Poincaré–Birkhoff–Witt isomorphism, Hochshild homology and Riemann–Roch theorem

Nikita Markarian

To Andrei Nikolaevich Tyurin
in occasion of his 60-th birthday

Introduction

An idea of considering a Lie algebra as a supermanifold with a closed vector field goes back to D. Quillen. For a Lie algebra g the function ring is the Grassman algebra $S^*(g^*[1])$ and the vector field is given by the expression $c_{ij}^k x^i x^j \partial_k$, where c_{ij}^k are the structure constants of the algebra. This point of view allows to translate many construction from Lie groups and algebras theory into the (super)geometrical language. For example, the structural constants may be treated as the Atiyah class of the tangent bundle (this analogy is exploited in [K]); function ring of the group provides a resolution of the diagonal. Another case where such supermanifolds (in other words, DG-algebras) arise is the case of complex non-affine manifolds. The function ring may be realized as the ring of $(0, q)$-forms, the vector field is $\overline{\partial}$.

Inspired by Lie groups and algebras theory we introduce in section 1 analogs of Poincaré–Birkhoff–Witt (PBW) isomorphism and of the exponential map for any smooth manifold (definition–propositions 1 and 2). Theorem 1 contains the main calculation of the paper. Under the analogy above, this result corresponds to the formula of left and right invariant vector fields on a Lie group in terms of exponential coordinates.

In section 2 we discuss the Riemann-Roch theorem. To write down the Riemann–Roch theorem one have to calculate a particular class from $H^*(\Omega^*)$, namely the Todd genus. In this calculation (lemma 11) theorem 1 is crucially used. Drawing the analogy, the Todd genus corresponds to the (left) invariant volume form written in exponential coordinates. In this section we also prove the holomorphic Lefschetz formula.

Section 3 is nearly independent of section 2 and partly duplicate results of it. The main character of this section is the duality isomorphism (44). Theorem 8 describes interrelation of this map with isomorphisms introduced in section 1. It seems that this result explains appearance of the Todd genus in the Riemann–Roch theorem and other related statements.

This paper would never have been written without numerous fruitful discussions with A. Beilinson, A. Bondal, B. Feigin and M. Kontsevich. It is pleasure for me to express them my deep gratitude. I am grateful to Max–Planck–Institut für Mathematik, Bonn for fine work conditions.

1 Poincaré–Birkhoff–Witt isomorphism

Notations Let M be a smooth complex manifold. By $\triangle$ we denote the diagonal in the direct product $M \times M$, by $\mathcal{O}_\triangle$ denote the structure sheaf of the diagonal. By p_1 and p_2 denote projections on the first and the second factors.

The tangent and cotangent bundles on the manifold under consideration we shall denote by Ω^1 and T. By $S(\Omega^1[1])$ and $S(T[-1])$ denote correspondingly $\bigoplus_i \Omega^i[i]$ and $\bigoplus_i \Lambda^i T[-i]$. Objects $p_1^*(-) \otimes \mathcal{O}_\triangle$

we shall denote by $-_\triangle$, where $-$ is T, Ω^1 and so on. We shall identify normal bundle to the diagonal with $T_\triangle$, conormal bundle with $\Omega^1_\triangle$ and so on.

By $\mathcal{D}^-(\mathcal{M})$ denote the derived category of bounded above complexes of sheaves on M .

1 Atiyah class

Let $\mathcal{O}_{\triangle^2}$ be the structure sheaf of the first neighborhood of the diagonal in $M \times M$. It fits into an exact sequence

$$0 \to \Omega^1_\triangle \to \mathcal{O}_{\triangle^2} \to \mathcal{O}_\triangle \to 0, \tag{1}$$

where the last arrow is a restriction map.

Take any object $E \in \mathcal{D}^-(\mathcal{M})$ and apply the functor $Rp_{2,*}(- \overset{L}{\otimes} p_1^*(E))$ to the exact sequence (1). In the resulting triangle

$$\Omega^1 \otimes E \to Rp_{2*}(E \overset{L}{\otimes} p_1^*(E)) \to E \overset{+1}{\to} \tag{2}$$

the middle term is called *the object of first jets* of E and is denoted by $j^1(E)$.

Definition 1 (see [A]) *For $E \in \mathcal{D}^-(\mathcal{M})$ morphism $\alpha(E) \in \mathrm{Ext}^1(E, E \otimes \Omega^1)$ from the triangle (2) is called* the Atiyah class *of E.*

The following lemma is needed for the sequel.

Lemma 1 *Let (E, d) be a complex of locally free quasicoherent sheaves on M every component of which is supplied with a connection ∇. Then the Atiyah class $\alpha(E)$ is represented by $\nabla \cdot d + d \cdot \nabla$.*

Proof Left to reader. $\square$

2 Poincaré–Birkhoff–Witt isomorphism

Consider the class

$$\alpha \in \mathrm{Ext}^1(\mathcal{O}_\triangle, \Omega^1_\triangle)$$

of extension (1). Define an exponent of this class as

$$e^\alpha = 1 + \alpha + \frac{\alpha^2}{2} + \ldots \in \mathrm{Hom}(\mathcal{O}_\triangle, S(\Omega^1[1])_\triangle). \tag{3}$$

Definition-proposition 1 *Let $\vdash : S(T[-1])_\triangle \otimes S(\Omega^1[1])_\triangle \to \mathcal{O}_\triangle$ be the contraction map. Then the composition of e^α and $\vdash$ gives an isomorphism between $S(T[-1])_\triangle$ and $\mathcal{E}xt(\mathcal{O}_\triangle, \mathcal{O}_\triangle)$. We refer to this isomorphism as* the Poincaré–Birkoff–Witt (PBW) isomorphism *and denote it by I_{PBW}.*

Proof Local check is enough. $\square$

Corollary $\mathrm{Ext}^n(\mathcal{O}_\triangle, \mathcal{O}_\triangle) = \bigoplus_{i+j=n} H^i_M(\Lambda^j T)$.

This isomorphism we also denote by I_{PBW}.

3 $\mathrm{Ext}^*(\mathcal{O}_\triangle, \mathcal{O}_\triangle)$ as $\mathrm{Ext}^*(\mathrm{id}, \mathrm{id})$

For any object $f \in \mathcal{D}^-(\mathcal{M} \times \mathcal{M})$ define a functor Φ_f from the category $\mathcal{D}^-(\mathcal{M})$ to itself by the formula

$$\Phi_f(-) = Rp_{2*}(f \overset{L}{\otimes} p_1^*(-)). \tag{4}$$

Any morphism of objects $x \in \mathrm{Hom}_{M\times M}(g,h)$ gives us a morphism of corresponding functors. We denote it by $\Theta_x : \Phi_g \to \Phi_h$.

An example where these notions naturally arise is related with the Atiyah class (see subsection 1). For any object $E \in \mathcal{D}^-(\mathcal{M})$ the Atiyah class defines a morphism $\alpha(E) \in \mathrm{Ext}^1(E, E \otimes \Omega^1)$. One may easily check that this is a morphism from identity functor to multiplication by cotangent bundle functor shifted by one. We denote this morphism by $\mathcal{A} \in \mathrm{Ext}^1(\mathrm{id}, - \otimes \Omega^1)$.

Lemma 2

$$\mathcal{A} = \Theta_\alpha$$

where $\alpha \in \mathrm{Ext}^1(\mathcal{O}_\triangle, \Omega^1_\triangle)$ corresponds to extension (1).

Proof By definition. □

As functor $\Phi_{\mathcal{O}_\triangle}$ is identity, any element $a \in \mathrm{Ext}^i(\mathcal{O}_\triangle, \mathcal{O}_\triangle)$ defines a morphism of functors $\Theta_a \in \mathrm{Ext}^i(\mathrm{id}, \mathrm{id})$. This point of view on higher automorphisms of a derived category is outlined in [B].

Lemma 3 *Morphism of functors $\Theta_{I_{PBW}(p)}$ for $p \in H^i(\Lambda^j T)$ acts on an element $E \in \mathcal{D}^-(\mathcal{M})$ as follows*

$$\Theta_{I_{PBW}(p)}(E) = e^{\alpha(E)} \vdash p \in \mathrm{Ext}^{i+j}(E, E), \tag{5}$$

where $e^{\alpha(E)} = 1 + \alpha(E) + \alpha(E)^2/2 + \ldots \in \bigoplus_i \mathrm{Ext}^i(E, E \otimes \Omega^i)$ and $\vdash$ is the contraction map.

Proof The lemma directly follows from the construction of PBW isomorphism and lemma 2. □

The reader may deduce commutativity of the diagram

$$\begin{array}{ccc} E & \xrightarrow{x} & F \\ \downarrow \Theta_{I_{PBW}(p)} & & \downarrow \Theta_{I_{PBW}(p)} \\ E & \xrightarrow{x} & F \end{array}$$

for any E, F and x immediately from properties of the Atiyah class.

4 Exponential map

When treating the object $\mathcal{E}xt(\mathcal{O}_\triangle, \mathcal{O}_\triangle)$ it is useful to consider its action on the torsion product $\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle$.

Introduce three natural maps connected with $\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle$:

$$\begin{array}{rcl} S & : & \mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle \to \mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle \\ 1 \otimes - & : & \mathcal{O}_\triangle \to \mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle \\ \mu & : & \mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle \to \mathcal{O}_\triangle \otimes \mathcal{O}_\triangle = \mathcal{O}_\triangle \end{array} \tag{6}$$

Here the first map is permutation of factors and the third map is a natural morphism from a left derived functor to the initial one.

Analog of the PBW isomorphism for $\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle$ is the next statement.

Definition-proposition 2 *Composition of maps*

$$\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle \xrightarrow{\mathrm{id}\otimes e^\alpha} \mathcal{O}_\triangle \overset{L}{\otimes} S(\Omega^1[1])_\triangle \xrightarrow{\mu} S(\Omega^1[1])_\triangle \tag{7}$$

provides an isomorphism between $\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle$ *and* $S(\Omega^1[1])_\triangle$. *We refer to this isomorphism as* the right exponential map *and denote it by* $\exp_R$. *By definition,* the left exponential map $\exp_L$ *equals* $\exp_R \circ S$.

Proof Local check is enough. □
Corollary $\mathrm{Tor}^n(\mathcal{O}_\triangle, \mathcal{O}_\triangle) = \bigoplus_{i-j=n} H^i_M(\Omega^j)$.
Remark One may derive the PBW isomorphism from the right exponential map by composing with $1 \otimes -$.

