A Short note on automorphisms of generalized Kac-Moody superalgebras

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The aim of this note is to give the decomposition of an arbitrary automorphism of a generalized Kac-Moody superalgebra. In order to do so, we show that Cartan subalgebras of generalized Kac-Moody superalgebras are conjugate under the action of inner automorphisms. This well-known result of Peterson-Kac [PK] for Kac-Moody algebras therefore remains valid in the more general context. This implies that not only the rank of a Cartan subalgebra of a generalized Kac-Moody superalgebra, but also the rank of of the Kac-Moody subalgebra generated by all the simple real root spaces, are invariants of the superalgebra. This result therefore puts an end to the speculation according to which the Monster vertex algebra is the tensor product of the vertex algebra of a Niemeier lattice and of the vertex algebra of the rank two even unimodular Lorentzian lattice.

1. A generalization of a Gabber-Kac result

The definitions of Kac-Moody algebras given independently by Kac [K1] and Moody [M] are different but were shown to be equivalent by Gabber and Kac [GK]. Both Kac's definition and Moody's definition can be generalized to define generalized Kac-Moody superalgebras [Ra]. We start by showing that in this more general context, both definitions remain equivalent. This was assumed to be true in [Ra] and we use a result from [Ra] in this paper.

We first fix some notation used throughout this paper.
As usual $G_{\overline{0}}$ and $G_{\overline{1}}$ will denote the even and odd part of the Lie superalgebra G.

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The base field $\mathbf{F}$ will be either $\mathbf{R}$ or $\mathbf{C}$. Let I be a finite set $\{1, ..., n\}$ or a countable one identified with $\mathbf{Z}_+$ and S be a subset of I. The set S indexes the generators contained in $G_{\overline{1}}$.
Let H be a $\mathbf{F}$-vector space with a non-degenerate symmetric bilinear form $(.,.)$, containing vectors h_i indexed by the set I.
Set $a_{ij} := (h_i, h_j)$ and assume that the matrix $A = (a_{ij})_{i,j \in I}$ is a generalized symmetric Cartan matrix, i.e.

$a_{ij} = a_{ji}$;
$a_{ij} \leq 0$ if $i \neq j$;
$\frac{2a_{ij}}{a_{ii}} \in \mathbf{Z}$ if $a_{ii} > 0$;
$\frac{a_{ij}}{a_{ii}} \in \mathbf{Z}$ if $a_{ii} > 0$ and $i \in S$.

The Lie superalgebra $\tilde{G} = \tilde{G}(A, S, H)$ is generated by the vector space H considered as an even abelian subalgebra and elements e_i, f_i, $i \in I$, satisfying the defining relations:
(i) $[e_i, f_j] = \delta_{ij} h_i$;
(ii) $[h, e_j] = (h, h_i) e_j$, $h \in H$;
(iii) $[h, f_j] = -(h, h_j) f_j$
(iv) $\deg e_i = \deg f_i = \overline{0}$ if $i \notin S$
(v) $\deg e_i = \deg f_i = \overline{1}$ if $i \in S$

Following Moody's definition [M], we define the generalized Kac-Moody algebra $G = G(A, S, H)$ by generators and relations.

Definition. *The generalized Kac-Moody algebra $G = G(A, S, H)$ with Cartan matrix A and Cartan subalgebra H is the Lie superalgebra generated by H, e_i, and f_i, $i \in I$ and with defining relations (i)-(v) and with the following extra ones:*
(vi) $(\mathrm{ad}\, e_i)^{\frac{-2a_{ij}}{a_{ii}}+1} e_j = 0 = (\mathrm{ad}\, f_i)^{\frac{-2a_{ij}}{a_{ii}}+1} f_j$ for all $i \in I$ such that $a_{ii} > 0$ and $j \in I$
(vii) $(\mathrm{ad}\, e_i)^{\frac{-a_{ij}}{a_{ii}}+1} e_j = 0 = (\mathrm{ad}\, f_i)^{\frac{-a_{ij}}{a_{ii}}+1} f_j$ for all $i \in S$ such that $a_{ii} > 0$ and $j \in I$
$(viii)$ $[e_i, e_j] = 0 = [f_i, f_j]$ if $a_{ij} = 0$.

