y_0\times\mathbb{R}}(a^{(0)},b^{(0)}),\\
&\textstyle\bigcup_{\Theta\in\Xi_{W_1}(a^{(1)},a(\epsilon))}\mathcal{M}_{V\times\mathbb{R}}(\Theta)\overset{\#_{W_1}}{\longrightarrow}\mathcal{M}^{W_1}(a^{(1)},a(\epsilon)),\\
&\textstyle\bigcup_{\Theta\in\Xi_{W_0}(a(\epsilon),a^{(0)})}\mathcal{M}_{V\times\mathbb{R}}(\Theta)\overset{\#_{W_0}}{\longrightarrow}\mathcal{M}^{W_0}(a(\epsilon),a^{(0)}),
\end{aligned}$$

where, for simplicity, we denoted a triple $([A,\psi],[\tilde{A},\tilde{\psi}],a_\infty)$ by Θ.

5 Proof of exactness and the surgery triangle

In this section, we will prove the main theorem of this paper. Notice that we only prove the refined exact triangle in Theorem 1.1 for $m=0$. The arguments for this case can be adapted to give an analogous proof for $m\neq 0$.

5.1 The chain homomorphisms

Recall that the moduli spaces for W_1 and W_0 which we consider are only the components of minimal energy (or dimension). They are smooth and oriented manifolds of the expected dimension, and they can be compactified according to Proposition 4.1. In particular, whenever one such moduli space is 0-dimensional, we have a counting of points with the orientation. Thus, as in [6], we can define the chain homomorphisms between the Floer chain groups of $(Y,\mu,\mathfrak{s}\otimes L_0)$, $(Y_1,\mathfrak{s}\otimes L_0)$ and $(Y_0,\mathfrak{s}\otimes L_{nk+p})$ as follows.

Definition 5.1. *We define the map $w^1_*: C_*(Y_1)\to C_*(Y,\mu)$ by assigning the matrix elements*

$$\langle a, w^1_*(a_1)\rangle = \#\mathcal{M}^{W_1}(a,a_1),$$

where the right hand side is a counting of points with the orientation if $\mathcal{M}^{W_1}(a,a_1)$ is 0-dimensional and non-empty, and it is $\langle a, w^1_*(a_1)\rangle = 0$ otherwise. Similarly, we define the map $w^0_*: C_*(Y,\mu)\to \oplus_k C_{(*)}(Y_0,\mathfrak{s}\otimes L_{nk+p})$, with the the matrix coefficients

$$\langle a_0, w^0_*(a)\rangle = \#\mathcal{M}^{W_0}(a,a_0)$$

if $\mathcal{M}^{W_0}(a,a_0))$ is 0-dimensional and non-empty, and $\langle a_0, w^0_*(a)\rangle = 0$ otherwise. Here the chain groups $C_{(*)}(Y_0,\mathfrak{s}\otimes L_{nk+p})$ are equipped with the lifting of the relative grading on $\mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$ as described in Proposition 3.2.

Notice that on W_1 there is a $\mathbb{Z}$-family of Spin^c structures which agree with $\mathfrak{s}\otimes L_0$ when restricted to the two ends. From Lemma 4.1 in [6], the choice of components with minimal energy and dimension essentially excludes other Spin^c structures when we consider the moduli space $\mathcal{M}^{W_1}(a_1,a)$ with fixed asymptotic values $a_1\in\mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0)$ and $a\in\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$.

In the case of $b_1(Y)>0$ and $c_1(\mathfrak{s}\otimes L_0)$ is non-torsion, when both $\mathcal{M}_{Y,\mu}(\mathfrak{s}\otimes L_0)$ and $\mathcal{M}_{Y_1}(\mathfrak{s}\otimes L_0)$ are $\mathbb{Z}_{2\ell}$-graded, with 2ℓ the multiplicity of $c_1(\mathfrak{s}\otimes L_0)$ in $H^2(Y,\mathbb{Z})/\text{Torsion}$, then Proposition 3.2 ensures that there exists a compatible grading on $\mathcal{M}_{Y_1}(\mathfrak{s}\otimes L_0)$ and a $\mathbb{Z}_{2\ell}$–lifting of the relative grading on any $\mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$. For those Spin^c structures $\mathfrak{s}\otimes L_0$ with torsion $c_1(\mathfrak{s}\otimes L_0)$, the corresponding Floer homology is defined to be

$$HF^{SW}_*(Y,\mathfrak{s}\otimes L_0,\mathbb{Z}[[t]])|t=0,$$

as described in [7]. Then it is easy to see that the choice of components with minimal energy and dimension makes w^1_* and w^0_* well-defined.

Then, with the help of the compactifications in Proposition 4.1, the proofs of Lemma 4.3 and Lemma 4.4 in [6] go through without any substantial change to give the following Lemma.

Lemma 5.2. *1. The maps w^i_* are chain homomorphisms.*

2. *Suppose given a_1 in $\mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0)$ and a_0 in $\mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$, such that the relative index induced from $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$ as in Proposition 3.2 is zero, then the composite map $w^0_* \circ w^1_*$ is given by*

$$\langle w^0_* \circ w^1_*(a_1), a_0\rangle = \#\mathcal{M}^W(a_1, a_0).$$

Here we use the same convention on the choice of the moduli spaces for $W = W_1\#_Y W_0$.

Thus, we have obtained a sequence of chain complexes induced by the surgery cobordisms:

$$0 \to C_*(Y_1, \mathfrak{s}\otimes L_0) \xrightarrow{w^1_*} C_*(Y,\mu, \mathfrak{s}\otimes L_0) \xrightarrow{w^0_*} \oplus_{k\in\mathbb{Z}} C_{(*)}(Y_0, \mathfrak{s}\otimes L_{nk+p}) \to 0,$$

for any fixed $p \in \{0, \cdots, n-1\}$. We shall prove that this sequence is exact and that the corresponding exact triangle is the surgery triangle, namely the connecting homomorphism is also defined via a surgery cobordism. The gluing theorems 4.3 for the admissible geometric limits on $V\times\mathbb{R}$ and the corresponding holomorphic triangles (or discs) will play a crucial role in the proof of these statements.

