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# STABILITY IN VOLUME COMPARISON PROBLEMS

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Abstract. A comparison problem for volumes of convex bodies asks whether inequalities $f_K(\xi) \le f_L(\xi)$ for all $\xi \in S^{n-1}$ imply that $\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)$, where $K, L$ are convex bodies in $\mathbb{R}^n$, and $f_K$ is a certain geometric characteristic of $K$. By stability in comparison problems we mean that there exists a constant $c$ such that for every $\varepsilon > 0$, the inequalities $f_K(\xi) \le f_L(\xi) + \varepsilon$ for all $\xi \in S^{n-1}$ imply that $(\mathrm{Vol}_n(K))^{\frac{n-1}{n}} \le (\mathrm{Vol}_n(L))^{\frac{n-1}{n}} + c\varepsilon$.

We prove such results in the settings of the affirmative parts of the Busemann-Petty and Shephard problems, where $f_K(\xi) = S_K(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp)$ is the section function or $f_K(\xi) = P_K(\xi) = \mathrm{Vol}_{n-1}(K|\xi^\perp)$ is the projection function, correspondingly. Here $\xi^\perp$ is the central hyperplane perpendicular to $\xi$, and $K|\xi^\perp$ is the orthogonal projection of $K$ to $\xi^\perp$. We also establish stability in the section case for arbitrary measures in place of the volume. The latter allows to extend to arbitrary measures the hyperplane inequality in dimensions $n \le 4$.

## 1. Introduction

A typical comparison problem for the volume of convex bodies asks whether inequalities

$$f_K(\xi) \le f_L(\xi), \qquad \forall \xi \in S^{n-1}$$

imply $\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)$ for any $K, L$ from a certain class of origin-symmteric convex bodies in $\mathbb{R}^n$, where $f_K$ is a certain geometric characteristic of $K$ and $\mathrm{Vol}_n$ is the $n$-dimensional volume.

If $f_K = S_K$ is the section function of $K$ defined by

$$S_K(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp), \qquad \xi \in S^{n-1},$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ orthogonal to $\xi$, the corresponding comparison question is the matter of the Busemann-Petty problem, raised in 1956 in [BP] and solved in the end of the 1990's as the result of a sequence of papers [LR], [Ba], [Gi], [Bo4], [L], [Pa], [G1], [G2], [Z1], [Z2], [K2], [K3], [Z3], [GKS] ; see [K4, p. 3] or [G3, p. 343] for the history of the solution. The answer is affirmative if $n \le 4$, and it is negative if $n \ge 5$.

Another example is the Shephard problem with $f_K = P_K$ being the projection function

$$P_K(\xi) = \mathrm{Vol}_{n-1}(K|\xi^\perp), \qquad \xi \in S^{n-1},$$

where $K|\xi^\perp$ is the orthogonal projection of $K$ to the hyperplane $\xi^\perp$. The Shephard problem was posed in 1964 in [Sh] and solved soon after that by Petty [Pe] and Schneider [S1]. The answer if affirmative only in dimension 2.

Since the answers to the Busemann-Petty and Shephard problems are negative in most dimensions, one may ask what information about the functions $S_K$ and $P_K$ does allow to compare the volumes in all dimensions. In the section case an answer to this question was given in [KYY]: for two origin-symmetric infinitely smooth bodies $K, L$ in $\mathbb{R}^n$ and $\alpha \in [n-4, n-1)$ the inequalities

$$(-\Delta)^{\alpha/2} S_K(\xi) \le (-\Delta)^{\alpha/2} S_L(\xi), \qquad \forall \xi \in S^{n-1} \tag{1}$$

imply that $\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)$, while for $\alpha < n-4$ this is not necessarily true. Here $\Delta$ is the Laplace operator on $\mathbb{R}^n$, and the fractional powers of the Laplacian are defined by

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} (|x|_2^\alpha \hat{f}(x))^\wedge,$$

where the Fourier transform is considered in the sense of distributions, $|x|_2$ stands for the Euclidean norm in $\mathbb{R}^n$, and the functions $S_K$ and $S_L$ are extended in (1) to homogeneous functions of degree -1 on the whole $\mathbb{R}^n$. This result contains the solution to the original Busemann-Petty problem as a particular case and means that one has to differentiate the section functions at least $n-4$ times in order to compare the $n$-dimensional volumes.

The situation is different for projections where a similar extension does not directly generalize the solution to Shephard's problem. Yaskin [Y] proved that for $\alpha \in [n, n+1)$ the inequalities

$$(-\Delta)^{\alpha/2} P_K(\xi) \ge (-\Delta)^{\alpha/2} P_L(\xi), \qquad \forall \xi \in S^{n-1} \tag{2}$$

imply that $\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)$, where the projection functions are extended to homogeneous functions of degree 1 on the whole $\mathbb{R}^n$. The latter result is no longer true for $\alpha \in [n-2, n)$, which would be a natural extension of the solution to the original Shephard's problem.

Zvavitch [Zv] found a remarkable generalization of the Busemann-Petty problem to arbitrary measures. Let $f$ be an even continuous positive function on $\mathbb{R}^n$, and denote by $\mu$ the measure on $\mathbb{R}^n$ with density $f$. For every closed bounded set $B \subset \mathbb{R}^n$ or $B \subset \xi^\perp$ define

$$\mu(B) = \int\limits_B f(x)\, dx.$$

The result of Zvavitch is that the answer to the Busemann-Petty problem remains the same if volume is replaced by any measure with positive continuous density in $\mathbb{R}^n$. In particular, if $n \le 4$, then for any convex origin-symmetric bodies $K$ and $L$ in $\mathbb{R}^n$ the inequalities

$$\mu(K \cap \xi^\perp) \le \mu(L \cap \xi^\perp), \qquad \forall \xi \in S^{n-1}$$

imply

$$\mu(K) \le \mu(L).$$

This is generally not true if $n \ge 5$, as also shown in [Zv].

In this article we study the flexibility of the results mentioned above. By *stability* in a comparison result we mean that there exists a constant $c$ such that for any $K, L$ from certain classes of convex bodies and every $\varepsilon > 0$ the inequalities

$$f_K(\xi) \le f_L(\xi) + \varepsilon, \qquad \forall \xi \in S^{n-1}$$

imply

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + c\varepsilon.$$

We also consider *separation* in comparison problems, where we are looking for a constant $c$ such that for any $K, L$ from certain classes of convex bodies and every $\varepsilon > 0$ the inequalities

$$f_K(\xi) \le f_L(\xi) - \varepsilon, \qquad \forall \xi \in S^{n-1}$$

imply

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-1}{n}} - c\varepsilon.$$

We first prove stability and separation for the section function $f_K = S_K$ under the additional assumption that $K$ is an intersection body. In the stability result the constant $c = 1$, but in the case of separation $c$ depends on the inradius of $K$ and on the dimension $n$. Since every origin-symmetric convex body in $\mathbb{R}^n$, $2 \le n \le 4$ is an intersection body, in these dimensions the results apply to arbitrary origin-symmetric convex bodies $K, L$.

