HYPERBOLIC MANIFOLDS
AND SPECIAL VALUES OF DEDEKIND ZETA-FUNCTIONS

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by

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§1. Introduction

A famous theorem, proved by Euler in 1734, is that the sum $\sum_{n=1}^{\infty} \frac{1}{n^{2m}}$ is a rational multiple of π^{2m} for all natural numbers m :

$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \ldots, \quad \sum_{1}^{\infty} \frac{1}{n^{12}} = \frac{691\ \pi^{12}}{638512875}, \quad \ldots .$$

This result was generalized some years ago by Klingen [3] and Siegel [5], who showed that for an arbitrary totally real number field K the value of the Dedekind zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \qquad \text{(sum over non-zero integral ideals } \mathfrak{a} \text{ of } K\text{)}$$

at a positive even integral argument $s = 2m$ can be expressed by a formula of the form

$$\zeta_K(2m) = \text{rational number} \times \frac{\pi^{2nm}}{\sqrt{D}},$$

where n and D denote the degree and discriminant of K, respectively. However, little is known about the numbers $\zeta_K(2m)$ for K not totally real. We will prove the following theorem which describes the nature of these numbers for $m = 1$.

THEOREM 1. *Let* $A(x)$ *be the real-valued function*

$$(1) \qquad A(x) = \int_0^x \frac{1}{1+t^2} \log \frac{4}{1+t^2}\, dt \qquad (x \in \mathbb{R})$$

(see Fig. 1). *Then the value of* $\zeta_K(2)$ *for an arbitrary number field* K *can be expressed by a formula of the form*

$$(2) \qquad \zeta_K(2) = \frac{\pi^{2r+2s}}{\sqrt{|D|}} \times \sum_{\nu} c_\nu\, A(x_{\nu,1}) \ldots A(x_{\nu,s}) \qquad (\text{finite sum}),$$

where D, r *and* s *denote the discriminant and numbers of real and complex places of* K, *respectively, the* c_ν *are rational, and the* $x_{\nu,j}$ *are real algebraic numbers.*

The proof will show that the $x_{\nu,j}$ can be chosen of degree at most 8 over K, and will in fact yield the following stronger statement: let $\sigma_1, \bar{\sigma}_1, \ldots, \sigma_s, \bar{\sigma}_s$ denote the distinct complex embeddings of K; then for any totally imaginary quadratic extension K_1/K and embeddings $\tilde{\sigma}_j : K_1 \to \mathbb{C}$ extending σ_j $(1 \le j \le s)$ there is a formula of the form (2) with $x_{\nu,j}\sqrt{-1}$ of degree ≤ 2 over $\tilde{\sigma}_j(K_1)$.

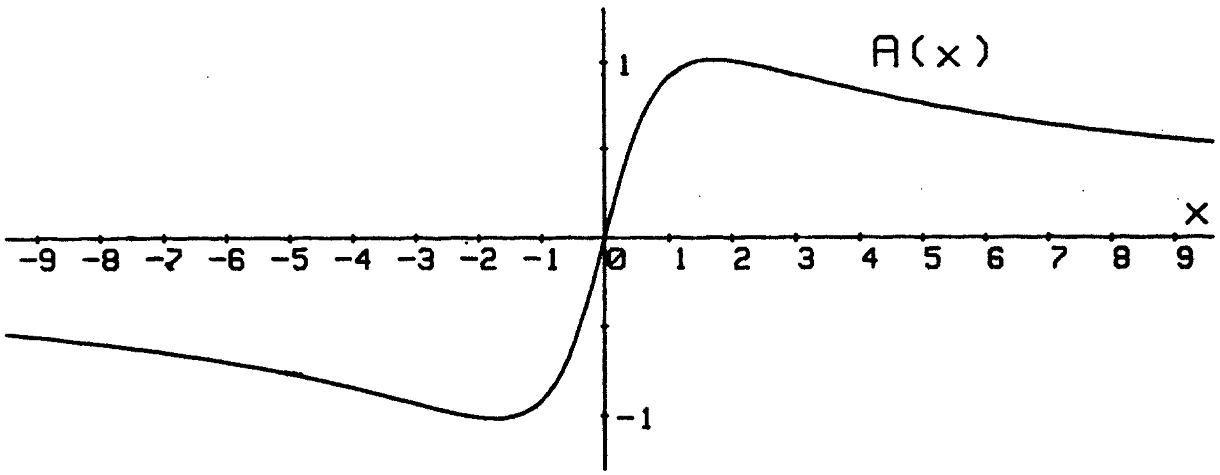


Figure 1

More picturesquely stated, the Klingen-Siegel theorem says that a single transcendental *number*, π^2, suffices to give the contribution of each real place of a field to the value of its zeta-function at $s = 2$, and our result says that a single transcendental *function*, $\pi^2 A(x)$, evaluated at algebraic arguments, suffices to give the contribution of each complex place.

The proof of Theorem 1 will be geometric, involving the interpretation of $\zeta_K(2)$ as the volume of a hyperbolic manifold (the function $A(x)$ is equivalent to the dilogarithm and Lobachevsky functions occurring in the formulas for the volumes of 3-dimensional hyperbolic tetrahedra). Since it is only $\zeta_K(2)$ which can be interpreted geometrically in this way, we did not get a formula for $\zeta_K(2m)$, $m > 1$. However, we conjecture that an analogous result holds here, namely:

CONJECTURE 1: *For each natural number* m *let* $A_m(x)$ *be the real-valued function*

$$(3) \qquad A_m(x) = \frac{2^{2m-1}}{(2m-1)!} \int_0^\infty \frac{t^{2m-1}\,dt}{x \sinh^2 t + x^{-1} \cosh^2 t} .$$

Then the value of $\zeta_K(2m)$ *for an arbitrary number field* K *equals* $\pi^{2m(r+s)}/\sqrt{|D|}$ *times a rational linear combination of products of* s *values of* $A_m(x)$ *at algebraic arguments*

The formulation of this conjecture, and the choice of A_m, are motivated by:

THEOREM 2: *Conjecture 1 holds if* K *is abelian over* $\mathbb{Q}$; *in fact, in this case the arguments* x *can be chosen of the form* $x = \cot\frac{\pi n}{N}$, *where* N *is the conductor of* K (the smallest natural number such that $K \subset \mathbb{Q}(e^{2\pi i/N})$). *For* $m = 1$, *the function defined by (3) agrees with the function* $A(x)$ *in Theorem 1.*

Theorems 1 and 2 and the Siegel-Klingen Theorem show that Conjecture 1 is true if K is totally real (i.e. $s = 0$), if $m = 1$, or if K is abelian, special cases of a sufficiently varied nature to make its truth in general very plausible. The proof of Theorem 2, given in §4, uses routine number-theoretical tools, and it is worth noting that, even for abelian fields, the geometrically proved Theorem 1 gives a stronger statement (for $m = 1$), namely that the arguments of $A(x)$ can be chosen to be of bounded degree over K. Thus, in the simplest case of imaginary quadratic fields ($r=0$, $s=1$), the proof of Theorem 2 gives

$$(4) \qquad \zeta_K(2) = \frac{\pi^2}{6\sqrt{|D|}} \sum_{0<n<|D|} \left(\frac{D}{n}\right) A\left(\cot \frac{\pi n}{|D|}\right),$$

where the arguments of $A(x)$ are of degree (D) over $\mathbb{Q}$, e.g. for $D=-7$ it gives

(5) $$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{\pi^2}{3\sqrt{7}} \left(A(\cot\tfrac{\pi}{7}) + A(\cot\tfrac{2\pi}{7}) + A(\cot\tfrac{4\pi}{7}) \right) ,$$

whereas the proof of Theorem 1 will lead to the formula

(6) $$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{2\pi^2}{7\sqrt{7}} \left(2A(\sqrt{7}) + A(\sqrt{7}+2\sqrt{3}) + A(\sqrt{7}-2\sqrt{3}) \right) ,$$

where now the arguments of $A(x)$, multiplied by $\sqrt{-1}$, are quadratic rather than cubic over K. In this connection we observe that the values of $A(x)$ at algebraic arguments satisfy many non-trivial linear relations over the rational numbers; I know of no direct proof, for instance, of the equality of the right-hand sides of equations (5) and (6).

We will discuss (6) and other examples of Theorem 1 later, after giving its proof.

§2. Proof of Theorem 1. Assume first that $s=1$, i.e. K is a field of degree $r+2$ with r real places and one complex place. Let B be a quaternion algebra over K which is ramified at all real places (i.e. $B\otimes_K\mathbb{R} \cong$ Hamiltonian quaternions for each real completion $\mathbb{R}$ of K), $\mathcal{O}$ an order in B, and Γ a torsion-free subgroup of finite index in the group $\mathcal{O}^1$ of units of $\mathcal{O}$ of reduced norm 1. Then choosing one of the two complex embeddings of K into $\mathbb{C}$ and an identification of $B\otimes_K\mathbb{C}$ with $M_2(\mathbb{C})$ gives an embedding of Γ into $SL_2(\mathbb{C})$ as a discrete subgroup and hence, identifying $SL_2(\mathbb{C})/\{\pm1\}$ with the group of isometries of hyperbolic 3-space $\mathcal{H}_3$, a free and properly discontinuous action of Γ on $\mathcal{H}_3$. The quotient $\mathcal{H}_3/\Gamma$ is smooth and is compact if $B \neq M_2(K)$ (which is automatic if $r > 0$ and can be assumed in any case) and its volume is well-known to be a rational multiple of $\zeta_K(2)/\pi^{2r+2}\sqrt{|D|}$ (see e.g. [7], IV,§1 or [1], 9.1(1)). We therefore have to show that this volume can be expressed as a rational linear combination of values of $A(x)$ at algebraic arguments x.

