Filter-Regularity and Cohen-Macaulay Multigraded Rees Algebras

Manfred Herrmann, Eero Hyry, Thomas Korb

Mathematisches Institut
der Universität zu Köln
Weyertal 86-90
50931 Köln
GERMANY

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

National Defence College
Santahamina
00860 Helsinki
FINLAND

MPI/95-80

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0. Introduction

Among the many problems related to the Cohen-Macaulayness of graded algebras, a question of C. Huneke and K. Smith – asking for the existence of arithmetic Macaulayfications – recently attracted special attention. Here a Noetherian local ring (A, m) of dimension d is said to have an arithmetic Macaulayfication if there is an ideal I such that the Rees algebra $R_A(I) = A[It]$ (where t is an indeterminate) is Cohen-Macaulay. The existence of arithmetic Macaulayfications can e.g. be shown for generalized Cohen-Macaulay rings (i.e. in particular Buchsbaum rings) using results of S. Goto (et al.) on the local cohomology of blow-up algebras with respect to a standard system of parameters. But in general, an arithmetic Macaulayfication does not have to exist for a given ring A – even if $\operatorname{Spec} A$ has a desingularization by $\operatorname{Proj} R_A(I)$ for some $I \subset A$. Actually, J. Lipman could show recently that $R_A(I)$ cannot be Cohen-Macaulay in this situation unless A is rational.

The existence resp. non-existence of an arithmetic Macaulayfication is a property of the ring A. The more classical problem in this direction is to find and describe sufficient conditions on ideals I – in a given class of rings A (as e.g. rings which are already Cohen-Macaulay) – which guarantee that $R_A(I)$ is Cohen-Macaulay. A weaker condition for a given ideal I is to ask for the Cohen-Macaulayness of some multi-Rees algebra $R_A(\mathbf{I}_r) = R_A(I, \ldots, I)$. This implies the Cohen-Macaulay property of the Rees algebra of some power of I, but not necessarily the Cohen-Macaulayness of $R_A(I)$ itself.

The Cohen-Macaulayness of $R_A(\mathbf{I}_r)$ can be characterized by means of the local cohomology of the form ring $gr_A(I) = R_A(I)/IR_A(I)$ (s. Theorem 1.1). In particular, all the a-invariants of $gr_A(I)$ have to be negative. In the case the ideal I is m-primary this leads us to consider the filter-regularity of sequences

$(a_1^*, \dots, a_d^*)$ where $a_1^*, \dots, a_d^*$ are the initial forms of a minimal reduction $a_1, \dots, a_d$ of I in $gr_A(I)$.

This paper is organized as follows: In section 1 we recall some easy consequences of the above mentioned Theorem 1.1. Moreover, we study the effect of the Cohen-Macaulayness of $R_A(\mathbf{I}_r)$ on the relationship between the depths of $R_A(I)$ and $gr_A(I)$.

Given an m-primary ideal $I \subset A$ and a minimal reduction $(a_1, \dots, a_d)$ of I we describe in section 2 the relationship between the filter-regularity of the sequence $(a_1^*, \dots, a_d^*)$ and the Cohen-Macaulay property of $R_A(\mathbf{I}_r)$. One interesting consequence is Theorem 2.8 which says that in the case depth $A \geq 2$ the Cohen-Macaulayness of $R_A(\mathbf{I}_r)$ implies that $gr_A(I)$ satisfies the Serre condition (S_2) and $R_A(I)$ the Serre condition (S_3). Another consequence is Theorem 2.13 where the situation that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for some $r > 0$ is characterized by conditions on I in the ground ring A.

In section 3 we then address the question how far we can improve for given I the Cohen-Macaulay property of $R_A(\mathbf{I}_r)$. For m-primary ideals we give an answer to this question in the following sense: Assuming that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay we characterize the Cohen-Macaulayness of $R_A(\mathbf{I}_s)$, where $s < r$, by means of "intersection conditions" similar to those Valabrega and Valla gave for the Cohen-Macaulayness of $gr_A(I)$ in [VV]. Finally we give a series of examples where $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for some $r \geq 2$, but the ordinary Rees algebra $R_A(I)$ is not.

1. Preliminaries and auxiliary results

We begin by fixing some notation and by recalling certain basic facts about multi-Rees rings. For details we refer to [HHR1], [HHR2] and [HHRT].

In the following we call $\mathbb{Z}^r$-graded rings and modules r-graded or simply multigraded. Rings are always assumed to be Noetherian and $\mathbb{N}^r$-graded. The norm of a multi-index $\mathbf{n} \in \mathbb{Z}^r$ is $|\mathbf{n}| = n_1 + \dots + n_r$. If $S = \bigoplus_{\mathbf{n} \in \mathbb{N}^r} S_{\mathbf{n}}$ is an r-graded ring, we denote $S^+ = \bigoplus_{\mathbf{n} \neq \mathbf{0}} S_{\mathbf{n}}$.

Let A be a ring and let $I_1, \dots, I_r \subset A$ be ideals. Set $\mathbf{I} = (I_1, \dots, I_r)$. The *multi-Rees ring* $R_A(\mathbf{I})$ is the r-graded ring

$$R_A(\mathbf{I}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^r} I_1^{n_1} \cdots I_r^{n_r}.$$

Furthermore, for every $i = 1, \dots, r$ the i:th *multi-form ring* is defined as

$$\begin{aligned} gr_A(\mathbf{I}; I_i) &= R_A(\mathbf{I})/I_i R_A(\mathbf{I}) \\ &= \bigoplus_{\mathbf{n} \in \mathbb{N}^r} I_1^{n_1} \cdots I_i^{n_i} \cdots I_r^{n_r} / I_1^{n_1} \cdots I_i^{n_i+1} \cdots I_r^{n_r}. \end{aligned}$$

We often identify $R_A(\mathbf{I})$ with the subring $A[I_1t_1,\ldots,I_rt_r]$ of $A[t_1,\ldots,t_r]$. If $\operatorname{ht} I_j > 0$ $(j=1,\ldots,r)$, we have $\dim R_A(\mathbf{I}) = \dim A + r$. Moreover, if A is local, $\dim gr_A(\mathbf{I};I_i) = \dim A + r - 1$ $(i=1,\ldots,r)$.

In the case $I_1 = \ldots = I_r = I$ we use the notation $\mathbf{I}_r$ for the r-tuple $(I,\ldots,I)$. We also denote $gr_A(\mathbf{I}_r) = gr_A(\mathbf{I}_r;I)$. The Cohen-Macaulay property of multi-Rees algebras $R_A(\mathbf{I}_r)$ and multi-form rings $gr_A(\mathbf{I}_{r+1})$ can be characterized in terms of the local cohomology of the usual Rees ring $R_A(I)$ and form ring $gr_A(I)$ as follows ([HHR1, Theorem 2.2], [HHRT, Proposition 1.6]).

1.1. Theorem. *Let (A,m) be a local ring of dimension d and let $I \subset A$ be an ideal of $\operatorname{ht} I > 0$. Let $\mathfrak{M}$ be the homogeneous maximal ideal of $R_A(I)$. Then the following conditions are equivalent for all $r \geq 1$:*

(1) $R_A(\mathbf{I}_r)$ is Cohen-Macaulay.

(2) $gr_A(\mathbf{I}_{r+1})$ is Cohen-Macaulay.

(3) $[\underline{H}^i_{\mathfrak{M}}(R_A(I)]_n = 0$ when $i < d+1$ and $n \notin \{-r+1,\ldots,-1\}$.

(4) $[\underline{H}^i_{\mathfrak{M}}(gr_A(I)]_n = 0$ when $i < d$ and $n \notin \{-r,\ldots,-1\}$,
$[\underline{H}^d_{\mathfrak{M}}(gr_A(I)]_n = 0$ when $n \geq 0$.

1.2. Corollary. *Let (A,m) be a local ring of dimension d and let $I \subset A$ be an ideal of $\operatorname{ht} I > 0$.*

1) If $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, then $R_A(\mathbf{I}_q)$ is Cohen-Macaulay for all $q \geq r$.

2) If $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, then $R_A(I^r)$ is Cohen-Macaulay.

Proof. Both claims follow directly from Theorem 1.1. Observe that in (2) $(R_A(I))^{(r)} = R_A(I^r)$ so that $(\underline{H}^i_{\mathfrak{M}}(R_A(I)))^{(r)} = \underline{H}^i_{\mathfrak{N}}(R_A(I^r))$ where $\mathfrak{M}$ and $\mathfrak{N}$ are the homogeneous maximal ideals of $R_A(I)$ and $R_A(I^r)$ respectively.

To formulate and to prove another consequence of Theorem 1.1 for m-primary ideals we first recall some facts about a-invariants and reduction numbers.