We also prove linear stability and separation for the projection function $f_K = P_K$ under the additional assumption that $L$ is a projection body. Here in the stability result the constant $c$ depends on $n$ and on the circumradius of $L$, while in the case of separation we have $c = \sqrt{1/e}$.

In order to remove the additional assumptions on the bodies and make the results work in general in higher dimensions, we prove stability and separation in the results from [KYY] and [Y] mentioned above. We consider the cases where

$$f_K = (-\Delta)^{\alpha/2} S_K, \qquad \alpha \in [n-4, n-1)$$

and

$$f_K = (-\Delta)^{\alpha/2} P_K, \qquad \alpha \in [n, n+1),$$

and $K, L$ are arbitrary infinitely smooth convex bodies in $\mathbb{R}^n$. In the stability case the constant $c$ for sections depends only on $\alpha$ and $n$, while for projections the constant also depends on the circumradius of $L$. In the separation case, $c$ depends only on $\alpha$ and $n$ for projections, and also depends on the inradius of $K$ for sections.

In Section 4 we prove stability in Zvavitch's result and apply it to extend the hyperplane inequality in dimensions $n \le 4$ to arbitrary measures. The hyperplane problem of Bourgain [Bo1], [Bo2] asks whether there exists an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le C \max_{\xi \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \xi^\perp). \tag{3}$$

The problem is still open, with the best-to-date estimate $C \sim n^{1/4}$ established by Klartag [Kl], who slightly improved the previous estimate of Bourgain [Bo3]. We refer the reader to recent papers [EK], [DP] for the history and current state of the hyperplane problem.

In the case where the dimension $n \le 4$, the inequality (3) can be proved with the best possible constant (see [G3, Theorem 9.4.11]):

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|} \max_{\xi \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \xi^\perp), \tag{4}$$

with equality when $K = B_2^n$ is the Euclidean ball. Here $|B_2^n| = \pi^{n/2}/\Gamma(1+n/2)$ is the volume of $B_2^n$. Note that the constant $C$ in (4) is less than 1; see Lemma 1. Inequality (4) follows from the affirmative answer to the Busemann-Petty problem in dimensions up to four, just let $L$ be the Euclidean ball in the formulation of the problem.

In Section 4 we prove that inequality (3) holds in dimensions up to four with arbitrary measure in place of the volume. Our extension of (4) is as follows: if $2 \le n \le 4$ and $K$ is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\mu(K) \le \frac{n}{n-1} \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ \mathrm{Vol}_n(K)^{1/n}. \tag{5}$$

By analogy with the volume case, one may expect that (5) immediately follows from Zvavitch's result. This is not so, the argument does not work in this case because the measure $\mu$ of sections of the Euclidean ball does not have to be a constant. Instead, to prove (5) we establish stability in the affirmative part of Zvavitch's result in the following sense: if $2 \leq n \leq 4$, $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $\varepsilon > 0$ so that for every $\xi \in S^{n-1}$,

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp) + \varepsilon,$$

then

$$\mu(K) \leq \mu(L) + \frac{n\varepsilon}{n-1} \mathrm{Vol}_n(K)^{1/n}.$$

To prove (5), we interchange $K$ and $L$ in the latter result and then put $L = \emptyset$; see Section 4.

In most cases we employ the techniques of the Fourier analytic approach to sections and projections that has recently been developed; see [K4] and [KY]. We use a more geometric Radon transform approach in the case $f_K = S_K$ to show the variety of methods; it is also possible to solve this case with the Fourier transform.

## 2. Stability and separation for sections

We say that a closed bounded set $K$ in $\mathbb{R}^n$ is a *star body* if every straight line passing through the origin crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the *Minkowski functional* of $K$ defined by

$$\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body $K$ is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \qquad x \in \mathbb{R}^n.$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of $K$ in the direction of $x$.

Writing the volume of $K$ in polar coordinates, one gets

$$\mathrm{Vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(\theta) d\theta = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{6}$$

The *spherical Radon transform* $R : C(S^{n-1}) \mapsto C(S^{n-1})$ is a linear operator defined by

$$Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) \ dx, \quad \xi \in S^{n-1}$$

for every function $f \in C(S^{n-1})$.

The polar formula (6) for the volume of a hyperplane section expresses this volume in terms of the spherical Radon transform (see for example [K4, p.15]):

$$S_K(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1} R(\|\cdot\|_K^{-n+1})(\xi). \tag{7}$$

The spherical Radon transform is self-dual (see [Gr, Lemma 1.3.3] ): for any functions $f, g \in C(S^{n-1})$

$$\int_{S^{n-1}} Rf(\xi)\ g(\xi)\ d\xi = \int_{S^{n-1}} f(\xi)\ Rg(\xi)\ d\xi. \tag{8}$$

The spherical Radon transform can be extended to measures. Let $\mu$ be a finite Borel measure on $S^{n-1}$. We define the spherical Radon transform of $\mu$ as a functional $R\mu$ on the space $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x).$$

By Riesz's characterization of continuous linear functionals on the space $C(S^{n-1})$, $R\mu$ is also a finite Borel measure on $S^{n-1}$. If $\mu$ has continuous density $g$, then by (8) the Radon transform of $\mu$ has density $Rg$.

The class of intersection bodies was introduced by Lutwak [L]. Let $K, L$ be origin-symmetric star bodies in $\mathbb{R}^n$. We say that $K$ is the intersection body of $L$ if the radius of $K$ in every direction is equal to the $(n-1)$-dimensional volume of the section of $L$ by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = \mathrm{Vol}_{n-1}(L \cap \xi^\perp). \tag{9}$$

All the bodies $K$ that appear as intersection bodies of different star bodies form *the class of intersection bodies of star bodies.*

Note that the right-hand side of (9) can be written in terms of the spherical Radon transform using (7):

$$\|\xi\|_K^{-1} = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} R(\|\cdot\|_L^{-n+1})(\xi).$$

It means that a star body $K$ is the intersection body of a star body if and only if the function $\|\cdot\|_K^{-1}$ is the spherical Radon transform of a continuous positive function on $S^{n-1}$. This allows to introduce a more general class of bodies. A star body $K$ in $\mathbb{R}^n$ is called an *intersection body* if there exists a finite Borel measure $\mu$ on the sphere $S^{n-1}$ so that $\|\cdot\|_K^{-1} = R\mu$ as functionals on $C(S^{n-1})$, i.e. for every continuous function $f$ on $S^{n-1}$,

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x)\ dx = \int_{S^{n-1}} Rf(x)\ d\mu(x). \tag{10}$$

Intersection bodies played the crucial role in the solution of the Busemann-Petty problem due to the following connection found by Lutwak [L]: if $K$ in an origin-symmetric intersection body in $\mathbb{R}^n$ and $L$ is any origin-symmetric star body in $\mathbb{R}^n$, then the inequalities $S_K(\xi) \leq S_L(\xi)$ for all $\xi \in S^{n-1}$ imply that $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)$, i.e. the answer to the Busemann-Petty problem in this situation is affirmative. For more information about intersection bodies, see [K4, Ch.4], [KY], [G3, Ch.8] and references there.

In this section we prove the stability of Lutwak's connection. First, we need some simple facts about the Γ-function.