The Lie superalgebra $\tilde{G}$ has a unique maximal ideal $\tilde{R}$ which intersects the Cartan subalgebra H trivially [K4]. Following Kac's definition in the Lie algebra case [K1], the generalized Kac-Moody algebra $G = G(A, S, H)$ is the quotient $\tilde{G}/\tilde{R}$. We show that both definitions are equivalent.

We first need a few more notations. Let G be the generalized Kac Moody superalgebra according to the above Definition. let Δ denote the set of roots, Δ^+ the set of positive roots, Δ_0 the set of even roots and Δ_0^+ the set of even positive roots.
Let $N^+ := \oplus_{\alpha \in \Delta_0^+} G_\alpha$ and $N_0^+ = N^+ \cap G_{\overline{0}}$.
Let W be the Weyl group (see [Ra] for details).

Lemma 1. *The even subalgebra $G_{\overline{0}}$ of G is a generalized Kac-Moody algebra.*

Proof. The Lie subalgebra N_0^+ is generated by root spaces G_α with $\alpha \in \Delta_0^+$ for which there does not exist any positive even roots $\beta_1, \beta_2 \in \Delta_0^+$ satisfying $\alpha = \beta_1 + \beta_2$. Let Π^0 be the set of all such positive even roots. Consider $\alpha, \beta \in \Pi^0$.

We first show that $(\alpha, \beta) \leq 0$.

If α is real then $(\alpha, \beta) \leq 0$. Otherwise, $(\alpha, \beta) > 0$, and so $\alpha - \beta$ is an even root. Hence $\beta = \alpha + (\beta - \alpha)$ or $\alpha = \beta + (\alpha - \beta)$, and so either α or β is not in Π^0, contradicting assumptions.

If α is imaginary, then by what precedes, $(\alpha, \alpha_i) \leq 0$ for all real roots α_i. If α_i is even, its a direct consequence of the above. Otherwise take $2\alpha_i$. Hence $(\alpha, \alpha_i) \leq 0$ for all $i \in I$. Therefore, $(\alpha, \beta) \leq 0$.

It remains to show that the roots α, β satisfy relation of the above type (vi) and $(viii)$.

If the root α is real, then there exists $w \in W$ such that $w(\alpha) = \alpha_i$. Therefore for any $e_\alpha \in G_\alpha$ and $e_\beta \in G_\beta$,

$$ade_\alpha^{\frac{-2(\alpha,\beta)}{(\alpha,\alpha)}+1}(e_\beta) = 0.$$

Finally suppose that α and β are both imaginary orthogonal roots and that $[e_\alpha, e_\beta] \neq 0$. Then as $(\alpha, \alpha_i) \leq 0$ for all $i \in I$, $(\alpha, \alpha_i) = 0$ for every root α_i in the support of β. Suppose that α_i is in the support of β but not in the support of α. Then $(\alpha_j, \alpha_i) = 0$ for every root α_j in the support of α. Hence $\alpha + \beta$ does not have a connected support which contradicts the fact that all roots have connected support [K4]. Hence α and β have the same support. Hence as the support of α is connected, all the roots in its support are real, and $(\alpha, \alpha_i) = 0$ for all α_i in the support of α. Hence the support of α generates an affine Lie superalgebra. It therefore follows from the structure of the affine Kac-Moody Lie superalgebras [K3] that $[e_\alpha, e_\beta] = 0$, contradicting assumptions.

Hence, we can conclude that $G_{\overline{0}}$ is a GKM algebra with simple root set Π^0. □

Proposition 2. *Any ideal of G intersecting the Cartan subalgebra H trivially is trivial.*

Proof. Without loss of generality, we assume that the Cartan matrix A is indecomposable. Let R be the unique maximal ideal of G having trivial intersection with the Cartan subalgebra H.

