The length of a shortest geodesic net on a closed Riemannian manifold.

Regina Rotman

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Abstract

In this paper we will estimate the smallest length of a minimal geodesic net on an arbitrary closed Riemannian manifold M^n in terms of the diameter of this manifold and its dimension. Minimal geodesic nets are critical points of the length functional on the space of immersed graphs into a Riemannian manifold. We prove that there exists a minimal geodesic net that consists of m geodesics connecting two points $p, q \in M^n$ of total length $\leq md$, where $m \in \{2, ..., (n+1)\}$ and d is the diameter of M^n. We also show that there exists a minimal geodesic net with at most $n+1$ vertices and $\frac{(n+1)(n+2)}{2}$ geodesic segments of total length $\leq (n+1)(n+2)FillRadM^n \leq (n+1)^2 n^n (n+2)\sqrt{(n+1)!}vol(M^n)^{\frac{1}{n}}$.

These results significantly improve one of the results of [NR2] as well as most of the results of [NR1].

Introduction

0.1 Stationary geodesic nets.

Definition 0.1 *Let Γ be a graph immersed into a Riemannian manifold M^n satisfying the following conditions:*
(1) each edge of Γ is a geodesic segment;
(2) the sum of unit vectors at each vertex tangent to the edges and directed from this vertex equals to zero.

We will then call Γ a stationary (or minimal) geodesic net in M^n.

Here, by a graph we mean a multigraph. That is, we allow loops and multiple edges between vertices.

The above conditions ensure that this immersed graph is a stationary point for the length functional on the space of immersed graphs. That is, let Γ be a stationary geodesic net on M^n, let v be a smooth vector field on M^n, let $\Phi_v(t)$ denote the corresponding 1-parameter family of diffeomorphisms of M^n, and $l_{v,\Gamma}(t) = \text{length}(\Phi_v(t)(\Gamma))$. Then the first variation of the length functional implies that $\frac{dl_{v,\Gamma}}{dt}(0) = 0$.

Note that our definition of a stationary geodesic net is slightly different from the definition of J. Hass and F. Morgan ([HM]), who asked that each edge is embedded and that different geodesic segments do not intersect.

Consider the geometric object that consists of two points in M^n and three minimal geodesic segments joining them. Suppose that the sum of the unit vectors tangent to these segments equals zero at each vertex. This object will be called a minimal (or stationary) θ-graph, (see fig. 1 (a)). It is one of the simplest examples of geodesic nets.

Minimal θ-graphs were first considered by J. Hass and F. Morgan, (see [HM]).

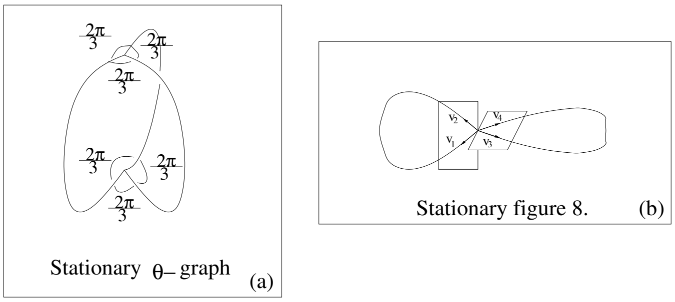

Figure 1: Geodesic Nets.

The conjecture of Hass and Morgan states that there exists a stationary θ-graph on any closed convex surface M in $\mathbf{R}^3$, such that its edges do not intersect or self-intersect. This conjecture had been proven for all convex surfaces in R^3 that are sufficiently close to the standard sphere, ([HM]).

In [NR2], among other results, we had shown that on any closed Riemannian manifold with a non-trivial second homology group there exists either a non-trivial θ-graph, or a closed geodesic, or a stationary figure 8 of length smaller than or equal to $3d$, (see fig. 1 (b)). Stationary figure 8 is a minimal geodesic net that consists of two geodesic loops with a common vertex p, such that the sum of the four unit vectors tangent to the loops at p and directed from p equals zero .

In this paper we will prove the theorem generalizing the above result. Let us begin with the following definition:

Definition 0.2 (Stationary m-cage.) *Let p, q be two distinct points of a Riemannian manifold M^n, joined by m geodesic segments, with an additional condition that the sums of unit vectors tangent to the above geodesic segments, directed from points p and q are equal to zero. We will call a graph defined in such a way a stationary m-cage. We will call a graph that consists of two vertices joined by m, not necessarily geodesic, segments, without any stationarity condition, simply an m-cage. Moreover, in the case when $p = q$ the graph that consists of $k \leq m-1$ geodesic loops starting and ending at p will be called a stationary m-cage, if the sum of all unit vectors tangent to non-constant geodesic segments and directed from p equals zero.*

A stationary m-cage is an example of a minimal geodesic net. Let us remark also that the length of the cage will be defined as the sum of lengths of individual segments.

Examples. Obviously, a closed geodesic is a stationary 2-cage, a minimal θ-graph will be an example of a stationary 3-cage, and so will be a stationary figure 8. In the latter case two points p and q coincide and the length of the third segment equals zero. Some examples of stationary 4-cages can be found on figure 2.