[The choice of B, $\mathcal{O}$ and Γ plays no role; the reader not familiar with quaternion algebras can take

$$(7)\qquad \Gamma = \left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \middle| \ a, b, c, d \in R\ ,\ a^2+b^2+c^2+d^2=1 \right\} \subset SL_2(\mathbb{C})\ ,$$

where $R \subset K \subset \mathbb{C}$ is the ring of integers of K or a subring of finite index (e.g. the ring $\mathbb{Z}[\alpha]$, where α is one of the two non-real roots of a polynomial f chosen as in the remark following the theorem) and $i = \sqrt{-1}$, corresponding to

$$\mathcal{O} = R + Ri + Rj + Rij \subset B = K + Ki + Kj + Kij \quad (i^2=j^2=-1,\ ij=-ji).$$

With this choice of B, the field K_1 occurring below can be taken to be $K(i)$.]

Choose a quadratic extension K_1 of K which is a splitting field for B, i.e. such that $B\otimes_K K_1 \cong M_2(K_1)$, and choose an embedding $K_1 \subset \mathbb{C}$ extending the chosen complex place of K and an identification of $B\otimes\mathbb{C}$ with $M_2(\mathbb{C})$ extending the isomorphism $B\otimes K_1 \cong M_2(K_1)$. Then $SL_2(K_1)$ is embedded into $SL_2(\mathbb{C})$ as a countable dense subgroup containing the discrete group Γ, and Γ acts on $\mathcal{H}_3$ preserving the dense set of points whose coordinates z, r in the standard representation of $\mathcal{H}_3$ as $\mathbb{C}\times\mathbb{R}_+$ belong to K_1. Hence if we choose a geodesic triangulation of $\mathcal{H}_3/\Gamma$ with sufficiently small simplices, then by moving the vertices slightly to lie on this dense set we can get a new geodesic triangulation whose vertices have coordinates which are algebraic and in fact lie in the chosen splitting field K_1 . To prove the theorem (still for $s=1$), it therefore suffices to show that the volume of a hyperbolic tetrahedron whose four vertices have coordinates belonging to a field $K_1 \subset \mathbf{C}$ can be expressed as a rational linear combination of values of $A(x)$ at arguments x of degree ≤ 4 over K_1 . In fact, we will show that it is a combination of at most 36 such values, with coefficients $\pm\frac{1}{4}$ or $\pm\frac{1}{2}$.

Let, then, $\Delta \subset \mathcal{H}_3$ be a tetrahedron with vertices $P_i = (z_i, r_i) \in K_1 \times (K_1 \cap \mathbb{R})_+^\times \subset \mathbf{C} \times \mathbb{R}_+^\times$ ($i=0,1,2,3$). The geodesic through P_0 and P_1 , continued in the direction from P_0 to P_1 , meets the ideal boundary $\mathbb{P}_1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ of $\mathcal{H}_3$ in a point of $\mathbb{P}_1(K_1)$, and by applying an element of $SL_2(K_1)$ (which does not change the volume of Δ) we may assume that this point is ∞ , i.e. that P_0 is vertically below P_1 . Then Δ is the difference of two tetrahedra with three vertices $P_i \in \mathcal{H}_3$ and one vertex at ∞ (Fig. 2). Such a tetrahedron is bounded by (parts of) three vertical planes and one hemisphere with base on $\mathbf{C} \times 0 \subset \partial(\overline{\mathcal{H}_3})$. Let P be the top point of this hemisphere. Looking down from infinity, we see a triangle and a point P ; drawing the straight lines from P to the vertices and the perpendiculars from P to the side of this triangle decomposes the triangle into six right triangles and the

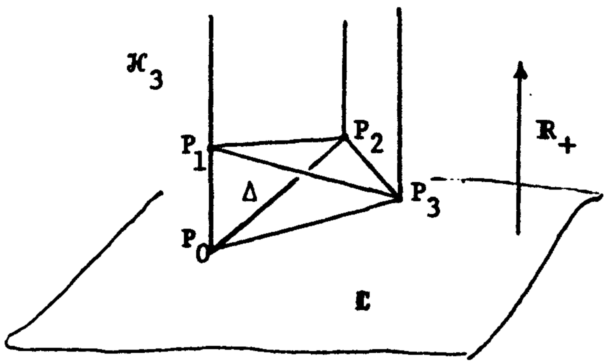


Figure 2

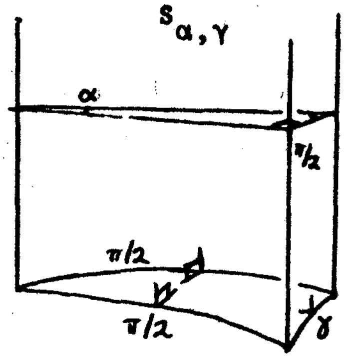


Figure 3

tetrahedron into six tetrahedra of the kind shown in Figure 3 (Fig. 4). The volume of the tetrahedron of Figure 3 is given by the formula

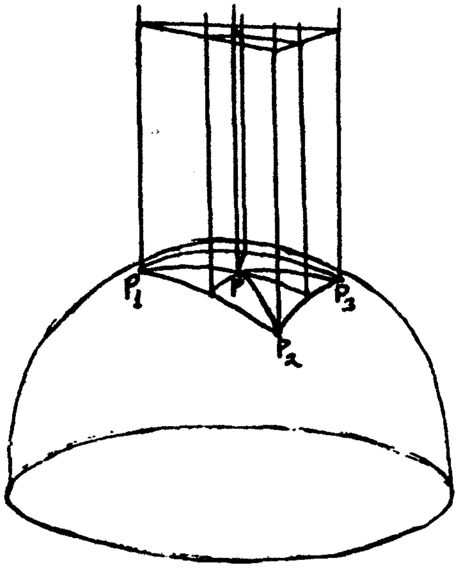


Figure 4

$$(8)\qquad \mathrm{Vol}\,(S_{\alpha,\gamma}) = \frac{1}{4}\left(\Lambda(\alpha+\gamma) + \Lambda(\alpha-\gamma) + 2\,\Lambda(\tfrac{\pi}{2}-\alpha)\right)$$

(cf. Chapter 7 of [6], by Milnor, p.), where $\Lambda(\theta)$ is the "Lobachevsky function" (actually introduced by Clausen in 1832, and discussed extensively in Chapter 4 of [ ]), defined by

$$(9)\qquad \Lambda(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^2} = -\int_0^{\theta}\log|2\sin t|\,dt \ .$$

From

$$\frac{d}{dx}\Lambda(\mathrm{arc\ cot}\ x) = -\frac{1}{1+x^2}\Lambda'(\mathrm{arc\ cot}\ x) = \frac{1}{2}\,\frac{1}{1+x^2}\log\frac{4}{1+x^2}$$

we deduce that

$$(10)\qquad A(x) = 2\,\Lambda(\mathrm{arc\ cot}\ x)\ .$$

Hence (8) is equivalent to

$$(11)\qquad \mathrm{Vol}\,(S_{\alpha,\gamma}) = \frac{1}{8}\left(A\left(\frac{1-ac}{a+c}\right) + A\left(\frac{1+ac}{a-c}\right) + 2A(a)\right)$$

$$(\ a = \tan\alpha\ ,\ c = \tan\gamma\ ),$$

so to complete the proof we need only check that the tangents of α and γ for the particular tetrahedra $S_{\alpha,\gamma}$ occurring in the decomposition of Figure 4 are algebraic and satisfy $a\sqrt{-1} \in K_1$, $c^2 \in K_1$ (so that the three arguments of $A(x)$ in (11), multiplied by $\sqrt{-1}$, are at most quadratic over K_1). This is a question of elementary analytic geometry. Let (Z,R) be the coordinates of the point P in Figure 4. Then the point $(Z,0)$ is at a distance R from each $P_i = (z_i, r_i)$, so

$$|z_i - Z|^2 + r_i^2 = R^2 \quad (i = 1,2,3).$$

This leads to the linear system of equations

$$\begin{pmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{pmatrix} \begin{pmatrix} Z \\ \bar{Z} \\ R^2 - |Z|^2 \end{pmatrix} = \begin{pmatrix} r_1^2 + |z_1|^2 \\ r_2^2 + |z_2|^2 \\ r_3^2 + |z_3|^2 \end{pmatrix} .$$

Since the numbers r_i and z_i belong to K_1, these imply that Z and R^2 belong to K_1. Referring to the picture, we see that the angle $\frac{\pi}{2} - \alpha$ is the argument of $\lambda = (z_j - z_i)/(Z - z_i) \in K_1$ for some i, j, from which $\sqrt{-1}\tan\alpha = \frac{\bar{\lambda}+\lambda}{\bar{\lambda}-\lambda} \in K_1$. We also find $\cos\gamma = \frac{D}{R}$ and hence $\tan^2\gamma = (R^2 - D^2)/D^2$, where D is the distance from Z to the line joining z_i and z_j, and a simple calculation shows that

$$D^2 = -\frac{1}{4}(-Z\bar{z}_i + Z\bar{z}_j + \bar{Z}z_i - \bar{Z}z_j + \bar{z}_i z_j - z_i \bar{z}_j)^2 / |z_i - z_j|^2 \in K_1 ,$$

as claimed. This completes the proof of the theorem for $s = 1$.