If G is a graded ring of dimension d defined over a local ring, the *a-invariants* of G are defined as

$$a_i(G) = \sup\{n \in \mathbb{Z} \mid [\underline{H}^i_{\mathfrak{M}}(G)]_n \neq 0\} \quad (i=1,\ldots,d),$$

where $\mathfrak{M}$ is the homogeneous maximal ideal of G.

Let (A,m) be a local ring and $I \subset A$ an ideal. Recall that an ideal $J \subset I$ is called a *reduction* of I if $I^{n+1} = JI^n$ for $n \gg 0$. The *reduction number* $r(I)$ is defined as the smallest integer n such that $I^{n+1} = JI^n$ for some minimal reduction $J \subset I$. If the ideal I is m-primary, a well known result of Trung [T2, Proposition 3.2] says that the reduction number $r(I)$ satisfies the inequality

$$a_d(gr_A(I)) + d \leq r(I) \leq \max_{0 \leq i \leq d}(a_i(gr_A(I)) + i).$$

1.3. Corollary. *Let (A,m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal. Then $R_A(\mathbf{I}_r)$ is Cohen-Macaulay if and only if the following conditions are satisfied:*

1) $[\underline{H}^i_{\mathfrak{M}}(gr_A(I)]_n = 0$ *when* $i < d$ *and* $n < -r$

2) $a_i(gr_A(I)) < 0$ *when* $i < d$

3) $r(I) \leq d-1$.

Next we consider the effect of the Cohen-Macaulayness of $R_A(\mathbf{I}_r)$ on the relationship between depth $R_A(I)$ and depth $gr_A(I)$. Inspired by results of Huckaba-Marley on "the expected depth inequality" for Rees and form rings:

$$\operatorname{depth} R_A(I) \geq \operatorname{depth} gr_A(I) + 1$$

(s. [HM]) we characterize the strict inequality in the case that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for some $r > 0$ (note that r may be arbitrarily large).

1.4. Proposition. *Let A be a local ring of dimension d. Let $I \subset A$ be an ideal of* $\operatorname{ht} I > 0$ *such that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay. Then*

(i) $\operatorname{depth} R_A(I) \geq \operatorname{depth} gr_A(I) + 1$.

(ii) $\operatorname{depth} R_A(I) > \operatorname{depth} gr_A(I) + 1$ *if and only if*

a) $g = \operatorname{depth} gr_A(I) < d$ *and*

b) $[\underline{H}^g_{\mathfrak{M}}(gr_A(I))]_n \cong \begin{cases} H^g_m(A), & n = -1 \\ 0, & n \neq -1. \end{cases}$

In this case also $\operatorname{depth} A = g$.

Proof. Put $R = R_A(I)$ and $G = gr_A(I)$.

(i) By Theorem 1.1 $a_i(G) < 0$ for $i = 0, \ldots, d$ so that the claim follows from [HM, Theorem 3.13].

(ii) Consider the long exact sequences of cohomology corresponding to the exact sequences

$$0 \longrightarrow R^+ \longrightarrow R \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow R^+(1) \longrightarrow R \longrightarrow G \longrightarrow 0.$$

By the cohomology sequence corresponding to the first sequence we have for all i the isomorphisms

$$[\underline{H}^i_{\mathfrak{M}}(R^+)]_n \cong [\underline{H}^i_{\mathfrak{M}}(R)]_n \quad (n \neq 0).$$

The cohomology sequence corresponding to the second sequence gives the exact sequence

$$[\underline{H}^g_{\mathfrak{M}}(R)]_n \longrightarrow [\underline{H}^g_{\mathfrak{M}}(G)]_n \longrightarrow [\underline{H}^{g+1}_{\mathfrak{M}}(R^+)]_{n+1} \longrightarrow [\underline{H}^{g+1}_{\mathfrak{M}}(R)]_n.$$

When $\operatorname{depth} R > g+1$, we have $[\underline{H}^g_{\mathfrak{M}}(R)]_n = [\underline{H}^{g+1}_{\mathfrak{M}}(R)]_n = 0$ so that we obtain an isomorphism

$$[\underline{H}^g_{\mathfrak{M}}(G)]_n \cong [\underline{H}^{g+1}_{\mathfrak{M}}(R^+)]_{n+1}.$$

If $n < -1$, this gives $[\underline{H}^g_{\mathfrak{M}}(G)]_n = 0$. If $n = -1$, we can use the cohomology sequence corresponding to the first sequence to see that $[\underline{H}^{g+1}_{\mathfrak{M}}(R^+)]_n = H^g_m(A)$. We have thus shown that in the case $\operatorname{depth} R > g+1$

$$[\underline{H}^g_{\mathfrak{M}}(G)]_n \cong \begin{cases} H^g_m(A), & n = -1 \\ 0, & n \neq -1. \end{cases}$$

Conversely, if $[\underline{H}^g_{\mathfrak{M}}(G)]_n = 0$ for $n < -1$, we have the monomorphisms

$$0 \longrightarrow [\underline{H}^{g+1}_{\mathfrak{M}}(R)]_{n+1} \longrightarrow [\underline{H}^{g+1}_{\mathfrak{M}}(R)]_n \quad (n < -1)$$

so that necessarily $\underline{H}^{g+1}_{\mathfrak{M}}(R) = 0$ and $\operatorname{depth} R > g+1$. The last remark follows easily by similar considerations.

1.5. Remark. The statement (i) of Proposition 1.4 is also true under the weaker assumption that $R_A(I^r)$ is Cohen-Macaulay (s. [KN, Lemma 2.9]).

The next proposition will show one more consequence of the situation (ii) in Proposition 1.4:

1.6. Proposition. *Let A be a local ring and let $I \subset A$ be an ideal of* $\operatorname{ht} I > 0$. *Let $\mathfrak{M}$ be the homogeneous maximal ideal of $R_A(I)$. Put $g = \operatorname{depth} gr_A(I)$ and assume that $[\underline{H}^g_{\mathfrak{M}}(gr_A(I))]_n = 0$ for $n \neq -1$. Then also $\operatorname{depth} gr_A(I^s) = g$ for all $s \geq 1$. Moreover, we have $\underline{H}^g_{\mathfrak{N}}(gr_A(I^s)) = \underline{H}^g_{\mathfrak{M}}(gr_A(I))$, where $\mathfrak{N}$ is the homogeneous maximal ideal of $R_A(I^s)$.*

Proof. We have the following so called approximation sequences (of $R_A(I^s)$-modules) introduced by Ribbe (see [R])

$$0 \longrightarrow I^{s-i+1}gr_A(I^s) \longrightarrow I^{s-i}gr_A(I^s) \longrightarrow (gr_A(I)(s-i))^{(s)} \longrightarrow 0 \quad (i = 1, \ldots, s).$$

By means of these sequences it is easy to see that $\operatorname{depth} gr_A(I^s) \geq g$. Therefore we only have to show that $\underline{H}^g_{\mathfrak{N}}(gr_A(I^s)) = \underline{H}^g_{\mathfrak{M}}(gr_A(I))$. From the long exact sequences of cohomology corresponding to the approximation sequences we get the sequences

$$\begin{aligned} 0 \longrightarrow [\underline{H}^g_{\mathfrak{N}}(I^{s-i+1}gr_A(I^s))]_n &\longrightarrow [\underline{H}^g_{\mathfrak{N}}(I^{s-i}gr_A(I^s))]_n \\ &\longrightarrow [\underline{H}^g_{\mathfrak{M}}(gr_A(I))]_{ns+s-i} \longrightarrow [\underline{H}^{g+1}_{\mathfrak{N}}(I^{s-i+1}gr_A(I^s))]_n. \end{aligned}$$

Suppose first that $n \neq -1$. Then $ns+s-i \neq -1$ so that by using the assumption $[\underline{H}^g_{\mathfrak{N}}(gr_A(I))]_{ns+s-i} = 0$ we get the isomorphisms

$$[\underline{H}^g_{\mathfrak{N}}(I^{s-i+1}gr_A(I^s))]_n \cong [\underline{H}^g_{\mathfrak{N}}(I^{s-i}gr_A(I^s))]_n \quad (i = 1, \ldots, s).$$

Now observe that $I^s gr_A(I^s) = 0$. It follows that for $n \neq -1$

$$[\underline{H}^g_{\mathfrak{N}}(gr_A(I^s))]_n \cong \ldots \cong [\underline{H}^g_{\mathfrak{N}}(I^s gr_A(I^s))]_n = 0.$$

Then suppose that $n = -1$. Since $I^s gr_A(I^s) = 0$, we obtain by choosing $i = 1$ in the above sequence the isomorphism

$$[\underline{H}^g_{\mathfrak{N}}(I^{s-1}gr_A(I^s))]_{-1} \cong [\underline{H}^g_{\mathfrak{M}}(gr_A(I))]_{-1}.$$

On the other hand, since $[\underline{H}^g_{\mathfrak{M}}(gr_A(I))]_{-i} = 0$ for $i = 2, \ldots, s$, we also get the isomorphisms

$$[\underline{H}^g_{\mathfrak{N}}(I^{s-i+1}gr_A(I^s))]_{-1} \cong [\underline{H}^g_{\mathfrak{N}}(I^{s-i}gr_A(I^s))]_{-1} \quad (i = 2, \ldots, s).$$

Putting this together we see that

$$[\underline{H}^g_{\mathfrak{N}}(gr_A(I^s))]_{-1} \cong [\underline{H}^g_{\mathfrak{M}}(gr_A(I))]_{-1}.$$

The claim has thus been proved.