0.2 Main results. In the next section we will prove the following theorem.

Theorem 0.3 *Let M^n be a closed Riemannian manifold of dimension n and of diameter d. Let q be the smallest number such that $\pi_q(M^n) \neq \{0\}$. Then there exists a non-trivial stationary m-cage, Cg^m on M^n, where $2 \leq m \leq (q+1)$, such that the length of Cg^m is bounded above by md. It follows that the smallest length of a stationary geodesic net in M^n is bounded above by $(q+1)d$ and, thus, by $(n+1)d$.*

Examples. When $q = 2$, Theorem 0.3 becomes Theorem 1 of [NR2] that was cited above. Suppose that M^n is diffeomorphic to S^3 or, more generally, $q = 3$. In other words, $\pi_1(M^n) = \pi_2(M^n) = \{0\}$, but $\pi_3(M^n) \neq \{0\}$. Then the conclusion of the theorem will be that there exists a geodesic net of length $\leq 4d$ of one of the following shapes: (a) two vertices joined by at most four geodesic segments, or (b) a "flower": two or three geodesic

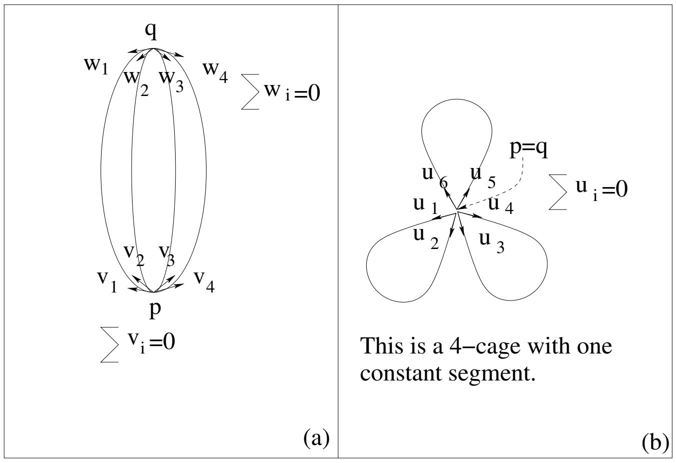

Figure 2: Examples of stationary 4-cages.

loops based at the same point p with the stationarity condition at that point, or (c) a closed geodesic.

Thus, Theorem 0.3 generalizes Theorem 1 of [NR2]. It also significantly improves Theorem 1 in [NR1], which states that under the same hypothesis as that of Theorem 0.3 above, there exists a non-trivial stationary 1-cycle on M^n that consists of at most $\frac{(q+2)!}{2}$ geodesic segments such that its length is at most $\frac{(q+2)!d}{4}$. For example, suppose M^n is diffeomorphic to S^3. Then the conclusion of this theorem would be that there must exist a non-trivial stationary 1-cycle that consists of at most 60 segments of length $\leq 30d$. However, if we assign a weight 2 to every edge in a stationary m-cage provided by Theorem 0.3, we obtain a stationary 1-cycle (i. e. a geodesic net that represents a cycle) of length at most $8d$.

While in [NR1] we have obtained the first curvature-free bounds for the length of a stationary 1-cycle in terms of diameter and, separately, in terms of volume of a manifold, presently, we have significantly improved the constants in Theorems 1 and 2 of [NR1] and, more importantly, obtained a better understanding of the shapes of graphs that must exist on M^n. The main idea behind the new proofs is the following: Consider a non-contractible sphere in M^n of smallest dimension. Unless there exists a minimal object of a certain shape of small length, we can construct a homotopy connecting this sphere with a point, thus, reaching a contradiction. The construction of the homotopy, in its turn, is based solely on the fact that curve/net shortening processes depend continuously on the initial object, assuming there is no critical points of a smaller length. Or looking at this idea from a different angle, the existence of the minimal objects is a reason, why the shortening processes are not continuous, and the spheres are not contractible. Note also, that in order to construct those homotopies, we need neither to foliate spheres by cycles or nets nor to make an explicit use of spaces of cycles or varifolds, (other than the existence of a flow that deforms it to a manifold in the absence of minimal objects), and this is precisely what allows us to better understand the geometry of the minimal objects in question and to obtain better estimates.

Note that by similar methods one can also prove the following estimate for the smallest length of a non-trivial stationary geodesic net in terms of the filling radius, defined by M. Gromov in [G].

Definition 0.4 **[G]** *Let M^n be an abstract manifold topologically imbedded into the Banach space of bounded Borel functions f on M^n, denoted as $X = L^\infty(M^n)$, where the imbedding of M^n into X is the map that assigns to each point p of M^n the distance function $p \longrightarrow f_p = d(p,q)$. Then the filling radius $FillRadM^n$ is the infimum of $\varepsilon > 0$, such that M^n bounds in the ε-neighborhood $N_\varepsilon(M^n)$, i.e. homomorphism $H_n(M^n) \longrightarrow H_n(N_\varepsilon(M^n))$ vanishes, where $H_n(M^n)$ denotes the singular homology group of dimension n with coefficients in $\mathbf{Z}$, when M is orientable, and with coefficients in $\mathbf{Z}_2$, when M is not orientable.*

Informally speaking, suppose M^n is isometrically imbedded into some metric space X, then, by the filling radius of M^n, subject to this imbedding, we mean the smallest ε, such that M^n bounds in the ε-neighborhood of M^n. For example, the filling radius of the standard 2-dimensional sphere in the Euclidean space $\mathbf{R^3}$ is exactly the radius of this sphere. Now, to define the filling radius $FillRadM^n$, we take the infimum over all the isometric imbeddings. It turns out that such infimum is achieved by isometrically imbedding M^n into the space $L^\infty(M^n)$ of bounded Borel functions on M^n. It was shown by M. Katz ([M]) that $FillRadM^n \leq \frac{d}{3}$, where d is the diameter of M^n.

The following result is due to M. Gromov, (see [G]).

Theorem 0.5 **[G]** *Let M^n be a closed connected Riemannian manifold. Then $FillRadM^n \leq gc(n)(vol(M^n))^{\frac{1}{n}}$, where $gc(n) = (n+1)n^n(n+1)!^{\frac{1}{2}}$ and $vol(M^n)$ denotes the volume of M^n.*

Here we prove the following theorem:

Theorem 0.6 *Let M^n be a closed Riemannian manifold of dimension n. Then there exists a stationary geodesic net with at most $(n+2)$ vertices and at most $\frac{(n+1)(n+2)}{2}$ geodesic segments of length $\leq (n+1)(n+2)FillRadM^n$, where $FillRadM^n$ denotes the filling radius of M^n. Its underlying graph can be obtained from the complete graph by performing one of the following operations finitely many (possibly zero) times: identifying two vertices and collapsing an edge to a point.*

The estimate of Katz in combination with Theorem 0.6 will lead us to the result that the length of a shortest geodesic net on a closed Riemannian manifold is bounded by $\frac{(n+1)(n+2)d}{3}$, which is certainly worse than the estimate of Theorem 0.3. Therefore, the main application of Theorem 0.6 would be to combine it with the volume estimate for the filling radius of M. Gromov, to obtain the volume estimate for the length of a shortest geodesic net.