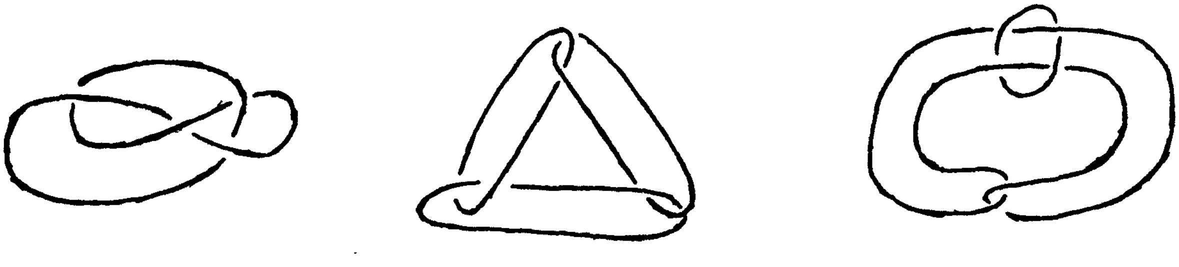
Now let s be arbitrary. We choose B, $\mathcal{O}$ and Γ as before (i.e. $B \neq M_2(K)$ a totally definite quaternion algebra over K, $\mathcal{O}$ an order in B, and $\Gamma \subset \mathcal{O}^1$ of finite index). The embeddings $\sigma_1, \ldots, \sigma_s \colon K \hookrightarrow \mathbb{C}$ give a map $\sigma \colon B \to M_2(\mathbb{C})$ such that $\sigma(\Gamma)$ is a discrete subgroup of $SL_2(\mathbb{C})^s$, and this gives a properly discontinuous, free action of Γ on $\mathcal{H}_3^s$. Let $M = \mathcal{H}_3^s/\Gamma$ denote the quotient; then M is a smooth, compact 3s-dimendional hyperbolic manifold whose volume is a rational multiple of $\zeta_K(2)/\pi^{2r+2s}\sqrt{|D|}$ (loc. cit.). We will show that M can be decomposed as the union (with multiplicities) of sets of the form $\pi(\Delta^{(1)} \times \cdots \times \Delta^{(s)})$, where $\pi: \mathcal{H}_3^s \to M$ is the projection and $\Delta^{(j)} \subset \mathcal{H}_3$ is a hyperbolic tetrahedron each of whose four vertices has both coordinates in $\tilde{\sigma}_j(K_1)$ (K_1 a splitting field of B over K, $\tilde{\sigma}_j$ as in the remark following Theorem 1). Th. by the calculation just given, $\mathrm{Vol}(\Delta^{(j)})$ is a rational linear combination of values $A(x)$ with $x\sqrt{-1}$ quadratic over $\tilde{\sigma}_j(K_1)$, and the desired result will follow.

Since M is compact, we can choose compact sets $F_1,\ldots,F_s \subset \mathcal{H}_3$ so large that $F_1 \times \cdots \times F_s$ contains a fundamental domain for the action of Γ on $\mathcal{H}_3^s$. We can clearly assume that F_j is triangulated by finitely many small tetrahedra $\Delta_a^{(j)}$ whose coordinates lie in the dense subset $\tilde{\sigma}_j(K_1) \times (\tilde{\sigma}_j(K_1) \cap \mathbb{R}_+)$ of $\mathcal{H}_3$; here "small" means that each product $\Delta_a = \Delta_{a_1}^{(1)} \times \cdots \times \Delta_{a_s}^{(s)}$ is mapped isomorphically onto its image in M by π. Hence M is covered by finitely many such products $\pi(\Delta_a)$, and by the principle of inclusion-exclusion

$$\mathrm{Vol}(M) = \sum_a \mathrm{Vol}(\Delta_a) - \sum_{a<b} \mathrm{Vol}(\Delta_a \cap \Delta_b) + \sum_{a<b<c} \mathrm{Vol}(\Delta_a \cap \Delta_b \cap \Delta_c) - \cdots,$$

where we have ordered the multi-indices a in some way. But each intersection $\Delta_a \cap \Delta_b \cap \ldots$ is itself a product $(\Delta_{a_1}^{(1)} \cap \Delta_{b_1}^{(1)} \cap \cdots) \times \cdots \times (\Delta_{a_s}^{(s)} \cap \Delta_{b_s}^{(s)} \cap \cdots)$, and each factor $\Delta_{a_j}^{(j)} \cap \Delta_{b_j}^{(j)} \cap \ldots$ can be further subdivided into small simplices with coordinates in $\eth_{\nu}(K_1)$, giving a decomposition of the type claimed. This completes the proof of Theorem 1.

§3. Numerical Examples. Various examples of arithmetic hyperbolic 3-manifolds with explicit triangulations are given in Thurston's notes [6]. Consider, for instance, the knot shown in Fig. 5(a). It was shown by Gieseking in 1912 that the complement M of this knot in S^3 can be triangulated by two 3-simplices (minus their vertices), the triangulation being such that six tetrahedron edges meet along each of the two 1-simplices of the triangulation. Hence, if the two 3-simplices are given the structure of ideal hyperbolic tetrahedra (= tetrahedra with vertices in $\partial\mathcal{H}_3$) with all dihedral angles equal to 60°, then M acquires a smooth hyperbolic structure with volume $2 \times 3\,\text{Л}(\frac{\pi}{3}) = 3\,A(\frac{1}{\sqrt{3}})$ (cf. (10); we have used the fact, proved in [6], that the volume of an ideal hyperbolic tetraheron with dihedral angles α, β, γ is $\text{Л}(\alpha) + \text{Л}(\beta) + \text{Л}(\gamma)$). On the other hand, Riley showed in 1975 that the same knot complement M has a fundamental group isomorphic to a subgroup Γ



(a) (b) (c)

Figure 5

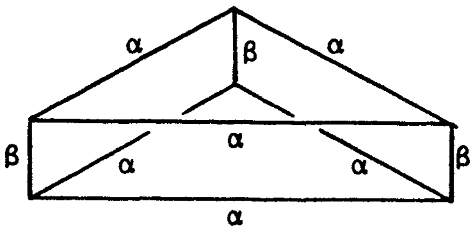
of $PSL_2(R)$ of index 12, where $R = \mathbb{Z} + \mathbb{Z}\,\frac{1+i\sqrt{3}}{2}$ is the ring of integers of $\mathbb{Q}(\sqrt{-3})$, so

$$\mathrm{Vol}(M) = \mathrm{Vol}(\mathcal{H}_3/\Gamma) = 12\,\mathrm{Vol}(\mathcal{H}_3/SL_2R) = 12 \times \frac{3\sqrt{3}}{4\pi^2}\,\zeta_{\mathbb{Q}(\sqrt{-3})}(2).$$

Comparing these formulas, we find

$$\zeta_{\mathbb{Q}(\sqrt{-3})}(2) = \frac{2\pi^2}{3\sqrt{3}}\,\text{Л}(\tfrac{\pi}{3}) = \frac{\pi^2}{3\sqrt{3}}\,A(\frac{1}{\sqrt{3}}).$$

This formula is not too interesting since it agrees with the formula (4) obtained by straight number-theoretical means (indeed, $\zeta_{\mathbb{Q}(\sqrt{-3})}(s)/\zeta(s) = 1 - 1/2^s + 1/4^s - \cdots$, which at $s = 2$ reduces to the series defining $\frac{4}{\sqrt{3}}\,\text{Л}(\frac{\pi}{3})$). However, if we take M instead to be the complement of one of the links is 5(b) or 5(c), then Thurston [6, pp. 6.38, 6.40] shows $\mathrm{Vol}(M) = 6\,\mathrm{Vol}(\mathcal{H}_3/SL_2R)$, where now R is the ring of integers of $\mathbb{Q}(\sqrt{-7})$. On the other hand, for the manifold of 5(b) he gives a decomposition into two pieces of the form



$$\alpha = \cos^{-1}(\frac{1}{2\sqrt{2}}),$$
$$\beta = \pi - 2\alpha,$$

and applying the volume formula on p. 7.16 of [6] we find that each of these pieces has volume

$$2A(\sqrt{7}) + A(\sqrt{7} + \sqrt{12}) + A(\sqrt{7} - \sqrt{12}).$$

Comparing these two formulas (and using the formula for $\mathrm{Vol}(\mathcal{H}_3/SL_2R)$), we obtain equation (6) of the introduction. This time, as we remarked at that point, the result is quite different from the formula (5) obtained number-theoretically; as a numerical check, we have the values

$$A(\sqrt{7}) \cong 0.962673014617 \qquad A(\cot\tfrac{\pi}{7}) \cong 1.004653150540$$
$$A(\sqrt{7}+\sqrt{12}) \cong 0.690148299958 \qquad A(\cot\tfrac{2\pi}{7}) \cong 0.826499033472$$
$$A(\sqrt{7}-\sqrt{12}) \cong -0.837664473558 \qquad A(\cot\tfrac{4\pi}{7}) \cong -0.307298022053$$

so that both (5) and (6) give the value $\zeta_{\mathbb{Q}(\sqrt{-7})}(2) \cong 1.89484144897$ to twelve places. (We shall say in a moment how to calculate $A(x)$ numerically.) Alternatively, we can compute $\zeta_{\mathbb{Q}(\sqrt{-7})}(2)$ independently and check (5) and (6) directly, rather than against one another. To do this, we note that $\mathbb{Q}(\sqrt{-7})$ has class number 1 and hence the norms of ideals are just the values of the norm from $x^2+xy+2y^2$, so

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(s) = \sum_{n=1}^{\infty} \frac{r(x^2+xy+2y^2,n)}{n^s},$$

where $r(Q,n)$ for a binary quadratic form Q is defined by

$$r(Q,n) = \#\{(x,y)\in\mathbb{Z}^2/\{\pm 1\} \mid Q(x,y) = n\}.$$

The series $\sum_{n=1}^{\infty} r(Q,n)/n^s$ is called an Epstein zeta-function and can be calculated by a well-known expansion which at $s=2$ becomes

$$(12)\quad \sum_{n=1}^{\infty} \frac{r(ax^2+bxy+cy^2,n)}{n^2} =$$

$$\frac{\pi^4}{90a^2} + 4\pi\zeta(3)\frac{a}{\delta^3} + \frac{8\pi}{\delta^2}\sum_{n=1}^{\infty}\left(\pi n + \frac{a}{\delta}\right)\sigma_{-3}(n)\, e^{\frac{-\pi n\delta}{a}} \cos\frac{\pi n b}{a}$$

with $\delta = \sqrt{4ac-b^2}$, $\sigma_{-3}(n) = \sum_{\substack{d\mid n\\ d\ge 1}} d^{-3}$, $\zeta(3) = \sum_{d\ge 1} d^{-3} = 1.202056903\ldots$.