**Lemma 1.** *For any* $n \in \mathbb{N}$, *the following inequalities hold:*

$$1 \leq \frac{\left(\Gamma(\frac{n}{2}+1)\right)^{\frac{n-1}{n}}}{\Gamma(\frac{n+1}{2})} \leq \sqrt{e},$$

$$\frac{\Gamma(\frac{n-1}{2})}{\left(\Gamma(\frac{n}{2})\right)^{\frac{n-1}{n}}} \leq \frac{n^{\frac{n-1}{n}} 2^{\frac{1}{n}}}{n-1}$$

*and*

$$\sqrt{\frac{n}{2}} \leq \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})} \leq \sqrt{\frac{n+1}{2}}.$$

**Proof :** For the first inequality see for example [KL, Lemma 2.1]. The second inequality is a simple modification of the lower estimate in the first, using the property $\Gamma(x+1) = x\Gamma(x)$ of the Γ-function.

The third inequality follows from log-convexity of the Γ-function (see [K4, p.30]):

$$\Gamma^2\left(\frac{n}{2}+1\right) \leq \Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{n+1}{2}\right) = \left(\frac{n+1}{2}\right)\Gamma^2\left(\frac{n+1}{2}\right),$$

and

$$\Gamma^2\left(\frac{n+1}{2}\right) \leq \Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}\right) = \frac{2}{n}\Gamma^2\left(\frac{n}{2}+1\right). \quad \square$$

**Theorem 1.** *Suppose that* $\varepsilon > 0$, $K$ *and* $L$ *are origin-symmetric star bodies in* $\mathbb{R}^n$, *and* $K$ *is an intersection body. If for every* $\xi \in S^{n-1}$

$$S_K(\xi) \leq S_L(\xi) + \varepsilon, \tag{11}$$

*then*

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + \varepsilon.$$

**Proof :** By (7), the condition (11) can be written as

$$R(\|\cdot\|_K^{-n+1})(\xi) \le R(\|\cdot\|_L^{-n+1})(\xi) + (n-1)\varepsilon,\ \forall \xi \in S^{n-1}. \tag{12}$$

Since $K$ is an intersection body, there exists a finite Borel measure $\mu$ on $S^{n-1}$ such that $\|\cdot\|_K^{-1} = R\mu$ as functionals on $C(S^{n-1})$. Together with (6), (12) and the definition of $R\mu$, the latter implies that

$$n\mathrm{Vol}_n(K) = \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_K^{-n+1}\ dx$$

$$= \int_{S^{n-1}} \|x\|_K^{-n+1}\ d(R\mu)(x) = \int_{S^{n-1}} R\left(\|\cdot\|_K^{-n+1}\right)(\xi)\ d\mu(\xi)$$

$$\le \int_{S^{n-1}} R\left(\|\cdot\|_L^{-n+1}\right)(\xi)\ d\mu(\xi) + (n-1)\varepsilon \int_{S^{n-1}} d\mu(\xi)$$

$$= \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_L^{-n+1}\ dx + (n-1)\varepsilon \int_{S^{n-1}} d\mu(x). \tag{13}$$

We estimate the first term in (13) using Hölder's inequality:

$$\int_{S^{n-1}} \|x\|_K^{-1}\|x\|_L^{-n+1}\ dx \le \left(\int_{S^{n-1}} \|x\|_K^{-n}\ dx\right)^{\frac{1}{n}} \left(\int_{S^{n-1}} \|x\|_L^{-n}\ dx\right)^{\frac{n-1}{n}}$$

$$= n\mathrm{Vol}_n(K)^{\frac{1}{n}}\mathrm{Vol}_n(L)^{\frac{n-1}{n}}. \tag{14}$$

We now estimate the second term in (13) adding the Radon transform of the unit constant function under the integral ($R1(x) = |S^{n-2}|$ for every $x \in S^{n-1}$), using again the fact that $\|\cdot\|_K^{-1} = R\mu$ and then applying Hölder's inequality:

$$(n-1)\varepsilon \int_{S^{n-1}} d\mu(x) = \frac{(n-1)\varepsilon}{|S^{n-2}|} \int_{S^{n-1}} R1(x)\ d\mu(x) \tag{15}$$

$$= \frac{(n-1)\varepsilon}{|S^{n-2}|} \int_{S^{n-1}} \|x\|_K^{-1}\ dx$$

$$\le \frac{(n-1)\varepsilon}{|S^{n-2}|} \left|S^{n-1}\right|^{\frac{n-1}{n}} \left(\int_{S^{n-1}} \|x\|_K^{-n}\ dx\right)^{\frac{1}{n}}, \tag{16}$$

where

$$\left|S^{n-2}\right| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \quad \text{and} \quad \left|S^{n-1}\right| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

are the surface areas of the unit spheres in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$, correspondingly.

We get that the quantity in (16) is equal to

$$\frac{(n-1)\Gamma(\frac{n-1}{2})}{2^{\frac{1}{n}}\left(\Gamma(\frac{n}{2})\right)^{\frac{n-1}{n}}}\varepsilon\left(n\mathrm{Vol}_n(K)\right)^{\frac{1}{n}} \leq n\varepsilon\left(\mathrm{Vol}_n(K)\right)^{\frac{1}{n}}$$

by the second inequality of Lemma 1.

Combining the latter inequality with (13) and (14),

$$n\mathrm{Vol}_n(K) \leq n\mathrm{Vol}_n(K)^{\frac{1}{n}}\mathrm{Vol}_n(L)^{\frac{n-1}{n}} + n\varepsilon\left(\mathrm{Vol}_n(K)\right)^{\frac{1}{n}}. \quad \Box$$

It is known that for $2 \leq n \leq 4$ every origin symmetric convex body in $\mathbb{R}^n$ is an intersection body (see [G2], [Z3], [GKS] or [K4, p. 73]). This means that the result of Theorem 1 holds in these dimensions for arbitrary origin-symmetric convex bodies $K, L$. Moreover, interchanging $K, L$ in Theorem 1, we prove

**Corollary 1.** *If $2 \leq n \leq 4$, then for any origin-symmetric convex bodies $K, L$ in $\mathbb{R}^n$,*

$$\left|\mathrm{Vol}_n(K)^{\frac{n-1}{n}} - \mathrm{Vol}_n(L)^{\frac{n-1}{n}}\right| \leq \|S_K - S_L\|_{C(S^{n-1})}.$$

We now prove the linear separation property of Lutwak's connection. Denote by

$$r(K) = \frac{\min_{\xi\in S^{n-1}}\rho_K(\xi)}{\mathrm{Vol}_n(K)^{1/n}}$$

the normalized inradius of $K$.