Since $G_{\overline{0}}$ is a generalized Kac-Moody algebra, the kernel of the invariant bilinear symmetric form $(.,.)_0$ induced by the Cartan matrix of $G_{\overline{0}}$ is contained in H [K4]. Let $(.,.)$ be the invariant bilinear symmetric form on G induced by the Cartan matrix A [K4] (with respect to the simple root set Π^0 of G_0) Then $(.,.)$ is consistent [K2] and non-trivial on $G_{\overline{0}}$. Hence its restriction to $G_{\overline{0}}$ is a non-trivial multiple of the form $(.,.)_0$. Therefore $(.,.)$ is non-degenerate on $G_{\overline{0}}$.

Let $x_\alpha \in G_\alpha$, where α is a positive root of G be an element generating the ideal R. Then $[x_\alpha, f_i] = 0$ for all $i \in I$. In particular, $[x_\alpha, x_{-\alpha}] = 0$ for all $x_{-\alpha} \in G_{-\alpha}$. It follows that $(x_\alpha, x_{-\alpha}) = 0$ since $[x_\alpha, x_{-\alpha}] = (x_\alpha, x_{-\alpha})h_\alpha$, where $(h, h_\alpha) = \alpha(h)$ for all $h \in H$ [K4]. As $(x_\alpha, y) = 0$ for any $y \in G_\beta$ for all $\beta \neq -\alpha$, x_α is therefore in the kernel K of $(.,.)$. Since $K \cap G_{\overline{0}} = 0$, $x_\alpha \in G_{\overline{1}}$. As $(.,.)$ is invariant, $[x_\alpha, x_\alpha] \in K$. Hence as $[x_\alpha, x_\alpha] \in G_{\overline{0}}$, $[x_\alpha, x_\alpha] = 0$. So $(\alpha, \alpha) = 0$. As $[x_\alpha, f_i] = 0$ for all $i \in I$, $(\alpha, \alpha_i) \leq 0$ for all $i \in I$. As x_α generates R, α cannot be a simple root. Thus the support of the root α being connected, all the simple roots in its support are real. Therefore the structure of affine Kac-Moody Lie superalgebras [K3] implies that there exits $i \in I$ such that $[x_\alpha, f_i] \neq 0$, contradicting assumptions. This proves that $R = 0$. □

Therefore the generalized Kac-Moody superalgebra G is the quotient of the Lie superalgebra $\tilde{G}$ by the ideal $\tilde{R}$.

2. Conjugacy of Cartan subalgebras of the Lie superalgebra G

Let $J = \{i \in I | a_{ii} > 0\}$ be the set indexing the real simple roots of G and G_J be the Kac-Moody sub-superalgebra of G generated by the elements e_i and f_i for $i \in J$.

Define U^+ (resp. U^-) as the Lie sub-superalgebra generated by the root spaces G_α where the support of the root $\alpha \in \Delta^+$ (resp. $-\alpha \in \Delta^+$) contains at least one imaginary (non-positive norm) simple root. Then:

Lemma 3. *Keeping the above notation,*

$$G = U^+ \oplus (G_J + H) \oplus U^-$$

as a direct sum of vector spaces.

Proof Using arguments similar to [J], this is a direct consequence of the elimination Theorem [Bou]. $\square$

Proposition 4. *All Cartan subalgebras of G are contained in the sub-superalgebra $G_J + H$.*

Proof. Without loss of generality, we may assume that the Cartan matrix of G is indecomposable. Suppose that K is a Cartan subalgebra of G not contained in $G_J + H$. By definition, K is an even subalgebra of G.