5.2 Proof of exactness

We first prove that w^1_* is injective and w^0_* is surjective, then we show that as in [6], $w^0_*\circ w^1_* = 0$. This, together with the specific properties of the maps w^1_* and w^0_* will be sufficient to prove the exactness in the middle term of the sequence:

$$0 \to C_*(Y_1, \mathfrak{s}\otimes L_0) \xrightarrow{w^1_*} C_*(Y,\mu, \mathfrak{s}\otimes L_0) \xrightarrow{w^0_*} \oplus_{k\in\mathbb{Z}} C_{(*)}(Y_0, \mathfrak{s}\otimes L_{nk+p}) \to 0.$$

We can partition the moduli space $\mathcal{M}_{Y,\mu}(\mathfrak{s}\otimes L_0)$ according to the decomposition (12) of Theorem 2.2,

$$\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0) \cong \mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0) \cup \bigcup_{k\in\mathbb{Z}} \mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p}),$$

where μ represents the surgery perturbation determined by f'_0 as in the proof of Theorem 2.2. Here f'_0 depends on a small parameter $\epsilon > 0$. That is, we identify $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$ with a collection of points

$$\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0) = \{a^{(1)}_i(\epsilon)\}_{i=1,\dots r} \cup \{a^{(0)}_j(\epsilon)\}_{j=r+1,\dots,s}, \tag{20}$$

so that, as we let $\epsilon \to 0$, the points $\{a_i^{(1)}(\epsilon)\}_{i=1,\dots r}$ get identified with the corresponding elements $\{a_i^{(1)}\}_{i=1,\dots r}$ in $\mathcal{M}^*_{Y_1}(\mathfrak{s}\otimes L_0)$ and, similarly, the points $\{a_j^{(0)}(\epsilon)\}_{j=r+1,\dots,s}$ get identified with the corresponding elements $\{a_j^{(0)}\}_{j=r+1,\dots,s}$ in $\cup_{k\in\mathbb{Z}}\mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$.

Lemma 5.3. *The coefficients of the maps w^1_* and w^0_* satisfy*

1. $\langle a_i^{(1)}(\epsilon), w^1_*(a_j^{(1)}\rangle = \delta_{ij}$;
2. $\langle a_i^{(0)}, w^0_*(a_j^{(0)}(\epsilon)\rangle = \delta_{ij}$.

Thus, w^1_* is injective and w^0_* is surjective.

Proof. Using the gluing Theorem 4.3, we can describe the moduli space $\mathcal{M}^{W_1}\big(a_i^{(1)}, a_j^{(1)}(\epsilon)\big)$ for two monopoles $a_i^{(1)}, a_j^{(1)}(\epsilon)$ of relative index 0. With our convention on the choice of components for these moduli spaces, we know that $\mathcal{M}^{W_1}\big(a_i^{(1)}, a_j^{(1)}(\epsilon)\big)$ is zero-dimensional, if non-empty, and obtained as the gluing of the admissible geometric limits on $V\times\mathbb{R}$ and the corresponding holomorphic triangles.

As we let $\epsilon\to 0$, it is easy to see that the holomorphic triangles degenerate to certain holomorphic discs, and the admissible geometric limits on $V\times\mathbb{R}$ for $\mathcal{M}^{W_1}\big(a_i^{(1)}, a_j^{(1)}(\epsilon)\big)$ are identified with the admissible geometric limits on $V\times\mathbb{R}$ for $\mathcal{M}_{Y_1\times\mathbb{R}}(a_i^{(1)}, a_j^{(1)})$. Since the relative index of $a_i^{(1)}, a_j^{(1)}$ is zero, $\mathcal{M}_{Y_1\times\mathbb{R}}(a_i^{(1)}, a_j^{(1)})$, being a zero-dimensional moduli space of minimal energy, is empty unless $a_i^{(1)} = a_j^{(1)}$, in which case $\mathcal{M}_{Y_1\times\mathbb{R}}(a_i^{(1)}, a_i^{(1)})$ consists of a unique solution. This proves that

$$\langle a_i^{(1)}(\epsilon), w^1_*(a_j^{(1)}\rangle = \delta_{ij}.$$

Similarly, we obtain $\langle a_i^{(0)}, w^0_*(a_j^{(1)}(\epsilon)\rangle = \delta_{ij}$. ☐

Now we prove the exactness in the middle term. We proceed as in [6] to show that $w^0_*\circ w^1_* = 0$, which, together with Lemma 5.3 is sufficient to establish the exact triangle. Again, we will use heavily the gluing theorem 4.3 to analyze the moduli spaces on the cobordisms. The following Lemma is the direct consequence of the results of Lemma 5.2 and Lemma 5.3.

Lemma 5.4. *Suppose given $a_i^{(1)}\in\mathcal{M}_{Y_1}(\mathfrak{s}\otimes L_0)$ and $a_j^{(0)}(\epsilon)\in\mathcal{M}_{Y,\mu}(\mathfrak{s}\otimes L_0)$ which corresponds to $a_j^{(0)}\in\cup_{k\in\mathbb{Z}}\mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_{nk+p})$. The coefficients of the* *composition map $w^0_* \circ w^1_*$ satisfy*

$$\langle a_j^{(0)}, w^0_* \circ w^1_*(a_i^{(1)})\rangle = \langle a_j^{(0)}(\epsilon), w^1_*(a_i^{(1)})\rangle + \langle a_j^{(0)}, w^0_*(a_i^{(1)}(\epsilon))\rangle.$$

Since the coefficient $\langle a_j^{(0)}(\epsilon), w^1_*(a_i^{(1)})\rangle$ is given by the counting of monopoles in $\mathcal{M}^{W_1}(a_i^{(1)}, a_j^{(0)}(\epsilon))$ with the orientation, and $\langle a_j^{(0)}, w^0_*(a_i^{(1)}(\epsilon))\rangle$ is given by the counting of monopoles in $\mathcal{M}^{W_0}(a_i^{(1)}(\epsilon), a_j^{(0)})$ with the orientation, the gluing Theorem 4.3, for the admissible geometric limits on $V \times \mathbb{R}$ and the corresponding holomorphic triangles, yields the following Lemma (cf. Theorem 6.9 [6]).