The series converges exponentially, and four terms of (12) suffice to compute $\zeta_{\mathbb{Q}(\sqrt{-7})}(2)$ to twelve places.

Finally, we consider the field $K = \mathbb{Q}(\sqrt{3+2\sqrt{5}})$ of degree 4 with $r = 2$, $s = 1$, $|D| = 275$ (this is the smallest discriminant for this r and s). Taking an appropriate Γ here gives a quotient) $\mathcal{H}_3/\Gamma$ which can be triangulated by a single tetrahedron Δ with angles as shown in Figure 6, while

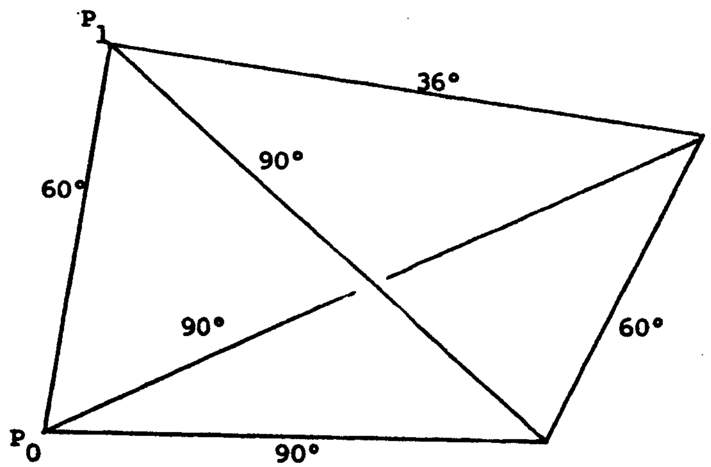


Figure 6

the arithmetic description of Γ leads to

$$\mathrm{Vol}(\mathcal{H}_3/\Gamma) = \frac{275^{3/2}}{2^7\pi^6}\,\zeta_K(2).$$

This example, due to Thurston, is discussed in Borel [1], p. 30. The group Γ has torsion, so $\mathcal{H}_3/\Gamma$ is only an "orbifold" rather than a smooth hyperbolic manifold; it is of special interest because it has the smallest known volume of any hyperbolic orbifold, arithmetic or otherwise. We can compute this volume either number-theoretically or topologically. The number-theoretical method uses the relation of K to the genus field of $\mathbb{Q}(\sqrt{-55})$

(we do not elaborate on this); this gives

(13)

$$\zeta_K(s) = \zeta_{\mathbb{Q}(\sqrt{5})}(s) \times \sum_{n=1}^{\infty} \frac{r(x^2+xy+14y^2,n) - r(4x^2+3xy+4y^2,n)}{n^s},$$

and since $\zeta_{\mathbb{Q}(\sqrt{5})}(2) = \frac{2\pi^4}{75\sqrt{5}}$ this permits us to calculate $\zeta_K(2)$ easily using equation (12) (in fact, very easily, since $e^{-\pi\sqrt{55}} < 10^{-10}$, so the series in (12) is negligible for $x^2 + xy + 14y^2$ and extremely rapidly convergent for $4x^2 + 3xy + 4y^2$). We find

$$\zeta_K(2) \cong \frac{2\pi^4}{75\sqrt{5}} (1.1193564009 - 0.2122647724) = 1.053742217, \text{ Vol} = 0.0390502856.$$

On the other hand, we can compute the volume of Δ geometrically by the method used in the proof of Theorem 1. If we choose P_0, P_1 as in Figure 6 and extend P_0P_1 to ∞ as in Figure 2, then because of the many right angles in Δ we can subdivide Δ into four simplices $S_{\alpha,\gamma}$ of the sort shown in Figure 3 rather than the usual twelve. Their angles can be computed in a straightforward way, and we find

$$\Delta = S_{\frac{\pi}{3},\theta} - S_{\beta,\theta-\frac{\pi}{3}} - S_{\frac{\pi}{6},\frac{\pi}{5}} + S_{\frac{\pi}{6}+\beta,\frac{\pi}{5}}$$

with

$$\theta = \operatorname{arc\,cot}\left(\frac{\sqrt{-3+2\sqrt{5}}}{3}\right), \qquad \beta = \operatorname{arc\,cot}\left(\frac{3+\sqrt{5}}{2}\sqrt{3} + \frac{7+3\sqrt{5}}{2\sqrt{2\sqrt{5}-3}}\right).$$

Now equation (8) gives a formula for $\text{Vol}(\Delta)$ as a sum of 12 values of $A(x)$ at (complicated!) algebraic arguments. Computing these values by the method given below, we find $\mathrm{Vol}(\Delta) \cong .039050286$, in agreement with (13).

We have discussed this last example in some detail because it shows how complicated the formula promised by Theorem 1 can be, even when the geometry of the hyperbolic manifold is very simple (in this case triangulated by a single, and very special, hyperbolic tetrahedron). In general, it is very hard to find examples of arithmetic hyperbolic manifolds for which one has both a good arithmetic and geometric description. Thus it is clear that getting actual formulas for $\zeta_K(2)$ by this method is usually impractical, so that, unless an arithmetical proof giving an explicit formula of the form (2) is found, Theorem 1 must be considered as of mostly theoretical interest.

Finally, we must say how to calculate $A(x)$, or equivalently (by (10)), the Lobachevsky function $Л(\theta)$. Neither the sum nor the integral in (9) are very convenient for numerical work, but there is a very rapidly convergent method. By periodicity, we can assume $|\theta| < \frac{\pi}{2}$. Then $Л(\theta)$ is given by

$$(14) \quad \frac{1}{\pi} Л(\pi t) = t(2N+1-\log|2\sin \pi t|) - \sum_{n=1}^{N} n \log\frac{n+t}{n-t} - \sum_{k=1}^{\infty}\Big(\zeta(2k) - \sum_{n=1}^{N}\frac{1}{n^{2k}}\Big)\frac{t^{2k+1}}{k+\frac{1}{2}}$$

for any $N \geq 0$. This formula, which is easily proved by differentiation, is a special case of the results of [2]. The series converges for $|t| \leq N+1$ and therefore converges very rapidly for $|t| \leq \frac{1}{2}$ and quite modest N. Taking $N = 4$ and breaking off the series at $k = 4$, we get the formula, suitable for use on a programmable pocket calculator,

$$\frac{1}{\pi}\,Л(\pi t) = t(9 - \log|2\sin\pi t|) - \sum_{n=1}^{4}\left(c_n t^{2n+1} + n\log\frac{n+t}{n-t}\right) + \varepsilon$$

with

n	1	2	3	4
c_n	.147548637158	.00142852188	.00002919407	.00000076258

and $|\varepsilon| < 1.1\ 10^{-11}$ for $|t| \le \frac{1}{2}$.

§4. <u>Proof of Theorem 2</u>. We begin by proving the special case (4), even though this is well-known (see e.g., Milnor [6], p.), since it illustrates the general case. Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field with discriminant $D < 0$ and $\chi(n) = (\frac{D}{n})$ the associated character. Then $\zeta_K(s)$ factors as $\zeta(s)L(s,\chi)$, where $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$, so $\zeta_K(2) = \frac{\pi^2}{6} L(2,\chi)$. The function $\chi(n)$ is odd and periodic with period $|D|$, so it has a Fourier sine expansion, well known to be

$$\chi(n) = \frac{1}{\sqrt{|D|}} \sum_{0<k<|D|} \chi(k) \sin \frac{2\pi kn}{|D|} .$$

Hence, by (9) and (10),

$$L(2,\chi) = \frac{2}{\sqrt{|D|}} \sum_{0<k<|D|} \chi(k) Л(\frac{\pi k}{|D|}) = \frac{1}{\sqrt{|D|}} \sum_{0<k<|D|} \chi(k) A(\cot \frac{\pi k}{|D|}) .$$