**Theorem 2.** *Let $K$ and $L$ be origin-symmetric star bodies in $\mathbb{R}^n$ and $\varepsilon > 0$. Assume that $K$ is an intersection body. If for every $\xi \in S^{n-1}$*

$$S_K(\xi) \leq S_L(\xi) - \varepsilon, \tag{17}$$

*then*

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} - \sqrt{\frac{2\pi}{n+1}}\, r(K)\varepsilon.$$

**Proof :** The proof goes along the same lines as that of Theorem 1, with the difference that now we need a lower estimate in place of the upper estimate (16). Similarly to (13) and (14), we get

$$n\mathrm{Vol}_n(K) \leq n\mathrm{Vol}_n(K)^{\frac{1}{n}}\mathrm{Vol}_n(L)^{\frac{n-1}{n}} - (n-1)\varepsilon\int_{S^{n-1}} d\mu(x). \tag{18}$$

Similarly to (15),

$$(n-1)\varepsilon\int_{S^{n-1}} d\mu(x) = \frac{(n-1)\varepsilon}{|S^{n-2}|}\int_{S^{n-1}} \|x\|_K^{-1}\, dx,$$

and, since $\|x\|_K^{-1} = \rho_K(x)$ for $x \in S^{n-1}$, using the definition of $r(K)$ we estimate the latter by

$$\geq \frac{\varepsilon(n-1)r(K)\mathrm{Vol}_n(K)^{\frac{1}{n}}\,|S^{n-1}|}{|S^{n-2}|}$$

$$= \varepsilon r(K) n \mathrm{Vol}_n(K)^{\frac{1}{n}} \frac{(n-1)\pi^{\frac{n}{2}}\Gamma(\frac{n-1}{2})}{n\pi^{\frac{n-1}{2}}\Gamma(\frac{n}{2})}$$

(we multipiled and divided by $n$ and now use $\Gamma(x+1) = x\Gamma(x)$ and the third inequality of Lemma 1)

$$= \varepsilon r(K) n \mathrm{Vol}_n(K)^{\frac{1}{n}} \sqrt{\pi} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)} \geq \varepsilon r(K) n \mathrm{Vol}_n(K)^{\frac{1}{n}} \sqrt{\frac{2\pi}{n+1}}.$$

Combining this with (18), we get

$$n\mathrm{Vol}_n(K) \leq n\mathrm{Vol}_n(K)^{\frac{1}{n}}\mathrm{Vol}_n(L)^{\frac{n-1}{n}} - n\varepsilon r(K)\mathrm{Vol}_n(K)^{\frac{1}{n}}\sqrt{\frac{2\pi}{n+1}}. \quad \square$$

We now pass to stability in the comparison result from [KYY]. The goal here is to establish stability of volume comparison in dimensions higher than 4 without the assumption that $K$ is an intersection body. We use the techniques of the Fourier approach to sections of convex bodies that has recently been developed; see [K4] and [KY].

The Fourier transform of a distribution $f$ is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function $\phi$ from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. For any even distribution $f$, we have $(\hat{f})^\wedge = (2\pi)^n f$.

If $K$ is a star body and $0 < p < n$, then $\|\cdot\|_K^{-p}$ is a locally integrable function on $\mathbb{R}^n$ and represents a distribution. Suppose that $K$ is infinitely smooth, i.e. $\|\cdot\|_K \in C^\infty(S^{n-1})$ is an infinitely differentiable function on the sphere. Then by [K4, Lemma 3.16], the Fourier transform of $\|\cdot\|_K^{-p}$ is an extension of some function $g \in C^\infty(S^{n-1})$ to a homogeneous function of degree $-n+p$ on $\mathbb{R}^n$. When we write $\left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)$, we mean $g(\xi),\ \xi \in S^{n-1}$. If $K, L$ are infinitely smooth star bodies, the following spherical version of Parseval's formula was proved in [K5] (see [K4, Lemma 3.22]): for any $p \in (-n, 0)$

$$\int_{S^{n-1}} \left(\|\cdot\|_K^{-p}\right)^\wedge(\xi) \left(\|\cdot\|_L^{-n+p}\right)^\wedge(\xi) = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}\, dx. \tag{19}$$

A distribution is called *positive definite* if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function $\phi$. The following was proved in [KYY]:

**Lemma 2.** ([KYY, Lemma 2.3]) *Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$. Assume that $\alpha \in [n-4, n-1)$, then $\|x\|_K^{-1} \cdot |x|_2^{-\alpha}$ is a positive definite distribution on $\mathbb{R}^n$.*

If $K$ is infinitely smooth, by Lemma 2 and [K4, Lemma 3.16], the Fourier transform $(|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge$ is an extension of a non-negative infinitely differentiable function on $S^{n-1}$ to the whole $\mathbb{R}^n$.

**Theorem 3.** *Let $\varepsilon > 0$, $\alpha \in [n-4, n-1)$, and let $K$ and $L$ be origin-symmetric infinitely smooth convex bodies in $\mathbb{R}^n$, $n \geq 4$, so that for every $\xi \in S^{n-1}$*

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi) + \varepsilon. \tag{20}$$

*Then*

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + c\varepsilon,$$

*where*

$$c = c(\alpha, n) = \frac{\sqrt{\pi}(n-1)\Gamma(\frac{n-\alpha-1}{2})}{2^{\alpha+\frac{1}{n}} n^{\frac{n-1}{n}} \Gamma(\frac{\alpha+1}{2}) \left(\Gamma(\frac{n}{2})\right)^{\frac{n-1}{n}}}.$$

**Proof :** It was proved in [K1] that

$$S_K(\xi) = \frac{1}{\pi(n-1)} (\|x\|_K^{-n+1})^\wedge(\xi), \qquad \forall \xi \in S^{n-1}. \tag{21}$$

Extending $S_K(\xi)$ to $\mathbb{R}^n$ as a homogeneous function of degree $-1$ and using the definition of fractional powers of the Laplacian we get

$$(-\Delta)^{\alpha/2} S_K(\theta) = \frac{1}{\pi(n-1)} (|x|_2^{\alpha}\|x\|_K^{-n+1})^\wedge(\theta),$$

therefore

$$\begin{aligned}(2\pi)^n n \mathrm{Vol}_n(K) &= (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-n+1}\|x\|_K^{-1} dx \\ &= (2\pi)^n \int_{S^{n-1}} (|x|_2^{-\alpha}\|x\|_K^{-1})(|x|_2^{\alpha}\|x\|_K^{-n+1}) dx \\ &= \int_{S^{n-1}} (|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(\theta)(|x|_2^{\alpha}\|x\|_K^{-n+1})^\wedge(\theta) d\theta \\ &= \pi(n-1) \int_{S^{n-1}} (|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(\theta)(-\Delta)^{\alpha/2} S_K(\theta) d\theta.\end{aligned}$$

Here we used Parseval's formula on the sphere (19). By Lemma 2, $(|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge$ is a non-negative function on $S^{n-1}$, and we can use (20) to estimate the latter quantity:

$$\leq \pi(n-1) \int_{S^{n-1}} (|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(\theta)(-\Delta)^{\alpha/2} S_L(\theta) d\theta$$

$$+\pi(n-1)\varepsilon\int_{S^{n-1}}(|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(\theta)d\theta. \tag{22}$$

Repeating the above calculation in the opposite order, we get that the first summand in (22) is equal to

$$(2\pi)^n\int_{S^{n-1}}\|x\|_L^{-n+1}\|x\|_K^{-1}dx \le (2\pi)^n n\mathrm{Vol}_n(K)^{\frac{1}{n}}\mathrm{Vol}_n(L)^{\frac{n-1}{n}} \tag{23}$$

by Hölder's inequality.