Combining our Theorem 0.6 with Theorem 0.5 we obtain the following corollary.

Corollary 0.7 *Let M^n be a closed Riemannian manifold. Then there exists a stationary geodesic net consisting of at most $(n+2)$ vertices and at most $\frac{(n+1)(n+2)}{2}$ segments of total length $\leq (n+1)(n+2)gc(n)(vol(M^n))^{\frac{1}{n}}$.*

Observe that in [NR1] we have shown the existence of a stationary 1-cycle with $\leq \frac{(n+2)!}{2}$ segments of length $\leq (n+2)!FillRadM^n$. Our present results sharply reduce the values of the constants and provide a much more specific information about the shape of a minimal object.

0.3 The scheme of the proofs of Theorems 0.3 and 0.6. The proof of Theorem 0.3 is given in the next section. In section 1.1 we describe the proof in the easier cases of $q = 1$ and $q = 2$ and sketch the basic ideas in the general case. Section 1.2 contains the formal proof.

The rough scheme of the proof goes as follows. We begin with a non-contractible map of a sphere of a smallest possible dimension. Assuming there is no "small" geodesic cages, we extend the map to a disc, thus reaching a contradiction. This extension process reduces to problem of "filling" m-cages by m-discs for all values of $m \leq q+1$, which can be performed by means of an inductive bootstrap procedure: For every m at the mth step we consider an m-cage and construct an $(m-1)$-sphere and m-disc "filling" this cage. A sphere is obtained by gluing discs, just as we glue $(m-1)$-dimensional simplices in the boundary of an m-simplex to obtain a sphere, (only, since one of those simplices is small, we treat it as a point). An m-disc is constructed by producing a 1-parameter family of $(m-1)$-spheres that start with the original sphere and end with a point. This family of spheres is created by contracting the original m-cage to a point, (using an assumption that there is no "small" geodesic cages) and at each time constructing an $(m-1)$-sphere, as it was discussed before. Here we also use a fact, that in the absence of minimal objects those spheres will change continuously. This proof uses a length shortening process for m-cages, which is an adaptation of a general length shortening process introduced in [NR1]. For the sake of completeness we discuss how to adapt the process of [NR1] to the case of m-cages in section 2.

In section 3 we prove Theorem 0.6. The proof is based on the combination of the ideas from the proof of Theorem 0.3 and the trick by M. Gromov from [G] involving filling M^n by a polyhedron W^{n+1} in $L^\infty(M^n)$, attempting to extend the identity map on M^n to W^{n+1} and obtaining a geodesic net as an obstruction to this extension.

1 The proof of Theorem 0.3.

1.1 The main ideas of the proof. First, let us describe the main ideas underlying the proof of Theorem 0.3.

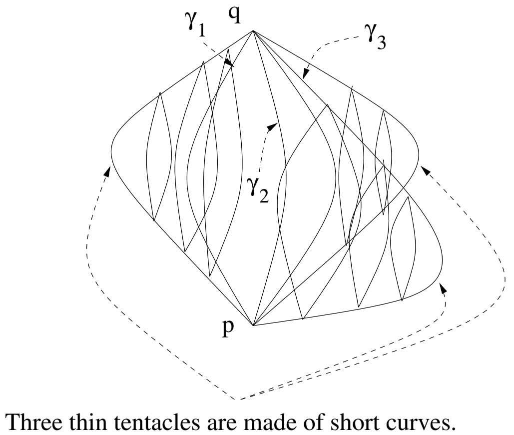

Figure 3: A thin-tentacled 2-sphere.

1.1.1 The case $q = 1$. Observe that a simple argument shows that if M^n is not simply connected then the length of a shortest closed geodesic on M^n is bounded above by $2d$, where d is the diameter of M^n: Consider a non-contractible loop $f : S^1 \longrightarrow M^n$. We subdivide S^1 into segments such that the diameter of each segment in the subdivision of $f(S^1)$ induced by f is smaller than some small δ. Let D^2 be the standard 2-disc triangulated as a cone over the triangulation of S^1. Assuming that the length of a shortest closed geodesic is $> 2d + \delta$ one can extend f to D^2 as follows. First, we map the center of the disc $\tilde{p}$ to an arbitrary point p of M^n. Next, we extend to the 1-skeleton of D^2 by mapping edges of the form $[\tilde{p}, \tilde{v}_i]$ to minimal geodesic segments $[p, v_i]$ connecting p with $v_i = f(\tilde{v}_i)$ of length $\leq d$. Thirdly, we extend to 2-skeleton by assigning to a 2-simplex $[\tilde{p}, \tilde{v}_i, \tilde{v}_j]$ a disc generated by a curve shortening homotopy connecting the image of its boundary (of length $\leq 2d + \delta$) with a point, thus reaching a contradiction. It is at the last stage that we use our assumption about nonexistence of short geodesics. Finally, we let δ go to zero.