2. Filter-regularity

We begin by recalling some facts about filter-regularity and a-invariants. Let G be a graded ring defined over a local ring G_0 and let $z_1, \ldots, z_r \in G$ be homogeneous elements. The sequence $(z_1, \ldots, z_r)$ is called *filter-regular* if

$$[(z_1, \ldots, z_{i-1}) : z_i]_n = [(z_1, \ldots, z_{i-1})]_n$$

for $n \gg 0$ $(i = 1, \ldots, r)$. If G_0 is Artinian, this means that the G-modules

$$(z_1, \ldots, z_{i-1}) : z_i/(z_1, \ldots, z_{i-1}) \quad (i = 1, \ldots, r)$$

have finite length or equivalently

$$(z_1, \ldots, z_{i-1}) : z_i \subset \bigcup_{n=0}^{\infty} (z_1, \ldots, z_{i-1}) : \mathfrak{M}^n \quad (i = 1, \ldots, r),$$

where $\mathfrak{M}$ is the homogeneous maximal ideal of G.

Following [AH] we say that the sequence $(z_1, \ldots, z_r)$ is $[t_1, \ldots, t_r]$-*regular* if

$$[(z_1, \ldots, z_{i-1}) : z_i]_n = [(z_1, \ldots, z_{i-1})]_n$$

for $n \geq t_i$ $(i = 1, \ldots, r)$. Also the value $-\infty$ is here allowed for t_i. We denote

$$\varrho_i(z_1, \ldots, z_r) = \inf\{n \in \mathbb{Z} \mid [(z_1, \ldots, z_{i-1}) : z_i]_n = [(z_1, \ldots, z_{i-1})]_n\}$$

for $i = 1, \ldots, r$ and call $[\varrho_1, \ldots, \varrho_r]$ the *filter-regularity* of $(z_1, \ldots, z_r)$.

2.1. Lemma. *Let G be a graded ring of dimension d defined over an Artinian local ring. Let $z \in G_r$ be a filter-regular element. Then*

$$a_{i+1}(G) + r \leq a_i(G/(z)) \leq \max(a_i(G), a_{i+1}(G) + r)$$

for all $i = 0, \ldots, d-1$. If $z\underline{H}^i_{\mathfrak{M}}(G) = 0$, then

$$a_i(G/(z)) = \max(a_i(G), a_{i+1}(G) + r).$$

Proof. The lemma is essentially [T2, Lemma 2.3]. For the convenience of the reader we give some arguments for the proof of the second part. Put $\overline{G} = G/(z)$. We have the exact sequences

$$0 \longrightarrow 0 : z \longrightarrow G \longrightarrow G/(0:z) \longrightarrow 0$$

and

$$0 \longrightarrow G/(0:z)(-r) \xrightarrow{\cdot z} G \longrightarrow \overline{G} \longrightarrow 0.$$

Since $0 : z = 0 : \mathfrak{M}^n$ for $n \gg 0$, we have $\dim_G(0 : z) = 0$, and the long exact sequence of cohomology corresponding to the first sequence implies that $\underline{H}^i_{\mathfrak{M}}(G/(0:z)) = \underline{H}^i_{\mathfrak{M}}(G)$ for $i > 0$. Consider then the long exact sequence of cohomology corresponding to the second sequence. Since $z\underline{H}^i_{\mathfrak{M}}(G) = 0$, we come to the sequence

$$0 \longrightarrow [\underline{H}^i_{\mathfrak{M}}(G)]_n \longrightarrow [\underline{H}^i_{\mathfrak{M}}(\overline{G})]_n \longrightarrow [\underline{H}^{i+1}_{\mathfrak{M}}(G)]_{n-r} \longrightarrow [\underline{H}^{i+1}_{\mathfrak{M}}(G)]_n,$$

which easily implies the claim.

Let G be a graded ring defined over an Artinian local ring. The ring G is said to be *generalized Cohen-Macaulay* if the localization $G_{\mathfrak{M}}$ is a generalized Cohen-Macaulay ring. By definition this means that the local cohomology modules $\underline{H}^i_{\mathfrak{M}}(G)$ $(i = 0, \ldots, d-1)$ have all finite length, which is equivalent to $[\underline{H}^i_{\mathfrak{M}}(G)]_n = 0$ for $n \ll 0$ $(i = 0, \ldots, d-1)$. It is also well known that this happens if and only if G is equidimensional and the localizations G_P are Cohen-Macaulay at all homogeneous prime ideals $P \neq \mathfrak{M}$.

Let $(z_1, \ldots, z_d)$ be a system of parameters of G. Recall the inequality

$$l(G/(z_1, \ldots, z_d)) - e(z_1, \ldots, z_d; G) \leq I(G),$$

where

$$I(G) = \sum_{i=0}^{d-1} \binom{d-1}{i} l_A(\underline{H}^i_{\mathfrak{M}}(G)).$$

If the equality holds, we say that $(z_1, \ldots, z_d)$ is a *standard system of parameters*. Equivalently,

$$(z_1, \ldots, z_d)\underline{H}^i_{\mathfrak{M}}(G/(z_1, \ldots, z_j)) = 0$$

for $i+j<d$. A homogeneous $\mathfrak{M}$-primary ideal $I \subset G$ is called a *standard ideal* if every system of parameters contained in I is standard. There exists $r \in \mathbb{N}$ such that every $I \subset \mathfrak{M}^r$ is standard. For more details about filter-regular sequences, generalized Cohen-Macaulay rings and standard systems of parameters we refer to [STC], [HIO] and [T1].

The following proposition is a modification of [T2, Proposition 2.2].

2.2. Proposition. *Let G be a graded ring defined over an Artinian local ring and let $(z_1,\ldots,z_d)$ be a filter-regular system of parameters consisting of homogeneous elements of degree r. Then*

$$\max\{\varrho_j(z_1,\ldots,z_d) \mid j=1,\ldots,i\} = \max\{a_j(G)+jr+1 \mid j=0,\ldots,i-1\}$$

for all $i=1,\ldots,d$. If, moreover, G is a generalized Cohen-Macaulay ring and $(z_1,\ldots,z_d)$ is a standard system of parameters, we have

$$\varrho_i(z_1,\ldots,z_d) = \max\{a_j(G)+jr+1 \mid j=0,\ldots,i-1\}.$$

In this case we especially get that $\varrho_{i-1}(z_1,\ldots,z_d) \le \varrho_i(z_1,\ldots,z_d)$ $(i=2,\ldots,d)$.

Proof. By induction on i. The case $i=1$ follows from the fact that for $n \gg 0$ $0:z_1 = 0:\mathfrak{M}^n = \underline{H}^0_{\mathfrak{M}}(G)$. Suppose $i>1$. If $z \in G$, denote $\overline{z} = z+(z_1) \in G/(z_1)$. Observe then that $\varrho_j(z_1,\ldots,z_d) = \varrho_{j-1}(\overline{z}_2,\ldots,\overline{z}_d)$ $(j=2,\ldots,d)$. The claim now follows from the induction hypothesis by Lemma 2.1.

2.3. Corollary. *Let G be a graded ring defined over an Artinian local ring and let $(z_1,\ldots,z_d)$ be a filter-regular system of parameters consisting of homogeneous elements of degree r. Then $a_i(G)<0$ for $i=1,\ldots,d-1$ if and only if $(z_1,\ldots,z_d)$ is $[0,r,\ldots,(d-1)r]$-regular.*

Proof. It follows by induction on i from Proposition 2.2 that the conditions $\varrho_j(z_1,\ldots,z_d) \le r(j-1)$ $(j=1,\ldots,i)$ and $a_j(G)<0$ $(j=1,\ldots,i-1)$ are equivalent for all $i=1,\ldots,d$.

We now want to apply these results for characterizing the situation where some multi-Rees algebra is Cohen-Macaulay.