To estimate the second summand in (22), we use the formula for the Fourier transform (in the sense of distributions; see [GS, p.194])

$$\left(|x|_2^{-n+\alpha+1}\right)^\wedge(\theta)=\frac{2^{\alpha+1}\pi^{\frac{n}{2}}\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{n-\alpha-1}{2})}|\theta|_2^{-\alpha-1}.$$

Again using Parseval's formula and then Hölder's inequality,

$$\int_{S^{n-1}}(|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(\theta)d\theta$$

$$=\frac{\Gamma(\frac{n-\alpha-1}{2})}{2^{\alpha+1}\pi^{\frac{n}{2}}\Gamma(\frac{\alpha+1}{2})}\int_{S^{n-1}}(|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(\theta)\left(|x|_2^{-n+\alpha+1}\right)^\wedge(\theta)d\theta$$

$$=\frac{(2\pi)^n\Gamma(\frac{n-\alpha-1}{2})}{2^{\alpha+1}\pi^{\frac{n}{2}}\Gamma(\frac{\alpha+1}{2})}\int_{S^{n-1}}\|x\|_K^{-1}\ dx$$

$$\le\frac{(2\pi)^n\Gamma(\frac{n-\alpha-1}{2})\left|S^{n-1}\right|^{\frac{n-1}{n}}}{2^{\alpha+1}\pi^{\frac{n}{2}}\Gamma(\frac{\alpha+1}{2})}\left(\int_{S^{n-1}}\|x\|_K^{-n}\ dx\right)^{\frac{1}{n}}$$

$$=\frac{(2\pi)^n\Gamma(\frac{n-\alpha-1}{2})\left|S^{n-1}\right|^{\frac{n-1}{n}}}{2^{\alpha+1}\pi^{\frac{n}{2}}\Gamma(\frac{\alpha+1}{2})}(n\mathrm{Vol}_n(K))^{\frac{1}{n}}$$

Combining this with (22) and (23), we get

$$(2\pi)^n n\mathrm{Vol}_n(K)\le(2\pi)^n n\mathrm{Vol}_n(K)^{\frac{1}{n}}\mathrm{Vol}_n(L)^{\frac{n-1}{n}}$$

$$+\frac{(2\pi)^n\varepsilon\pi(n-1)n^{\frac{1}{n}}\Gamma(\frac{n-\alpha-1}{2})\left|S^{n-1}\right|^{\frac{n-1}{n}}}{2^{\alpha+1}\pi^{\frac{n}{2}}\Gamma(\frac{\alpha+1}{2})}(\mathrm{Vol}_n(K))^{\frac{1}{n}},$$

which implies the result.

□

For $\alpha<n-4$ the statement of Theorem 3 is no longer true, simply because the comparison result itself does not hold, as shown in [KYY].

The corresponding separation result looks as follows:

**Theorem 4.** *Let* $\varepsilon > 0$, $\alpha \in [n-4, n-1)$, $K$ *and* $L$ *be origin-symmetric infinitely smooth convex bodies in* $\mathbb{R}^n$, $n \geq 4$, *so that for every* $\xi \in S^{n-1}$

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi) - \varepsilon. \tag{24}$$

*Then*

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} - c\varepsilon,$$

*where*

$$c = r(K) \frac{\pi(n-1)\Gamma(\frac{n-\alpha-1}{2})}{n 2^\alpha \Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n}{2})}.$$

**Proof :** Following the proof of Theorem 3, we get

$$(2\pi)^n n \mathrm{Vol}_n(K) \leq (2\pi)^n n \mathrm{Vol}_n(K)^{\frac{1}{n}} \mathrm{Vol}_n(L)^{\frac{n-1}{n}}$$
$$-\pi(n-1)\varepsilon \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge(\theta) d\theta.$$

The difference with the proof of Theorem 3 is that now we have to estimate

$$\int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge(\theta) d\theta$$

from below. In the same way as in Theorem 3 we write this integral as

$$\frac{(2\pi)^n \Gamma(\frac{n-\alpha-1}{2})}{2^{\alpha+1} \pi^{\frac{n}{2}} \Gamma(\frac{\alpha+1}{2})} \int_{S^{n-1}} \|x\|_K^{-1} \, dx.$$

The latter integral is greater or equal to $r(K) \left(\mathrm{Vol}_n(K)\right)^{\frac{1}{n}} |S^{n-1}|$. The result follows.

$\square$

## 3. Stability and separation for projections

We need several more definitions from convex geometry. We refer the reader to [S2] for details.

The *support function* of a convex body $K$ in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max_{\{\xi \in \mathbb{R}^n : \|\xi\|_K = 1\}} (x, \xi), \quad x \in \mathbb{R}^n.$$

If $K$ is origin-symmetric, then $h_K$ is a norm on $\mathbb{R}^n$.

The *surface area measure* $S(K, \cdot)$ of a convex body $K$ in $\mathbb{R}^n$ is defined as follows: for every Borel set $E \subset S^{n-1}$, $S(K, E)$ is equal to Lebesgue measure of the part of the boundary of $K$ where normal vectors belong to $E$. We usually consider bodies with absolutely continuous surface area measures. A convex body $K$ is said to have the *curvature function*

$$f_K : S^{n-1} \to \mathbb{R},$$

if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to Lebesgue measure $\sigma_{n-1}$ on $S^{n-1}$, and

$$\frac{dS(K,\cdot)}{d\sigma_{n-1}} = f_K \in L_1(S^{n-1}),$$

so $f_K$ is the density of $S(K,\cdot)$.

By the approximation argument of [S2, Th. 3.3.1], we may assume in the formulation of Shephard's problem that the bodies $K$ and $L$ are such that their support functions $h_K,\ h_L$ are infinitely smooth functions on $\mathbb{R}^n \setminus \{0\}$. Using [K4, Lemma 3.16] we get in this case that the Fourier transforms $\widehat{h_K},\ \widehat{h_L}$ are the extensions of infinitely differentiable functions on the sphere to homogeneous distributions on $\mathbb{R}^n$ of degree $-n-1$. Moreover, by a similar approximation argument (see also [GZ, Section 5]), we may assume that our bodies have absolutely continuous surface area measures. Therefore, in the rest of this section, $K$ and $L$ are convex symmetric bodies with infinitely smooth support functions and absolutely continuous surface area measures.