5 Standard resolution

Let A be the algebra of functions on the manifold M (for non-affine manifold one should consider the formal neighborhood of the diagonal). By definition, *the standard resolution* (see [NB, §3 , n^o 8]) of $\mathcal{O}_\triangle$ on $M \times M$, that is a resolution $p: B(A) \to A$ of A considered as a A-A-bimodule looks as follows:

$$B_{-n}(A) = \underbrace{A \otimes A \ \otimes \cdots \otimes A}_{\text{n+2 factors}}, \quad n \geq 0$$
$$d(a_0 \otimes a_1 \otimes a_2 \otimes \cdots) = a_0 a_1 \otimes a_2 \otimes \cdots - a_0 \otimes a_1 a_2 \otimes \cdots + \ldots .$$

Here $B_n(A)$ is equipped with a structure of left (right) free A-module by the action of A on the first (the last) factor.

Describe the Atiyah class of the diagonal in terms of the standard resolution.

Lemma 4 *The Atiyah class of the diagonal* $\alpha(\mathcal{O}_\triangle) \in \mathrm{Hom}(B(A), B(A) \otimes \Omega^1[1])$ *is given by the formula*

$$\begin{aligned} &\alpha(\mathcal{O}_\triangle): a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes a_{i+1} \mapsto \\ &\quad a_0 da_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes a_{i+1} + (-1)^i a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{i+1} da_i . \end{aligned} \tag{8}$$

Proof Complex $B(A)$ is equipped with an obvious flat connection. Applying lemma 1 we get the Atiyah class of $B(A)$. □

The Atiyah class $\alpha(\mathcal{O}_\triangle)$ nearly coincides with class α.

Lemma 5 *Class* α *of the extension* (1) *acts on the standard resolution as follows*

$$\alpha: a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes a_{i+1} \mapsto a_0 da_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes a_{i+1} . \tag{9}$$

Proof Consider the decomposition $\Omega^1_{M\times M} = p_1^*\Omega^1_M \oplus p_2^*\Omega^1_M$. Denote by $\alpha_{1,2}(\mathcal{O}_\triangle)$ the components of $\alpha(\mathcal{O}_\triangle)$ under this decomposition (they corresponds to the first and the second summands in (8)). One may easily shows, that each of this component is equal to class α of the extension (1) under a proper isomorphism between the conormal bundle and $p_{1,2}^*\Omega^1$. We identify conormal bundle with $p_1^*\Omega^1$, so $\alpha = \alpha_1(\mathcal{O}_\triangle)$. □

Substituting the lemma in the definition of PBW isomorphism we get the next statement.
Corollary PBW isomorphism $I_{PBW}: \bigoplus_i \Lambda^i T_\triangle[-i] \to \mathcal{H}om(B(A), A)$ looks as follows

$$\begin{aligned} &I_{PBW}(\partial_1 \wedge \partial_2 \wedge \ldots \wedge \partial_k): \\ &\quad a_0 \otimes a_1 \otimes \cdots \otimes a_{k+1} \mapsto \tfrac{1}{k!} \sum_{\sigma \in \Sigma_k} (-1)^{\mathrm{sgn}\,\sigma} a_0 a_{k+1} \partial_{\sigma(1)} a_1 \cdots \partial_{\sigma(k)} a_k , \end{aligned} \tag{10}$$

where ∂_i are vector fields.

Thus our definition of PBW isomorphism for an affine manifold coincides with the Hochshild–Kostant–Rosenberg isomorphism (see e. g. [Ko]).

Complexes $B(A)\otimes B(A)$ and $A\otimes B(A)$ represent one and the same object $\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle \in \mathcal{D}^-(\mathcal{M}\times\mathcal{M})$, as a homotopical equivalence is defined by the map $\mathrm{id}\otimes p$. Describe explicitly an inverse map.

Lemma 6 *A map*

$$\gamma\colon a_0\otimes(a_1\otimes\cdots\otimes a_n)\mapsto \sum_{i=0}^{n} a_0\otimes(a_1\otimes\cdots\otimes a_i)\otimes(a_{i+1}\otimes\cdots\otimes a_n)$$

is a morphism of complexes $\gamma\colon A\otimes B(A)\to B(A)\otimes B(A)$ *and is right-inverse to* $\mathrm{id}\otimes p$.

Proof Straightforward check. □

6 Hochshild complex

Using the standard resolution we see that the direct image $p_{1*}(\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle)$ is represented by the complex of A-modules $\mathcal{H}$

$$\begin{aligned}
&\mathcal{H}_{-n} = \underbrace{A\otimes A\ \otimes\cdots\otimes A}_{\text{n+1 factors}}, \quad n\ge 0;\\
&d(a_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes a_n)\\
&\qquad = a_0a_1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes a_n - a_0\otimes a_1a_2\otimes\cdots\otimes + \ldots\\
&\qquad -(-1)^n a_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1}a_n + (-1)^n a_0a_n\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1}
\end{aligned}$$

known as *the Hochshild complex.*

The Hochshild complex is equipped with the following important operations

$$\begin{array}{rcl}
S & : & \mathcal{H}\to\mathcal{H}\\
\Delta & : & \mathcal{H}\to\mathcal{H}\otimes\mathcal{H}\\
\times & : & \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}
\end{array} \tag{11}$$

(all morphisms and tensor products are over A!) given by the formulas

$$\begin{aligned}
&S(a_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes a_n)\\
&\qquad\qquad = (-1)^n a_0\otimes a_n\otimes a_{n-1}\otimes\cdots\otimes a_2\otimes a_1\\
&\Delta(a_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes a_n)\\
&\qquad\qquad = \sum_{i=0}^{n} a_0\otimes(a_1\otimes\cdots\otimes a_i)\otimes(a_{i+1}\otimes\cdots a_n)\\
&a_0\otimes a_1\otimes\cdots\otimes a_p\times a_0\otimes a_{p+1}\otimes\cdots\otimes a_{p+q}\\
&\qquad = \sum_{\sigma\in\Sigma_{p+q}\text{ is }p\text{-}q\text{-shuffle}} a_0a_0'\otimes a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(p)}\otimes a_{\sigma(p+1)}\otimes\cdots\otimes a_{\sigma(p+q)}
\end{aligned}$$

The map S is the direct image of S from (6), the map Δ is closely connected with $1\otimes -$ by lemma 6, the map $\times$ is closely connected with μ. These operations provide the Hochshild complex with the structure of Hopf algebra over A.

By definition-proposition 2, the Hochshild complex is quasi-isomorphic to $S(\Omega^1[1])$. The corollary of lemma 4 gives us an explicit form of this equivalence defining by the following well-known formula:

$$\exp_*\colon\ \mathcal{H}\ni a_0\otimes a_1\otimes\cdots\otimes a_n\mapsto a_0\,da_1\wedge\ldots\wedge da_n\in\Omega^n. \tag{12}$$

(We do not make difference between left and right exponential map because they coincide up to sign.)

It is well known and easy to show that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{H}\otimes\mathcal{H} & \xrightarrow{\times} & \mathcal{H} \\ \downarrow \exp_*\otimes\exp_* & & \downarrow \exp_* \\ S(\Omega^1[1])\otimes S(\Omega^1[1]) & \xrightarrow{\wedge} & S(\Omega^1[1]) \end{array} \tag{13}$$

PBW isomorphism supplies an isomorphism $p_{1*}\,\mathcal{E}xt(\mathcal{O}_\triangle,\mathcal{O}_\triangle) = S(T[-1])$. The direct image of action of $\mathcal{E}xt(\mathcal{O}_\triangle,\mathcal{O}_\triangle)$ on $\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle$ by acting on the second factor gives us an action of $S(T[-1])$ on $S(\Omega^1[1])$. To describe this action it is enough to describe the direct image of the map $1\otimes\alpha$. Below we describe this morphism using the Hochshild complex. The reader may try to translate our arguments into resolution-independent language.

Theorem 1 *Let $\alpha_L \in \mathcal{H}om(\mathcal{H},\mathcal{H}\otimes\Omega^1)$ acts on the Hochshild complex as follows*

$$\alpha_L(a_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_n) = a_0 da_1\otimes a_2\otimes\cdots\otimes a_n$$

(by lemma 5, α_L represents the direct image of the map $1\otimes\alpha$).

Let

$$\begin{array}{rcl} \alpha(\Omega^1) & \in & H^1(\mathcal{E}nd(\Omega^1)\otimes\Omega^1) \\ \mathrm{ad} & \in & H^1(\mathcal{E}nd(S(\Omega^1[1]))\otimes\Omega^1) \end{array}$$

be the Atiyah classes of Ω^1 and $S(\Omega^1[1])$, let $e^{\alpha(\Omega^1)} \in H^*(\mathcal{E}nd(\Omega^1)\otimes S(\Omega^1[1]))$ be an exponent of the Atiyah class. Let

$$\imath : S(\Omega^1[1]) \to \mathcal{E}nd(S(\Omega^1[1]))$$

be the map that takes an element of the Grassmann algebra to the multiplication by this element. Let $\mathcal{E}nd(\Omega^1)\otimes\mathcal{E}nd(S(\Omega^1[1]))$ acts on $\mathcal{E}nd(S(\Omega^1[1]))\otimes\Omega^1$ as follows: the first factor acts on the second one and vice versa. Put

$$\mathcal{L}_L = (\imath\, e^{\alpha(\Omega^1)} - \mathrm{id}\otimes\mathrm{id})^{-1}\cdot\mathrm{ad} \in H^*(\mathcal{E}nd(S(\Omega^1[1]))\otimes\Omega^1). \tag{14}$$

(The reader will have no difficulty in showing that this expression is well defined.)

Then the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\alpha_L} & \mathcal{H}\otimes\Omega^1 \\ \downarrow \exp_* & & \downarrow \exp_*\otimes\mathrm{id} \\ S(\Omega^1[1]) & \xrightarrow{\mathcal{L}_L} & S(\Omega^1[1])\otimes\Omega^1 \end{array}$$

Proof First let us introduce some notation. Put $\alpha_R = -S\circ\alpha_L\circ S$. By $\tilde\alpha$ denote a map $\mathcal{H}\to\Omega^1$ that is equal to $\tilde\alpha$ on $\mathcal{H}_{-1}$ and is equal to zero on other terms of the complex. Then

$$\begin{aligned} \alpha_L &= (\tilde\alpha\otimes\mathrm{id})\circ\Delta \\ \alpha_R &= (\mathrm{id}\otimes\tilde\alpha)\circ\Delta \end{aligned} \tag{15}$$

(in the second formula one should be careful with signs, $\tilde\alpha$ is considered as an odd element). If $\Delta^2(a\in\mathcal{H}) = \sum_i a_i^{(1)}\otimes a_i^{(2)}\otimes a_i^{(3)} \in \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$, then put by definition $\mathrm{Ad}\,a = \sum_i a_i^{(2)}\otimes(a_i^{(1)}\times S a_i^{(3)})$.