Recall first the following. Let (A,m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal. If $a \in I^q \setminus I^{q+1}$, put $a^* = a+I^{q+1} \in [gr_A(I)]_q$. Then $(b_1,\ldots,b_d)$ is a minimal reduction of I^q if and only if $(b_1^*,\ldots,b_d^*)$ is a system of parameters of $gr_A(I)$.

2.4. Proposition. *Let (A,m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal. Then $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for some $r>0$ if and only if $r(I) \le d-1$ and one of the following equivalent conditions holds:*

(1) *The sequence* $(a_1^*, \dots, a_d^*)$ *is* $[0, 1, \dots, d-1]$*-regular for some minimal reduction* $(a_1, \dots, a_d)$ *of* I *and the ring* $gr_A(I)$ *is generalized Cohen-Macaulay.*

(2) *The sequence* $(b_1^*, \dots, b_d^*)$ *is* $[0, q, \dots, (d-1)q]$*-regular for all minimal reductions* $(b_1, \dots, b_d)$ *of every* I^q, $q > 0$, *and the ring* $gr_A(I)$ *is equidimensional.*

Proof. Put $G = gr_A(I)$. If $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, G is generalized Cohen-Macaulay. By [STC, Satz 2.5] G is generalized Cohen-Macaulay if and only if every system of parameters of G is filter-regular. By Corollary 2.3 a system of parameters consisting of homogeneous elements of degree q is $[0, q, \dots, q(d-1)]$-regular if and only if $a_i(G) < 0$ for $i = 1, \dots, d-1$. It is therefore enough to show that if (2) holds, then $gr_A(I)$ is generalized Cohen-Macaulay. Let $P \in \operatorname{Proj} G$. Denote $h = \operatorname{ht} P$. By prime avoidance it is possible to find a system of parameters $(z_1, \dots, z_d)$ of G consisting of homogeneous elements of certain degree q such that $z_1, \dots, z_h \in P$ (s. [BH, Proposition 1.5.11]). There is then a minimal reduction $(b_1, \dots, b_d)$ of I^q such that $z_i = b_i^*$ $(i = 1, \dots, d)$. By assumption the sequence $(z_1, \dots, z_d)$ is filter-regular. Because $G^+ \not\subset P$, we get that the sequence $(z_1, \dots, z_d)$ is regular in G_P. Hence G_P is Cohen-Macaulay. Because G is equidimensional, we get that it is generalized Cohen-Macaulay.

2.5. Corollary. *Let* (A, m) *be a formally equidimensional local ring of dimension* d *and let* $I \subset A$ *be an* m*-primary ideal such that* $\operatorname{Proj} R_A(I)$ *is Cohen-Macaulay. Then* $R_A(\mathbf{I}_r)$ *is Cohen-Macaulay for some* $r > 0$ *if and only if* $r(I) \leq d-1$ *and there exists a minimal reduction* $(a_1, \dots, a_d)$ *of* I *such that* $(a_1^*, \dots, a_d^*)$ *is* $[0, \dots, d-1]$*-regular.*

Proof. The Cohen-Macaulay property of $\operatorname{Proj} R_A(I)$ is equivalent to that of $\operatorname{Proj} gr_A(I)$. Since A is formally equidimensional, we know that this is in turn equivalent to $gr_A(I)$ being generalized Cohen-Macaulay (see [HIO, Corollary 18.24 and Lemma 43.3]). We can thus apply Proposition 2.4 (1) to get the claim.

2.6. Remark: Without any assumption on A the Cohen-Macaulayness of $\operatorname{Proj} R_A(I)$ does not imply that $gr_A(I)$ is generalized Cohen-Macaulay as the following example ([HIO, Example 40.5]) shows: Let $A = k[[x, y, z]]/(x) \cap (y, z)$, where k is a field and $I = m$ the maximal ideal of A. Then $G = gr_A(m) = k[x, y, z]/(x) \cap (y, z)$. Since G_x, G_y and G_z are Cohen-Macaulay, we see that $\operatorname{Proj} gr_A(I)$ is Cohen-Macaylay. But it is easy to check that $\underline{H}^i_{\mathfrak{M}}(G)$ is not of finite length so that G is not generalized Cohen-Macaulay. Hence there is no Cohen-Macaulay multi-Rees ring $R_A(\mathbf{m}_r)$. This can also be seen in the following way. Assume that $R_A(\mathbf{m}_r)$ would be Cohen-Macaulay for some $r \geq 1$. By Proposition 2.4 there would then exists a $[0, 1]$-regular sequence (f_1, f_2) on G. Since $G_0 = k$, we would necessarily have $[(f_1) : (f_2)]_0 = 0$. This would mean that G is Cohen-Macaulay, which is not the case.

Next we show that the filter-regularity of the sequence $(a_1^*, \dots, a_d^*)$ is usually better than $[0, \dots, d-1]$:

2.7. Proposition. *Let (A, m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal such that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay. Let $(a_1, \dots, a_d)$ be a minimal reduction of I. Put $g = \operatorname{depth} G$. Then $\varrho_i(a_1^*, \dots, a_d^*) = -\infty$ for $i = 1, \dots, g$. If, moreover, $g < \operatorname{depth} A$, we get $\varrho_{g+1}(a_1^*, \dots, a_d^*) \leq g - 1$ and $\varrho_{g+2}(a_1^*, \dots, a_d^*) \geq \varrho_{g+1}(a_1^*, \dots, a_d^*) + 2$.*

Proof. Since $(a_1^*, \dots, a_d^*)$ is filter-regular and $a_i(gr_A(I)) = -\infty$ for $i = 0, \dots, g-1$, it follows immediately from Proposition 2.2 that $\varrho_i(a_1^*, \dots, a_d^*) = -\infty$ for $i = 1, \dots, g$. In the case $g < \operatorname{depth} A$, we know by [H2, Theorem 5.2] (s. also [K]) that $a_g(gr_A(I)) < a_{g+1}(gr_A(I))$. By Theorem 1.1 $a_{g+1}(gr_A(I)) < 0$ so that using Proposition 2.2 again we see that

$$\varrho_{g+1}(a_1^*, \dots, a_d^*) = a_g(gr_A(I)) + g + 1 \leq g - 1$$

and

$$\varrho_{g+2}(a_1^*, \dots, a_d^*) = a_{g+1}(gr_A(I)) + g + 2 \geq \varrho_{g+1}(a_1^*, \dots, a_d^*) + 2.$$

2.8. Theorem. *Let (A, m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal such that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay. Let $(a_1, \dots, a_d)$ be a minimal reduction of I. If $\operatorname{depth} A \geq 2$, we have*

$$\varrho_1(a_1^*, \dots, a_d^*) = \varrho_2(a_1^*, \dots, a_d^*) = -\infty.$$

In particular, $gr_A(I)$ has (S_2) and $R_A(I)$ has (S_3).

Proof. The first statement is obvious by Proposition 2.7. We then see that $\operatorname{depth} gr_A(I) \geq 2$. By Proposition 1.4 $\operatorname{depth} R_A(I) \geq 3$. Since $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, $gr_A(I)$ is generalized Cohen-Macaulay. Because I is m-primary, it follows easily that also $R_A(I)$ is generalized Cohen-Macaulay. This implies the second statement.

2.9. Remark. It is said in [HHR2, Appendix, Proposition 4.7] that the assumptions $d \geq 2$, $\operatorname{grade} I \geq 2$, $R_A(\mathbf{I}_r)$ is Gorenstein and $\underline{H}^d_{\mathfrak{M}}(R_A(I)) = 0$, where $\mathfrak{M}$ is the homogeneous maximal ideal of $R_A(I)$, imply that $gr_A(I)$ has the property (S_2). The above theorem shows that this is true under much weaker assumptions, if I is assumed to be m-primary.

In the following Theorem 2.13 we want to characterize the situation that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for some $r > 0$ in terms of conditions in A. For that we first prove some lemmas which show how the filter-regularity in $gr_A(I)$ can be expressed my means of "intersection conditions" on the minimal reductions of I.