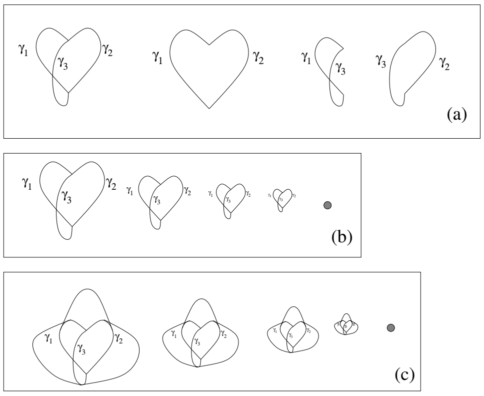

Figure 4: Deforming a 2-sphere to a point

1.1.2 The case $q = 2$. Next, consider the case of $q = 2$, that is the case of a simply connected manifold M^n with a non-trivial second homotopy group. This is the case considered in [NR2]. The idea of the proof was to consider a subdivision of S^2 by three meridians, connecting the North and the South poles, into three 2-cells. One can introduce a notion of a "thin-tentacled" 2-sphere in M (see fig. 3), which is a non-contractible map of the subdivided S^2 into M that takes the 0-skeleton into some two points p, q and the 1-skeleton into three curves $\gamma_1, \gamma_2, \gamma_3$ each of length at most d, where d is the diameter of M. The 2-skeleton is mapped by considering three pairs of curves: $\gamma_1 \cup -\gamma_2, \gamma_2 \cup -\gamma_3$ and $\gamma_3 \cup -\gamma_1$, (see fig. 4(a)) and contracting each curve to a point by a curve shortening process. Next, we try to continuously deform the 1-skeleton $\gamma_1 \cup \gamma_2 \cup \gamma_3$ of the thin-tentacled sphere to a point in a way that decreases its total length (see fig. 4(b)). This leads to a 1-parameter family $\theta_\tau, \tau \in [0,1]$ of (not stationary) θ-graphs, that begins with the original θ-graph and ends with a point, (see fig. 4(b)). At each time τ, we can then consider three digons and apply the curve shortening process to each of them, (see fig. 4(a)). Assuming that there are no periodic geodesics of length $\leq 3d$, each of those digons can be contracted to a point without the length increase, resulting in three discs, which, in turn, can be glued together to form a 2 sphere S^2_τ. As the length of θ-graph becomes small, the length of each geodesic segment becomes small, thus, the lengths of all three digons become small. Therefore all three discs forming a thin-tentacled sphere S^2_τ become small as well, and the resulting sphere converges to a point. Therefore, if M is diffeomorphic to S^2 then deforming the 1-skeleton together with the assumption $l(M) > 3d$ will also deform M to a point along itself, (see fig. 4(c)) thus reaching a contradiction. That S^2_τ changes continuously with τ is due to the fact that the curve shortening process depends continuously on a curve, unless there is a short closed geodesic present. To illustrate this assertion, consider the standard 2-sphere. Let E be its equator that divides this sphere into northern and southern hemispheres. When one applies, let's say, Birkhoff Curve Shortening Process to parallels $\sigma_N(t)$ and $\sigma_S(t)$ that are close to E in the northern and the southern hemisphere, respectively, one sees that $\sigma_N(t)$ contracts to the north pole and $\sigma_S(t)$ contracts to the south pole of S^2. So there is, of course, no continuity of a curve shortening process with respect to the original curve, but only due to the existence of a closed geodesic E. A similar situation occurs with geodesic nets.

1.1.3 The general case. The idea of the proof in the case of an arbitrary q is the following. Given a non-stationary m-cage, and assuming there are no "small" stationary i-cages for $i \leq m$, it is possible to "fill" this m-cage with an $(m-1)$-dimensional sphere and an m-dimensional disc. The procedure of filling m-cages is a bootstrap procedure, that is in order to "fill" an m-cage with a sphere, we need to be able to "fill" its $(m-1)$-subcages with a disc. A sphere is then obtained by gluing those discs. And, in order to "fill" an $(m-1)$-cage with a disc, it is necessary to be able to "fill" some auxiliary $(m-1)$-cages with spheres. Formally speaking, the "filling" is a continuous map from the space of m-cages to the space of maps of S^{m-1} (or D^m) to M^n, so that the m-cage will be an image of m meridians of $S^{m-1} = \partial D^m$.

More specifically, here is what we do. Let us first assume, for the sake of the exposition that $q = 3$. Consider a non-contractible map $f : S^3 \longrightarrow M^n$, from a finely triangulated sphere to M^n. Assuming that the conclusion of Theorem 0.3 is not satisfied, we will extend this map to the 4-dimensional disc D^4, triangulated as a cone over S^3.