Consider the series of commutative diagrams.

1. This diagram is commutative by lemma 1:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\alpha_L-\alpha_R} & \mathcal{H}\otimes\Omega^1 \\ \downarrow \exp_* & & \downarrow \exp_* \\ S(\Omega^1[1]) & \xrightarrow{\mathrm{ad}} & S(\Omega^1[1])\otimes\Omega^1 \end{array} \tag{16}$$

2. This diagram is commutative for the same reason:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathrm{Ad}} & \mathcal{H}\otimes\mathcal{H} \\ \downarrow \exp_* & & \downarrow \exp_* \\ S(\Omega^1[1]) & \xrightarrow{e^{\mathrm{ad}}} & S(\Omega^1[1])\otimes S(\Omega^1[1]) \end{array} \tag{17}$$

3. Figures in the squares on this diagram show the reasons for their commutativity:

$$\begin{array}{ccccccc} \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H}\otimes\mathcal{H} & \xrightarrow{\mathrm{Ad}\otimes\mathrm{id}} & \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} & \xrightarrow{\mathrm{id}\otimes\times} & \mathcal{H}\otimes\mathcal{H} \\ \downarrow \alpha_L & (15) & \downarrow & (17) & \downarrow & (13) & \downarrow \\ \Omega^1\otimes\mathcal{H} & \xrightarrow{\mathrm{id}\otimes\exp_*} & \Omega^1\otimes S(\Omega^1[1]) & \xrightarrow{e^{\alpha(\Omega^1)}} & \Omega^1\otimes S(\Omega^1[1])\otimes S(\Omega^1[1]) & \xrightarrow{\mathrm{id}\otimes\wedge} & \Omega^1\otimes S(\Omega^1[1]) \end{array} \tag{18}$$

Here the unmarked vertical arrows represent maps which are equal to $\tilde{\alpha}$ on the first factor and are equal to $\exp_*$ on other ones.

An easy calculation gives us an identity

$$(\mathrm{id}\otimes\times)\circ(\mathrm{Ad}\otimes\mathrm{id})\circ\Delta = \sigma\circ\Delta, \tag{19}$$

where σ denotes a permutation of factors (with signs!) in the tensor product. Therefore moving from the upper left corner of diagram (18) to the lower right one by the upper row of arrows and the lower one and using (15), we have

$$\imath\, e^{\alpha(\Omega^1)}\circ\exp_*\circ\alpha_L = \exp_*\circ\alpha_R \tag{20}$$

Combining (20) with (16) we obtain the statement of the theorem. $\square$

2 Riemann–Roch theorem

Notations In this section M is a compact manifold of dimension n. By ω denote the canonical bundle. By $\omega_\triangle$ denote the sheaf $p_1^*\omega\otimes\mathcal{O}_\triangle = p_2^*\omega\otimes\mathcal{O}_\triangle$ on $M\times M$. By a complex of sheaves we mean a bounded complex of coherent sheaves.

1 Serre duality

Let x be a point of a manifold M and $\mathcal{O}_x$ be the structure sheaf of this point. Let $z_1, z_2, \ldots, z_n$ be local coordinates in a neighborhood U of x. Define *the Koszul complex* of $\mathcal{O}(U)$-modules as follows:

$$\begin{array}{l} K_\bullet = \mathcal{O}(U)\otimes\Lambda^* < \delta z_1, \ldots, \delta z_n >; \\ d(\delta z_i) = z_i. \end{array} \tag{21}$$

Here $< - >$ denotes the vector space generated by $-$, action of d is defined by Leibniz rule and $\deg \delta z_i = -1$. The Koszul complex is a free resolution of the sheaf $\mathcal{O}_x$.

Define morphisms $e_x \in \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_x)$ and $f_x \in \mathrm{Ext}^n(\mathcal{O}_x, \omega)$ explicitly through the Koszul complex:

$$\begin{array}{l} e_x\colon\ \mathcal{O}\to K_0\to K_\bullet \\ f_x\colon\ K_\bullet\to K_n\to\omega. \end{array}$$

Here the first arrow is the isomorphism and the last one takes $\delta z_1\wedge\ldots\wedge\delta z_n$ to $dz_1\wedge\ldots\wedge dz_n$. The reader will easily check independence of this definition on coordinates.

For a compact manifold morphisms e_x and f_x are dual to each other, moreover, the duality is defined by the composition. This follows from Serre duality. We present this theorem in the form we need in the sequel. The proof and generalizations one may find e. g. in [H].

Theorem 2 (Serre duality) *For any complex of sheaves E over M there exists a linear function* Tr (trace) *on the space* $\mathrm{Ext}^n(E, E\otimes\omega)$*:*

$$\mathrm{Tr}:\quad \mathrm{Ext}^n(E, E\otimes\omega) \to \mathbf{C},$$

such that for any two complexes of sheaves E and F the composition map

$$\mathrm{Ext}^*(E,F)\otimes \mathrm{Ext}^{n-*}(F, E\otimes\omega) \xrightarrow{\circ} \mathrm{Ext}^n(E, E\otimes\omega) \xrightarrow{\mathrm{Tr}} \mathbf{C}$$

defines a pairing that is degenerate and supercommutative:

$$\mathrm{Tr}(a\cdot b) = (-1)^{\deg a}(b\otimes\omega[n]\cdot a). \tag{22}$$

Moreover, for morphisms e_x and f_x defined above

$$\mathrm{Tr}(f_x e_x) = 1. \tag{23}$$

By Serre duality, $\dim \mathrm{Ext}^n(\mathcal{O},\omega) = \dim \mathrm{Ext}^0(\mathcal{O},\mathcal{O}) = 1$. An element of $\mathrm{Ext}^n(\mathcal{O},\omega)$ that is taken to 1 by Tr we denote by v. By (23), $v = f_x e_x$ for any point $x\in M$.

The following lemma is needed for the sequel.

Lemma 7 *Let E be a complex of sheaves and $s\in \mathrm{Ext}^n(E, E\otimes\omega)$. For a finite-dimensional graded vector space $V=\bigoplus V_i$ and a grading-preserving operator $A=\bigoplus A_i$ consider the morphism $\bigoplus s\otimes A_i \in \mathrm{Ext}^n(\bigoplus E\otimes V_i[i], \bigoplus E\otimes V_i[i]\otimes\omega)$. Then*

$$\mathrm{Tr}(\bigoplus s\otimes A_i) = \mathrm{str}\, A\cdot \mathrm{Tr}(s),$$

where str *denotes* supertrace*:*

$$\mathrm{str}\, A = \sum_i (-1)^i \,\mathrm{tr}\, A_i.$$

Proof Let $v^1, v^2, \dots$ be a union of bases of vector spaces V_i and $v_1, v_2, \dots$ be the dual basis of the dual vector space. By v^k also denote the multiplication operator by the vector from $\bigoplus \mathrm{Ext}^{-i}(E, E\otimes V_i[i])$, and by v_k denote the contraction operator from $\bigoplus \mathrm{Ext}^i(E\otimes V_i[i], E)$. Then

$$\begin{aligned}\mathrm{Tr}(\textstyle\bigoplus s\otimes A_i) &= \mathrm{Tr}(Av^i\cdot s\cdot v_i)\\ &= \mathrm{Tr}((-1)^{\deg v_i} Av_i\cdot v^i\cdot s)\\ &= \textstyle\sum_i (-1)^i\,\mathrm{tr}\, A_i\cdot \mathrm{Tr}(s).\end{aligned}$$

□

2 $\mathrm{Ext}^*(\mathrm{id}, S)$ and Chern characters

Lemma 8 *An isomorphism is defined*

$$I_{PBW}: S(\Omega^1[1])_\triangle = \mathcal{E}xt(\mathcal{O}_\triangle, \omega_\triangle[\dim M]).$$

Proof This statement is a "twisted" version of definition-proposition 1 and the proof is the same. We have

$$\mathcal{E}xt(\mathcal{O}_\triangle, \omega_\triangle[\dim M) = S(T[-1])_\triangle\otimes p_1^*\omega[\dim M] = S(\Omega^1[1])_\triangle.$$

□

Corollary $\mathrm{Ext}^i_{M\times M}(\mathcal{O}_\triangle,\omega_\triangle)=\bigoplus_{j+k=i}H^j_M(\Omega^{n-k})$.

This isomorphism we also denote by I_{PBW}.

The functor $-\otimes\omega[n]$ is called *the Serre functor*. Obviously, $S=\Phi_{\omega_\triangle[n]}$ (for definition of Φ see subsection 1.3). Any element of $\mathrm{Ext}^i(\mathcal{O}_\triangle,\omega_\triangle)$ defines a morphism of functors $\mathrm{Ext}^{i-n}(\mathrm{id},S)$. Describe this morphism explicitly.

Lemma 9 *Morphism of functors $\Theta_{I_{PBW}(\Omega)}$ for $\Omega\in H^i(\Omega^j)$ acts on a complex of sheaves E as follows*

$$\Theta_{I_{PBW}(\Omega)}(E)=\mathrm{top}(e^{\alpha(E)}\wedge\Omega)\in\mathrm{Ext}^{i-j}(E,SE),\tag{24}$$

where $e^{\alpha(E)}\in\bigoplus_i\mathrm{Ext}^i(E,E\otimes\Omega^i)$ is the exponent of the Atiyah class and top *is the projection onto the forms of top weight.*

Proof See lemma 3. □

Definition 2 Chern character Ch_E *of a complex of sheaves E on M is a linear function on the space* $\mathrm{Ext}^0(\mathrm{id},S)$ *defined by the formula:*

$$\mathrm{Ch}_E(a)=\mathrm{Tr}(a_E).\tag{25}$$

The reader may easily check that this definition is connected with the usual definition of Chern character $\mathrm{ch}(E)\in\bigoplus_i H^i(\Omega^i)$ as follows:

$$\mathrm{Ch}_E(\Theta_{PBW(l)})=\int l\wedge\mathrm{ch}(E),\tag{26}$$

where $l\in\bigoplus_i H^i(\Omega^i)$ and $\int$ is the Poincaré pairing on the space $\bigoplus_i H^i(\Omega^i)$.

3 Riemann–Roch theorem

Consider morphisms e_x and f_x introduced in subsection 1 for all points x simultaneously.

Lemma 10 *There exist unique morphisms*

$$e\in\mathrm{Ext}^0_{M\times M}(\mathcal{O},\mathcal{O}_\triangle)$$
$$f\in\mathrm{Ext}^n_{M\times M}(\mathcal{O}_\triangle,p_2^*\omega)$$

such that their restrictions on $p_1^{-1}(x)$ for any x are equal to e_x and f_x respectively.