Lemma 5.5. *For small enough ϵ and large $r \geq r_0$, there is an orientation reversing diffeomorphism*

$$\mathcal{M}^{W_1(r)}\big(a_i^{(1)}, a_j^{(0)}(\epsilon)\big) \cong \mathcal{M}^{W_0(r)}\big(a_i^{(1)}(\epsilon), a_j^{(0)}\big).$$

Hence, we have $w^0_* \circ w^1_* = 0$.

The Lemma corresponds to the fact that, in the two cases, the same triangles are counted with the reverse orientation.

With these Lemmata at hand, the arguments in Section 6.5 [6] yield the following exact triangle for any $p \in \{0, \cdots, n-1\}$:

$$\begin{tikzcd} HF_*^{SW}(Y_1, \mathfrak{s} \otimes L_0) \arrow[r, "w^1_*"] & HF_*^{SW}(Y, \mathfrak{s} \otimes L_0) \arrow[dl, "w^0_*"'] \\ \bigoplus_{k\in\mathbb{Z}} HF_{(*)}^{SW}(Y_0, \mathfrak{s} \otimes L_{nk+p}) \arrow[u, "\Delta_{(*)}"'] & \end{tikzcd} \tag{21}$$

After a possible shift of relative index, we obtain the general exact triangle for any $m, p \in \{0, \cdots, n-1\}$:

$$\begin{tikzcd} HF_*^{SW}(Y_1, \mathfrak{s} \otimes L_0) \arrow[r, "w^1_*"] & HF_*^{SW}(Y, \mathfrak{s} \otimes L_0) \arrow[dl, "w^0_*"'] \\ \bigoplus_{k\in\mathbb{Z}} HF_{(*)}^{SW}(Y_0, \mathfrak{s} \otimes L_{nk+p}) \arrow[u, "\Delta_{(*)}"'] & \end{tikzcd}$$

This completes the proof of our main result, Theorem 1.1, except for the claim that the connecting homomorphisms $\Delta_{(*)}$ are induced by the surgery cobordism connecting Y_0 to Y_1. We shall discuss this statement in the next subsection.

5.3 The surgery triangle

To give a precise description of the connecting homomorphism $\Delta_{(*)}$, we need to study the discrepancy between the boundary operator ∂_Y of the Floer complex $C_*(Y, \mathfrak{s}\otimes L_0, \mu)$ and the operator $\partial_{Y_1} \oplus \bigoplus_{k\in\mathbb{Z}} \partial_{Y_0,k}$ on

$$C_*(Y_1, \mathfrak{s}\otimes L_0) \oplus \bigoplus_{k\in\mathbb{Z}} C_{(*)}(Y_0, \mathfrak{s}\otimes L_{nk+p}).$$

Let us identify again the points of $\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0)$ as in (20) with

$$\mathcal{M}^*_{Y,\mu}(\mathfrak{s}\otimes L_0) = \{a_i^{(1)}(\epsilon)\}_{i=1,\dots r} \cup \{a_j^{(0)}(\epsilon)\}_{j=r+1,\dots,s}.$$

Then we have the following Lemma which gives the connecting homomorphism $\Delta_{(*)}$.

Lemma 5.6. *Suppose given a cycle in $\sum_i x_i a_i^{(0)}$ in $C_{(*)}(Y_0, \mathfrak{s}\otimes L_{nk+p})$. The image of $\sum_i x_i a_i^{(0)}$ under the connecting homomorphism Δ is given by*

$$\Delta(\sum_i x_i a_i^{(0)}) = \sum_{i,j} x_i \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_i^{(0)}(\epsilon), a_j^{(1)}(\epsilon))\big) a_j^{(1)}.$$

Proof. (Lemma 7.1, Lemma 7.2 in [6]) Using the diagram chasing, Lemma 5.3 and Lemma 5.5, we see that we have

$$\begin{aligned}
&\partial_{Y,\mu}(\textstyle\sum_i x_i a_i^{(0)}(\epsilon)) \\
=\ & \textstyle\sum_{i,j} x_i \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_i^{(0)}(\epsilon), a_j^{(1)}(\epsilon))\big) a_j^{(1)}(\epsilon) \\
& - \textstyle\sum_{i,j,k} x_i \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_i^{(0)}(\epsilon), a_j^{(1)}(\epsilon))\big) \#\big(\mathcal{M}^{W_0}(a_j^{(1)}(\epsilon), a_k^{(0)})\big) a_k^{(0)}(\epsilon) \\
=\ & \textstyle\sum_{i,j} x_i \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_i^{(0)}(\epsilon), a_j^{(1)}(\epsilon))\big) a_j^{(1)}(\epsilon) \\
& + \textstyle\sum_{i,j,k} x_i \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_i^{(0)}(\epsilon), a_j^{(1)}(\epsilon))\big) \#\big(\mathcal{M}^{W_1}(a_j^{(1)}, a_k^{(0)})(\epsilon)\big) a_k^{(0)}(\epsilon).
\end{aligned}$$