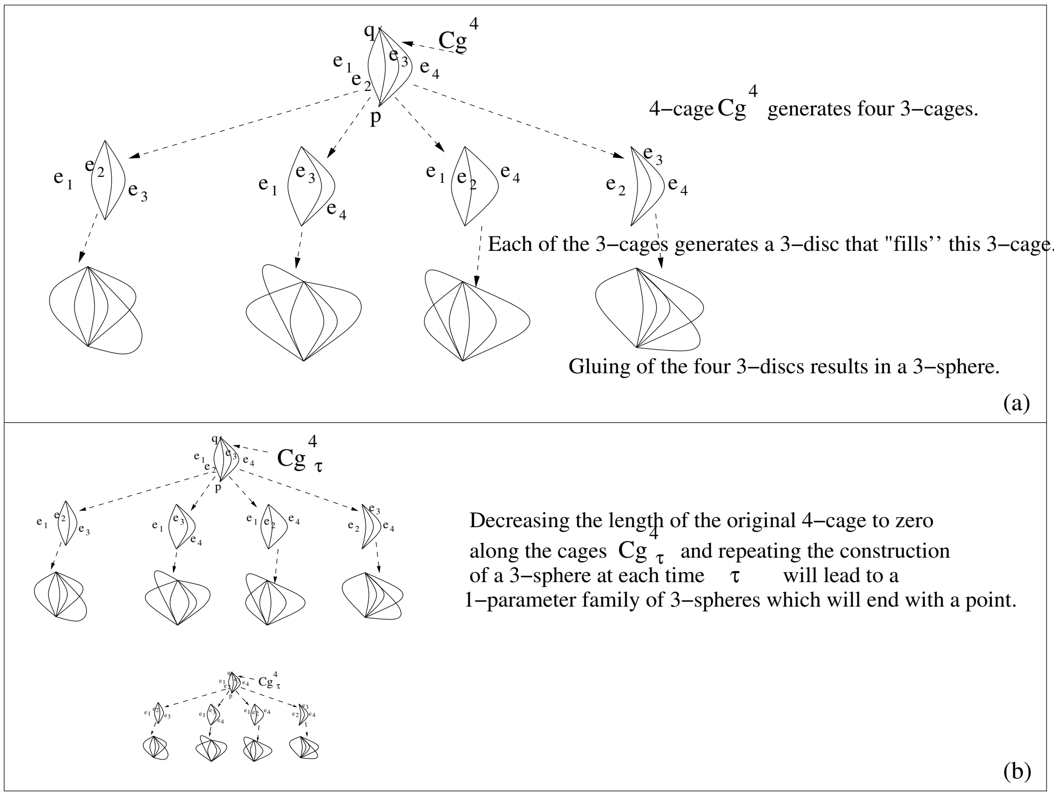

Figure 5: Extending to 3-skeleton.

Step 1. As in case of $q = 2$, we will begin by extending to 0, 1-skeleta, by mapping the center of the disc $\tilde{p}$ to an arbitrary point p of a manifold and the edges of the form $[\tilde{p}, \tilde{v}_i]$ to minimal geodesic segments $[p, v_i]$, connecting p with corresponding vertices of the triangulation induced by f. To extend to 2-skeleton, we need to extend to every 2-simplex $[\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_j}]$. Its boundary is mapped to a closed curve of small length, which, in the absence of "short" closed geodesics, can be homotoped to a point without the length increase, resulting in a 2-disc $[p, v_{i_1}, v_{i_2}]$. We map the 2-simplex to the 2-disc.

Step 2. To extend to 3-skeleton, consider an arbitrary 3-simplex $\tilde{\sigma}_i^3 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$. The image of its boundary is a 2-dimensional sphere that is obtained by gluing 1-small disc, that we, for convenience, will treat as a point and 3 large discs. If we ignore this small disc and consider the natural CW structure of this sphere, then what we have is a 3-cage, "filled" by a 2-sphere, that is a 2-sphere that is obtained by gluing three 2-discs that arise out of this cage. To extend to 3-simplex, we need to contract this 2-sphere to a point, or to "fill" the 3-cage by a 3-disc $[p, v_{i_1}, v_{i_2}, v_{i_3}]$. This 3-disc is obtained by constructing a 1-parameter family of 2-spheres that begins with the original sphere and ends with a point, as it had been described in the introduction.

Step 3. Finally, consider an arbitrary 4-simplex $\tilde{\sigma}_i^4 = [\tilde{p}, \tilde{v}_{i_1}, ..., \tilde{v}_{i_4}]$. Its boundary is mapped to a 3-sphere that is obtained by gluing five 3-discs: one of them, $[v_{i_1}, ..., v_{i_4}]$ is very small and can be treated as a point q. Thus, this sphere is essentially constructed by taking two points p and q, joining them by four geodesic segments $e_1, ..., e_4$, (thus, obtaining a 4-cage Cg^4), considering four triples of these segments and "filling" them by four 3-discs, (see fig. 5 (a)). We now want to construct a 4-disc that "fills" the 4-cage. In order to do that we construct a 1-parameter family $S_\tau^3, \tau \in [0,1]$ of 3-spheres that starts with our sphere and ends with a point. This is done as follows: apply the curve shortening process to Cg^4 to obtain a 1-parameter family of cages Cg_τ^4. At each time τ, take four triplets of segments and for each of the triplet construct a 3-disc that "fills" this triplet as it was done in Step 2. Assuming that there are no "short" geodesic 4-cages S_1^3 will be a point, (see fig. 5 (b)).

1.2 Proof of Theorem 0.3.

Proof of Theorem 0.3. It is well known that if $\pi_1(M^n) \neq \{0\}$, then there exists a non-contractible closed geodesic of length $\leq 2d$, (as it was shown in 1.1.1). Therefore, we can assume that $q \geq 2$. Assume that $\pi_i(M^n) = \{0\}$ for $i = 1, ..., q-1$, and that $\pi_q(M^n) \neq \{0\}$. Let $f : S^q \longrightarrow M^n$ be a non-contractible map from the standard sphere to a manifold M^n. Assume that S^q has been triangulated in such a way that the diameter of each simplex in the induced triangulation is less than some small $\delta > 0$. Assuming that the length of a smallest stationary m-cage is greater than md for $m = 2, ..., (q+1)$, we will extend f to D^{q+1}, the standard disc, triangulated as a cone over S^q, thus, obtaining a contradiction.

Our extension procedure will be inductive on skeleta of D^{q+1}. Let us begin with 0-skeleton. It consists of only one additional point, namely, the center of the disc, that will be denoted as $\tilde{p}$. We will assign to this point an arbitrary point $p \in M^n$. Next we will continue with the 1-skeleton. To an arbitrary 1-simplex of the form $[\tilde{p}, \tilde{v}_i]$, where $\tilde{v}_i$ is a vertex of S^q we will assign a minimal geodesic segment $[p, v_i]$, connecting p with $v_i = f(\tilde{v}_i)$. To extend to the 2-skeleton, we use the assumption that the length of a shortest closed geodesic is $> 2d+\delta$. Consider an arbitrary 2-simplex $\tilde{\sigma}_i^2 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}]$. Its boundary $[\tilde{v}_{i_1}, \tilde{v}_{i_2}] - [\tilde{p}, \tilde{v}_{i_2}] + [\tilde{p}, \tilde{v}_{i_1}]$ is mapped to a closed curve of length $\leq 2d+\delta$. Let us use the Birkhoff Curve Shortening Process to contract this curve to a point. (We can assume without loss of generality that there is no closed geodesic of length $\leq 2d$, since otherwise the theorem immediately follows). We will map the above simplex to a (possibly singular) disc σ_i^2 that is generated by the above homotopy.