Proof Morphism e is simply the unit section of the sheaf $\mathcal{O}_\triangle$. To define f consider the Leray spectral sequence:

$$H^*_M(Rp_{1*}(\mathcal{E}xt(\mathcal{O}_\triangle,p_2^*\omega)))\Rightarrow\mathrm{Ext}^*_{M\times M}(\mathcal{O}_\triangle,p_2^*\omega).$$

This sequence degenerates in the second term because $Rp_{1*}(\mathcal{E}xt(\mathcal{O}_\triangle,p_2^*\omega))$ is a (shifted) line bundle. Above we define a canonical trivialization of this bundle and the unit section gives us the morphism we need. □

Morphisms e and f define morphisms of functors $\Theta_e\in Ext^0(\Phi_{\mathcal{O}},\mathrm{id})$ and $\Theta_f\in\mathrm{Ext}^n(\mathrm{id},\Phi_{p_2^*\omega})$. The reader will easily check that

$$\Phi_{\mathcal{O}}(-)=\mathcal{O}\otimes\mathrm{Ext}^*(\mathcal{O},-)$$

and

$$\Phi_{p_2^*\omega}=S\Phi_{\mathcal{O}}[-n].$$

Taking compositions of Θ_e and Θ_f and calculating the trace we get Riemann–Roch theorem.

Theorem 3 (Riemann–Roch theorem) *There exists an element*

$$\mathrm{Td} \in \mathrm{Ext}^n(\mathcal{O}_\triangle, \omega_\triangle),$$

which is called Todd genus, *such that for any complex of sheaves E*

$$\sum (-1)^i \dim H^i(E) = \mathrm{Ch}_E(\Theta_{\mathrm{Td}}).$$

In other words (see the remark after definition 2),

$$\sum (-1)^i \dim H^i(E) = \int \mathrm{ch}(E) \wedge I_{PBW}(\mathrm{Td}).$$

Proof Put $\mathrm{Td} = ef$ where e and f are defined in lemma 10. Then

$$\begin{aligned}\mathrm{Ch}_E(\Theta_{\mathrm{Td}}) &= \mathrm{Tr}(\Theta_{ef}(E)) = \mathrm{Tr}(\Theta_e(\Phi_{\mathcal{O}}(E))\Theta_f(E)) \\ &= \mathrm{Tr}(\Theta_f(E)\Theta_e(\Phi_{\mathcal{O}}(E))) = \mathrm{Tr}(\Theta_{fe}(\Phi_{\mathcal{O}}(E))).\end{aligned}$$

Consider the product $fe \in H^n(p_1^*\mathcal{O} \otimes p_2^*\omega) = \sum_i H^i(\mathcal{O}) \otimes H^{n-i}(\omega)$. Only summands with the first factor of degree 0 contribute in the trace. By the remark after theorem 2, the only such a summand has the form $1 \otimes p_2^* v$, where $v \in H^n(\omega)$ is defined in the remark. Thus $\Theta_{fe}(E) \in \mathrm{Ext}^n(\mathcal{O} \otimes H^*(E), \omega \otimes H^*(E))$ is a multiplication by v. By lemma 7,

$$\mathrm{Tr}(\Theta_{fe}(E)) = \mathrm{str}\ \mathrm{id}_{H^*(E)} \cdot \mathrm{Tr}(v) = \sum (-1)^i \dim H^i(E).$$

□

4 Holomorphic Lefschetz formula

Before calculating the Todd genus let us consider an example of applying the trick, which we use in the proof of Riemann–Roch theorem, in another analogous situation.

Let $\varphi: M \to M$ be an automorphism of M such that its fixed points are non-degenerate. Let E be a complex of sheaves on M. Suppose a morphism $g: \varphi_* E \to E$ is given (note that $\varphi_* = (\varphi^{-1})^*$, because φ is an automorphism). Then g acts on cohomologies of E as follows:

$$H^*(E) \xrightarrow{\varphi_*} H^*(\varphi_* E) \xrightarrow{g} H^*(E)$$

Denote this action by $g_H \in \mathrm{End}(H^*(E))$.

For a complex of sheaves $\mathcal{F}$ introduce its fiber at a point x by $\mathcal{F}_x = \mathcal{F} \overset{L}{\otimes} \mathcal{O}_x$, where $\mathcal{O}_x$ is the structure sheaf of the point. $\mathcal{F}_x$ is a module over $\mathcal{O}_x$ that is a vector space. For a fixed point $x \in \mathrm{Fix}\,\varphi$ vector spaces E_x and $\varphi_* E_x$ are naturally isomorphic. This yields an action of g on the fiber E_x. Denote this action by $g_x \in \mathrm{End}(E_x)$.

Supertraces of two actions described are related by the following theorem (see e. g. [GH]).

Theorem 4 (Holomorphic Lefschetz formula)

$$\mathrm{str}\, g_H = \sum_{x \in \mathrm{Fix}\,\varphi} \frac{\mathrm{str}\, g_x}{\det(1 - \varphi'(x)^{-1})},$$

where $\varphi'(x)$ is the derivative of φ at x.

Proof Let

$$\mathcal{O}_\Gamma = (\varphi \times \mathrm{id})^* \mathcal{O}_\triangle$$

be the structure sheaf of the graph of the map φ. Obviously, $\Phi_{\mathcal{O}_\Gamma} = \varphi_*$. Introduce notations:

$$\begin{aligned} e_\Gamma &= (\varphi \times \mathrm{id})^* e : \ \mathcal{O} \to \mathcal{O}_\Gamma \\ f_\Gamma &= (\varphi \times \mathrm{id})^* f : \ \mathcal{O}_\Gamma \to p_2^* \omega. \end{aligned}$$

We shall show that left-hand and right-hand sides of Lefschetz formula are equal to $\mathrm{Tr}(g\Theta_{e_\Gamma}\Theta_f(E))$.

L.H.S. Consider a diagram:

$$\begin{array}{ccccccccc} \mathcal{O}\otimes H & \xrightarrow{\Theta_e} & E & \xrightarrow{\Theta_f} & S\mathcal{O}\otimes H & \xrightarrow{\Theta_{e_\Gamma}} & S\varphi_* E & \xrightarrow{g} & SE \\ \uparrow g_H & & \uparrow g & & & & & & \\ \mathcal{O}\otimes H & \xrightarrow{\Theta_{e_\Gamma}} & \varphi_* E & & & & & & \end{array}$$

where by H is denoted spaces $H^*(\varphi_* E) = H^*(E)$. We have

$$\begin{aligned} \mathrm{Tr}(g\Theta_{e_\Gamma}\Theta_f) &= \mathrm{Tr}(\Theta_f g \Theta_{e_\Gamma}) = \mathrm{Tr}(\Theta_f \Theta_e g_H) \\ &= \mathrm{Tr}(\Theta_{fe} g_H) = \mathrm{Tr}(v g_H) = \mathrm{str}\, g_H. \end{aligned} \tag{27}$$

R.H.S. Consider the morphism $e_\Gamma f \in \mathrm{Ext}^n(\mathcal{O}_\triangle, \mathcal{O}_\Gamma \otimes p_2^*\omega)$. By the hypothesis of the theorem,

$$\mathrm{supp}\ \mathcal{E}xt(\mathcal{O}_\triangle, \mathcal{O}_\Gamma \otimes p_2^*\omega) = \triangle \cap \Gamma = \{(x,x) | x \in \mathrm{Fix}\,\varphi\}$$

is a set of isolated points. Consider the morphism $e_\Gamma f$ in a neighborhood of any such point x. By a straightforward calculation (adduced below), we get in a neighborhood of x

$$e_\Gamma f = \frac{1}{\det(1-\varphi'(x)^{-1})} (p_2^* f_x \otimes \mathcal{O}_\Gamma)(p_2^* e_x \otimes \mathcal{O}_\triangle), \tag{28}$$

where e_x and f_x are defined before theorem 2. This local statement leads to the global result:

$$e_\Gamma f = \sum_{x\in \mathrm{Fix}\,\varphi} \frac{1}{\det(1-\varphi'(x)^{-1})} (p_2^* f_x \otimes \mathcal{O}_\Gamma)(p_2^* e_x \otimes \mathcal{O}_\triangle),$$

Now we have

$$\begin{aligned} \mathrm{Tr}(g\Theta_{e_\Gamma}\Theta_f) = \mathrm{Tr}(g\Theta_{e_\Gamma f}) &= \sum_{x\in\mathrm{Fix}\,\varphi} \tfrac{1}{\det(1-\varphi'(x)^{-1})}\,\mathrm{Tr}(g f_x e_x) \\ = \sum_{x\in\mathrm{Fix}\,\varphi} \tfrac{1}{\det(1-\varphi'(x)^{-1})}\,\mathrm{Tr}(e_x g f_x) &= \sum_{x\in\mathrm{Fix}\,\varphi} \tfrac{1}{\det(1-\varphi'(x)^{-1})}\,\mathrm{Tr}(e_x f_x g_x) \\ = \sum_{x\in\mathrm{Fix}\,\varphi} \tfrac{1}{\det(1-\varphi'(x)^{-1})}\,\mathrm{Tr}(v g_x) &= \sum_{x\in\mathrm{Fix}\,\varphi} \tfrac{\mathrm{str}\, g_x}{\det(1-\varphi'(x)^{-1})} \end{aligned} \tag{29}$$

By composing (27) and (29) we conclude the proof. □

Calculation Prove (28).

Let $z_1,\ldots,z_n$ be local coordinates on M in a neighborhood of the fixed point x. Denote $X_i = p_1^* z_i$ and $Y_i = p_2^* z_i$.

Write down Koszul resolutions of the sheaves $\mathcal{O}_\triangle$, $\mathcal{O}_\Gamma$ and the structure sheaf of the point $\mathcal{O}_{(x,x)}$ on $M\times M$:

$$\begin{aligned} &K_\bullet(\mathcal{O}_\triangle) = \mathcal{O}\otimes\Lambda^* < \delta(Y_1 - X_1),\ldots,\delta(Y_n - X_n) > \\ &K_\bullet(\mathcal{O}_\Gamma) = (\varphi\times\mathrm{id})^* K_\bullet(\mathcal{O}_\triangle) = \mathcal{O}\otimes\Lambda^* < \delta(Y_1 - \varphi(X),\ldots,\delta(Y_n - \varphi(X)) > \\ &K_\bullet(\mathcal{O}_{(x,x)}) = \mathcal{O}\otimes\Lambda^* < \delta(X_1),\ldots,\delta(X_n),\delta(Y_1),\ldots,\delta(Y_n) > . \end{aligned}$$

Here notations are the same as in (21). We treat δ as a linear operation: $\delta(X_i - Y_i) = \delta(X_i) - \delta(Y_i)$ and so on.