By comparing this expression with

$$\begin{aligned}
& w^1_*(\textstyle\sum_i x_i \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_i^{(0)}(\epsilon), a_j^{(1)}(\epsilon))\big) a_j^{(1)}) \\
=\ & \textstyle\sum_j x_j \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_j^{(0)}(\epsilon), a_i^{(1)}(\epsilon))\big) a_i^{(1)}(\epsilon) \\
& + \textstyle\sum_{i,j,k} x_i \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_i^{(0)}(\epsilon), a_j^{(1)}(\epsilon))\big) \#\big(\mathcal{M}^{W_1}(a_j^{(1)}, a_k^{(0)}(\epsilon)) a_k^{(0)}(\epsilon)\big),
\end{aligned}$$

we obtain

$$\Delta(\sum_i x_i a_i^{(0)}) = \sum_{i,j} x_i \#\big(\mathcal{M}_{Y\times\mathbb{R}}(a_i^{(0)}(\epsilon), a_j^{(1)}(\epsilon)\big) a_j^{(1)}.$$

This completes the proof. □

In next proposition, we show that the connecting homomorphism in the exact sequence can also be described as a map $w_{(*)}$, induced by a surgery cobordism $\bar{W}_2$ connecting Y_0 to Y_1, which satisfies $W_1\#_Y W_0 = W_2\#\bar{\mathbb{CP}}^2$. The resulting diagram

$$C_*(Y_1, \mathfrak{s}\otimes L_0) \overset{w_*^1}{\to} C_*(Y, \mathfrak{s}\otimes L_0, \mu) \overset{w_*^0}{\to} \oplus_k C_{(*)}(Y_0, \mathfrak{s}\otimes L_{nk+p}) \overset{w_{(*)}}{\to} C_*(Y_1, \mathfrak{s}\otimes L_0)[-1]$$

is therefore a distinguished triangle, the surgery triangle.

Proposition 5.7. *(Proposition 7.3 [6]) The connecting homomorphism $\Delta_{(*)}$ in the exact triangle is given by the following expression,*

$$\Delta_{(*)}(\sum_i x_i a_i^{(0)}) = w_{(*)}(\sum_i x_i a_i^{(0)}),$$

for any cycle $\sum_i x_i a_i^{(0)}$ in $\oplus_k C_{(*)}(Y_0, \mathfrak{s} \otimes L_{nk+p})$ for any fixed $p \in \{0, \cdots, n-1\}$, where

$$w_{(*)} : \oplus_k C_{(*)}(Y_0, \mathfrak{s} \otimes L_{nk+p}) \to C_*(Y_1, \mathfrak{s} \otimes L_0)[-1]$$

is the homomorphism defined by counting solutions in the zero-dimensional components of the moduli spaces

$$\mathcal{M}^{\bar{W}_2}(a_j^{(0)}, a_i^{(1)}),$$

over the cobordism $\bar{W}_2$.

Proof. The argument is exactly the same as in the proof of Proposition 7.3 in [6], therefore we shall omit the proof here. □

6 Seiberg–Witten and Casson–Walker invariant

In this section, we derive the relation between the topologically invariant version of the Seiberg-Witten invariant and the Casson-Walker invariant for rational homology 3-spheres. Together with the equivalence between the Casson-Walker invariant and the theta invariant introduced by Ozsváth and Szabó in [10], our result proves their conjecture relating the Seiberg-Witten invariant and their theta invariant.

Let Y be a rational homology 3-sphere with a smoothly embedded knot K representing a torsion element of order n in $H_1(Y, \mathbb{Z})$,

$$\frac{|H_1(Y, \mathbb{Z})|}{|\mathrm{Torsion}(H_1(Y - \nu(K), \mathbb{Z}))|} = n,$$

and endowed with the canonical framing (m, l) in $\nu(K) \cong D^2 \times S^1$. Let p and q be relatively prime integers. The Dehn surgery with coefficient $p/q \in \mathbb{Q} \cup \{\infty\}$ on K gives rise to another closed manifold $Y_{p/q}$.

Denote by $\mathrm{Spin}^c(V)$ the set of equivalence classes of Spin^c structures on $V = Y \backslash \nu(K)$ with trivial restriction to the boundary T^2. Then, for any $Y_{p/q}$, there is a surjective map:

$$\iota_{Y_{p/q}} : \qquad \mathrm{Spin}^c(Y_{p/q}) \to \mathrm{Spin}^c(V),$$

where, for any $\mathfrak{s} \in \mathrm{Spin}^c(Y_{p/q})$, $\iota_{Y_{p/q}}(\mathfrak{s})$ is given by the restriction to $V \subset Y_{p/q}$. The fiber of $\iota_{Y_{p/q}}$ is given by a cyclic group generated by the Poincaré dual of the core of $Y_{p/q} \backslash V$. Both V and $\nu(K)$ are equipped with a metric with a cylindrical end modeled on T^2. Formally, for $\iota_Y(\mathfrak{s})$ with $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, we identify the fiber of ι_Y with the following set of Spin^c structures

$$\iota_Y^{-1}(\iota_Y(\mathfrak{s}) = \bigcup_{m=0,\cdots,n-1} \{\mathfrak{s} \otimes L_m | c_1(L_m) = mPD([K]) \in H^2(Y, \mathbb{Z})\}.$$

Similarly the fiber of $\iota_{Y_{p/q}}$ is given by

$$\bigcup_{m=0,\cdots,np-1} \{\mathfrak{s} \otimes L_m | c_1(L_m) = mPD([K]) \in H^2(Y_{p/q}, \mathbb{Z})\},$$

and the fiber of ι_{Y_0} is given by

$$\bigcup_{m \in \mathbb{Z}} \{\mathfrak{s} \otimes L_m | c_1(L_m) = mPD([K]) \in H^2(Y_0, \mathbb{Z})\}.$$

Here we use the same notation $\mathfrak{s}$ on $Y_{p/q}$ (Y, or Y_0) as the corresponding Spin^c structure obtained by gluing $\mathfrak{s} \in \mathrm{Spin}^c(V)$ with the trivial Spin^c structure on $\nu(K)$ by the trivial gauge transformation on T^2. We hope this notation will not cause any confusion.