Next we will extend to 3-skeleton as follows: consider an arbitrary 3-simplex $\tilde{\sigma}_i^3 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$. The boundary of this simplex is mapped to the following spherical cycle $[v_{i_1}, v_{i_2}, v_{i_3}] - [p, v_{i_2}, v_{i_3}] + [p, v_{i_1}, v_{i_3}] - [p, v_{i_1}, v_{i_2}]$. For the sake of simplicity of the exposition, let us treat $[v_{i_1}, v_{i_2}, v_{i_3}]$ as a point that we will denote by q (see the Remark following the proof). So, what we have is a sphere obtained by connecting the points p and q by minimal geodesic segments e_1, e_2, e_3 and then contracting each of the digons $\alpha_i = e_i - e_{i \bmod 3+1}$, $i = 1, 2, 3$ to a point by the Birkhoff Curve Shortening Process. Let us use the assumption that the length of any stationary 3-cage, (i.e. θ-graph) is $> 3d$. Therefore, the above θ-graph is contractible to a point without the length increase (See the description of a length shortening process for m-cages in Section 2 below). Let us denote the family of θ-graphs generated by this homotopy as θ_t. Each θ_t gives rise to three digons $\alpha_{it}, i = 1, 2, 3$. Each of those digons is contractible to a point by Birkhoff Curve Shortening Process, assuming there are no "short" geodesics. This gives us a sphere $(S_i^2)_t$ at each time t. We claim, that those spheres change continuously with t. This is due to the fact, that in the absence of closed geodesics, the Birkhoff Curve Shortening Process depends continuously on a curve. Therefore, as θ-graph is being contracted to a point, the sphere is being contracted to a point as well. We will map $\tilde{\sigma}_i^3$ to the (possibly singular) disc σ_i^3 generated by the above family of spheres.

Next, let us extend to 4-skeleton: consider an arbitrary 4-simplex $\tilde{\sigma}_i^4 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, ..., \tilde{v}_{i_4}]$. Its boundary is mapped to the spherical cycle $\Sigma_{j=0}^4 (-1)^j [v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_4}]$, where $v_{i_0} = p$. Let us again assume that $[v_{i_1}, ..., v_{i_4}]$ is so small that it can be treated as a point denoted by q. So, this spherical cycle then consists of two points p and q, four geodesic segments $[p, v_{i_j}], j = 1, 2, 3, 4$, that we will denote as $e_j, j = 1, 2, 3, 4$, six discs of dimension 2 obtained by contracting six digons to a point, and four discs of dimension 3 obtained by contracting four 2-spheres to a point, (in order to contract these spheres to a point we use the previous step of the construction involving contracting 3-cages to a point). Assuming that there is no "small" stationary 4-cages, it is possible to use a curve shortening process described in Section 2 below to contract 4-cage to a point. As it is being contracted to a point, it generates the family of spheres S_t^3 that starts with the spherical cycle that corresponds to the boundary of the above's simplex and that ends with a point. Let us denote the 4-cage at time t as Cg_t^4. S_t^3 is constructed by performing the above process for Cg_t^4, namely, by contracting each of the digons of Cg_t^4 to a point using the Birkhoff Curve Shortening Process, and then by contracting each of the four resulting 2-spheres. These spheres are being contracted using the previous step of construction. To do so, we contract the corresponding 3-cages that are obtained from Cg_t^4 by forgetting one of its four segments. Then we glue the resulting four 3-discs thus obtaining a 3-sphere. (These discs correspond to four of five faces of the boundary of the standard 4-simplex. We assign a point to the fifth disc.) This 1-parameter family of 3-spheres can be regarded as a 4-disc σ_i^4. We will map $\tilde{\sigma}_i^4$ to this disc.

Now suppose we have extended in such a manner to k-skeleton of D^{q+1}, $k \leq q$. It follows that, assuming there is no "small" $k+1$-cages, we can extend to $k+1$-skeleton of D^{q+1} as follows. Let $\tilde{\sigma}_i^{k+1} = [\tilde{p}, \tilde{v}_{i_1}, ..., \tilde{v}_{i_{k+1}}]$ be an arbitrary $(k+1)$-simplex. Its boundary is mapped to $\Sigma_{j=0}^{k+1}(-1)^j[v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_{k+1}}]$, where $v_{i_0} = p$. Once again, let us, for the sake of the exposition, treat $[v_{i_1}, ..., v_{i_{k+1}}]$ as a point that we will denote by q.

Let us denote the edges $[p, v_{i_j}]$ as $e_j, j = 1, ..., k+1$. Assuming that there are no "small" stationary $(k+1)$-cages, the cage obtained from p, q by joining it with $(k+1)$ geodesic segments e_j is contractible to a point by a length decreasing process along cages Cg_t^{k+1} described in Section 2 below. Corresponding to each such cage, we can construct a sphere S_t^k that starts with the original spherical cycle and ends with a point. Here we use the previous step of the induction. Thus, we obtain a disc σ_i^{k+1}, and we can extend the map to $(k+1)$ skeleton of D^{q+1}. We can continue until we reach the dimension $q+1$ obtaining a contradiction.