Now let K be an arbitrary abelian field. Then $\zeta_K(s)$ is the product of $[K:\mathbb{Q}]$ L-series $L(s,\chi)$, where the χ are primitive Dirichlet characters whose conductors f divide the conductor N of K. If χ is an even character, then $\chi(n)$ has a Fourier expansion

$$\chi(n) = \frac{1}{G_\chi} \sum_{k=1}^{f} \chi(k) \cos \frac{2\pi kn}{f}$$

where G_χ (defined by setting $n = 1$ in this formula) is a certain algebraic integer, the Gauss sum attached to χ. Therefore

$$L(2m,\chi) = \frac{\pi^{2m}}{G_\chi} \sum_{k=1}^{f} \chi(k) b_{m,k,f} \qquad (\chi \text{ even})$$

where $b_{m,k,f} = \pi^{-2m} \sum_{n=1}^{\infty} \frac{1}{n^{2m}} \cos \frac{2\pi kn}{f}$, which is known to be a rational number $(b_{m,k,f} = \frac{2^{2m-1}}{(2m)!} B_{2m}(\frac{k}{f})$, where B_r denotes the r-th Bernoulli polynomial). If χ is an odd character, then instead

$$\chi(n) = \frac{i}{G_\chi} \sum_{k=1}^{f-1} \chi(k) \sin \frac{2\pi kn}{f} ,$$

(where again G_χ is defined by setting $n = 1$). But

$$\begin{aligned} \frac{(2m-1)!}{2^{2m-1}} \sum_{n=1}^{\infty} \frac{\sin 2n\theta}{n^{2m}} &= 2 \sum_{n=1}^{\infty} \sin 2n\theta \int_0^{\infty} e^{-2nt}\, t^{2m-1}\, dt \\ &= 2 \int_0^{\infty} \operatorname{Im}\Big(\sum_{n=1}^{\infty} e^{2in\theta-2nt}\Big) t^{2m-1}\, dt \\ &= \sin 2\theta \int_0^{\infty} \frac{t^{2m-1}\,dt}{\cosh 2t - \cos 2\theta} \\ &= \int_0^{\infty} \frac{t^{2m-1}\,dt}{\cosh^2 t \tan\theta - \sinh^2 t \cot\theta} , \end{aligned}$$

and comparing this with the definition of $A_m(x)$ (eq.(3)) we find

$$A_m(\cot\theta) = \sum_{n=1}^{\infty} \frac{\sin 2n\theta}{n^{2m}}$$

(which in view of (9) and (10) proves that $A_1 = A$) and

$$L(2m,\chi) = \frac{i}{G_\chi} \sum_{k=1}^{f-1} \chi(k) A_m(\cot \frac{\pi k}{f}) \qquad (\chi \text{ odd}).$$

Since K is abelian, it is either totally real $(r = [K:\mathbb{Q}]$, $s = 0)$ or totally imaginary $(r = 0,\ s = \frac{1}{2}[K:\mathbb{Q}])$. In the first case all of the χ are even, so

$$\zeta_K(2m) = \frac{\pi^{2m[K:\mathbf{Q}]}}{\prod_\chi G_\chi} \prod_\chi \Big(\sum_{k=1}^{f_\chi} \chi(k) b_{m,k,f_\chi} \Big) ,$$

and this has the form $\frac{\pi^{2mr}}{\sqrt{|D|}} \times$ (rational number) because $\prod_\chi G_\chi = \sqrt{D}$, $D > 0$, and the set of χ is closed under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. (We could also have appealed to the Klingen-Siegel theorem.) In the second case half of the χ are even and half are odd, so

$$\zeta_K(2m) = \frac{\pi^{2ms} i^s}{\prod_\chi G_\chi} \prod_{\chi \text{ even}} \Big(\sum_{k=1}^{f_\chi} \chi(k) b_{m,k,f_\chi} \Big) \prod_{\chi \text{ odd}} \Big(\sum_{k=1}^{f_\chi - 1} \chi(k) A_m(\cot \frac{\pi k}{f_\chi}) \Big).$$

The factor in front equals $\pi^{2ms}/\sqrt{|D|}$ because $\prod G_\chi = \sqrt{D}$ and $(-1)^s D > 0$; the second factor is rational for the same reason as before, and for the same reason the third factor is a rational (in fact, integral) linear combination of products of s values of $A_m(x)$ at arguments $x = \cot \frac{\pi n}{N}$. This completes the proof.

§5. Partial zeta-functions and decomposition of the volume. The zeta-function $\zeta_K(s)$ splits up naturally into h summands $\zeta_K(\mathcal{A},s)$, where h is the class number of K and for each ideal class $\mathcal{A}$ the *partial zeta-function* $\zeta_K(A,s)$ is defined as $\sum_{\mathfrak{a}\in\mathcal{A}} N(\mathfrak{a})^{-s}$. From a number-theoretical point of view, these partial zeta-functions are just as good as Dedekind zeta-functions, so it is natural to make

CONJECTURE 2: *Conjecture 1 remains true with* $\zeta_K(2m)$ *replaced by* $\zeta_K(\mathcal{A},2m)$ *for any ideal class* $\mathcal{A}$ *of* K.

This conjecture can be verified in some cases. For instance, if $K = \mathbb{Q}(\sqrt{D})$ is an imaginary quadratic field with class number 2, then the theory of genera gives

$$\zeta_K(\mathcal{A}_0,s) + \zeta_K(\mathcal{A}_1,s) = \zeta_K(s) = \zeta(s)L(s,\chi_D) ,$$

$$\zeta_K(\mathcal{A}_0,s) - \zeta_K(\mathcal{A}_1,s) = L(s,\chi_{D_1})L(s,\chi_{D_2}) ,$$

where $\mathcal{A}_0$ and $\mathcal{A}_1$ denote the trivial and non-trivial ideal classes and $D_1 > 0 > D_2$ are fundamental discriminants with $D_1 \cdot D_2 = D$; the proof of Theorem 2 shows that for $s = 2m$ the right-hand side of both expressions is $\pi^{2m}|D|^{-\frac{1}{2}}$ times a rational linear combination of numbers $A_m(\cot\frac{\pi n}{|D|})$. Similar formulas hold for any imaginary quadratic field with one class per genus. A less trivial example is provided by the field $\mathbb{Q}(\sqrt{-55})$, whose class group is cyclic of order 4 ; here we can verify Conjecture 2 for $m = 1$ using eq. (13).

In the proof of Theorem 1, we obtained $\zeta_K(2)$ as (essentially) the volume of $\mathcal{H}_3^s/\Gamma$, where Γ is a torsion-free group without paraboli elements contained in a totally definite quaternion algebra over K . However, the proof works even in the presence of elliptic or parabolic elements and for quaternion algebras not ramified at the real places of K , except that then we have to take quotients of $\mathcal{H}_2^t \times \mathcal{H}_3^s$ $(0 \le t \le r)$ and these may be non-compact. In particular, we can take $\Gamma = SL_2(\mathcal{O}_K)$ acting on $\mathcal{H}_2^r \times \mathcal{H}_3^s$ (Hilbert modular group), in which case the quotient X has h cusps, but still has finite volume given as a simple multiple of $\zeta_K(2)$ (cf. [1], 7.4(1)). The fact that X has exactly the same number of cusps as the number of summands $\zeta_K(\mathcal{A},2)$ into which $\zeta_K(2)$ naturally decomposes suggests a possible geometric interpretation of Conjecture 2 for $m=1$: it may be possible to break up X into h pieces, each containing one cusp, in such a way that the volumes of the individual pieces are proportional to the $\zeta_K(A,2)$; then if the pieces can be triangulated by simplices with algebraic coordinates, Conjecture 2 follows. There are in fact various natural decompositions of X into h neighborhoods of cusps (these will be described explicitly in the next section for the case $r=0$, $s=1$), but I have not been able to ascertain whether any of them gives the right volumes.

§6. Geometrical decomposition of $\zeta_{\mathbb{Q}(\sqrt{-d})}(2)$. In this section we will prove the following sharpening of Theorem 1 for imaginary quadratic fields.

THEOREM 3. *Let* K *be an imaginary quadratic field of discriminant* $-d$. *Then* $\zeta_K(2)$ *can be written as a finite sum*

$$\zeta_K(2) = \frac{\pi^2}{2d^{3/2}} \sum_\nu \left(A\left(\frac{1-a_\nu c_\nu}{a_\nu + c_\nu}\right) + A\left(\frac{1-a_\nu c_\nu}{a_\nu - c_\nu}\right) + 2A(a_\nu) \right)$$

with $a_\nu \in \frac{1}{\sqrt{d}}\mathbb{Q}$, $c_\nu^2 \in \mathbb{Q}$.