2.10. Lemma. *Let G be a graded ring of dimension d defined over a local ring and let $(z_1, \dots, z_d)$ be a filter- regular system of parameters consisting of homogeneous elements of degree r. Let $k \in \{1, \dots, d\}$. Then*

$$[(z_1, \dots, z_{i-1}) : z_i]_n = [(z_1, \dots, z_{i-1})]_n \qquad (i = 1, \dots, k)$$

if and only if

$$[H_1(z_1,\dots,z_i)]_{n+r}=0 \qquad (i=1,\dots,k).$$

Proof. We use induction on k. The case $k=1$ is clear, because $[H_1(z_1)]_{n+r}=[0:z_1]_n$. Let $k>1$. The claim then follows by the induction hypothesis from the exact sequence

$$[H_1(z_1,\dots,z_{k-1})]_{n+r}\longrightarrow[H_1(z_1,\dots,z_k)]_{n+r}\longrightarrow\frac{[(z_1,\dots,z_{k-1}):z_k]_n}{[(z_1,\dots,z_k)]_n}\longrightarrow 0$$

coming from the sequence

$$[H_1(z_1,\dots,z_{k-1})]_n \xrightarrow{\cdot\pm z_k} [H_1(z_1,\dots,z_{k-1})]_{n+r}\longrightarrow[H_1(z_1,\dots,z_k)]_{n+r}\longrightarrow[H_0(z_1,\dots,z_{k-1})]_n \xrightarrow{\cdot\pm z_k}[H_0(z_1,\dots,z_{k-1})].$$

2.11. Lemma. *Let (A,m) be a local ring of dimension d and let $I\subset A$ be an m-primary ideal. Let $(b_1,\dots,b_d)$ be a minimal reduction of I^q. Let $n\in\mathbb{N}$ and $k\in\{1,\dots,d\}$. If*

$$[(b_1^*,\dots,b_{i-1}^*):b_i^*]_n=[(b_1^*,\dots,b_{i-1}^*)]_n$$

for $i=1,\dots,k$, then the following conditions are satisfied for each $i=1,\dots,k$

1) $(b_1,\dots,b_i)I^n\cap I^{n+q+1}=(b_1,\dots,b_i)I^{n+1}$
2) $((b_1,\dots,b_{i-1})I^n:b_i)\cap I^n\subset(b_1,\dots,b_{i-1})I^{n-q}+I^{n+1}$.

Conversely, if these conditions hold for some $i\in\{1,\dots,k\}$, we get that

$$[(b_1^*,\dots,b_{i-1}^*):b_i^*]_n=[(b_1^*,\dots,b_{i-1}^*)]_n.$$

Proof. Suppose first that

$$[(b_1^*,\dots,b_{i-1}^*):b_i^*]_n=[(b_1^*,\dots,b_{i-1}^*)]_n$$

for $i=1,\dots,k$. Let $i\in\{1,\dots,k\}$. Denote $R=R_A(I)=A[It]$ and $G=gr_A(I)$. Consider the elements $b_1t^q,\dots,b_it^q\in R_q$. From the long exact sequence of Koszul homology corresponding to the exact sequence

$$0\longrightarrow R^+(1)\longrightarrow R\longrightarrow G\longrightarrow 0$$

we get the exact sequence

$$H_1(b_1t^q,\dots,b_it^q;G)\longrightarrow H_0(b_1t^q,\dots,b_it^q;R^+(1))\longrightarrow H_0(b_1t^q,\dots,b_it^q;R).$$

By Lemma 2.10 we now have

$$[H_1(b_1t^q,\ldots,b_it^q;G)]_{n+q}=0.$$

In degree $n+q$ the above sequence then yields a monomorphism

$$0\longrightarrow\frac{I^{n+q+1}}{(b_1,\ldots,b_i)I^{n+1}}\longrightarrow\frac{I^{n+q}}{(b_1,\ldots,b_i)I^n}.$$

It then follows that we must have

$$(b_1,\ldots,b_i)I^n\cap I^{n+q+1}=(b_1,\ldots,b_i)I^{n+1}.$$

Let us then show that if 1) holds for some $i\in\{1,\ldots,k\}$, then 2) is equivalent to

$$[(b_1^*,\ldots,b_{i-1}^*):b_i^*]_n=[(b_1^*,\ldots,b_{i-1}^*)]_n.$$

One immediately sees that the last condition is equivalent to

$$(((b_1,\ldots,b_{i-1})I^n+I^{n+q+1}):b_i)\cap I^n=(b_1,\ldots,b_{i-1})I^{n-q}+I^{n+1}. \qquad (*)$$

Let us now prove that this is equivalent to 2). Take

$$x\in(((b_1,\ldots,b_{i-1})I^n+I^{n+q+1}):b_i)\cap I^n.$$

Then $b_ix=y+z$, where $y\in(b_1,\ldots,b_{i-1})I^n$ and $z\in I^{n+q+1}$. By 1)

$$z\in(b_1,\ldots,b_i)I^n\cap I^{n+q+1}=(b_1,\ldots,b_i)I^{n+1}.$$

It follows that

$$x\in((b_1,\ldots,b_{i-1})I^n:b_i)\cap I^n+I^{n+1}.$$

Hence

$$(((b_1,\ldots,b_{i-1})I^n+I^{n+q+1}):b_i)\cap I^n=((b_1,\ldots,b_{i-1})I^n:b_i)\cap I^n+I^{n+1}.$$

Then $(*)$ becomes

$$((b_1,\ldots,b_{i-1})I^n:b_i)\cap I^n\subset(b_1,\ldots,b_{i-1})I^{n-q}+I^{n+1}$$

as desired.

2.12. Lemma. *Let (A,m) be a local ring of dimension d and let $I\subset A$ be an m-primary ideal. Let $(b_1,\ldots,b_d)$ be a minimal reduction of I^q. Let $t_1\le\ldots\le t_d$. Then $(b_1^*,\ldots,b_d^*)$ is $[t_1,\ldots,t_d]$-regular if and only if the following conditions are satisfied for $i=1,\ldots,d$ and $n\ge t_i$:*

1) $(b_1,\ldots,b_i)I^n\cap I^{n+q+1}=(b_1,\ldots,b_i)I^{n+1}$
2) $((b_1,\ldots,b_{i-1})I^n:b_i)\cap I^n=(b_1,\ldots,b_{i-1})I^{n-q}$.

Proof. It follows from Lemma 2.11 that $(b_1^*, \ldots, b_d^*)$ being $[t_1, \ldots, t_d]$-regular is equivalent to the conditions

1') $(b_1, \ldots, b_i)I^n \cap I^{n+q+1} = (b_1, \ldots, b_i)I^{n+1}$

2') $((b_1, \ldots, b_{i-1})I^n : b_i) \cap I^n \subset (b_1, \ldots, b_{i-1})I^{n-q} + I^{n+1}$

for $i = 1, \ldots, d$ and $n \geq t_i$. One easily sees by induction on n that these are further equivalent to the conditions

1") $(b_1, \ldots, b_i)I^{t_i} \cap I^{n+q+1} = (b_1, \ldots, b_i)I^{n+1}$

2") $((b_1, \ldots, b_{i-1})I^{t_{i-1}} : b_i) \cap I^n \subset (b_1, \ldots, b_{i-1})I^{n-q} + I^{n+1}$

for $i = 1, \ldots, d$ and $n \geq t_i$. It is thus enough to show that 2") implies 2) for $i = 1, \ldots, d$ and $n \geq t_i$. By 2")

$((b_1, \ldots, b_{i-1})I^{t_{i-1}} : b_i) \cap I^n \subset (b_1, \ldots, b_{i-1})I^{n-q} + ((b_1, \ldots, b_{i-1})I^{t_{i-1}} : b_i) \cap I^{n+1}$

for every $n \geq t_i$ so that

$$\begin{aligned}((b_1, \ldots, b_{i-1})I^{t_{i-1}} : b_i) \cap I^n &\subset \bigcap_{k \geq 1} \left((b_1, \ldots, b_{i-1})I^{n-q} + I^{n+k}\right) \\ &= (b_1, \ldots, b_{i-1})I^{n-q}.\end{aligned}$$

2.13. Theorem. *Let (A, m) be a formally equidimensional local ring of dimension d and let $I \subset A$ be an m-primary ideal. Then $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for some $r > 0$ if and only if $r(I) \leq d - 1$ and all minimal reductions $(b_1, \ldots, b_d)$ of every I^q, $q > 0$, satisfy the following conditions for $i = 1, \ldots, d$ and $n \geq q(i-1)$:*

a) $(b_1, \ldots, b_i)I^n \cap I^{n+q+1} = (b_1, \ldots, b_i)I^{n+1}$

b) $((b_1, \ldots, b_{i-1})I^n : b_i) \cap I^n = (b_1, \ldots, b_{i-1})I^{n-q}$.

Proof. Because A is formally equidimensional, we get that $gr_A(I)$ is equidimensional. The claim then follows from Proposition 2.4 (2) and Lemma 2.12.

The next proposition provides us some insight in the "intersection conditions" of Theorem 2.13 a) (cf. [T2, Lemma 5.1]):

2.14. Proposition. *Let (A, m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal. Let $(b_1, \ldots, b_d)$ be a minimal reduction of I^q. Suppose that $gr_A(I)$ and hence also A are generalized Cohen-Macaulay. If $(b_1, \ldots, b_d)$ is a standard system of parameters of A, then*

$$\sum_{n=0}^{\infty} l((b_1, \ldots, b_d)I^n \cap I^{n+q+1}/(b_1, \ldots, b_d)I^{n+1}) \leq I(gr_A(I)) - I(A)$$

and the equality holds if and only if $(b_1^*, \ldots, b_d^*)$ is a standard system of parameters of $gr_A(I)$.