□

Remark. Let us consider a sphere in the manifold M^n obtained by taking a small 2-simplex $[v_{i_1}, v_{i_2}, v_{i_3}]$ and a point p, connecting p with each v_{i_j} by a minimal geodesic segment $e_j, j = 1, 2, 3$, and finally, by contracting each of the closed curves $e_j + [v_{i_j}, v_{i_{j \bmod 3+1}}] - e_{j \bmod 3+1}$, where $j = 1, 2, 3$ to a point, (see fig. 6 (a)). We claim that for all practical purposes $[v_{i_1}, v_{i_2}, v_{i_3}]$ can be treated as a point q. Simply take $q \in [v_{i_1}, v_{i_2}, v_{i_3}]$. Consider the boundary $\partial[v_{i_1}, v_{i_2}, v_{i_3}] = [v_{i_2}, v_{i_3}] - [v_{i_1}, v_{i_3}] + [v_{i_1}, v_{i_2}]$. Let us denote each of the segments $[v_{i_j}, v_{i_{j \bmod 3+1}}]$ as $s_j, j = 1, 2, 3$. Without loss of generality, we can assume that $s_1 + s_2 + s_3$ can be contracted to q in $[v_{i_1}, v_{i_2}, v_{i_3}]$ without the length increase. Moreover, each of the vertices, v_{i_j} will trace a trajectory σ_j of length $\leq \varepsilon(\delta)$, such that ε approaches 0 as δ approaches 0. Let us denote the images of s_j under the homotopy as s_{j_t}, and the trajectories traced at the time t as σ_{j_t} Then instead of curves $e_j + [v_{i_j}, v_{i_{j \bmod 3+1}}] - e_{j \bmod 3+1}$ consider new curves $e_j + \sigma_{j_t} + s_{j_t} - \sigma_{j \bmod 3+1_t} - e_{j \bmod 3+1}$ of length $\leq 2d + 2\varepsilon(\delta) + 3\delta$, (see fig. 6 (b), (c)). Each of those curves is contractible to a point without the length increase, assuming there is no short geodesics. Moreover, at $t = 1$ we will obtain the sphere that is constructed as follows: take two points p and q and connect them with three segments $e_j^* = e_j + \sigma_j$, $j = 1, 2, 3$. Then take three digons $e_j^* - e_{j \bmod 3+1}^*$ and contract them to the point, (see fig. 6 (d). Thus, the initial and the final 2-spheres are homotopic. We can eventually let δ go to 0. The same idea works in higher dimension as well.

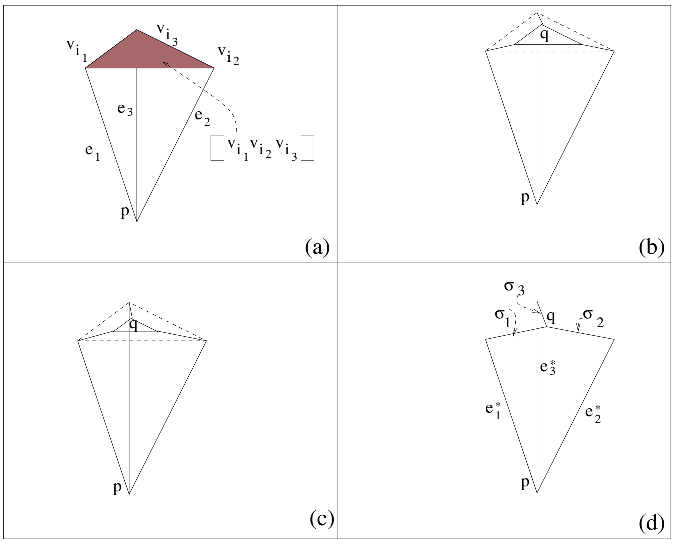

Figure 6: A small 2-simplex can be ignored.

2 Length shortening proces for m-cages.

In this section we will describe a length shortening process for m-cages. A similar length shortening process for curves was introduced by G. Birkhoff and is described in detail in section 2 of [C]. Consider the length functional on the space C_L^m of the immersed m-cages of length $\leq L$. One can construct a flow on C_L^m that decreases the length functional, assuming there is no stationary m-cages of length $\leq L$. Note that closed curves and points can also be regarded as m-cages. We claim that in such a case there exists a deformation retraction of C_L^m to M^n, such that the length functional decreases along the trajectory of the deformation. Consider an m-cage consisting of two vertices a and b and m curves $\alpha_i, i = 1, ..., m$ that join those vertices.

The length shortening process we will describe is very similar to the Birkhoff Curve Shortening Process.

We will begin by replacing the curves α_i's by piecewise geodesics. This is done by subdividing each of the curves into many equal "small" segments, each of length $\leq injrad(M^n)/4$, where $injrad(M^n)$ denotes the injectivity radius of M^n, and then replacing each small segment by the minimal geodesic segment. Clearly, the original m-cage and the new piecewise geodesic m-cage will be homotopic by a length-decreasing homotopy. Moreover, this homotopy will continuously depend on the initial cage. (This observation is analogous to the starting point of Birkhoff Curve Shortening Process, (see [C])).

Thus, we find a deformation retraction of C_L^m to a finite dimensional space that we will denote FC_L^m, such that the length of an arbitrary edge does not increase during this deformation.

FC_L^m can be regarded as a subset of $(M^n)^N$ for a sufficiently large N.

Let $Cg^m \in FC_L^m$. We can define a vector of steepest descent tangent to FC_L^m at Cg^m. It will be defined as follows: consider all the vertices, (i.e. non-smooth points of m-cage). There will be many vertices, where two geodesic segments come together and two points a and b, where m geodesic segments come together. If $a = b$, there will be one point where $\leq 2m$ tangent vectors come together.

At each vertex consider the sum of the diverging unit vectors tangent to geodesic segments meeting at this vertex, (see fig. 7). This collection of vectors tangent to M^n constitutes the vector of the steepest descent for Cg^m. Note also, that it will also "work" for m-cages that are sufficiently close to Cg^m. That is, for any m-cage, sufficiently close to Cg^m, if we parallel transport our vector to that m-cage, we will obtain a vector such that the first variation of the length functional in the direction of this vector will be negative. Now choosing an appropriate locally finite partition of unity we can construct a vector field on FC_L^m such that the first variation of the length functional in the direction of this field is negative and FC_L^m deforms to FC_0^m in a finite time.