One may suppose that φ is linear in chosen coordinates. The reader will have no difficulty in showing that three morphisms appeared in (28) acts on the Koszul complexes as follows:

$$\begin{array}{rcl} e_\Gamma f & : & K_{-n}(\mathcal{O}_\triangle) \ni \delta(Y_1 - X_1) \wedge \ldots \wedge \delta(Y_n - X_n) \mapsto \\ & & dY_1 \wedge \ldots \wedge dY_n \in K_0(\mathcal{O}_\Gamma) \otimes p_2^*\omega \\ p_2^* e_x \otimes \mathcal{O}_\triangle & : & K_{-k}(\mathcal{O}_\triangle) \ni \delta(Y_{i_1} - X_{i_1}) \wedge \ldots \wedge \delta(Y_{i_k} - X_{i_k}) \mapsto \\ & & (\delta(Y_{i_1}) - \delta(X_{i_1})) \wedge \ldots \wedge (\delta(Y_{i_k}) - \delta(X_{i_k})) \in K_{-k}(\mathcal{O}_{(x,x)}) \\ p_2^* f_x \otimes \mathcal{O}_\Gamma & : & K_{-n}(\mathcal{O}_{(x,x)}) \ni a \mapsto \end{array}$$
$$\frac{a \wedge \delta(Y_1 - \varphi_1(X)) \wedge \ldots \wedge \delta(Y_n - \varphi_n(X))}{\delta Y_1 \wedge \ldots \wedge \delta Y_n \wedge \delta(Y_1 - \varphi_1(X)) \wedge \ldots \wedge \delta(Y_n - \varphi_n(X))} \, dY_1 \wedge \ldots \wedge dY_n \in K_0(\mathcal{O}_\Gamma) \otimes p_2^*\omega$$

It follows that

$$\begin{aligned} & (p_2^* f_x \otimes \mathcal{O}_\Gamma)(p_2^* e_x \otimes \mathcal{O}_\triangle)\, \delta(Y_1 - X_1) \wedge \ldots \wedge \delta(Y_n - X_n) \\ = & \frac{\delta(Y_1 - X_1) \wedge \ldots \wedge \delta(Y_n - X_n) \wedge \delta(Y_1 - \varphi_1(X)) \wedge \ldots \wedge \delta(Y_n - \varphi_n(X))}{\delta Y_1 \wedge \ldots \wedge \delta Y_n \wedge \delta(Y_1 - \varphi_1(X)) \wedge \ldots \wedge \delta(Y_n - \varphi_n(X))} \, dY_1 \wedge \ldots \wedge dY_n \\ = & \det(1 - \varphi'(x)^{-1})\, dY_1 \wedge \ldots \wedge dY_n = \det(1 - \varphi'(x)^{-1})\, e_\Gamma f\, \delta(Y_1 - X_1) \wedge \ldots \wedge \delta(Y_n - X_n). \end{aligned}$$

5 Todd genus

Return to Riemann–Roch theorem. Recall that theorem 3 states existence of a particular class $\mathrm{Td} \in \mathrm{Ext}^n(\mathcal{O}_\triangle, \mathcal{O}_\triangle \otimes p_2^*\omega)$, which is constructed in the proof. It remains to find an element from $\bigoplus_i H^i(\Omega^i)$ that corresponds to Td under PBW isomorphism (see corollary of lemma 8). Before do it prove two lemmas.

Let us multiply Td by $\mathcal{O}_\triangle$, that is introduce the morphism

$$\mathrm{id} \otimes \mathrm{Td} \in \mathrm{Ext}^n(\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle, \mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle \otimes p_2^*\omega).$$

Consider the direct image of this morphism $p_{2*}(\mathrm{id} \otimes \mathrm{Td}) \in \mathrm{Ext}^n(S(\Omega^1[1]), S(\Omega^1[1]) \otimes \omega)$, where we identify $S(\Omega^1[1])$ with $p_{2*}(\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle)$ by the right exponential map (see definition-proposition 2).

Lemma 11 *Let $i\colon \omega[n] \to S(\Omega^1[1])$ be the inclusion of top weight forms. Then*

$$p_{2*}(\mathrm{id} \otimes \mathrm{Td}) \circ i = 1 \otimes \mathrm{id}.$$

Proof By the proof of theorem 3, $\mathrm{Td} = ef$. Local calculation shows that

$$\mathrm{Im}(p_{2*}(\mathrm{id} \otimes \mathrm{Td}) \circ i - 1 \otimes \mathrm{id}) \subset \bigoplus_{i=1}^{n} \Omega^i[i].$$

On the other hand

$$p_{2*}(\mathrm{id} \otimes ef) = p_{2*}(\mathrm{id} \otimes e) p_{2*}(\mathrm{id} \otimes f).$$

But $p_{2*}(\mathrm{id} \otimes e) \in \mathrm{Ext}^0(\mathcal{O}, S(\Omega^1[1]))$ is simply an inclusion of $\mathcal{O}$ into $S(\Omega^1[1])$. It follows that image of $p_{2*}(\mathrm{id} \otimes \mathrm{Td})$ lies in $\mathcal{O}$, so $\mathrm{Im}(p_{2*}(\mathrm{id} \otimes \mathrm{Td}) \circ i - 1 \otimes \mathrm{id}) = 0$. $\square$

Consider the object $\mathcal{E}nd(S(\Omega^1[1]))$. By $\imath\colon S(\Omega^1[1]) \to \mathcal{E}nd(S(\Omega^1[1]))$ and $\imath\colon \Lambda^i T[-i] \to \mathcal{E}nd(S(\Omega^1[1]))$ denote maps taking a form to the operator of multiplication by the form and a polyvector field to the operator of contraction by this field respectively. The standard fact from Clifford algebras theory claims that the composition map

$$S(\Omega^1[1]) \otimes \Lambda^* T \xrightarrow{\imath \otimes \imath} \mathcal{E}nd(S(\Omega^1[1])) \otimes \mathcal{E}nd(S(\Omega^1[1])) \xrightarrow{\circ} \mathcal{E}nd(S(\Omega^1[1])) \qquad (30)$$

is an isomorphism. By D_i denote image of $S(\Omega^1[1])\otimes\Lambda^i T[-i]$ in $\mathcal{E}nd(S(\Omega^1[1]))$:

$$\mathcal{E}nd(S(\Omega^1[1]))=\bigoplus_i D_i. \tag{31}$$

Equip $S(\Omega^1[1])$ with a non-degenerate supersymmetric pairing taking values in $\omega[n]$ by the formula

$$<-,->=\mathrm{top}(-\wedge -),$$

where top means the projection on the forms of top weight.

The pairing defines *the conjugation operation* $+$ on $\mathcal{E}nd(S(\Omega^1[1]))$ by the formula

$$<La,b>=<a,L^+b>, \tag{32}$$

where a and b are local sections. Note that

$$\begin{array}{l} L^{++}=L \\ (L_1L_2)^+=(-1)^{\deg L_1\deg L_2}L_2^+L_1^+. \end{array} \tag{33}$$

Using isomorphism (30) we have

$$(\imath\Omega^i\cdot\imath\Lambda^jT)^+=(-1)^{ij}(-1)^j\imath\Lambda^jT\cdot\imath\Omega^i. \tag{34}$$

For the family of operators $\mathcal{L}_L$ depending on the tangent bundle, which was introduced in theorem 1, conjugate operators have a very particular form.

Lemma 12 *Put*

$$\mathrm{td}=\det(f(\alpha(T))\in\bigoplus_i H^i(\Omega^i),$$

where $f(z)=z/(1-e^z)$ and $\alpha(T)\in H^1(\Omega^1\otimes\mathcal{E}nd(T))$ is the Atiyah class of the tangent bundle. Then

$$\mathcal{L}_L^+=-\imath\mathrm{td}\cdot\mathcal{L}_L\cdot\imath\mathrm{td}^{-1}.$$

Proof Consider $\mathcal{E}nd(S(\Omega^1[1]))$ as a Lie superalgebra. Then D_1 (see (31)) is a subalgebra. $\mathcal{E}nd(S(\Omega^1[1]))$ is analogous to differential operators algebra, D_1 is analogous to vector fields Lie algebra. $S(\Omega^1[1])$ is analogous to functions ring and D_1 acts on it by derivations.

Consider an analog of volume form $S^n_{S(\Omega^1[1])}D_1$, that is the top antisymmetric power of D_1 over $S(\Omega^1[1])$. $S^n_{S(\Omega^1[1])}D_1$ is a one-dimensional $S(\Omega^1[1])$-module. Consider the element $\mathrm{Id}\in H^0(T\otimes\Omega^1)\subset H^1(D_1\otimes\Omega^1)$. Then $\mathrm{Id}^n\in H^n(S^n_{S(\Omega^1[1])}D_1\otimes\Omega^n)$ gives a generator of $S^n_{S(\Omega^1[1])}D_1$.

D_1 naturally acts on $S^n_{S(\Omega^1[1])}D_1$. A straightforward calculation gives the commutative diagram

$$\begin{array}{ccc} S(\Omega^1[1]) & \xrightarrow{\mathrm{Id}^n} & S^n_{S(\Omega^1[1])}D_1\otimes\omega \\ \downarrow L^+ & & \downarrow -L \\ S(\Omega^1[1]) & \xrightarrow{\mathrm{Id}^n} & S^n_{S(\Omega^1[1])}D_1\otimes\omega \end{array} \tag{35}$$

where $L\in D_1$

Recall that in theorem 1 a family of operators $\mathcal{L}_L$ on $S(\Omega^1[1])$ were introduced. They describe the direct image of action of the Atiyah class of the diagonal on $\mathcal{O}_\triangle\overset{L}{\otimes}\mathcal{O}_\triangle$ by acting on the second factor. Introduce also $\mathcal{L}_R=\mathcal{L}_L-\mathrm{ad}$. This family describes the direct image of action of the Atiyah class on $\mathcal{O}_\triangle\overset{L}{\otimes}\mathcal{O}_\triangle$ by acting on the first factor. Indeed, these families commute with each other.