Proof. Denote $\mathbf{b} = (b_1, \ldots, b_d)$ and $\mathbf{b}^* = (b_1^*, \ldots, b_d^*)$. Write

$$\begin{aligned} l(A/I^{N+q}) &= \sum_{n=0}^{q-1} l(I^n/I^{n+1}) + \sum_{n=q}^{N+q-1} l(I^n/I^{n+1}) \\ &= \sum_{n=0}^{q-1} l(I^n/I^{n+1}) + \sum_{n=0}^{N-1} l(I^{n+q}/I^{n+q+1}) \end{aligned}$$

and

$$l(A/\mathbf{b}I^N) = l(A/\mathbf{b}) + \sum_{n=0}^{N-1} l(\mathbf{b}I^n/\mathbf{b}I^{n+1}).$$

For $N \gg 0$ we have $I^{N+q} = \mathbf{b}I^N$ so that

$$\sum_{n=0}^{q-1} l(I^n/I^{n+1}) + \sum_{n=0}^{N-1} l(I^{n+q}/I^{n+q+1}) = l(A/\mathbf{b}) + \sum_{n=0}^{N-1} l(\mathbf{b}I^n/\mathbf{b}I^{n+1}).$$

It follows that

$$l(A/\mathbf{b}) = \sum_{n=0}^{q-1} l(I^n/I^{n+1}) + \sum_{n=0}^{\infty}(l(I^{n+q}/I^{n+q+1}) - l(\mathbf{b}I^n/\mathbf{b}I^{n+1})).$$

Because

$$\begin{aligned} l(gr_A(I)/\mathbf{b}^*) &= \sum_{n=0}^{q-1} l(I^n/I^{n+1}) + \sum_{n=q}^{\infty} l(I^n/\mathbf{b}I^{n-q} + I^{n+1}) \\ &= \sum_{n=0}^{q-1} l(I^n/I^{n+1}) + \sum_{n=q}^{\infty}(l(I^n/I^{n+1}) - l(\mathbf{b}I^{n-q} + I^{n+1}/I^{n+1})) \\ &= \sum_{n=0}^{q-1} l(I^n/I^{n+1}) + \sum_{n=q}^{\infty}(l(I^n/I^{n+1}) - l(\mathbf{b}I^{n-q}/\mathbf{b}I^{n-q} \cap I^{n+1})) \\ &= \sum_{n=0}^{q-1} l(I^n/I^{n+1}) + \sum_{n=0}^{\infty}(l(I^{n+q}/I^{n+q+1}) - l(\mathbf{b}I^n/\mathbf{b}I^n \cap I^{n+q+1})), \end{aligned}$$

we get

$$\begin{aligned} l(gr_A(I)/\mathbf{b}^*) - l(A/\mathbf{b}) &= \sum_{n=0}^{\infty}(l(\mathbf{b}I^n/\mathbf{b}I^{n+1}) - l(\mathbf{b}I^n/\mathbf{b}I^n \cap I^{n+q+1})) \\ &= \sum_{n=0}^{\infty} l(\mathbf{b}I^n \cap I^{n+q+1}/\mathbf{b}I^{n+1}). \end{aligned}$$

Now observe that $e(\mathbf{b}^*; gr(A)) = e(\mathbf{b}; A)$. Then

$$\begin{aligned} I(gr(A)) &\geq l(gr_A(I)/\mathbf{b}^*) - e(\mathbf{b}^*; gr(A)) \\ &= l(gr_A(I)/\mathbf{b}^*) - l(A/\mathbf{b}) + l(A/\mathbf{b}) - e(\mathbf{b}; A) \\ &= \sum_{n=0}^{\infty} l(\mathbf{b}I^n \cap I^{n+q+1}/\mathbf{b}I^{n+1}) + I(A) \end{aligned}$$

and the equality holds if and only if $(b_1^*, \ldots, b_d^*)$ is standard system of parameters of $gr_A(I)$.

3. Testing the Cohen-Macaulay property by length functions

Let (A, m) be a local ring and $I \subset A$ an m-primary ideal. In this section we compare the Cohen-Macaulay property of $R_A(\mathbf{I}_r)$ and $R_A(\mathbf{I}_s)$, where $s < r$. Theorem 3.6 gives a necessary and sufficient condition for the Cohen-Macaulayness of $R_A(\mathbf{I}_s)$ if $R_A(\mathbf{I}_r)$ is Cohen-Macaulay.

Let G be a graded ring of dimension d defined over an Artinian local ring B. Suppose that G is generalized Cohen-Macaulay and $(z_1, \ldots, z_d)$ is a homogeneous system of parameters of G. Denote

$$l_n^i(z_1, \ldots, z_d) = l_B([(z_1, \ldots, z_{i-1}) : z_i]_n / [(z_1, \ldots, z_{i-1})]_n) \quad (i = 1, \ldots, d)$$

and

$$h_p^i(G) = l_B([\underline{H}^i_{\mathfrak{M}}(G)]_p) \quad (i = 0, \ldots, d),$$

where $\mathfrak{M}$ is the homogeneous maximal ideal of G.

3.1. Lemma. *Let G be a graded ring of dimension d defined over an Artinian local ring A. Suppose G is generalized Cohen-Macaulay and $(z_1, \ldots, z_d)$ is a standard system of parameters of G consisting of homogeneous elements of degree r. Then*

$$l_n^i(z_1, \ldots, z_d) = \sum_{j=0}^{i-1} \binom{i-1}{j} h^j_{n-rj}(G)$$

for all $i = 1, \ldots, d$ and $n \in \mathbb{N}$.

Proof. The lemma is well known, but in a lack of a suitable reference we sketch a proof. Observe first that

$$(z_1, \ldots, z_{i-1}) : z_i / (z_1, \ldots, z_{i-1}) = \underline{H}^0(G/(z_1, \ldots, z_{i-1}))$$

so that $l_n^i(z_1,\dots,z_d) = h_n^0(G/(z_1,\dots,z_{i-1}))$. Let us prove by induction on i that the following more general formula holds for all $i = 1,\dots,d$ and $k = 0,\dots,d-i$:

$$h_n^k(G/(z_1,\dots,z_{i-1})) = \sum_{j=k}^{i+k-1} \binom{i-1}{j-k} h^j_{n-r(j-k)}(G).$$

The case $i = 1$ being trivial assume $i > 1$. Denote $G^j = G/(z_1,\dots,z_{j-1})$ and $K^j = (z_1,\dots,z_{j-1}) : z_j/(z_1,\dots,z_{j-1})$ $(j = 1,\dots,d)$. We then have the exact sequences

$$0 \longrightarrow K^{i-1} \longrightarrow G^{i-1} \longrightarrow G^{i-1}/K^{i-1} \longrightarrow 0$$

and

$$0 \longrightarrow G^{i-1}/K^{i-1}(-r) \xrightarrow{\cdot z_{i-1}} G^{i-1} \longrightarrow G^i \longrightarrow 0$$

Since $\dim K^{i-1} = 0$, the long exact sequence of cohomology corresponding to the first sequence implies that $\underline{H}^j_{\mathfrak{M}}(G^{i-1}/K^{i-1}) = \underline{H}^j_{\mathfrak{M}}(G^{i-1})$ for $j > 0$. Because $z_{i-1}\underline{H}^j_{\mathfrak{M}}(G^{i-1}) = 0$, the cohomology sequence corresponding to the second sequence gives the exact sequence

$$0 \longrightarrow [\underline{H}^k_{\mathfrak{M}}(G^{i-1})]_n \longrightarrow [\underline{H}^k_{\mathfrak{M}}(G^i)]_n \longrightarrow [\underline{H}^{k+1}_{\mathfrak{M}}(G^{i-1})]_{n-r} \longrightarrow 0.$$

Thus

$$h_n^k(G^i) = h_n^k(G^{i-1}) + h_{n-r}^{k+1}(G^{i-1})$$

and we can use the induction hypothesis to get the claim.

The lemma shows that in the case $(z_1,\dots,z_d)$ is standard we may in fact denote $l_n^i(z_1,\dots,z_d)$ by $l_n^i(G)$.

3.2. Remark. Note especially the following consequence of Lemma 3.1. Since

$$0 = l_n^i(G) = \sum_{j=0}^{i-1} \binom{i-1}{j} h^j_{n-rj}(G)$$

for $n < 0$, we obtain that $[\underline{H}^j_{\mathfrak{M}}(G)]_p = 0$ for $p < -rj$.