Let $\mathfrak{s}$ be a Spin^c structure on Y. By the result of [2] on the moduli space of finite energy monopoles on $(V, \iota_Y(\mathfrak{s}))$, we know that the irreducible part, denoted $\mathcal{M}^*_V(\mathfrak{s})$, is a smooth, oriented 1-dimensional manifold. The asymptotic values along the cylindrical end define a continuous map:

$$\partial_\infty : \qquad \mathcal{M}^*_V(\mathfrak{s}) \to \chi_0(T^2, V),$$

where $\chi_0(T^2, V)$ is a $\mathbb{Z} \times \mathbb{Z}_n$-covering of the character torus $\chi(T^2)$. Sometime it is convenient to compose the above asymptotic value map with this covering map and define a boundary value map:

$$\partial_\infty : \qquad \mathcal{M}^*_V(\mathfrak{s}) \to \chi(T^2). \tag{22}$$

Notice that the reducible part $\chi(V)$ of the moduli space on $(V, \iota_Y(\mathfrak{s}))$ is an embedded circle $\chi(V) \subset \chi_0(T^2, V)$ under the asymptotic value map. This becomes a circle of multiplicity n in $\chi(T^2)$. There is a "bad point" in $\chi(T^2)$, given by the flat connections such that the corresponding twisted Dirac operator has a non-trivial kernel. We can endow $\chi(T^2)$ with a coordinate system (u, v) defined by the holonomy around the longitude l and the meridian m, respectively, so that the bad point corresponds to $(u, v) = (1, 1)$. The reducible circle $\chi(V)$, with the holonomy around the longitude l of order n, is given by $u = u(\mathfrak{s})$, with $u(\mathfrak{s}) \in \{0, 2/n, \cdots, 2(n-1)/n\}$. After a suitable perturbation, and a corresponding shift of coordinates, as discussed in [2], we can assume that the bad point does not lie on any of the n possible circles $u = u(\mathfrak{s})$ of reducibles $\chi(V)$.

From the result in [2], we know that, under the map ∂_∞ in (22), the boundary points $\partial(\mathcal{M}^*_V(\mathfrak{s}))$ are either mapped to the bad point in $\chi(T^2)$ or mapped to the reducible circle $u = u(\mathfrak{s})$ on $\chi(T^2)$.

Let $\chi(\nu(K) \subset Y_{p/q})$ be the reducible circle on $\chi(T^2)$, which maps to a closed curve on $\chi(T^2)$ with slope p/q in the (u, v)-coordinates: parallel to $pv = qu$. Looking at the induced Spin structure on $T^2 \subset Y_{p/q}$, we know that the curve $\chi(\nu(K) \subset Y_{p/q})$ goes through $(0, 1)$ if q is odd or goes through $(0, 0)$ if q is even, cf.[2]. Again, after a suitable perturbation, and the corresponding shift of coordinates, we can assume that this p/q-curve is away from the bad point on $\chi(T^2)$ and does not meet $u = u(\mathfrak{s})$ along the coordinate line $v = 0$. Then we know that $u = u(\mathfrak{s})$ intersects $\chi(\nu(K) \subset Y_{p/q})$ inside $\chi(T^2)$ at p points, which are denoted by $\theta_1, \cdots, \theta_p$, ordered according the orientation of $u = u(\mathfrak{s}) \subset \chi(T^2)$. They can be lifted to pn points in $\chi_0(T^2, V)$. We denote these points by $\theta_1^{(k)}, \cdots, \theta_p^{(k)}$, $(k = 0, 1, \cdots, n-1)$ according to the order. Denote by θ_0 the intersection point of $u = u(\mathfrak{s})$ with $v = 0$ in $\chi(T^2)$. This can be lifted to n-points $\theta_0^{(0)}, \theta_0^{(1)}, \cdots, \theta_0^{(n-1)}$ on $\chi_0(T^2, V)$. Moreover, we can assume that the map ∂_∞ in (22) is transverse to the curves $u = u(\mathfrak{s})$, $v = 0$ and $\chi(\nu(K) \subset Y_{p/q})$, by a suitable perturbation of the Seiberg-Witten equations on V as in [2]. We can also assume that the image $\partial_\infty \mathcal{M}^*_V(\mathfrak{s})$ does not meet the points $\theta_0, \theta_1, \cdots \theta_p$ in $\chi(T^2)$, again by suitable perturbation, as discussed in [2].

Let I be any open interval in

$$\chi(V) = \{u = u(\mathfrak{s})\} \subset \chi_0(T^2, V).$$

We denote by $SF_I(\partial\!\!\!/^V)$ the complex spectral flow of the Dirac operator on V twisted with the path of reducible connections I on V. From the analysis in [2] and [4], we know that

$$\#\big(\partial_\infty|_{\partial\mathcal{M}^*_V(\mathfrak{s})}\big)^{-1}(I) = SF_I(\partial\!\!\!/^V). \tag{23}$$

For convenience, we define

$$SF_{[\theta_i,\theta_j]}(\partial\!\!\!/^V) = \sum_{k=0}^{n-1} SF_{[\theta_i^{(k)},\theta_j^{(k)}]}(\partial\!\!\!/^V). \tag{24}$$

With this notation understood, we can state the following proposition relating the Seiberg-Witten invariants on $Y_{p/q}$, Y and Y_0.