Proof. We will describe geometric decompositions of $X = \mathcal{H}_3/SL_2(\mathcal{O}_K)$ into h pieces, each of which is in a canonical way a union of finitely many tetrahedra $S_{\alpha,\gamma}$ as in Figure 3 with α, γ satisfying

$$(15) \qquad \tan\alpha \in \frac{1}{\sqrt{d}}\mathbb{Q} \ , \qquad \tan^2\gamma \in \mathbb{Q} \ .$$

In view of equation (11) and the formula $\mathrm{Vol}(X) = \frac{d^{3/2}}{4\pi^2}\zeta_K(2)$, this will prove the theorem. The decomposition of X will depend on the choice of positive weights $C_A \in \mathbb{Q}$ for each ideal class A of K; since only the ratios of the C_A matter, we will normalize by taking $C_{[\mathcal{O}_K]} = 1$. The cusps of the action of $\Gamma = SL_2(\mathcal{O}_K)$ on H_3 are the points of $\mathbb{P}^1(K) \subset \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} = \partial H_3$, and the Γ-equivalence classes of cusps are mapped bijectively onto the ideal class group Cl_K by sending $\kappa = (a:b) \in \mathbb{P}^1(K)$ to the ideal class of the ideal (a,b) (greatest common divisor of a, b). We write $[\kappa]$ for the Γ-equivalence class of κ or for the corresponding element of Cl_K. If $P = (z,r) \in \mathbb{C} \times \mathbb{R}_+ = H_3$, we set

$$d(P,\kappa) = C_{[\kappa]}\, N((a,b))^{-1}\, \frac{|bz-a|^2 + |b|^2 r^2}{r} \ ,$$

the "distance" from the point P to the cusp $\kappa = (a:b)$ (if $\kappa = \infty$ this is simply $\frac{1}{r}$). This is clearly well-defined (i.e. independent of the choice of a and b) and satisfies the invariance property $d(\gamma P, \gamma\kappa) = d(P,\kappa)$ for $\gamma \in \Gamma$. It follows that we have a Γ-invariant decomposition

$$H_3 = \bigcup_{\kappa \in \mathbb{P}^1(K)} Y_\kappa \ , \qquad Y_\kappa = \{ P \mid d(P,\kappa) \le d(P,\lambda) \quad \forall\, \lambda \in \mathbb{P}^1(K) \} ,$$

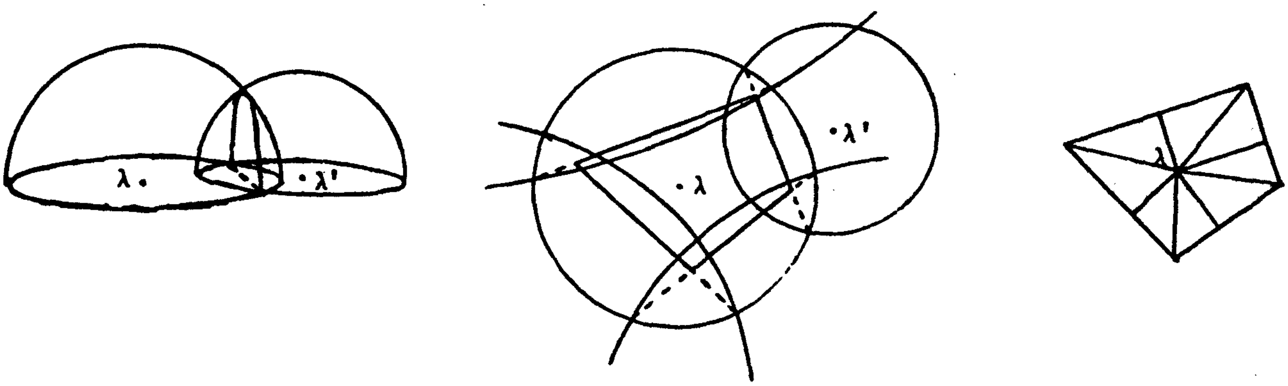
the union being disjoint except for the boundaries of the Y_κ. The Γ-invariance implies that the image of Y_κ in X depends only on $[\kappa]$ and equals Y_κ/Γ_κ (Γ_κ = stabilizer of κ in Γ):

$$X = \bigcup_{A \in Cl_K} X_A \ , \qquad X_{[\kappa]} \cong Y_\kappa/\Gamma_\kappa \ .$$

We will now describe the geometry of a typical region Y_κ and show that Y_κ/Γ_κ is a union of finitely many $S_{\alpha,\gamma}$ subject to (15). (The method of finding a fundamental domain for H_3/Γ we are in the process of describing goes back to Picard Hurwitz and Bianchi and has been given by several other authors.)

We first transform by an element of $SL_2(K)$ to map κ to infinity. For each $\lambda \in K$ the set $H_\lambda = \{P \in H_3 \mid d(P,\kappa) = d(P,\lambda)\}$ is a hemisphere with center λ

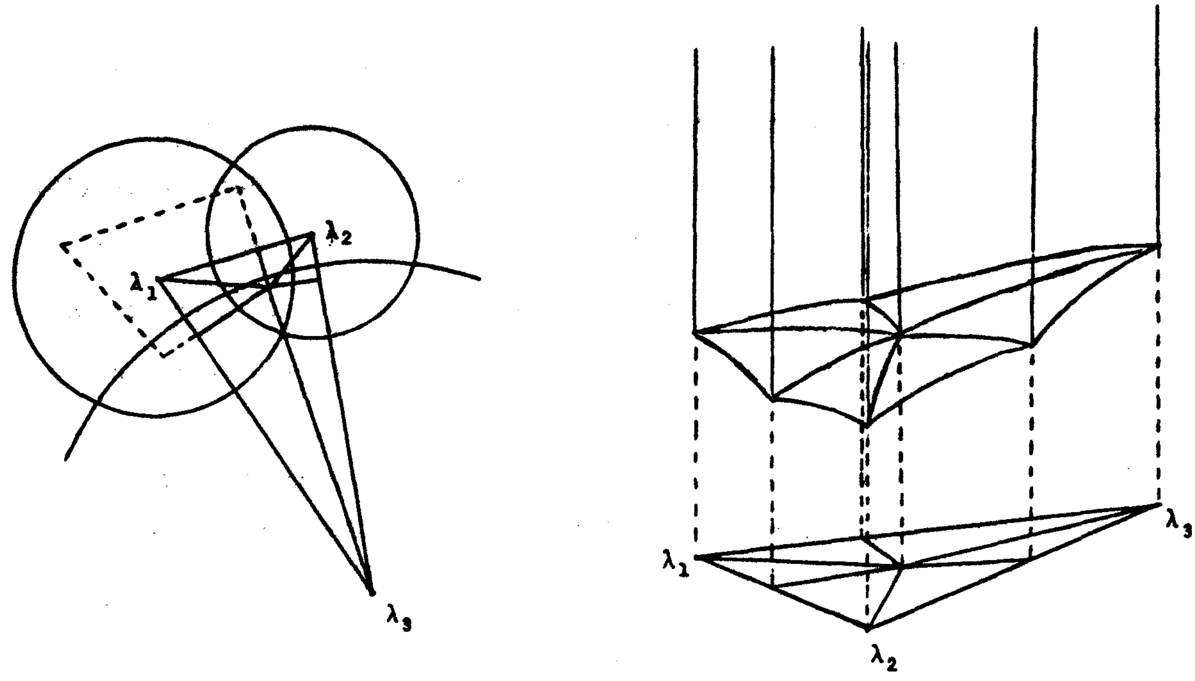
and radius $r(\lambda)$ satisfying $r(\lambda)^2 \in \mathbb{Q}$ (because we chose $C_A \in \mathbb{Q}$); Y_κ is the part of H_3 lying above all of these hemispheres. The stabilizer Γ_κ is a free rank 2 module contained in $K \subset \mathbb{C}$ (actually a fractional ideal in the class A^{-2}, where $A \leftrightarrow [\kappa]$) and acting on H_3 by translations; this action preserves $\bigcup_\lambda H_\lambda$ and there are only finitely many λ modulo Γ_κ for which H_λ contributes to ∂Y_κ. Any two H_λ meet, if at all, along a vertical semicircle (Fig. 7a). Looking from ∞, we see the hemispheres as circles and their intersections as the line segments joining the two intersection points of two circles. Thus the part of H_λ contributing to ∂Y_κ looks from above like a polygon (Fig. 7b) and we have a decomposition of $\mathbb{C}/\Gamma_\kappa$ into finitely many such polygons and of Y_κ/Γ_κ into cylinders whose cross-section is a polygon and whose base is part of a hemisphere. Connecting the center λ of H_λ by line segments to the vertices and by perpendiculars to the sides of the corresponding polygon (cf. Fig. 7c) gives a decomposition of each n-gon into $2n$ right triangles and a decomposition



(a) (b) (c)

Fig. 7

of the corresponding cylinder into $2n$ standard tetrahedra $S_{\alpha,\gamma}$; we will give formulas for α and γ implying (15) in a moment. We have now given a triangulation of the torus $\mathbb{C}/\Gamma_\kappa$ into finitely many right triangles; these can be recombined 6 at a time (Fig. 8a) to give a triangulation of $\mathbb{C}/\Gamma_\kappa$ with vertices at the λ (this triangulation is dual to the original decomposition of $\mathbb{C}/\Gamma_\kappa$ into polygons centered at the λ) and a corresponding decomposition of Y_κ/Γ_κ



(a) (b)

Fig. 8

into "standard pieces" bounded by three vertical sides and six hemispherical right triangles (Fig. 8b).* Each such "standard piece" is described by six positive real numbers: $A_{ij} = |\lambda_i - \lambda_j|^2$ $(1 \le i < j \le 3)$, the squares of the lengths of the sides of the triangle, and $a_i = r(\lambda_i)^2$ $(1 \le i \le 3)$, the squares of the radii of the hemispheres; it is a union of six standard tetrahedra $S_{\alpha,\gamma}$ and its volume is given by

$$F(A_{23},A_{13},A_{12};a_1,a_2,a_3) \;=\; \sum_{\{i,j,k\}=\{1,2,3\}} \mathrm{Vol}\,(S_{\alpha_{ijk},\gamma_{ijk}}) \;,$$