For the following let (A, m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal such that $r(I) \leq d-1$. Suppose that $gr_A(I)$ is generalized Cohen-Macaulay and that the sequence $(a_1^*,\dots,a_d^*)$ is $[0,\dots,d-1]$-regular for some minimal reduction $(a_1,\dots,a_d) \subset I$. Suppose, moreover, that for a certain r the sequence $(a_1^{*r},\dots,a_d^{*r})$ is a standard system of parameters of $gr_A(I)$. It follows from Corollary 1.3 by the preceeding remark and Corollary 2.3 that $R_A(\mathbf{I}_s)$ is always Cohen-Macaulay for $s \geq r(d-1)$. For example, in the case $gr_A(I)$ is Buchsbaum, we can take $r = 1$ and get that $R_A(\mathbf{I}_s)$ is Cohen-Macaulay for all $s \geq d-1$. The following proposition answers the question, when $R_A(\mathbf{I}_r)$ itself is Cohen-Macaulay. We first we need a lemma.

3.3. Lemma. *Let (A, m) be a local ring and $I \subset A$ an m-primary ideal. Put $G = gr_A(I)$. If $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, the ring G is a generalized Cohen-Macaulay ring and the ideal $(G^+)^r \subset G$ a standard ideal.*

Proof. If $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, we know by Theorem 1.1 that also $gr_A(\mathbf{I}_{r+1})$ is Cohen-Macaulay. Put $Q = G^+$. We now have

$$\begin{aligned} R_G(\mathbf{Q}_r) = \bigoplus_{n_1,\ldots,n_r \geq 0} Q^{n_1+\ldots+n_r} &= \bigoplus_{n_1,\ldots,n_r \geq 0} \Big(\bigoplus_{k \geq n_1+\ldots+n_r} G_k \Big) \\ &= \bigoplus_{n_1,\ldots,n_r \geq 0} \Big(\bigoplus_{n_{r+1} \geq 0} G_{n_1+\ldots+n_{r+1}} \Big) \\ &= \bigoplus_{n_1,\ldots,n_{r+1} \geq 0} G_{n_1+\ldots+n_{r+1}} \\ &= gr_A(\mathbf{I}_{r+1}). \end{aligned}$$

By Corollary 1.2 this implies that also $R_G(Q^r)$ is Cohen-Macaulay. But we then know by [HIO, Theorem 45.7] that Q^r is a standard ideal.

3.4. Proposition. *Let (A, m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal. Let $(a_1, \ldots, a_d)$ be a minimal reduction of I. Then $R_A(\mathbf{I}_r)$ is Cohen-Macaulay if and only if the following conditions hold:*

1) *$r(I) \leq d - 1$*
2) *The sequence $(a_1^*, \ldots, a_d^*)$ is $[0, \ldots, d-1]$-regular.*
3) *$gr_A(I)$ is a generalized Cohen-Macaulay ring and $(a_1^{*r}, \ldots, a_d^{*r})$ a standard system of parameters.*
4) *$l^d_{ir-q}(gr_A(I)) = \binom{d-1}{i} l^{i+1}_{ir-q}(gr_A(I))$ $(i = 1, \ldots, d-1,\ q = 1, \ldots, r)$.*

Proof. Put $G = gr_A(I)$. If 3) holds, 2) is by Proposition 2.3 equivalent to $a_i(G) < 0$ for $i = 0, \ldots, d-1$. By Corollary 1.3 it is then enough to assume that 1), 2) and 3) hold and show that 4) is equivalent to $h^i_p(G) = 0$ for $p < -r$ and $i = 1, \ldots, d-1$. Let $0 < q \leq r$. By Lemma 3.1

$$l^{i+1}_{ir-q}(G) = \sum_{j=0}^{i} \binom{i}{j} h^j_{(i-j)r-q}(G) = h^i_{-q}(G)$$

for all $i = 0, \ldots, d-1$. Moreover,

$$l^d_{ir-q}(G) = \sum_{j=0}^{d-1} \binom{d-1}{j} h^j_{(i-j)r-q}(G)$$

$$= \binom{d-1}{i} h^i_{-q}(G) + \sum_{j=i+1}^{d-1} \binom{d-1}{j} h^j_{(i-j)r-q}(G)$$
$$= \binom{d-1}{i} l^{i+1}_{ir-q}(G) + \sum_{j=i+1}^{d-1} \binom{d-1}{j} h^j_{(i-j)r-q}(G).$$

It follows that 4) is equivalent to $h^j_{(i-j)r-q}(G) = 0$ for $j = 0, \ldots, d-1$, $i < j$ and $0 < q \leq r$. Since always $h^j_p(G) = 0$ for $p < -jr$, this is the same as $h^j_p(G) = 0$ for $p < -r$. The claim has thus been proved.

3.5. Lemma. *Let (A, m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal such that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay. Let $(a_1, \ldots, a_d)$ be a minimal reduction of I. Let $\mathfrak{M}$ be the homogeneous maximal ideal of $R_A(\mathbf{I}_r)$. Suppose $i < d$ and $p \in \{1, \ldots, r\}$. Then*

$$[\underline{H}^i_{\mathfrak{M}}(gr_A(I))]_{-p} = 0$$

if and only if the following conditions are satisfied:

1) $(a_1^r, \ldots, a_d^r)I^{ir-p} \cap I^{(i+1)r-p+1} = (a_1^r, \ldots, a_d^r)I^{ir-p+1}$
2) $((a_1^r, \ldots, a_{d-1}^r)I^{ir-p} : a_d^r) \cap I^{ir-p} \subset (a_1^r, \ldots, a_{d-1}^r)I^{(i-1)r-p} + I^{ir-p+1}$.

If this is the case, we also have

1) $(a_1^r, \ldots, a_k^r)I^{ir-p} \cap I^{(i+1)r-p+1} = (a_1^r, \ldots, a_k^r)I^{ir-p+1}$
2) $((a_1^r, \ldots, a_{k-1}^r)I^{ir-p} : a_k^r) \cap I^{ir-p} \subset (a_1^r, \ldots, a_{k-1}^r)I^{(i-1)r-p} + I^{ir-p+1}$

for $k < d$

Proof. By using Lemma 3.1 as in the proof of Proposition 3.4 we see that

$$l^k_{ir-p}(gr_A(I)) = \begin{cases} 0 & \text{if } k < i+1 \\ \binom{k-1}{i} h^i_{-p}(gr_A(I)) & \text{if } k \geq i+1. \end{cases}$$

Hence $h^i_{-p}(gr_A(I)) = 0$ if and only if $l^d_{ir-p}(gr_A(I)) = 0$. Moreover, if this is the case we have $l^k_{ir-p}(gr_A(I)) = 0$ for $k = 1, \ldots, d$. The claim then follows from Lemma 2.11.

If $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, we know by Theorem 2.13 that the following conditions are satisfied for $i = 1, \ldots, d$ and $n \geq (d-1)r$:

a) $(a_1^r, \ldots, a_d^r)I^n \cap I^{n+r+1} = (a_1^r, \ldots, a_d^r)I^{n+1}$
b) $((a_1^r, \ldots, a_{d-1}^r)I^n : a_d^r) \cap I^n = (a_1^r, \ldots, a_{d-1}^r)I^{n-r}$.

It turns out in the following Theorem 3.6 that in order to $R_A(\mathbf{I}_s)$ be Cohen-Macaulay for some $s < r$ similar conditions must also hold for certain $n < (d-1)r$.

3.6. Theorem. *Let (A, m) be a local ring of dimension d and let $I \subset A$ be an m-primary ideal. Let $(a_1, \ldots, a_d)$ be a minimal reduction of I. If $R_A(\mathbf{I}_r)$ is Cohen-Macaulay and $s < r$, then $R_A(\mathbf{I}_s)$ is Cohen-Macaulay if and only if the following conditions hold for $i = 1, \ldots, d-1$ and $p = s+1, \ldots, r$*

1) $(a_1^r, \ldots, a_d^r) \cap I^{(i+1)r-p+1} = (a_1^r, \ldots, a_d^r)I^{ir-p+1}$

2) $((a_1^r, \ldots, a_{d-1}^r)I^{ir-p} : a_d^r) \cap I^{ir-p} \subset (a_1^r, \ldots, a_{d-1}^r)I^{(i-1)r-p} + I^{ir-p+1}$.

Proof. As $R_A(\mathbf{I}_r)$ is Cohen-Macaulay Theorem 1.1 implies that $R_A(\mathbf{I}_s)$ is Cohen-Macaulay if and only if $h^i_{-p}(gr_A(I)) = 0$ for $i = 1, \ldots, d-1$ and $p = s+1, \ldots, r$. By Lemma 3.5 this is equivalent to the conditions

1') $(a_1^r, \ldots, a_d^r)I^{ir-p} \cap I^{(i+1)r-p+1} = (a_1^r, \ldots, a_d^r)I^{ir-p+1}$

2') $((a_1^r, \ldots, a_{d-1}^r)I^{ir-p} : a_d^r) \cap I^{ir-p} \subset (a_1^r, \ldots, a_{d-1}^r)I^{(i-1)r-p} + I^{ir-p+1}$.