Proposition 6.1. *Consider generic compatible small perturbations of the Seiberg-Witten equations on $Y_{p/q}$, Y and Y_0, such that the map ∂_∞ as in (22) is transverse to the curves $u = u(\mathfrak{s})$, $v = 0$ and $\chi(\nu(K) \subset Y_{p/q})$ and misses the points $\theta_0, \theta_1, \cdots, \theta_p$ in $\chi(T^2)$. Then we have the following relation:*

$$\begin{aligned} & \textstyle\sum_{k=0}^{pn-1} SW_{Y_{p/q}}(\mathfrak{s}\otimes L_k, g_{Y_{p/q}}) \\ = \quad & \textstyle p\sum_{k=0}^{n-1} SW_Y(\mathfrak{s}\otimes L_k, g_Y) + q\sum_{k\in\mathbb{Z}} SW_{Y_0}(\mathfrak{s}\otimes L_k) \\ & \textstyle +\sum_{i=1}^{p} SF_{[\theta_0,\theta_i]}(\partial\!\!\!/^V). \end{aligned}$$

Proof. By the gluing theorem for 3-dimensional monopoles as in [2], we have

$$\bigcup_{k=0}^{pn-1} \mathcal{M}^*_{Y_{p/q}}(\mathfrak{s}\otimes L_k) = \mathcal{M}^*_V(\mathfrak{s}) \times_{\chi(T^2)} \chi(\nu(K) \subset Y_{p/q}),$$

where $\chi(\nu(K) \subset Y_{p/q})$ is the p/q-curve on $\chi(T^2)$. Thus, we obtain

$$\sum_{k=0}^{pn-1} SW_{Y_{p/q}}(\mathfrak{s}\otimes L_k, g_{Y_{p/q}}) = \#\big(\mathcal{M}^*_V(\mathfrak{s}) \times_{\chi(T^2)} \chi(\nu(K) \subset Y_{p/q})\big). \tag{25}$$

Notice that the set $\{\theta_1^{(k)}, \cdots, \theta_p^{(k)} : k = 0, 1, \cdots, n-1\}$ consists of the unique reducible monopole for each $(Y_{p/q}, \mathfrak{s} \otimes L_k)$.

Similarly, we have

$$\sum_{k=0}^{n-1} SW_Y(\mathfrak{s} \otimes L_k, g_Y) = \#\big(\mathcal{M}_V^*(\mathfrak{s}) \times_{\chi(T^2)} \{v = 0\}\big). \tag{26}$$

Here the reducible set consists of $\{\theta_0^{(0)}, \theta_0^{(1)}, \cdots, \theta_0^{(n-1)}\}$.

In order to avoid the circle of reducibles on $(Y_0, \mathfrak{s} \otimes L_0)$, we need to introduce a small perturbation such that $\chi(\nu(K) \subset Y_0)$ on $\chi(T^2)$ is a small parallel shifting of $u = u(\mathfrak{s})$ such that the bad point is not contained in the narrow strip bounded by these two parallel curves. We denote this small shift of $u = u(\mathfrak{s})$ by $u = u(\mathfrak{s}) + \eta$, where η is a sufficiently small positive number. This can be achieved by a perturbation of the equations as in [2]. Then we have

$$\sum_{k \in \mathbb{Z}} SW_{Y_0}(\mathfrak{s} \otimes L_k) = \#\big(\mathcal{M}_V^*(\mathfrak{s}) \times_{\chi(T^2)} \{u = u(\mathfrak{s}) + \eta\}\big). \tag{27}$$

In order to compare the three countings in (25) – (27), we need to choose an oriented 2-chain C in $\chi(T^2)$ whose boundary 1-chain is given by

$$\begin{aligned} & \chi(\nu(K) \subset Y_{p/q}) - p\chi(\nu(K) \subset Y) - q\chi(\nu(K) \subset Y_0) \\ = \; & \chi(\nu(K) \subset Y_{p/q}) - p\{v = 0\} - q\{u = u(\mathfrak{s}) + \eta\}, \end{aligned}$$

and such that C does not contain the bad point in $\chi(T^2)$. Then, counting the boundary points of $\partial_\infty^{-1}(C)$, as a 0-chain, we obtain

$$\begin{aligned} & \#\big(\partial_\infty^{-1}(\chi(\nu(K) \subset Y_{p/q})\big) \\ = \; & p\#\big(\partial_\infty^{-1}(\{v = 0\})\big) + q\#\big(\partial_\infty^{-1}(\{u = u(\mathfrak{s}) + \eta\})\big) \\ & +\#\big(\partial_\infty |_{\partial(\mathcal{M}_V^*(\mathfrak{s}))}\big)^{-1}(C). \end{aligned} \tag{28}$$

As C does not contain the bad points, we know that the possible points of $\partial_\infty(\partial(\mathcal{M}_V^*(\mathfrak{s}))) \cap C$ all lie on the curve $u = u(\mathfrak{s})$, which misses the points $\theta_0, \theta_1, \cdots, \theta_p$. It is easy to see that C covers the intervals of $u = u(\mathfrak{s})$ between two consecutive points θ_i with different multiplicities: the multiplicities are $p, p-1, \cdots, 1, 0$, for the intervals

$$[\theta_0, \theta_1], [\theta_1, \theta_2], \cdots, [\theta_{p-1}, \theta_p], [\theta_p, \theta_0],$$

respectively. By the identity (23) and the definition (24), we know that

$$\#\big(\partial_\infty|_{\partial(\mathcal{M}^*_V(\mathfrak{s}))}\big)^{-1}(C) = \sum_{i=1}^{p} SF_{[\theta_0,\theta_i]}(\not{\partial}^V). \tag{29}$$

Combining all the identities in (25), (26), (27), (28) and (29), we obtain the proof of the proposition. □

The Seiberg-Witten invariant for any rational homology 3-sphere depends on metric and perturbation (cf.[4]). We now consider the correction term (5) as defined in the introduction. We have the following proposition relating the correction terms for $Y_{p/q}$ and Y.

Proposition 6.2. *1. For any rational homology 3-sphere Y with a* Spin^c *structure $\mathfrak{s}$ and a Riemannian metric g_Y,*

$$S\hat{W}_Y(\mathfrak{s}) = SW_Y(\mathfrak{s}, g_Y) - \xi(\mathfrak{s}, g_Y)$$

is a well-defined topological invariant.