(16) $$\tan\alpha_{123} \;=\; \frac{A_{12}(A_{13}+A_{23}-A_{12}+a_1+a_2-a_3) + (A_{23}-A_{13})(a_1-a_2)}{\Delta(A_{12}+a_1-a_2)} \;,$$

$$\tan\gamma_{123} \;=\; \frac{A_{12}+a_1+a_2}{2\sqrt{A_{12}a_1}} \;,$$

$$\Delta \;=\; \sqrt{(A_{12}+A_{13}+A_{23})^2 - 2(A_{12}^2+A_{13}^2+A_{23}^2)} \;,$$

where $\mathrm{Vol}(S_{\alpha,\gamma})$ is given by (11). Note that

$$\Delta \;=\; 4\times\text{area of the triangle } \lambda_1\lambda_2\lambda_3 \;=\; 2\,\mathrm{Im}(\lambda_1\bar\lambda_2+\lambda_2\bar\lambda_3+\lambda_3\bar\lambda_1) \;,$$

so that in our case $\lambda_i \in K$ we have $\Delta \in \mathbb{Q}\cdot\sqrt{d}$, and since the a_i and A_{ij}

* It may happen that λ in Fig. 7b falls outside the polygon or that the central point in Fig. 8b falls outside the triangle $\lambda_1\lambda_2\lambda_3$, in which case

are rational this proves (15). We have thus proved even more than Theorem 3: $\frac{d^{3/2}}{4\pi^2}\zeta_K(2)$ can be written not only as a finite combination of $S_{\alpha,\gamma}$ with α, γ satisfying (15) but as a finite sum of the function F defined by (16) with arguments A_{ij}, $a_i \in \mathbb{Q}$ and $\Delta \in \mathbb{Q}\sqrt{d}$. (Since F is homogeneous of degree 0 in its 6 arguments, we can even take A_{ij}, a_i and $\Delta/\sqrt{d}$ in $\mathbb{Z}$.) Moreover, this decomposition is canonical if $h=1$ and depends only on the choice of the C_A in general.

We end with some examples. For $d=7$ we have $h=1$, so there is only one region $Y_K/\Gamma_K = Y_\infty/\mathcal{O}_K$; the corresponding decomposition is shown in Fig. 9 and gives the formula

$$\frac{7^{3/2}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \mathrm{Vol}\ X = 2F(2,2,1;1,1,1)$$

$$\left(= 2\ \mathrm{Vol}\ S_{\tan^{-1}\frac{3}{\sqrt{7}},\frac{\pi}{3}} + 4\ \mathrm{Vol}\ S_{\tan^{-1}\frac{1}{\sqrt{7}},\frac{\pi}{4}}\right).$$

For $K=\mathbb{Q}(\sqrt{-23})$ we have $h=3$. Choosing $C_A = 1$ for all three ideal classes gives the triangulations of $\mathbb{C}/\Gamma_K$ shown in Fig. 10 for the principal class A_0 and a non-principal ideal class A_1. This gives

$$\begin{aligned}\mathrm{Vol}\,(X_{A_0}) &= 2F(1,6,6;4,2,2)+2F(4,3,2;2,2,1)+2F(1,8,6;4,2,2)\\ &= .609313\ldots + .971546\ldots + .637795\ldots \approx 2.2186552639,\\ \mathrm{Vol}\,(X_{A_1}) &= 2F(2,3,4;2,2,1)+2F(3,2,6;2,2,1)\\ &= .979093\ldots + 1.136175\ldots \approx 2.1152684701,\end{aligned}$$

not the same as the values

$$\frac{23^{3/2}}{4\pi^2}\zeta_{A_0}(2) \approx 3.4066738851, \qquad \frac{23^{3/2}}{4\pi^2}\zeta_{A_1}(2) \approx 1.5212591595$$

obtained using (12). Thus the hope expressed in §5 is not fulfilled for the geometric decomposition of X corresponding to the obvious choice $C_A = 1$. Another natural choice is

$$C_A = \min\{N\mathfrak{b} \mid \mathfrak{b}\text{ integral},\ \mathfrak{b}\in A^{-1}\}$$

which corresponds to

$$Y_\infty = \{ (z,r) \in H_3 \mid |bz-a|^2 + |b|^2 r^2 \geq 1 \quad \forall\ a,b \in \mathcal{O}_K ,\ \text{not both } 0 \}$$

$$= \{ (z,r) \mid |bz-a|^2 + |b|^2 r^2 \geq N(\mathfrak{a}) \ \text{ for all principal ideals } \mathfrak{a} \text{ of } K \text{ and all } a,b \in \mathfrak{a} ,\ \text{not both } 0 \}$$

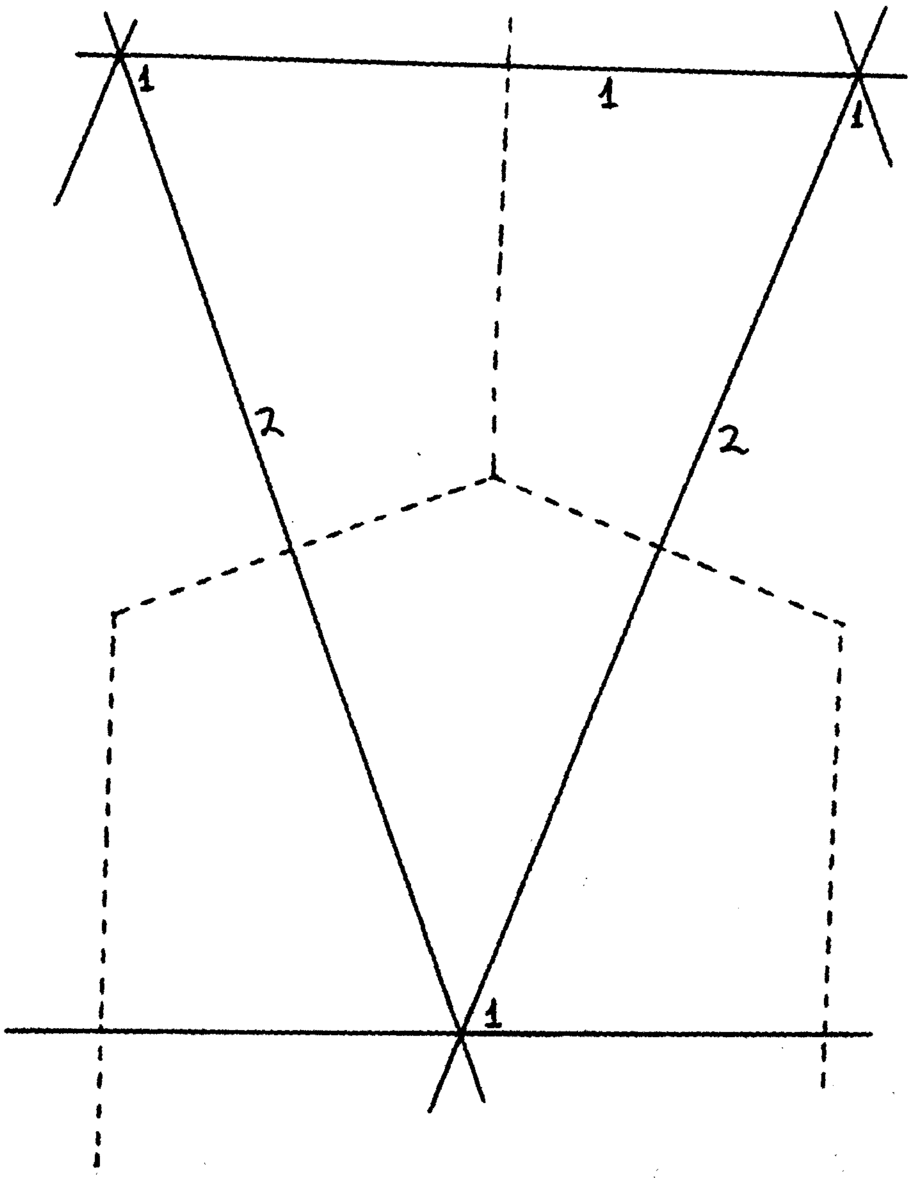
(the choice $C_A = 1$ for all A corresponds to Y_∞ defined by the same formula but without the word "principal"). Here we find the decompositions of $\mathbb{C}/\Gamma_K$ shown in Figure 11 for A_0 and A_1 and the corresponding volumes

$$\begin{aligned} \mathrm{Vol}(X_{A_0}) &= 4F(1,8,12;8,2,1)+2F(1,6,6;4,1,1)+2F(1,64,72;64,8,8) \\ &\quad +2F(1,8,12;16,8,8)+2F(2,3,4;4,4,2)+2F(8,9,8;8,8,1) \\ &= .958015\ldots + 1.024692\ldots + .190774\ldots + .112422\ldots \\ &\quad + .373173\ldots + .704497\ldots \approx 3.3635757982 , \\ \mathrm{Vol}(X_{A_1}) &= F(2,3,4;2,2,2) \approx 1.5428082030 , \end{aligned}$$

again differing (though this time by very little!) from the zeta-values. As a check, we can verify that the equation