In according with theorem 1 families of operators $\mathcal{L}_L$ and $\mathcal{L}_R$ lie into D_1. On may easily check that the element $\mathcal{L}_R^n \in H^n(S^n_{S(\Omega^1[1])}D_1 \otimes \omega)$ gives a generator of $S^n_{S(\Omega^1[1])}D_1$ as $S(\Omega^1[1])$-module. Moreover,

$$\mathcal{L}_L \cdot \mathcal{L}_R^n = 0,$$

because $\{L_L, L_R\} = 0$. Thus by diagram (35),

$$\mathcal{L}_L^+((\mathrm{Id}^n)^{-1}\mathcal{L}_R^n) = 0.$$

A calculation (by using formulas $\alpha(T) = -\,\alpha(\Omega^1)$ and (14)) shows

$$(\mathrm{Id}^n)^{-1}\mathcal{L}_R^n = \mathrm{td}.$$

It follows that

$$\mathcal{L}_L^+ \mathrm{td} = 0.$$

Finally we have for local sections a and b

$$\begin{aligned} < \mathcal{L}_L a,\ b >&= \mathrm{top}(\mathcal{L}_L a \cdot b) = \mathrm{top}(\mathcal{L}_L a \cdot \mathrm{td}^{-1}\ b\ \mathrm{td}) \\ &= \mathrm{top}(\mathcal{L}_L(a \cdot \mathrm{td}^{-1}\ b)\ \mathrm{td}) - (-1)^{\deg a}\,\mathrm{top}(a \cdot \mathcal{L}_L(\mathrm{td}^{-1}\ b)\ \mathrm{td}) \\ &=< \mathcal{L}_L(a\ \mathrm{td}^{-1}\ b),\ \mathrm{td} > -(-1)^{\deg a} < a,\ \mathrm{td}\ \mathcal{L}_L(\mathrm{td}^{-1}\ b) > \\ &=< a\ \mathrm{td}^{-1}\ b,\ \mathcal{L}_L^+ \mathrm{td} > -(-1)^{\deg a} < a,\ \mathrm{td}\ \mathcal{L}_L(\mathrm{td}^{-1}\ b) > \\ &= -(-1)^{\deg a} < a,\ \mathrm{td}\ \mathcal{L}_L(\mathrm{td}^{-1}\ b) > . \end{aligned}$$

This concludes the proof. $\square$

Theorem 5 (Riemann–Roch theorem, part 2) *The Todd genus* Td *introduced in theorem 3 has the form*

$$\mathrm{Td} = I_{PBW}(\det f(\alpha(T))),$$

where $f(z) = z/(1 - e^{-z})$ *and* $\alpha(T) \in H^1(\Omega^1 \otimes \mathcal{E}nd(T))$ *is the Atiyah class of the tangent bundle.*

Proof Let $x \in \bigoplus_i H^i(\Lambda^{n-i}T \otimes \omega)$ be the required element that is

$$\mathrm{Td} = I_{PBW}(x).$$

By definition of PBW isomorphism and $\mathcal{L}_L$, we have

$$p_{2*}(\mathrm{id} \otimes \mathrm{Td}) = e^{\mathcal{L}_L} \vdash x, \tag{36}$$

where $e^{\mathcal{L}_L} \in H^0(\mathcal{E}nd(S(\Omega^1[1])) \otimes S(\Omega^1[1]))$ and $\vdash: \Lambda^i T \otimes \Omega^i \to \mathcal{O}$ is the contraction map.

Strategy of the proof is to reconstruct Td by its action on $p_{2*}(\mathcal{O}_\triangle \overset{L}{\otimes} \mathcal{O}_\triangle)$ that is by

$$p_{2*}(\mathrm{id} \otimes \mathrm{Td}) \in H^n(\mathcal{E}nd(S(\Omega^1[1])) \otimes \omega).$$

Introduce two isomorphisms $f_{r,l}: \mathcal{E}nd(S(\Omega^1[1])) \to S(\Omega^1[1]) \otimes \Lambda^* T$. We call this isomorphisms *the left* and *the right forms*. The left form f_l is given by composition map (30) and f_r differs by the permutation of factors:

$$f_l^{-1}\colon S(\Omega^1[1]) \otimes \Lambda^* T \xrightarrow{\sigma} \Lambda^* T \otimes S(\Omega^1[1]) \xrightarrow{\imath \otimes \imath} \mathcal{E}nd(S(\Omega^1[1])) \otimes \mathcal{E}nd(S(\Omega^1[1])) \xrightarrow{\circ} \mathcal{E}nd(S(\Omega^1[1])),$$

where σ is the factors permutation (with signs!). By $|_0$ denote the *constant term* map

$$|_0\colon S(\Omega^1[1]) \otimes \Lambda^* T \to \Lambda^* T$$

defined by the projection of $S(\Omega^1[1])$ to $\mathcal{O}$.

By lemma 11,

$$f_r(p_{2*}(\mathrm{id}\otimes \mathrm{Td}))|_0 = \mathrm{Id}^n \in H^0(\Lambda^n T\otimes\omega),$$

where $\mathrm{Id}\in H^0(T\otimes\Omega^1)$ is the canonical element. On the other hand, by definition of PBW isomorphism (see remark after definition-proposition 2),

$$f_l(p_{2*}(\mathrm{id}\otimes I_{PBW}(-)))|_0 = -;$$

in particular,

$$f_l(p_{2*}(\mathrm{id}\otimes \mathrm{Td}))|_0 = x.$$

By (34) we have

$$f_l(-^+) = I f_r(-),$$

where I multiplies $\Lambda^i T$ by $(-1)^i$. By (36) and lemma 12 we have

$$\begin{aligned}(p_{2*}(\mathrm{id}\otimes \mathrm{Td}))^+ &= (e^{\mathcal{L}_L}\vdash x)^+ = e^{\mathcal{L}_L^+}\vdash x\\ &= e^{-\mathrm{td}\mathcal{L}_L\mathrm{td}^{-1}}\vdash x = \mathrm{td}\,(e^{\mathcal{L}_L^+}\vdash Ix)\,\mathrm{td}^{-1}\\ &= \mathrm{td}\,(p_{2*}(\mathrm{id}\otimes I_{PBW}(Ix)))\,\mathrm{td}^{-1}.\end{aligned}$$

So

$$f_r(p_{2*}(\mathrm{id}\otimes\mathrm{Td})) = I f_l(\mathrm{td}\,(p_{2*}(\mathrm{id}\otimes I_{PBW}(Ix)))\,\mathrm{td}^{-1}).$$

Taking constant terms of both sides we get

$$\mathrm{Id}^n = I(\mathrm{td}\vdash Ix),$$

where $\vdash : S(\Omega^1[1])\otimes\Lambda^*T\to\Lambda^*T$ is the contraction map. This equality implies the statement of the theorem. $\square$

3 Hochshild homology and relative Riemann–Roch theorem

Notations Let M be a smooth complex manifold. By $\Delta : M\to M\times M$ denote the diagonal embedding. By $\mathcal{O}_\triangle$ and $\omega_\triangle$ denote the sheaves $\Delta_*\mathcal{O}$ and $\Delta_*\omega$ respectively, where ω is the canonical bundle. We identify $\omega_\triangle$ with $\mathcal{O}_\triangle\otimes p_2^*\omega$ and $\omega_\triangle\otimes\omega_{M\times M}^{-1}$ with $\mathcal{O}_\triangle\otimes p_2^*\omega^{-1}$ in natural way, where p_2 is the projection on the second factor. By $S(\Omega^1[1])$ denote differential forms, more exactly $S(\Omega^1[1]) = \oplus_i\Omega^i[i]$.

By S denote the functor $-\otimes\omega[\dim M]$. Below functors f^*, $\otimes$, f_* and $f_!$ mean Lf^*, $\overset{L}{\otimes}$, Rf_* and $Rf_!$.

By $\mathcal{D}^b(M)$ denote the derived category of bounded complexes of coherent sheaves on M.

1 Relative Serre duality

For a proper morphism of smooth manifolds $f : Y\to X$ *define* functor $f_!$ from $\mathcal{D}^b(Y)$ to $\mathcal{D}^b(X)$ by the formula

$$f_! = S_X^{-1}\circ f_*\circ S_Y.$$

By relative Serre duality, functor $f_!$ is left adjoint to f^*. We cite here this statement without proof, referring to [H].

Theorem 6 (Relative Serre duality) *For a proper morphism of smooth manifolds $f : Y \to X$ and objects $E \in \mathcal{D}^\flat(Y)$ and $F \in \mathcal{D}^\flat(X)$ a bifunctorial isomorphism*

$$D_f : \mathrm{Ext}(E, f^*F) \to \mathrm{Ext}(f_!E, F) \tag{37}$$

is defined. For a pair of morphisms $Z \xrightarrow{f} Y \xrightarrow{g} X$,

$$D_{g\circ f} = D_g \circ D_f. \tag{38}$$

For a smooth closed embedding $f : Y \to X$, isomorphism D_f is defined by the canonical isomorphism

$$\mathcal{E}xt(f_*\mathcal{O}_Y, \mathcal{O}_X) \cong S_X^{-1} f_* S_Y \mathcal{O}_Y. \tag{39}$$

Remark In theorem 2 the trace function

$$\mathrm{Tr} : \mathrm{Ext}(E, SE) \to \mathbf{C}$$

were introduced for an object $E \in \mathcal{D}^\flat(X)$ on a compact manifold X. For the structure sheaf the trace is equal to the image of $1 \in \mathrm{Ext}(\mathcal{O}_X, p_X^*\mathcal{O}_{pt})$ under the duality isomorphism

$$D_{p_X} : \mathrm{Ext}(\mathcal{O}_X, p_X^*\mathcal{O}_{pt}) \to \mathrm{Ext}(p_{X!}\mathcal{O}_X, \mathcal{O}_{pt}),$$

where $p_X : X \to pt$ is the morphism of the manifold to a point. For a general object E consider a morphism from E to SE as an element $x \in \mathrm{Ext}(\mathcal{E}nd(E), S\mathcal{O})$. Then the trace of this morphism equals

$$\mathrm{Tr}(x \cdot \mathrm{id}),$$

where $x \cdot \mathrm{id} \in \mathrm{Ext}(\mathcal{O}, S\mathcal{O})$ and $\mathrm{id} \in \mathrm{Ext}(\mathcal{O}, \mathcal{E}nd(E))$ is the unit section. In other words, the trace is equal to the value of the linear function $D_{p_X}(S^{-1}x) \in \mathrm{Ext}(\mathrm{End}(E), \mathcal{O}_{pt})$ on the unit section.