The following version of Parseval's formula was proved in [KRZ] (see also [K4, Lemma 8.8]):

$$\int_{S^{n-1}} \widehat{h_K}(\xi)\widehat{f_L}(\xi)\ d\xi = (2\pi)^n \int_{S^{n-1}} h_K(x) f_L(x)\ dx. \tag{25}$$

The volume of a body can be expressed in terms of its support function and curvature function:

$$\mathrm{Vol}_n(K) = \frac{1}{n}\int_{S^{n-1}} h_K(x) f_K(x)\ dx. \tag{26}$$

If $K$ and $L$ are two convex bodies in $\mathbb{R}^n$ the *mixed volume* $V_1(K,L)$ is equal to

$$V_1(K,L) = \frac{1}{n}\lim_{\varepsilon\to+0} \frac{\mathrm{Vol}_n(K+\epsilon L) - \mathrm{Vol}_n(K)}{\varepsilon}.$$

We use the following first Minkowski inequality (see [K4, p.23]): for any convex bodies $K, L$ in $\mathbb{R}^n$,

$$V_1(K,L) \geq \mathrm{Vol}_n(K)^{(n-1)/n}\mathrm{Vol}_n(L)^{1/n}. \tag{27}$$

The mixed volume can also be expressed in terms of the support and curvature functions:

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(x) f_K(x)\ dx. \tag{28}$$

Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$. The *projection body* $\Pi K$ of $K$ is defined as an origin-symmetric convex body in $\mathbb{R}^n$

whose support function in every direction is equal to the volume of the hyperplane projection of $K$ to this direction: for every $\theta \in S^{n-1}$,

$$h_{\Pi K}(\theta) = \mathrm{Vol}_{n-1}(K|\theta^\perp). \tag{29}$$

If $L$ is the projection body of some convex body, we simply say that $L$ is a projection body.

Both Petty [Pe] and Schneider [S1] in their solutions of the Shephard problem (see the introduction) used the connection with projection bodies: if the body $L$ (with greater projections) is a projection body then the answer to the question of the Shephard problem is affirmative for any body $K$. We now prove the stability of this connection.

Define the normalized circumradius of $L$ by

$$R(L) = \frac{\max_{\xi \in S^{n-1}} \rho_L(\xi)}{\mathrm{Vol}_n(L)^{\frac{1}{n}}}.$$

**Theorem 5.** *Suppose that $\varepsilon > 0$, $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $L$ is a projection body. If for every $\xi \in S^{n-1}$*

$$P_K(\xi) \le P_L(\xi) + \varepsilon, \tag{30}$$

*then*

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + \sqrt{\frac{2\pi}{n}}\ R(L)\varepsilon.$$

**Proof :** It was proved in [KRZ] that

$$P_K(\xi) = -\frac{1}{\pi}\widehat{f_K}(\theta), \qquad \forall \xi \in S^{n-1}, \tag{31}$$

where $f_K$ is extended from the sphere to a homogeneous function of degree $-n-1$ on the whole $\mathbb{R}^n$, and the Fourier transform $\widehat{f_K}$ is the extension of a continuous function $P_K$ on the sphere to a homogeneous of degree 1 function on $\mathbb{R}^n$.

Therefore, the condition (30) can be written as

$$-\frac{1}{\pi}\widehat{f_K}(\xi) \le -\frac{1}{\pi}\widehat{f_L}(\xi) + \varepsilon, \qquad \forall \xi \in S^{n-1}. \tag{32}$$

It was also proved in [KRZ] that an infinitely smooth origin-symmetric convex body $L$ in $\mathbb{R}^n$ is a projection body if and only if $\widehat{h_L} \le 0$ on the sphere $S^{n-1}$. Therefore, integrating (32) with respect to a negative density,

$$\int_{S^{n-1}} \widehat{h_L}(\xi)\widehat{f_L}(\xi)\ d\xi \ge \int_{S^{n-1}} \widehat{h_L}(\xi)\widehat{f_K}(\xi)\ d\xi + \pi\varepsilon \int_{S^{n-1}} \widehat{h_L}(\xi)\ d\xi.$$

Using this, (26) and (25), we get

$$\begin{aligned}(2\pi)^n n\mathrm{Vol}_n(L) &= (2\pi)^n \int_{S^{n-1}} h_L(x) f_L(x)\ dx = \int_{S^{n-1}} \widehat{h_L}(\xi)\widehat{f_L}(\xi)\ d\xi \\ &\geq \int_{S^{n-1}} \widehat{h_L}(\xi)\widehat{f_K}(\xi)\ d\xi + \pi\varepsilon \int_{S^{n-1}} \widehat{h_L}(\xi)\ d\xi \\ &= (2\pi)^n \int_{S^{n-1}} h_L(x) f_K(x)\ dx + \pi\varepsilon \int_{S^{n-1}} \widehat{h_L}(\xi)\ d\xi. \qquad (33)\end{aligned}$$

We estimate the first summand from below using the first Minkowski inequality:

$$(2\pi)^n \int_{S^{n-1}} h_L(x) f_K(x)\ dx \geq (2\pi)^n n \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}} \left(\mathrm{Vol}_n(K)\right)^{\frac{n-1}{n}}. \qquad (34)$$

To estimate the second summand in (33), note that, by (31), the Fourier transform of the curvature function of the Euclidean ball

$$\widehat{f_2}(\xi) = -\pi \mathrm{Vol}_{n-1}(B_2^{n-1}) = -\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}, \qquad \forall \xi \in S^{n-1},$$

where $B_2^{n-1}$ is the unit Euclidean ball in $\mathbb{R}^{n-1}$. Therefore,

$$\begin{aligned}\pi\varepsilon \int_{S^{n-1}} \widehat{h_L}(\xi)\ d\xi &= -\varepsilon \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}} \int_{S^{n-1}} \widehat{h_L}(\xi)\widehat{f_2}(\xi)\ d\xi \\ &= -(2\pi)^n \varepsilon \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}} \int_{S^{n-1}} h_L(x) f_2(x)\ dx \\ &= -(2\pi)^n \varepsilon \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}} \int_{S^{n-1}} h_L(x)\ dx \\ &\geq -(2\pi)^n \varepsilon \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}} R(L) \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}} \left|S^{n-1}\right| \\ &= -(2\pi)^n n\varepsilon \frac{\sqrt{\pi} R(L) \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}} \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)},\end{aligned}$$

where we again used Parseval's formula, the fact that $f_2 = 1$, and a simple estimate $h_L(x) \leq R(L) \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}}$.

Combining this with (33) and (34), and using the third inequality of Lemma 1, we get

$$\begin{aligned}(2\pi)^n n\mathrm{Vol}_n(L) \geq{}& (2\pi)^n n \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}} \left(\mathrm{Vol}_n(K)\right)^{\frac{n-1}{n}} \\ &-(2\pi)^n n \sqrt{\frac{2\pi}{n}} R(L) \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}} \varepsilon,\end{aligned}$$

which finishes the proof.

□

We now prove the corresponding separation result.