We only need to prove that 1') implies 1). We use descending induction on p.

Suppose $p = r$. Since $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, also $R_A(I^r)$ is Cohen-Macaulay. Let $\mathfrak{N}$ be the homogeneous maximal ideal of $gr_A(I^r)$. By [TI] we have

$$[\underline{H}^i_{\mathfrak{N}}(gr_A(I^r))]_n = \begin{cases} 0 & \text{if } n \neq -1 \\ H^i_m(A) & \text{if } n = -1 \end{cases}$$

for $i < d$. This implies that

$$\begin{aligned} I(gr_A(I^r)) &= \sum_{i=0}^{d-1} \binom{d-1}{i} l_A([\underline{H}^i_{\mathfrak{N}}(gr_A(I^r))]_{-1}) \\ &= \sum_{i=0}^{d-1} \binom{d-1}{i} l_A(H^i_m(A)) = I(A). \end{aligned}$$

It then follows from Proposition 2.14 that

$$(a_1^r, \ldots, a_d^r)I^{kr} \cap I^{(k+2)r} = (a_1^r, \ldots, a_d^r)I^{(k+1)r}$$

for $k \geq 0$. From this it is easy to see that also

$$(a_1^r, \ldots, a_d^r) \cap I^{(k+2)r} = (a_1^r, \ldots, a_d^r)I^{(k+1)r}$$

for $k \geq 0$. By 1') we then obtain

$$(a_1^r, \ldots, a_d^r)I^{(i-1)r+1} = (a_1^r, \ldots, a_d^r)I^{(i-1)r} \cap I^{ir+1} = (a_1^r, \ldots, a_d^r) \cap I^{ir+1}$$

so that 1) holds if $p = r$.

Then suppose $p < r$. By the induction hypothesis we have

$$(a_1^r, \ldots, a_d^r) \cap I^{ir-p+r} = (a_1^r, \ldots, a_d^r) I^{ir-p}.$$

Hence

$$(a_1^r, \ldots, a_d^r) I^{ir-p+1} = (a_1^r, \ldots, a_d^r) I^{ir-p} \cap I^{ir-p+r+1} = (a_1^r, \ldots, a_d^r) \cap I^{ir-p+r+1}.$$

3.7. Remark. If A is Cohen-Macaulay in Theorem 3.6, then 1) implies 2).

In order to see this let us first show that if 1) holds, then

$$(a_1^r, \ldots, a_{d-1}^r) \cap I^{ir-p} = (a_1^r, \ldots, a_{d-1}^r) I^{(i-1)r-p}$$

for $i = 1, \ldots, d-1$ and $p = s+1, \ldots, r$. Use induction on i. The case $i = 1$ being clear assume $i > 1$. Take

$$x \in (a_1^r, \ldots, a_{d-1}^r) \cap I^{ir-p} \subset (a_1^r, \ldots, a_d^r) \cap I^{ir-p} = (a_1^r, \ldots, a_d^r) I^{(i-1)r-p}.$$

Then

$$x = \sum_{j=1}^{d-1} \lambda_j a_j^r + \lambda_d a_d^r,$$

where $\lambda_1, \ldots, \lambda_d \in I^{(i-1)r-p}$. Since $(a_1^r, \ldots, a_d^r)$ is a regular sequence, we get by the inductive assumption that

$$\lambda_d \in (a_1^r, \ldots, a_{d-1}^r) \cap I^{(i-1)r-p} = (a_1^r, \ldots, a_{d-1}^r) I^{(i-2)r-p}.$$

This implies the above claim. Let us now show that 2) is satisfied. We obtain

$$\begin{aligned}((a_1^r, \ldots, a_{d-1}^r) I^{ir-p} : a_d^r) \cap I^{ir-p} &\subset ((a_1^r, \ldots, a_{d-1}^r) : a_d^r) \cap I^{ir-p} \\ &\subset (a_1^r, \ldots, a_{d-1}^r) \cap I^{ir-p} \\ &\subset (a_1^r, \ldots, a_{d-1}^r) I^{(i-1)r-p}.\end{aligned}$$

As an application of Theorem 3.6 we give the following corollary.

3.8. Corollary. *Let (A, m) be a local ring of dimension two and let $I \subset A$ be an m-primary ideal such that $R_A(\mathbf{I}_r)$ is Cohen-Macaulay. Let (a_1, a_2) be a minimal reduction of I. Then $R_A(\mathbf{I}_{r-1})$ is Cohen-Macaulay if and only if the following implication holds:*

$$\lambda_1 a_1^r + \lambda_2 a_2^r \in I^{r+1} \Rightarrow \lambda_1, \lambda_2 \in I.$$

Proof. According to Theorem 3.6 $R_A(\mathbf{I}_{r-1})$ Cohen-Macaulay means that $(a_1^r, a_2^r) \cap I^{r+1} = (a_1^r, a_2^r)I$ and $(a_1^r) : a_2^r \subset I$, $(a_2^r) : a_1^r \subset I$. These conditions are clearly equivalent to the implication mentioned in the claim.

3.9. Remark. If A is a local Cohen-Macaulay ring of dimension two, $R_A(\mathbf{I}_r)$ Cohen-Macaulay implies always that $R_A(I)$ is Cohen-Macaulay. If A has dimension three, the same holds for $I = m$. Since $r(I) \leq 1$ and $r(m) \leq 2$ in these cases, this follows from [VV] and [S].

3.10. Remark. If A is not Cohen-Macaulay, $I = m$ and $\dim A = 2$, it may happen that $R_A(\mathbf{m}_r)$ is not Cohen-Macaulay for any $r \geq 1$. As an example consider $A = k[[s^2, s^3, st, t]]$, where s and t are indeterminates. Now (s^2, t) is a minimal reduction of m. Since $s^3 \in (s^2 m : t) \cap m$, but $s^3 \notin (s^2)$, we see from Theorem 2.13 b) that $R_A(\mathbf{m}_r)$ cannot be Cohen-Macaulay for any $r \geq 1$.

Next we want to mention a class of examples where $R_A(\mathbf{I}_r)$ is Cohen-Macaulay for some $r \geq 1$, but $R_A(I)$ is not Cohen-Macaulay.

3.11. Example. Let (A, m) be a local generalized Cohen-Macaulay ring of dimension d. Let $I \subset A$ be a standard parameter-ideal. By [HIO, Theorem 40.10]

$$[\underline{H}^i_{\mathfrak{M}}(gr_A(I))]_n = \begin{cases} H^i_m(A), & \text{if } n = -i \\ 0 & , \text{ otherwise} \end{cases} \qquad (i < d),$$

$$[\underline{H}^d_{\mathfrak{M}}(gr_A(I))]_n = 0, \quad \text{if } n > -d,$$

where $\mathfrak{M}$ is the homogeneous maximal ideal of $R_A(I)$. Let $1 \leq r < d$. By [G1] it is always possible to find a local Buchsbaum ring (A, m) with $d \geq 3$, $\operatorname{depth} A > 0$ and $H^r_m(A) \neq 0$, $H^i_m(A) = 0$ for $r < i < d$. Then $R_A(\mathbf{I}_r)$ is Cohen-Macaulay, but $R_A(\mathbf{I}_{r-1})$ is not Cohen-Macaulay.

3.12. Example. Let (A, m) be a d-dimensional local Buchsbaum ring of maximal embedding dimension. By [T1, Proposition 5.11]

$$[\underline{H}^i_{\mathfrak{M}}(gr_A(I))]_n = \begin{cases} H^i_m(A), & \text{if } n = -(i-1) \\ 0 & , \text{ otherwise} \end{cases} \qquad (i < d),$$

$$[\underline{H}^d_{\mathfrak{M}}(gr_A(I))]_n = 0, \quad \text{if } n > 1 - d,$$

where $\mathfrak{M}$ is the homogeneous maximal ideal of $R_A(I)$. Let $1 \leq r < d-1$. By [G2] it is always possible to find a local Buchsbaum ring (A, m) of maximal embedding dimension such that $d \geq 4$, $\operatorname{depth} A > 0$ and $H_m^{r+1}(A) \neq 0$, $H_m^i(A) = 0$ for $r+1 < i < d$. Then $R_A(\mathbf{m}_r)$ is Cohen-Macaulay, but $R_A(\mathbf{m}_{r-1})$ is not Cohen-Macaulay.

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M. Herrmann
T. Korb
Mathematisches Institut
der Universität zu Köln
Weyertal 86-90
D-50931 Köln
Germany

Eero Hyry
National Defence College
Santahamina
00860 Helsinki
Finland