2 Inverse image of $\mathrm{Ext}(\mathrm{id}, S)$'s

For a proper morphism $f : Y \to X$ introduce morphisms of functors

$$\begin{array}{c} \mathcal{E}_f : f^*f_* \to \mathrm{id} \\ \mathcal{F}_f : \mathrm{id} \to f^*f_! \end{array} \tag{40}$$

by formulas

$$\begin{array}{c} \mathcal{E}_f(E) = C_f^{-1}(\mathrm{id}_{f_*E}) \\ \mathcal{F}_f(E) = D_f^{-1}(\mathrm{id}_{f_!E}) \end{array} \tag{41}$$

where C_f denotes the isomorphism $\mathrm{Ext}(f^*-, -) \to \mathrm{Ext}(-, f_*-)$ and D is from theorem 6.

Definition 3 *Let $f : Y \to X$ be a proper morphism of smooth manifolds and $a \in \mathrm{Ext}_X(\mathrm{id}, S)$ be a morphism from the identity functor to S in $\mathcal{D}^\flat(X)$. Then by* the inverse image *of a we call an element*

$$f^*a \in \mathrm{Ext}_Y(\mathrm{id}, S),$$

such that $S^{-1}f^*a(E)$ is the composition

$$S^{-1}E \xrightarrow{\mathcal{F}_f(S^{-1}E)} f^*f_!S^{-1}E \longrightarrow f^*f_*E \xrightarrow{\mathcal{E}_f(E)} E, \tag{42}$$

where the middle arrow is the inverse image of the morphism $S^{-1}a(f_*E) \in \mathrm{Ext}(S^{-1}f_*E, f_*E) = \mathrm{Ext}(f_!S^{-1}E, f_*E)$.

An important property of the inverse image is the compatibility with the Chern character. Recall (definition 2), that the Chern character of an object $E \in \mathcal{D}^\flat(X)$ on a compact manifold X is a linear function on $\mathrm{Ext}_X(\mathrm{id}, S)$ defined by the formula

$$\mathrm{Ch}_E(a) = \mathrm{Tr}\, a(E). \tag{43}$$

Proposition 1 *Let $f : Y \to X$ be a morphism of compact smooth manifolds, $E \in \mathcal{D}^\flat(Y)$ and $a \in \mathrm{Ext}_X(\mathrm{id}, S)$. Then*

$$\mathrm{Ch}_{f_*E}(a) = \mathrm{Ch}_E(f^*a).$$

Proof Introduce a map

$$\pi : f_!\, \mathcal{E}xt(E, S^{-1}E) \to \mathcal{E}xt(f_*E, f_!S^{-1}E)$$

corresponding under D to the map

$$\mathcal{E}xt(E, S^{-1}E) \to \mathcal{E}xt(f^*f_*E, f^*f_!S^{-1}E)$$

that is the left multiplication by $\mathcal{E}_f(E)$ and the right multiplication by $\mathcal{F}_f(S^{-1}E)$ (compare with (42)).

Calculate the trace of the composition of maps

$$S^{-1}\mathcal{O} \xrightarrow{f_*\mathrm{id}} f_!\, \mathcal{E}xt(E, S^{-1}E) \xrightarrow{\pi} \mathcal{E}xt(f_*E, f_!S^{-1}E) \xrightarrow{S^{-1}a(f_*E)} \mathcal{O}$$

in two ways. Here the first arrow

$$f_*\mathrm{id} \in \mathrm{Ext}(S^{-1}\mathcal{O}, f_!\, \mathcal{E}nd(E)) = \mathrm{Ext}(\mathcal{O}, f_*\, \mathcal{E}nd(E))$$

is the direct image of the unit section and the last arrow

$$S^{-1}a(f_*E) \in \mathrm{Ext}(\mathcal{E}xt(f_*E, f_!S^{-1}E), \mathcal{O}) = \mathrm{Ext}(S^{-1}f_*E, f_*E)$$

is due to the action of the morphism of functors a on f_*E.

It is easy to see that $f_*\mathrm{id} \cdot \pi = S^{-1}\mathrm{id}_{f_*E}$, where $\mathrm{id}_{f_*E} \in \mathrm{Ext}(\mathcal{O}, \mathcal{E}nd(E))$ is the unit section. Consequently (see the remark after theorem 6 and (43)),

$$\mathrm{Tr}(Sf_*\mathrm{id} \cdot S\pi \cdot a(f_*E)) = \mathrm{Tr}(\mathrm{id} \cdot a(f_*E)) = \mathrm{Ch}_{f_*E}(a).$$

On the other hand, by definition,

$$\pi \cdot S^{-1}a(f_*E) \in \mathrm{Ext}(f_!\, \mathcal{E}xt(E, S^{-1}E), \mathcal{O})$$

corresponds under D to

$$S^{-1}f^*a(E) \in \mathrm{Ext}(\mathcal{E}xt(E, S^{-1}E), f^*\mathcal{O}) = \mathrm{Ext}(S^{-1}E, E).$$

Thus, by naturalness of D, $\mathrm{Tr}(Sf_*\mathrm{id} \cdot S\pi \cdot a(f_*E))$ is equal to the value of the linear function

$$D_{p_X}(S\pi \cdot a(f_*E)) \in \mathrm{Ext}(p_{X!}(f_!\, \mathcal{E}xt(E, S^{-1}E)), \mathcal{O}_{pt})$$

on the section $f_*\mathrm{id}$ (here $p_X : X \to pt$ and $p_Y : Y \to pt$). By (38) for $p_Y = p_X \circ f$, it is equal to the value of the function

$$D_{p_Y}(f^*a(E)) \in \mathrm{Ext}(p_{Y!}\, \mathcal{E}xt(E, S^{-1}E)), \mathcal{O}_{pt}) = \mathrm{Ext}(p_{Y*}\, \mathcal{E}nd(E), \mathcal{O}_{pt})$$

on the unit section, that is equal to $\mathrm{Ch}_E(f^*a)$ (see the remark after theorem 6 and (43)). $\square$

3 Hochshild homology

For a manifold M the space $HH_M = \mathrm{Tor}_{M\times M}(\mathcal{O}_\triangle, \mathcal{O}_\triangle)$ is called *the Hochshild homology* of M. For any morphism $f: Y \to X$ the inverse image

$$f^*_{HH} : HH_X \to HH_Y$$

is obviously defined.

Theorem 6 supplies *the duality isomorphism*

$$D_\Delta : HH_M = \mathrm{Ext}(\mathcal{O}_M, \Delta^*\Delta_*\mathcal{O}_M) \to \mathrm{Ext}(\Delta_!\mathcal{O}_M, \Delta_*\mathcal{O}_M) = \mathrm{Ext}(\mathcal{O}_\triangle, \omega_\triangle[\dim M]). \tag{44}$$

In section 2.2 we introduced the map $\Theta : \mathrm{Ext}(\mathcal{O}_\triangle, \omega_\triangle[\dim M]) \to \mathrm{Ext}_M(\mathrm{id}, S)$.

Theorem 7 *For a proper morphism $f: Y \to X$ of smooth manifolds and $a \in HH_X$ the following holds*

$$\Theta_{D_\Delta(f^*_{HH}a)} = f^*\Theta_{D_\Delta(a)}.$$

Proof To prove the theorem take advantage of the following lemma.

Lemma 13 *For a proper morphism $f: Y \to X$ of smooth manifolds consider the commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\Delta} & Y\times Y \\ \downarrow f & & \downarrow F \\ X & \xrightarrow{\Delta} & X\times X, \end{array} \tag{45}$$

where $F = f\times f$.

1.

$$\begin{aligned} \Phi(F^*\Delta_*\mathcal{O}_X) &= f^*f_* \\ \Phi(F^*\Delta_!\mathcal{O}_X) &= f^*S^{-1}f_*, \end{aligned}$$

where Φ denotes the functor from $\mathcal{D}^\flat(Y)$ to itself corresponding to the object on $Y\times Y$ (see section 1.3).

2. *Let $\varepsilon \in \mathrm{Ext}(F^*\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_Y)$ and $\phi \in \mathrm{Ext}(\Delta_!\mathcal{O}_Y, F^*\Delta_!\mathcal{O}_X)$ be the images of $1 \in \mathrm{Ext}(\mathcal{O}_Y, \mathcal{O}_Y) = \mathrm{Ext}(f^*\mathcal{O}_X, \mathcal{O}_Y) = \mathrm{Ext}(\mathcal{O}_Y, f^*\mathcal{O}_X)$ under the composition of maps*

$$\begin{aligned} &\mathrm{Ext}(f^*\mathcal{O}_X, \mathcal{O}_Y) \to \mathrm{Ext}(O_X, f_*\mathcal{O}_Y) \xrightarrow{\Delta_*} \\ &\quad \mathrm{Ext}(\Delta_*O_X, \Delta_*f_*\mathcal{O}_Y) \to \mathrm{Ext}(\Delta_*O_X, F_*\Delta_*\mathcal{O}_Y) \to \mathrm{Ext}(F^*\Delta_*O_X, \Delta_*\mathcal{O}_Y) \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Ext}(O_Y, f^*\mathcal{O}_X) \to \mathrm{Ext}(f_!O_Y, \mathcal{O}_X) \xrightarrow{\Delta_!} \\ &\quad \mathrm{Ext}(\Delta_!f_!\mathcal{O}_Y, \Delta_!\mathcal{O}_X) \to \mathrm{Ext}(F_!\Delta_!O_Y, \Delta_!\mathcal{O}_X) \to \mathrm{Ext}(\Delta_!O_Y, F^*\Delta_!\mathcal{O}_X), \end{aligned}$$

where unmarked arrows designate natural isomorphisms.

Then

$$\Theta_\varepsilon = \mathcal{E}_f \qquad \in \mathrm{Ext}(\mathrm{id}, f^*f_*)$$

and

$$\Theta_\phi = \mathcal{F}_fS^{-1} \in \mathrm{Ext}(S^{-1}, f^*f_!S^{-1}).$$

Proof The first statement is obvious.

To prove the second one we have to show that $C_f(\Theta_\varepsilon(E)) = \mathrm{id}_{f_*E}$ and $D_f(\Theta_\phi(E)) = \mathrm{id}_{f_!E}$ for any $E \in \mathcal{D}^\flat(Y)$, where $C_f : \mathrm{Ext}(f^*-,-) \to \mathrm{Ext}(-, f_*-)$ is the natural isomorphism and D_f is the duality isomorphism (theorem 6). By naturalness of C and D, this equalities are equivalent to following ones

$$C_f(\mathrm{id}_{f^*f_*E}) \cdot f_*\Theta_\varepsilon(E) = \mathrm{id}_{f_*E} \tag{46}$$

and

$$f_!\Theta_\phi(E) \cdot D_f(\mathrm{id}_{f^*f_!E}) = \mathrm{id}_{f_!E}. \tag{47}$$

Prove them.