2. For any relatively prime integers p and q, a positive integer n, and $u \in \{0, 2/n, \ldots, 2(n-1)/n\}$, we have that

$$\sum_{k=0}^{pn-1} \xi_{Y_{p/q}}(\mathfrak{s}\otimes L_k, g_{Y_{p/q}}) - p\sum_{k=0}^{n-1} \xi_Y(\mathfrak{s}\otimes L_k, g_Y) - \sum_{i=1}^{p} SF_{[\theta_0,\theta_i]}(\not{\partial}^V)$$

is independent of the manifold Y and depends only on p, q, n, and $u(\mathfrak{s}) \in \{0, 2/n, \ldots, 2(n-1)/n\}$.

Proof. Claim (1) follows from the wall-crossing formulae in [4] and the Atiyah-Patodi-Singer index theorem. The proof of claim (2) is analogous to the proof of Proposition 7.9 in [10]. We adapt their arguments to our situation. We write the standard surgery cobordism between S^3 and the Lens space $L(p,q)$ as

$$W(S^3, L(p,q)) = \big([0,1]\times S^1 \times D^2\big) \cup_{[0,1]\times S^1\times S^1} X_{p/q},$$

Then the surgery cobordism between Y and $Y_{p/q}$ can be identified as

$$W_{p/q} = \big([0,1]\times V\big) \cup_{[0,1]\times S^1\times S^1} X_{p/q}.$$

We fix a metric on $W_{p/q}$ which respects the product structure $[0,1]\times V$ and $[0,1]\times S^1\times S^1$, and agrees with g_Y and $g_{Y_{p/q}}$ on the boundaries Y and $Y_{p/q}$, respectively.

For a Spin^c structure $\mathfrak{s}\otimes L_i^{(m)}$ in $\{\mathfrak{s}\otimes L_k : k=0,\cdots,pn-1\}$ on $Y_{p/q}$, whose reducible monopole corresponds to $\theta_i^{(m)}$ (with $i\in\{1,\cdots,p\}$ and $m\in\{0,\cdots n-1\}$), we consider the Spin^c structure $\mathfrak{s}\otimes L_m$ on Y whose reducible monopole is $\theta_0^{(m)}$. Then we claim that

$$\xi_{Y_{p/q}}(\mathfrak{s}\otimes L_i^{(m)}, g_{Y_{p/q}}) - \xi_Y(\mathfrak{s}\otimes L_m, g_Y) - SF_{[\theta_0^{(m)},\theta_i^{(m)}]}(\not{\partial}^V) \tag{30}$$

is independent of Y and depends only on p,q,n and on $u(\mathfrak{s})\in\{0,2/n,\cdots,2(n-1)/n\}$.

To prove this claim, we choose a Spin^c structure $\tilde{\mathfrak{s}}$ on $W_{p/q}$ whose restriction to Y and $Y_{p/q}$ is given by $\mathfrak{s}\otimes L_m$ and $\mathfrak{s}\otimes L_i^{(m)}$, respectively, and such that $c_1(\tilde{\mathfrak{s}})^2=1$. On $(W_{p/q},\tilde{\mathfrak{s}})$, we choose a connection A, whose restriction to $V\times[0,1]$ is the path of reducibles connecting $\theta_0^{(m)}$ to $\theta_i^{(m)}$ along the curve $\chi(V)\subset\chi_0(T^2,V)$. Then we have

$$\begin{aligned}
&\xi_{Y_{p/q}}(\mathfrak{s}\otimes L_i^{(m)}, g_{Y_{p/q}}) - \xi_Y(\mathfrak{s}\otimes L_m, g_Y)\\
=\ & Ind_{\mathbb{C}}(\not{D}_A^{W_{p/q}}) - \Big(\frac{c_1(\tilde{\mathfrak{s}})^2-\sigma(W_{p/q})}{8}\Big)\\
=\ & Ind_{\mathbb{C}}(\not{D}_A^{W_{p/q}})\\
=\ & Ind_{\mathbb{C}}(\not{D}_A^{[0,1]\times V}) + Ind_{\mathbb{C}}(\not{D}_A^{X_{p/q}})
\end{aligned} \tag{31}$$

where the third equality follows from the splitting principle for the index, as the Dirac operator has no kernel on the various boundaries and corners [1]. Notice that we have

$$Ind_{\mathbb{C}}(\not{D}_A^{[0,1]\times V}) = SF_{[\theta_0^{(m)},\theta_i^{(m)}]}(\not{\partial}^V),$$

and the connection $A|_{X_{p/q}}$ extends to connection A_0 on $W(S^3,L(p,q))$ by a flat connection, whose index on $[0,1]\times S^1\times D^2$ satisfies

$$Ind_{\mathbb{C}}(\not{D}_{A_0}^{[0,1]\times S^1\times D^2}) = 0.$$

In fact, we can choose the metric on $W(S^3,L(p,q))$ with a positive scalar curvature metric on $[0,1]\times S^1\times D^2$. Therefore, we have

$$Ind_{\mathbb{C}}(\not{D}_A^{X_{p/q}}) = Ind_{\mathbb{C}}(\not{D}_{A_0}^{W(S^3,L(p,q))}),$$

which depends only on p, q, n and $u(\mathfrak{s})$, and so does the quantity

$$\xi_{Y_{p/q}}(\mathfrak{s} \otimes L_i^{(m)}, g_{Y_{p/q}}) - \xi_Y(\mathfrak{s} \otimes L_m, g_Y) - SF_{[\theta_0^{(m)}, \theta_i^{(m)}]}(\not{\partial}^V). \tag{32}$$

When summing the identity (32) over $i \in \{1, \cdots, p\}$ and $m \in \{0, \cdots, n-1\}$, notice that the term $\xi_Y(\mathfrak{s} \otimes L_m, g_Y)$ is independent of $i \in \{1, \cdots, p\}$, hence we obtain the proof of the claim (2) by using the definition (24). □

With these two propositions in place, we now have the following surgery formula for the modified version of the Seiberg-Witten invariant.