**Theorem 6.** *Suppose that $\varepsilon > 0$, $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $L$ is a projection body. If for every $\xi \in S^{n-1}$*

$$P_K(\xi) \le P_L(\xi) - \varepsilon, \tag{35}$$

*then*

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-1}{n}} - \frac{\varepsilon}{\sqrt{e}}.$$

**Proof :** Similarly to the proof of Theorem 5, we get (33), but with negative sign in front of $\varepsilon$ :

$$(2\pi)^n n \mathrm{Vol}_n(L) \ge (2\pi)^n \int_{S^{n-1}} h_L(x) f_K(x)\ dx - \pi\varepsilon \int_{S^{n-1}} \widehat{h_L}(\xi)\ d\xi. \tag{36}$$

The difference with the proof of Theorem 5 is that now we need an upper estimate for

$$\pi\varepsilon \int_{S^{n-1}} \widehat{h_L}(\xi)\ d\xi = -(2\pi)^n \varepsilon \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}} \int_{S^{n-1}} h_L(x) f_2(x)\ dx.$$

Using the first Minkowski inequality (27), the latter is

$$\le -(2\pi)^n n\varepsilon \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}} \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}} \left(\mathrm{Vol}_n(B_2^n)\right)^{\frac{n-1}{n}}$$

$$= -(2\pi)^n n\varepsilon \frac{\Gamma(\frac{n+1}{2})}{\left(\Gamma(\frac{n}{2}+1)\right)^{\frac{n-1}{n}}} \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}} \le -\frac{(2\pi)^n n\varepsilon}{\sqrt{e}} \left(\mathrm{Vol}_n(L)\right)^{\frac{1}{n}}$$

by the first inequality of Lemma 1. In conjunction with (36) and (27), (28), this implies the result.

□

We end this section by formulating the stability version of the result from [Y] mentioned in the introduction, which treats projections in arbitrary dimension without the additional assumption that $L$ is a projection body. The proof follows the lines of the proof of Theorem 5 with changes corresponding to those in the proof of Theorem 3; we leave this proof to the willing reader, as well as the separation result in this case. Let us just mention that one has to use the fact that for every $\alpha \in [n, n+1)$ the distribution $|x|_2^{-\alpha} h_L(x)$ is positive definite, which is explained in [Y].

**Theorem 7.** *Let $\varepsilon > 0,\ \alpha \in [n, n+1)$, $K$ and $L$ be origin-symmetric infinitely smooth convex bodies in $\mathbb{R}^n$, $n \geq 3$, so that for every $\xi \in S^{n-1}$*

$$(-\Delta)^{\alpha/2} P_L(\xi) \leq (-\Delta)^{\alpha/2} P_K(\xi) + \varepsilon.$$

*Then*

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + c\varepsilon,$$

*where*

$$c = \frac{\Gamma(\frac{n-\alpha+1}{2})\, |S^{n-1}|\, R(L)}{2^{\alpha+1} \pi^{\frac{n}{2}} \Gamma(\frac{\alpha+1}{2}) n}.$$

Note that this is no longer true if $\alpha < n$, because the underlying comparison result fails, as shown in [Y].

## 4. A hyperplane inequality for arbitrary measures

We start with stability in Zvavitch's result, and for this we need some more facts about distributions. L. Schwartz's generalization of Bochner's theorem (see, for example, [GV, p.152]) states that a distribution is positive definite if and only if it is the Fourier transform of a tempered measure on $\mathbb{R}^n$. Recall that a (non-negative, not necessarily finite) measure $\mu$ is called tempered if

$$\int_{\mathbb{R}^n} (1 + |x|_2)^{-\beta}\ d\mu(x) < \infty$$

for some $\beta > 0$.

If $0 < p < n$, then $\|\cdot\|_K^{-p}$ is a locally integrable function on $\mathbb{R}^n$ and represents an even homogeneous of degree $-p$ distribution. If this distribution is positive definite for some $p \in (0, n)$, then its Fourier transform is a tempered measure which is at the same time a homogeneous distribution of degree $-n + p$. One can express such a measure in polar coordinates:

**Proposition 1.** *(*[K4, Corollary 2.26]*) Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$ and $p \in (0, n)$. The function $\|\cdot\|_K^{-p}$ represents a positive definite distribution on $\mathbb{R}^n$ if and only if there exists a finite Borel measure $\mu_0$ on $S^{n-1}$ so that for every even test function $\phi$,*

$$\int_{\mathbb{R}^n} \|x\|_K^{-p} \phi(x)\ dx = \int_{S^{n-1}} \left( \int_0^\infty t^{p-1} \hat{\phi}(t\xi) dt \right) d\mu_0(\xi).$$

It was proved in [GKS] (see [K4, Corollary 4.9]) that

**Proposition 2.** *If $2 \leq n \leq 4$ and $K$ is any origin-symmetric convex body in $\mathbb{R}^n$, then the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution.*

For any continuous function $f$ on the sphere $S^{n-1}$ denote by $f \cdot r^{-n+1}$ the extension of $f$ to an even homogeneous function of degree $-n+1$ on the whole $\mathbb{R}^n$. It was proved in [K4, Lemma 3.7] that the Fourier transform of $f \cdot r^{-n+1}$ is equal to another continuous function $g$ on $S^{n-1}$ extended to an even homogeneous of degree $-1$ function $g \cdot r^{-1}$ on the whole $\mathbb{R}^n$ (in fact, $g$ is the spherical Radon transform of $f$, up to a constant). This is why we can remove smoothness conditions in the Parseval formula on the sphere [K4, Corollary 3.23] and formulate it as follows:

**Proposition 3.** *Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$. Suppose that $\|\cdot\|_K^{-1}$ is a positive definite distribution, and let $\mu_0$ be the finite Borel measure on $S^{n-1}$ that corresponds to $\|\cdot\|_K^{-1}$ by Proposition 1. Then for any even continuous function $f$ on $S^{n-1}$,*

$$\int_{S^{n-1}} (f \cdot r^{-n+1})^\wedge(\theta)\ d\mu_0(\theta) = \int_{S^{n-1}} \|\theta\|_K^{-1} f(\theta)\ d\theta. \tag{37}$$

Finally, we need a formula from [Zv], expressing the measure of a section in terms of the Fourier transform. This formula generalizes the corresponding result for volume; see (21).

**Proposition 4.** ([Zv]) *Let $K$ be an origin-symmetric star body in $\mathbb{R}^n$, then, for every $\xi \in S^{n-1}$,*

$$\mu(K \cap \xi^\perp) = \frac{1}{\pi} \left( |x|_2^{-n+1} \int_0^{|x|_2/\|x\|_K} t^{n-2} f\left(\frac{tx}{|x|_2}\right) dt \right)^\wedge (\xi),$$

*where the Fourier transform of the function of $x \in \mathbb{R}^n$ in the right-hand side is a continuous homogeneous of degree $-1$ function on $\mathbb{R}^n \setminus \{0\}$.*

The following elementary fact was used by Zvavitch [Zv] in his generalization of the Busemann-Petty problem.

**Lemma 3.** *Let $a, b > 0$ and let $\alpha$ be a non-negative function on $(0, \max\{a, b\}]$ so that the integrals below converge. Then*

$$\int_0^a t^{n-1}\alpha(t)dt - a\int_0^a t^{n-2}\alpha(t)dt \le \int_0^b t^{n-1}\alpha(t)dt - a\int_0^b t^{n-2}\alpha(t)dt. \tag{38}$$

**Proof :** The inequality (38) is equivalent to

$$a\int_a^b t^{n-2}\alpha(t)dt \le \int_a^b t^{n-1}\alpha(t)dt.$$

Note that the latter inequality also holds in the case $a \ge b$.

□

The measure of a body can be expressed in polar coordinates as follows:

$$\mu(K) = \int_K f(u)\ du = \int_{S^{n-1}} \left( \int_0^{\|x\|_K^{-1}} t^{n-1} f(tx) dt \right) dx. \tag{39}$$

We are ready to prove the stability.