Since K is a Cartan subalgebra, it acts on G in a locally finite way. Let

$$x = u + v + y$$

be an element of K, where $u \in U^+$, $v \in U^-$ and $y \in G_J + H$, which is not in $G_J + H$. We first assume that $0 \neq u$. So there exists positive roots $\beta_1, ..., \beta_r \in \Delta$ containing imaginary simple roots in their support and root vectors $u_i \in G_{\beta_i}$ such that $u = \sum_{i=1}^r u_i$.

Suppose that there exists a simple root α_i such that for some $j = 1, ...r$, $(\beta_j, \alpha_i) < 0$.

Since K is an even subalgebra, all roots β_j are even. So the roots β_j, $1 \leq j \leq r$ are all imaginary since they contain imaginary simple roots in their support and cannot thus be conjugate to a simple root under the action of the Weyl group. This implies that $(adu_j)^n e_i \neq 0$ for all $n \in \mathbf{Z}_+$ [Ra].

We claim that there are infinitely many vectors $(adx)^n(e_i)$, $n \in \mathbf{Z}_+$ which are linearly independent. Let $k \in \{1, ..., r\}$ be such that β_k is of maximum height with the property that $\beta_k + \alpha_i$ is a root. Then for each positive integer n, $(adu)^n(e_i)$ is the sum of root vectors, exactly one of which corresponds to the root $n\beta_i + \alpha_i$.

This contradicts the fact that K acts in a locally finite way.

Next suppose that $(\beta_j, \alpha_i) \geq 0$ for all $j = 1, ..., r$ and all $i \in I$. If α_i is imaginary, then this forces $(\beta_j, \alpha_i) = 0$. Hence if α_i is in the support of β_j, then α_i is orthogonal to all the simple roots distinct from α_i in the support of β_j. Since the support of a root is connected, we can deduce that the roots β_j are simple imaginary roots for all $j = 1, ..., r$. Furthermore they are orthogonal to all the imaginary simple roots. Since β_j is simple, $(\beta_j, \alpha_i) \leq 0$ for any simple root α_i. Hence the simple roots β_j are orthogonal to all simple roots. As the Cartan matrix is assumed to be indecomposable, it follows that $j = 1$ and $|I| = 1$ and G is a Heisenberg algebra of rank 1. Therefore the Cartan subalgebra H is unique and in particular $K = H$, contradicting assumptions.

So $u = 0$. If $v \neq 0$, then similar arguments lead to a contradiction. It follows that $x \in G_J + H$ and this forces the Cartan subalgebra K to be contained in $G_J + H$, contradicting assumptions and proving the result. $\square$

Theorem 5. *All Cartan subalgebras of G are conjugate under the action of the derived Lie group of G_J.*

Proof. As all Cartan subalgebras are contained in $G_J + H$ by Proposition 4 and G_J is a Kac-Moody algebra, this is a consequence of the conjugacy Theorem of Peterson-Kac [PK] (which can easily be generalized to Kac-Moody superalgebras). $\square$

The following is an immediate consequence of this result.

Corollary 6. *The rank of the Lie superalgebra is invariant and there is a uniquely defined generalized Cartan matrix for G.*

Remark. Since the Cartan matrix uniquely defines the generalized Kac-Moody superalgebra G, it follows that the rank of the Kac-Moody sub-superalgebra G_J

generated by the elements e_i, f_i, $i \in J$ is also an invariant of the Lie superalgebra G.

We next give two applications of Theorem 5.

a) The Monster vertex algebra

The crucial first step towards the proof of the Moonshine Theorem was the construction of a **Z**-graded module V for the Monster simple group [FLM]. It has a rich but complex algebraic structure. One of its most important characteristics is that it is a vertex algebra [FLM], [Bor1]. However the structure of a vertex algebra on it is very hard to see. It would therefore be useful to have a different construction of the Monster vertex algebra, which is the tensor vertex algebra of V and the lattice vertex algebra corresponding to the even unimodular Lorentzian lattice of rank 2 $II_{1,1}$. In particular this may help to find an adequate integral form on V for modular moonshine questions [Ry] [BR] [Bor4].