Theorem 6.3. *Given any two relatively prime integers p and q, a positive integer n and $u \in \{0, 2/n, 2(n-1)/n\}$, there is a rational valued function $s(p, q, n, u)$, depending only on p, q, n and u, satisfying the following property. Let Y be a rational homology 3-sphere with a smoothly embedded knot and a canonical framing (m, l) such that $\nu(K) \cong D^2 \times S^1$. Assume that K represents a torsion element of order n in $H_1(Y, \mathbb{Z})$. Let $\mathfrak{s}$ be a* Spin^c *structure on Y. Then we have*

$$\begin{aligned} & \textstyle\sum_{k=0}^{pn-1} S\hat{W}_{Y_{p/q}}(\mathfrak{s} \otimes L_k) \\ = \quad & p \textstyle\sum_{k=0}^{n-1} S\hat{W}_Y(\mathfrak{s} \otimes L_k,) + q \sum_{k \in \mathbb{Z}} SW_{Y_0}(\mathfrak{s} \otimes L_k) \\ & + s(p, q, n, u). \end{aligned}$$

Proof. Following from Proposition 6.2, we know that

$$\sum_{k=0}^{pn-1} \xi_{Y_{p/q}}(\mathfrak{s} \otimes L_k, g_{Y_{p/q}}) - p \sum_{k=0}^{n-1} \xi_Y(\mathfrak{s} \otimes L_k, g_Y) - \sum_{i=1}^{p} SF_{[\theta_0, \theta_i]}(\not{\partial}^V) \tag{33}$$

depends only on p, q, n and $u = u(\mathfrak{s}) \in \{0, 2/n, \cdots, 2(n-1)/n\}$, which is denoted by $s(p, q, n, u)$. By subtracting (33) from the surgery formula for the Seiberg-Witten invariants in Proposition 6.1, we obtain the proof of this theorem. □

Now we can establish the equivalence between the modified version of the Seiberg-Witten invariant $S\hat{W}$ and the Casson-Walker invariant for rational homology 3-spheres.

Theorem 6.4. *For any rational homology 3-sphere Y, we have*

$$\sum_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} S\hat{W}_Y(\mathfrak{s}) = \frac{1}{2} |H_1(Y, \mathbb{Z})| \lambda(Y)$$

where $\lambda(Y)$ is the Casson-Walker invariant.

Proof. We first derive the surgery formula for the invariant $\sum_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} S\hat{W}_Y(\mathfrak{s})$ from Theorem 6.3 and the Seiberg-Witten invariant for Y_0 (a rational homology $S^1 \times S^2$, i.e., $b_1(Y_0) = 1$) (see [8] [3]):

$$
\begin{aligned}
& \sum_{\mathfrak{s}\in \mathrm{Spin}^c(Y_{p/q})} S\hat{W}_{Y_{p/q}}(\mathfrak{s}) \\
= \ & p\sum_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} S\hat{W}_Y(\mathfrak{s}) + q\sum_{i=0}^{\infty} a_j j^2 + |H_1(Y,\mathbb{Z})| s(p,q,n)
\end{aligned} \tag{34}
$$

where $s(p,q,n) = \sum_u s(p,q,n,u)/n$ and a_j is the coefficient of the symmetrized Alexander polynomial of Y_0,

$$A(t) = |\mathrm{Torsion}(H_1(Y_0,\mathbb{Z}))| + \sum_{j=1}^{\infty} a_j(t^j + t^{-j})$$

normalized such that

$$A(1) = |\mathrm{Torsion}(H_1(Y_0,\mathbb{Z}))|.$$

Set $\bar{\lambda}(Y) = \frac{1}{2}|H_1(Y,\mathbb{Z})|\lambda(Y)$ as the normalized Casson-Walker invariant. Then the surgery formula in [11] for $\bar{\lambda}(Y)$ can be expressed as (cf. [10]):

$$
\begin{aligned}
\bar{\lambda}(Y_{p/q}) &= p\bar{\lambda}(Y) + q\sum_{i=0}^{\infty} a_j j^2 \\
&\quad + |H_1(Y,\mathbb{Z})|\left(\frac{q(n^2-1)}{12n^2} - \frac{ps(p,q)}{2}\right).
\end{aligned} \tag{35}
$$

Here $s(p,q)$ is the Dedekind sum of relatively prime integers p and q (cf. [11]). Comparing (34) and (35), we only need to show that

$$s(p,q,n) = \frac{q(n^2-1)}{12n^2} - \frac{ps(p,q)}{2}. \tag{36}$$

Since $s(p,q,n)$ is independent of the manifold Y, we can choose some examples that can be computed explicitly, and use them to identify the coefficient $s(p,q,n)$. The Lens space $L(p,q)$ can be obtained by a p/q-surgery on an unknot in S^3. The calculation of Nicolaescu [9] for $L(p,q)$ gives us that

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(L(p,q))} S\hat{W}_{L(p,q)}(\mathfrak{s}) = -\frac{ps(p,q)}{2}.$$

This implies that (36) holds for $n = 1$. Now we can prove (36) by induction on n. This is exactly the same argument as in the proof of Theorem 7.5 in [10] on the equivalence of their theta invariant and the Casson-Walker invariant. The example is the Seifert manifold $M(n,1;-n,1;q,-p)$, obtained by p/q surgery on a knot of order n in $L(n,1)\#\overline{L(n,1)}$. By Kirby calculus it is possible to show that $M(n,1;-n,1;q,-p)$ can be obtained as $(-n)$-surgery on a knot in the Lens space $L(pn-q,q)$, and can be obtained as a sequence of surgeries on knots of order less than n, see the proof of Theorem 7.5 in [10] for details. $\square$

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