It can easily be checked that $f_*f^*f_*E = f_*E \otimes f_*\mathcal{O}$, the map $C_f(\mathrm{id}_{f^*f_*E}) \in \mathrm{Ext}(f_*E, f_*E \otimes f_*\mathcal{O})$ takes a local section x to $x \otimes 1$, and $f_*\Theta_\varepsilon(E) \in \mathrm{Ext}(f_*E \otimes f_*\mathcal{O}, f_*E)$ takes a local section $x \otimes y$ to $x \cdot y$ (by $\cdot$ we mean the action of $f_*\mathcal{O}$ on f_*E). Thus, the composition of this two morphisms is the identity and (46) is proved.

For $E \in \mathcal{D}^\flat(Y)$ denote $\mathcal{E}xt(E, \mathcal{O})$ by $E^\vee$. As it follows from theorem 6 (by using the functorial property of D), the following holds:

$$f_*(E^\vee) = (f_!E)^\vee.$$

The reader will have no difficulty in showing that equality (47) is obtained of (46) by substituting in it $E^\vee$ except E and applying $\mathcal{E}xt(-, \mathcal{O})$ to it. Thus (46) leads (47). This concludes the proof. $\square$

Return to the theorem. It is easy to see that to get the inverse image of $a \in HH_X = \mathrm{Ext}(\mathcal{O}_X, \Delta^*\Delta_*\mathcal{O}_X)$ one should take $f^*a \in \mathrm{Ext}(\mathcal{O}_Y, f^*\Delta^*\Delta_*\mathcal{O}_X) = \mathrm{Ext}(\mathcal{O}_Y, \Delta^*F^*\Delta_*\mathcal{O}_X)$ (see diagram (45)) and apply an element of

$$\mathrm{Ext}(F^*\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_Y) = \mathrm{Ext}(\Delta_*\mathcal{O}_X, F_*\Delta_*\mathcal{O}_Y) = \mathrm{Ext}(\Delta_*\mathcal{O}_X, \Delta_*f_*\mathcal{O}_Y)$$

which is the direct image of $1 \in \mathrm{Ext}(\mathcal{O}_X, f_*\mathcal{O}_Y)$. By definition, this element is map ε from lemma 13. Thus, the inverse image of $a \in HH_X = \mathrm{Ext}(\mathcal{O}_X, \Delta^*\Delta_*\mathcal{O}_X)$ is equal to the composition

$$\mathcal{O}_Y \xrightarrow{f^*a} f^*\Delta^*\Delta_*\mathcal{O}_X = \Delta^*F^*\Delta_*\mathcal{O}_X \xrightarrow{\Delta^*\varepsilon} \Delta^*\Delta_*\mathcal{O}_Y. \tag{48}$$

Consider $D_\Delta(a) \in \mathrm{Ext}(\Delta_!\mathcal{O}_X, \Delta_*\mathcal{O}_X)$. By naturalness of D, a is equal to the composition

$$\mathcal{O}_X \xrightarrow{D_\Delta^{-1}(\mathrm{id}_{\Delta_!\mathcal{O}_X})} \Delta^*\Delta_!\mathcal{O}_X \xrightarrow{\Delta^*D_\Delta(a)} \Delta^*\Delta_*\mathcal{O}_X.$$

Substituting in (48), we get

$$\begin{aligned} f^*_{HH}a &= f^*D_\Delta^{-1}(\mathrm{id}_{\Delta_!\mathcal{O}_X}) \cdot f^*\Delta^*D_\Delta(a) \cdot \Delta^*\varepsilon \\ &= f^*D_\Delta^{-1}(\mathrm{id}_{\Delta_!\mathcal{O}_X}) \cdot \Delta^*F^*D_\Delta(a) \cdot \Delta^*\varepsilon. \end{aligned}$$

By naturalness of D,

$$D_\Delta(f^*_{HH}a) = \Delta_!f^*D_\Delta^{-1}(\mathrm{id}_{\Delta_!\mathcal{O}_X}) \cdot F^*D_\Delta(a) \cdot \varepsilon.$$

It is easy to see, that $\Delta_!f^*D_\Delta^{-1}(\mathrm{id}_{\Delta_!\mathcal{O}_X})$ is equal to map ϕ from lemma 13. Thus, $D_\Delta(f^*_{HH}a)$ is equal to the composition

$$\Delta_!\mathcal{O}_Y \xrightarrow{\phi} F^*\Delta_!\mathcal{O}_X \xrightarrow{F^*D_\Delta(a)} F^*\Delta_*\mathcal{O}_X \xrightarrow{\varepsilon} \Delta_*\mathcal{O}_Y. \tag{49}$$

Applying lemma 13 to (49) and comparing with (42) we get the statement of the theorem. $\square$

4 Relative Riemann–Roch theorem

By the corollary of definition–proposition 2 for a smooth manifold M the isomorphism

$$\exp_* : HH_M \to H^*_M(\Omega^*).$$

is defined. It is well known, that for any morphism of smooth manifolds $f : Y \to X$ the following diagram is commutative

$$\begin{array}{ccc} HH_Y & \stackrel{\exp_*}{\longrightarrow} & H^*_Y(\Omega^*) \\ \uparrow f^*_{HH} & & \uparrow f^*_{\Omega^*} \\ HH_X & \stackrel{\exp_*}{\longrightarrow} & H^*_X(\Omega^*), \end{array} \tag{50}$$

where $f^*_{\Omega^*}$ denotes the "usual" inverse image of differential forms (in fact, (50) may be considered as a definition of the "usual" inverse image).

Besides in lemma 8 we introduced an isomorphism

$$I_{PBW} : H^*_M(\Omega^*) \to \mathrm{Ext}_{M\times M}(\mathcal{O}_\triangle, \omega_\triangle[\dim M]).$$

For a compact manifold M and $E \in \mathcal{D}^\flat(M)$,

$$\mathrm{Ch}_E(\Theta_{I_{PBW}(l)}) = \int l \wedge \mathrm{ch}(E), \tag{51}$$

where $l \in H^*(\Omega^*)$, $\int$ is the Poincaré pairing on $H^*(\Omega^*)$ and ch is the usual Chern character (see remark after definition 2).

Finally, in the previous subsection we define the duality isomorphism $D_\Delta : HH_M \to \mathrm{Ext}_{M\times M}(\mathcal{O}_\triangle, \omega_\triangle[\dim M])$.

Theorem 8 *For a smooth manifold M the following diagram is commutative*

$$\begin{array}{ccc} HH_M & \stackrel{D_\Delta}{\longrightarrow} & \mathrm{Ext}(\mathcal{O}_\triangle, \omega_\triangle[\dim M]) \\ \downarrow \exp_* & & \uparrow I_{PBW} \\ H^*_M(\Omega^*) & \stackrel{\mathrm{td}_M}{\longleftarrow} & H^*_M(\Omega^*), \end{array}$$

where td_M *means the multiplication by the Todd genus* $\mathrm{td} \in \bigoplus_i H^i(\Omega^i)$ *(see e. g. theorem 5).*

Proof Below we identify $\Delta^*\Delta_*\mathcal{O}_M$ with $\oplus_i\Omega^i[i]$ and $\Delta_*\Delta^*\Delta_*\mathcal{O}_M = \mathcal{O}_\triangle \otimes \mathcal{O}_\triangle$ with $\Delta_* \oplus_i \Omega^i[i]$ by $\exp_*$.

Consider the canonical morphism

$$D_\Delta^{-1}(\mathrm{id}_{\Delta_!\mathcal{O}}) \in \mathrm{Ext}(\mathcal{O}, \Delta^*\Delta_!\mathcal{O}) = \mathrm{Ext}(\mathcal{O}, S^{-1}\Delta^*\Delta_*\mathcal{O}).$$

Local analysis and (39) shows that $S^{-1}D_\Delta^{-1}(\mathrm{id}_{\Delta_!\mathcal{O}})$ is the embedding of shifted form of the highest weight $\Omega^n[n]$ into $\oplus_i\Omega^i[i]$ as a direct summand.

By naturalness of D, the duality isomorphism D_Δ^{-1} for $v \in \mathrm{Ext}(\Delta_!\mathcal{O}, \Delta_*\mathcal{O})$ is given by the composition

$$\mathcal{O} \stackrel{D_\Delta^{-1}(\mathrm{id}_{\Delta_!\mathcal{O}})}{\longrightarrow} \Delta^*\Delta_!\mathcal{O} \stackrel{\Delta^* v}{\longrightarrow} \Delta^*\Delta_*\mathcal{O}.$$

It means, that $D_\Delta^{-1}(v)$ is equal to the result of action of v on

$$1 \in \mathrm{Ext}_M(\mathcal{O}, \oplus_i\Omega^i[i] \otimes \omega^{-1}[-n]) = \mathrm{Ext}_{M\times M}(\mathcal{O}, \mathcal{O}_\triangle \otimes \mathcal{O}_\triangle \otimes p_2^*\omega^{-1})$$

(we mean the "twisted" version of action of $\mathrm{Ext}(\mathcal{O}_\triangle, \mathcal{O}_\triangle)$ on $\mathcal{O}_\triangle \otimes \mathcal{O}_\triangle$). By the proof of theorem 5, result of such an action for $v = I_{PBW}(\alpha)$ equals $\alpha \cdot \mathrm{td}$. This proves the theorem. □

Combining lemma 1, theorem 7, theorem 8, (50) and (51) we get the following statement.

Theorem 9 (Relative Riemann–Roch theorem) *For a morphism of smooth compact manifolds* $f : Y \to X$, $E \in \mathcal{D}^\flat(Y)$ *and* $l \in H^*_X(\Omega^*)$ *the following holds*

$$\int \mathrm{td}_X \wedge l \wedge \mathrm{ch}(f_* E) = \int \mathrm{td}_Y \wedge f^* l \wedge \mathrm{ch}(E).$$

References

[A] Atiyah M. F., Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. vol.85 (1957) No.1, pp. 181–207.

[B] Bondal A., Non-commutative deformations and Poisson brackets on projective spaces, preprint MPI/93-67.

[NB] Bourbaki, N. Algèbre, Chapitre 10, Algèbre homologique (Eléments de mathématique), Paris, Masson, 1980.

[GH] Griffiths Ph., Harris J., Principles of algebraic geometry, New York, Wiley, 1978.

[H] Hartshorne R., Residues and duality, Heidelberg, Springer, 1966.

[K] Kapranov M., Rozansky–Witten invariants via Atiyah classes, Compositio Math. 115 (1999), no. 1, 71–113.

[Ko] Kontsevich M., Deformation quantization of Poisson manifolds, I, q-alg/9709040.

e-mail: nikita@mccme.ru