Unlike the Monster vertex algebra, the Fake Monster vertex algebra has a nice construction. It is the lattice vertex algebra associated to the even unimodular non-degenerate Lorentzian lattice $II_{25,1}$ of rank 26 [Bor1], [Bor2]. Now, an important property of the Monster vertex algebra is that the Virasoro algebra acts on it with central charge 26. Similarly, an important aspect of a lattice vertex algebra constructed from an even non-degenerate lattice is that the Virasoro algebra acts on it with central charge equal to the rank of the lattice. Therefore there were speculations that the Monster vertex algebra would also have a construction derived from lattices in the following manner [Bor5]: For a lattice L, let V_L be the corresponding vertex algebra [Bor1]. Let L be a Niemeier lattice [CS]. So by definition the lattice L is positive definite of rank 24. Then the Virasoro algebra acts on the vertex algebra $V_L \otimes V_{II_{1,1}}$ with central charge 26 [Bor1].

Due to Theorem 5, the Monster vertex algebra cannot be constructed as a tensor product vertex algebra of this form, contrary to what had been thought.

Corollary 7. *Let $II_{1,1}$ be the even unimodular rank 2 Lorentzian lattice and L a Niemeier lattice. Then $V_L \otimes V_{II_{1,1}}$ is not isomorphic to the Monster vertex* *algebra.*

Proof. Let L be a Niemeier lattice, the direct sum lattice $L \oplus II_{1,1}$ is even unimodular Lorentzian of rank 26 . Up to isomorphism there is a unique lattice $II_{25,1}$ with this property [CS] and $V_{II_{25,1}}$ is the root lattice of the Fake Monster Lie algebra [Bor2]. Therefore $V = V_L \otimes V_{II_{1,1}}$ is isomorphic to the Fake Monster vertex algebra. Let G be the generalized Kac-Moody algebra which can be constructed from as a sub-quotient of V according to the method in [Bor1]. Then G is the Monster Lie algebra [Bor3] and is isomorphic to the Fake Monster Lie algebra. Hence by Corollary 6, they both have the same Cartan matrix (up to reordering). Therefore G cannot be the Monster Lie algebra. $\square$

b) Automorphisms of generalized Kac-Moody superalgebras

We finally give the decomposition of an arbitrary automorphism of the generalized kac-Moody superalgebra G. Let ω be the Chevalley automorphism of period 4 of G [K3]. It is the Chevalley involution if G is a generalized Kac-Moody algebra. We remind the reader that a diagram automorphism θ of G satisfies by $\theta e_i = e_{\dot{\theta}(i)}$ and $\theta f_i = f_{\dot{\theta}(i)}$ for all $i \in I$, where $\dot{\theta}$ is a bijection on the indexing set I such that $a_{\dot{\theta} i, \dot{\theta} j} = a_{i,j}$ and $\dot{\theta}(i) \in S$ if and only if $i \in S$.

Corollary 8. *Let ϕ be an automorphism of the Lie superalgebra G. Then there exists an inner automorphism ϕ_1 and a diagram automorphism ϕ_2 of G such that $\phi = \omega^i \phi_1 \phi_2$ for some integer $0 \leq i \leq 3$.*

Proof. Since $\phi(H)$ is a Cartan subalgebra of G, there exists an inner automorphism ϕ_1 such that $\psi_1 = \phi_1^{-1}\phi$ fixes H. Let Π be a base of H. Then $\psi_1(\Pi)$ is a base of H. All bases of H are conjugate under the action of $\pm W$ [K4]. The Weyl group elements are given by inner automorphisms ([FH]: the proof is the same in the infinite dimension and superalgebra case). Therefore for some integer $0 \leq i \leq 3$, the automorphism $\omega^i \psi_1$ fixes the base Π. Hence as this automorphism leaves the bilinear form invariant, it is a diagram automorphism.
$\square$

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