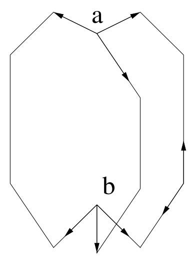

Figure 7: Length Shortening Process for θ-graph.

This process is a very much simplified version of the process described in paper [NR1], (see the proof of a Morse-theoretic type lemma for the space of 1-cycles made of at most k segments, (Lemma 3) in [NR1], in which we show that, assuming there are no non-trivial stationary 1-cycles in the space of 1-cycles Γ_k^x made of at most k segments of length $\leq x$, then the space Γ_k^0 of 1-cycles of 0 length is a deformation retract of Γ_k^x). All the technical difficulties that arise during this deformation were dealt with in [NR1]. One can find it summarized for θ-graphs in [NR2], (see section 3: Length-decreasing process for θ-graphs). During this length shortening process, it can happen that the length of one of the edges becomes 0 and the two points a and b coincide. We will then have to move this unique vertex in the direction of the sum of all unit vectors tangent to geodesic segments and diverging from this vertex. Another difficulty is that despite the fact that the total length of each cage decreases, the distance between two neighboring vertices can increase. We want this distance to remain smaller than $injradM^n$. Otherwise we will not be able to connect the endpoints by a unique geodesic segment. Therefore, to resolve this difficulty, we apply the flow only for the time $t = \frac{injradM^n}{4}$. Then we stop, divide each segment into equal segments of length $\leq \frac{injradM^n}{4}$ and replace it by a piecewise geodesic curve, as it was done in the beginning.

Then we apply the flow again for $t = \frac{injradM^n}{4}$ etc.

Under this curve shortening process the m-cage converges either to a stationary m-cage, (possibly degenerate, where two vertices coincide and lenghts of one or more segments equal zero), or to a point.

3 The proof of Theorem 0.6.

The proof of 0.6 is very similar to that of Theorem 0.3, except that instead of contracting m-cages, we will be contracting 1-skeletons of simplices. The spheres and discs are then built out of those 1-skeletons in a similar fashion.

Proof of Theorem 0.6. Let us begin by assuming that the shortest length of a minimal geodesic length is $> (n+1)(n+2)FillRadM^n$. By the definition of the filling radius of M^n, M^n bounds in the $(FillRadM^n + \delta)$-neighborhood of M^n in $L^\infty(M^n)$. Let W be a chain, such that $M^n = \partial W$, when M^n is orientable and $M^n = \partial W$ mod 2, when M^n is not orientable. Moreover, let W fill M^n in the $(FillRadM^n + \delta)$-neighborhood of M^n. WLOG we can assume that W is a polyhedron, (see [G]). Let W and M^n be triangulated in such a way that the diameter of any simplex in this triangulation is smaller than some small $\delta > 0$.

One can show that there exists a singular $(n+1)$-chain on M^n, such that the boundary of this chain is homologous to the boundary of W, which would be a contradiction. This chain is constructed by induction on the dimension of skeleta of W.

Let us begin with 0-skeleton. To each vertex $\tilde{w}_i \in W$ we will assign a vertex $w_i \in M^n$, that is closest to $\tilde{w}_i$. That is $d(\tilde{w}_i, w_i) \leq FillRadM^n + \delta$. Next, to extend to 1-skeleton, we will assign to each edge of the form $[\tilde{w}_i, \tilde{w}_j] \subset W \backslash M^n$ a minimal geodesic segment $[w_i, w_j]$ connecting w_i and w_j of length $\leq 2FillRadM^n + 3\delta$. Now, let us go to 2-skeleton. Let $\tilde{\sigma}^2_{i_0,i_1,i_2} = [\tilde{w}_{i_0}, \tilde{w}_{i_1}, \tilde{w}_{i_2}]$ be an arbitrary 2-simplex. Its boundary is mapped to a closed curve of length $\leq 6FillRadM^n + 9\delta$. Assuming there are no closed geodesics of smaller length, (since we consider them as minimal geodesic nets) we can contract this curve to a point without the length increase. Moreover, this curve shortening homotopy can be arranged to depend continuously on a curve, in the absence of "short" closed geodesics. We will map $\tilde{\sigma}^2_{i_0,i_1,i_2}$ to a surface that is generated by above mentioned homotopy, that we will denote as $\sigma^2_{i_0,i_1,i_2}$.

Next let us go to 3-skeleton. Consider an arbitrary 3-simplex $\tilde{\sigma}^3_{i_0,i_1,i_2,i_3} = [\tilde{w}_{i_0}, ..., \tilde{w}_{i_3}]$. By the previous step of the induction, its boundary is mapped to the following chain: $\Sigma_{j=0}^{3}(-1)^j \sigma^2_{i_i,\dots,\hat{i}_j,\dots,i_3}$. Consider its 1-skeleton. It will be a (not geodesic) net, that we will denote by K_i. Let us apply a length shortening process for nets to continuously deform it to a point. (We will not explicitly describe this length shortening process, but it can be found in [NR1] and it is very similar to the length shortening process for m-cages). At each time t during this deformation, we can use the net $(K_i)_t$ to construct a 2-dimensional sphere S_t^2 in a way that is analogous to the similar construction in the proof of Theorem 0.3. This 1-parameter family of 2-spheres can be regarded as a 3-disc that we will denote as $\sigma^3_{i_0,\dots,i_3}$. We will assign it to $\tilde{\sigma}^3_{i_0,\dots,i_3}$. We can continue in a similar fashion until we reach the $(n+1)$-skeleton of W, thus constructing a singular chain on M^n, that has the fundamental class $[M^n]$ as its boundary, and therefore, arriving at a contradiction.

$\square$

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R. Rotman
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 2E4
Canada
e-mail: rina@math.toronto.edu

Department of Mathematics
Penn State University
University Park, PA 16802
USA
rotman@math.psu.edu