**Theorem 8.** *Let $f$ be an even positive continuous function on $\mathbb{R}^n$, $2 \le n \le 4$, $\mu$ is the measure with density $f$, $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $\varepsilon > 0$. Suppose that for every $\xi \in S^{n-1}$,*

$$\mu(K \cap \xi^\perp) \le \mu(L \cap \xi^\perp) + \varepsilon. \tag{40}$$

*Then*

$$\mu(K) \le \mu(L) + \frac{n\varepsilon}{n-1} \mathrm{Vol}_n(K)^{1/n}. \tag{41}$$

**Proof :** First, we rewrite the condition (40) using Proposition 4:

$$\left( |x|_2^{-n+1} \int_0^{\frac{|x|_2}{\|x\|_K}} t^{n-2} f\left(\frac{tx}{|x|_2}\right) dt \right)^\wedge (\xi)$$

$$\le \left( |x|_2^{-n+1} \int_0^{\frac{|x|_2}{\|x\|_L}} t^{n-2} f\left(\frac{tx}{|x|_2}\right) dt \right)^\wedge (\xi) + \pi\varepsilon$$

for each $\xi \in S^{n-1}$.

We integrate the latter inequality over $S^{n-1}$ with respect to the measure $\mu_0$ corresponding to the positive definite homogeneous of degree $-1$ distribution $\|\cdot\|_K^{-1}$ by Proposition 1:

$$\int_{S^{n-1}} \left( |x|_2^{-n+1} \int_0^{\frac{|x|_2}{\|x\|_K}} t^{n-2} f\left(\frac{tx}{|x|_2}\right) dt \right)^\wedge (\xi)\, d\mu_0(\xi)$$

$$\le \int_{S^{n-1}} \left( |x|_2^{-n+1} \int_0^{\frac{|x|_2}{\|x\|_L}} t^{n-2} f\left(\frac{tx}{|x|_2}\right) dt \right)^\wedge (\xi)\, d\mu_0(\xi) + \pi\varepsilon \int_{S^{n-1}} d\mu_0(\xi),$$

and now apply the spherical Parseval formula, Proposition 3:

$$\int_{S^{n-1}} \|x\|_K^{-1} \left( \int_0^{\|x\|_K^{-1}} t^{n-2} f(tx)dt \right) dx$$

$$\leq \int_{S^{n-1}} \|x\|_K^{-1} \left( \int_0^{\|x\|_L^{-1}} t^{n-2} f(tx)dt \right) dx + \pi\varepsilon \int_{S^{n-1}} d\mu_0(\xi). \tag{42}$$

By Lemma 3 with $a = \|x\|_K^{-1}$, $b = \|x\|_L^{-1}$ and $\alpha(t) = f(tx)$,

$$\int_0^{\|x\|_K^{-1}} t^{n-1} f(tx)\ dt - \|x\|_K^{-1} \int_0^{\|x\|_K^{-1}} t^{n-2} f(tx)\ dt$$

$$\leq \int_0^{\|x\|_L^{-1}} t^{n-1} f(tx)dt - \|x\|_K^{-1} \int_0^{\|x\|_L^{-1}} t^{n-2} f(tx)dt, \quad \forall x \in S^{n-1}.$$

Integrating over $S^{n-1}$ we get

$$\int_{S^{n-1}} \left( \int_0^{\|x\|_K^{-1}} t^{n-1} f(tx)dt \right) dx - \int_{S^{n-1}} \|x\|_K^{-1} \left( \int_0^{\|x\|_K^{-1}} t^{n-2} f(tx)dt \right) dx \tag{43}$$

$$\leq \int_{S^{n-1}} \left( \int_0^{\|x\|_L^{-1}} t^{n-1} f(tx)dt \right) dx - \int_{S^{n-1}} \|x\|_K^{-1} \left( \int_0^{\|x\|_L^{-1}} t^{n-2} f(tx)dt \right) dx.$$

Adding inequalities (42) and (43) we get

$$\int_{S^{n-1}} \left( \int_0^{\|x\|_K^{-1}} t^{n-1} f(tx)dt \right) dx$$

$$\leq \int_{S^{n-1}} \left( \int_0^{\|x\|_L^{-1}} t^{n-1} f(tx)dt \right) dx + \pi\varepsilon \int_{S^{n-1}} d\mu_0(\xi),$$

and, by the polar formula (39), the latter can be written as

$$\mu(K) \leq \mu(L) + \pi\varepsilon \int_{S^{n-1}} d\mu_0(\xi).$$

It remains to estimate the integral in the right-hand side of the latter inequality. For this we use the formula for the Fourier transform (in the sense of distributions; see [GS, p.194])

$$\left(|x|_2^{-n+1}\right)^\wedge(\xi) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n-1}{2})}|\xi|_2^{-1}.$$

Again using Parseval's formula, Proposition 3,

$$\pi\varepsilon\int_{S^{n-1}} d\mu_0(\xi) = \frac{\pi\varepsilon\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\int_{S^{n-1}}\left(|\cdot|_2^{-n+1}\right)^\wedge(\xi)d\mu_0(\xi)$$

$$= \frac{\pi\varepsilon\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\int_{S^{n-1}}\|x\|_K^{-1}dx \le \frac{\pi\varepsilon\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\left(\int_{S^{n-1}}\|x\|_K^{-n}dx\right)^{1/n}\left|S^{n-1}\right|^{\frac{n-1}{n}},$$

where we also used Hölder's inequality. Here

$$\left|S^{n-1}\right| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

is the surface area of the unit sphere $S^{n-1}$. Now use the polar formula for volume (6) and apply Lemma 1 to get the desired estimate.

□

Theorem 8 does not hold true in dimensions greater than four, simply because the answer to the Busemann-Petty problem in these dimensions is negative. Note that in the case of the volume ($f \equiv 1$), the result of Theorem 8 is weaker than what is provided by Theorem 1. In fact, in the case of volume the estimate of Theorem 8 follows from the estimate of Theorem 1 by the Mean Value Theorem applied to the function $h(t) = t^{n/(n-1)}$.

Interchanging $K$ and $L$ in Theorem 8, we get

**Corollary 2.** *Under the conditions of Theorem 8, we have*

$$|\mu(K) - \mu(L)|$$

$$\le \frac{n}{n-1}\max_{\xi\in S^{n-1}}\left|\mu(K\cap\xi^\perp) - \mu(L\cap\xi^\perp)\right|\max\left(\mathrm{Vol}_n(K)^{\frac{1}{n}}, \mathrm{Vol}_n(L)^{\frac{1}{n}}\right). \tag{44}$$

Now to prove inequality (5) simply put $L = \emptyset$ in Corollary 2:

$$\mu(K) \le \frac{n}{n-1}\max_{\xi\in S^{n-1}}\mu(K\cap\xi^\perp)\ \mathrm{Vol}_n(K)^{1/n}.$$

The author does not know whether the inequalities of Corollary 1 and Corollary 2 hold in higher dimensions (maybe with an extra absolute constant in the right-hand side). Note that this question includes the hyperplane conjecture as a particular case.

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