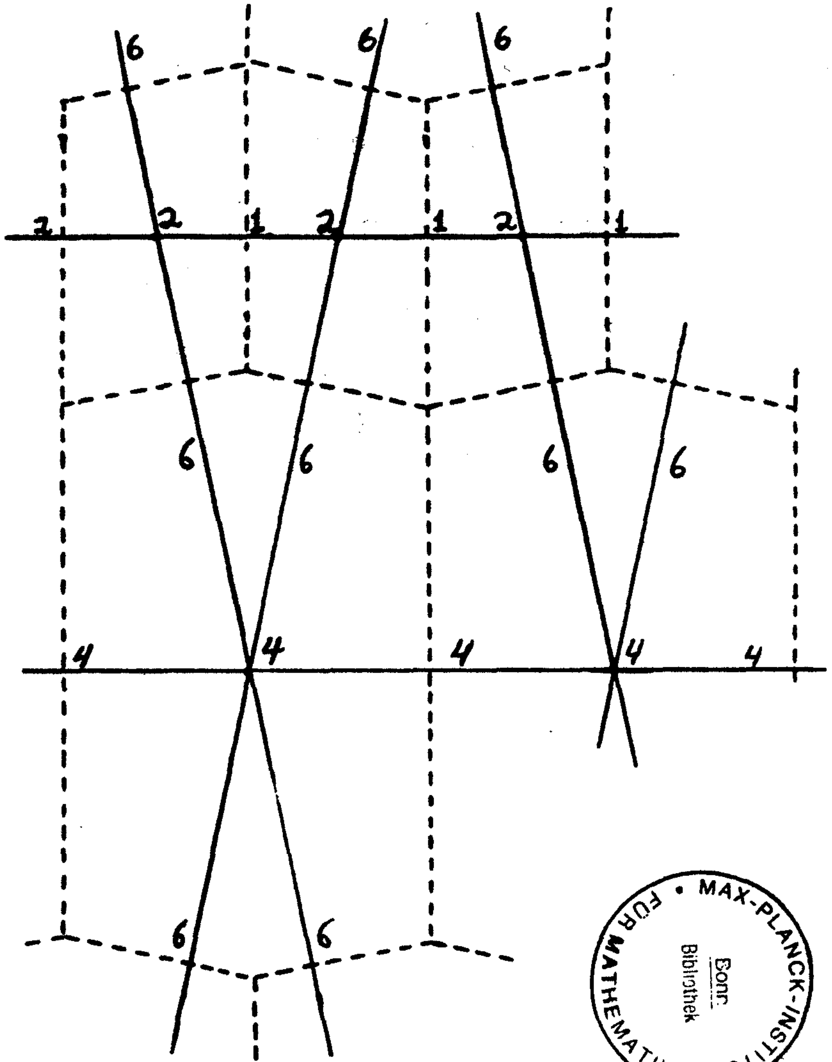
$$\mathrm{Vol}(X_{A_0}) + 2\,\mathrm{Vol}(X_{A_1}) = \frac{23^{3/2}}{4\pi^2}\zeta_{A_0}(2) + 2\,\frac{23^{3/2}}{4\pi^2}\zeta_{A_1}(2)$$

holds numerically for both decompositions described. It would be of some interest to compute the unique real number C_{A_1} (holding $C_{A_0} = 1$ fixed) making $\mathrm{Vol}(X_{A_i}) = \frac{23^{3/2}}{4\pi^2}\zeta_{A_i}(2)$ and look whether it appears to be rational.

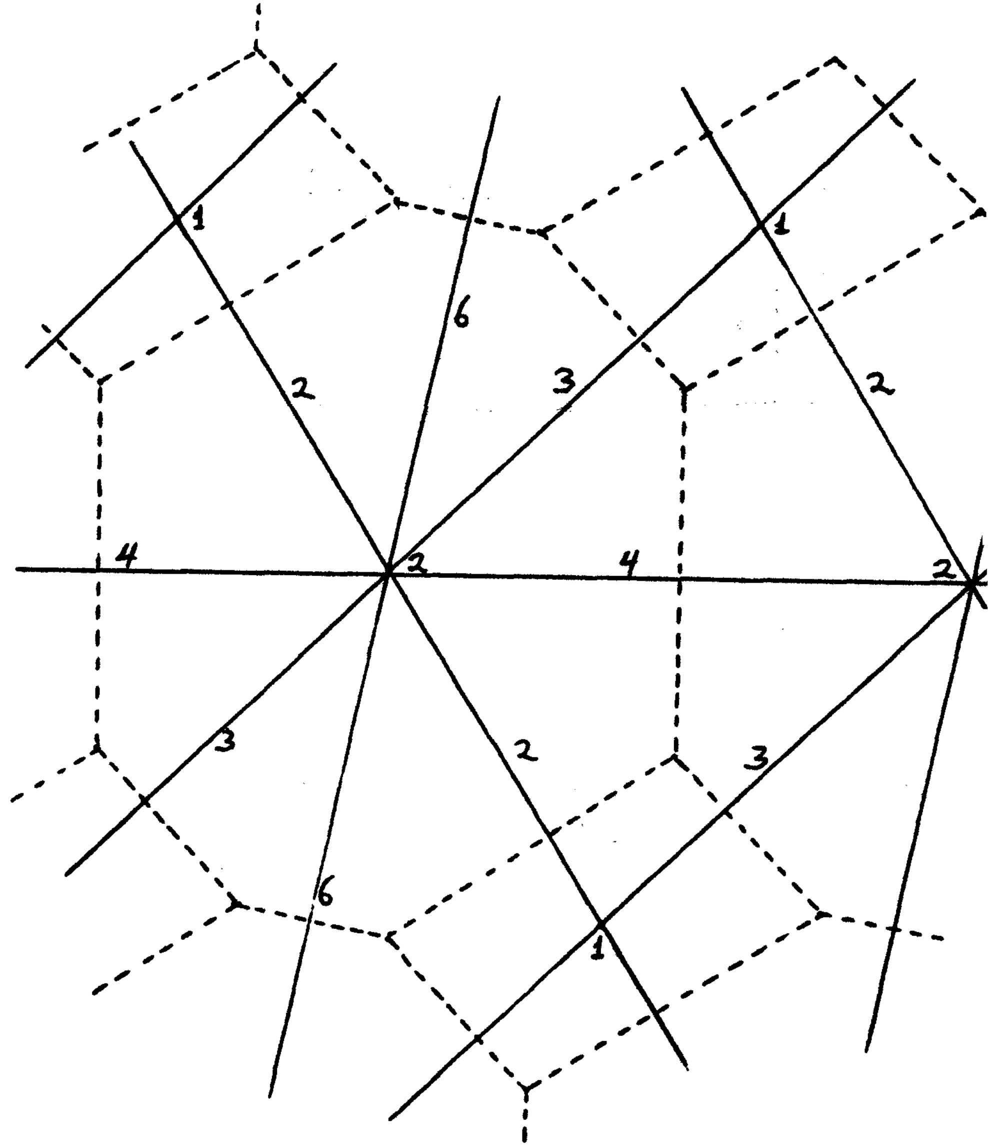


The dotted lines show the polygons around the points λ, the solid lines are the edges of the triangles with vertices λ. The numbers at the vertices are the $a_i = r(\lambda_i)^2$; the numbers on the edges are the $A_{ij} = |\lambda_i - \lambda_j|^2$.

Figure 9. <u>The triangulation for</u> $\mathbb{Q}(\sqrt{-7})$

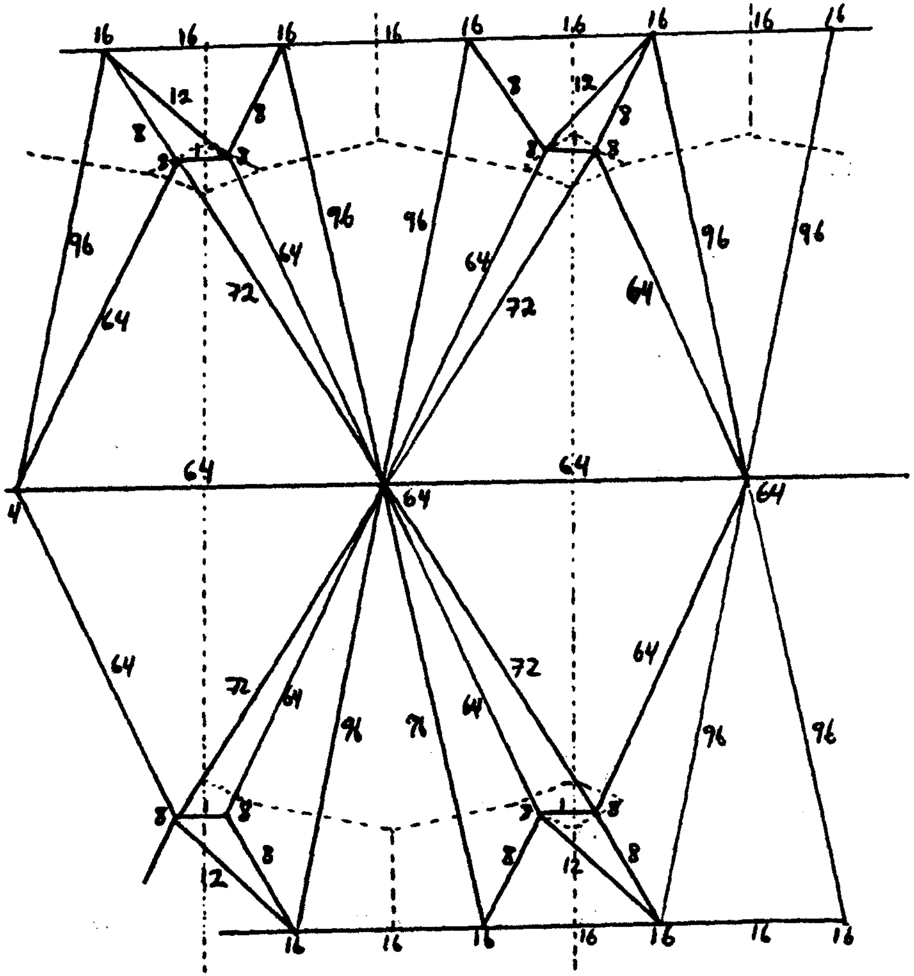


Principal cusp

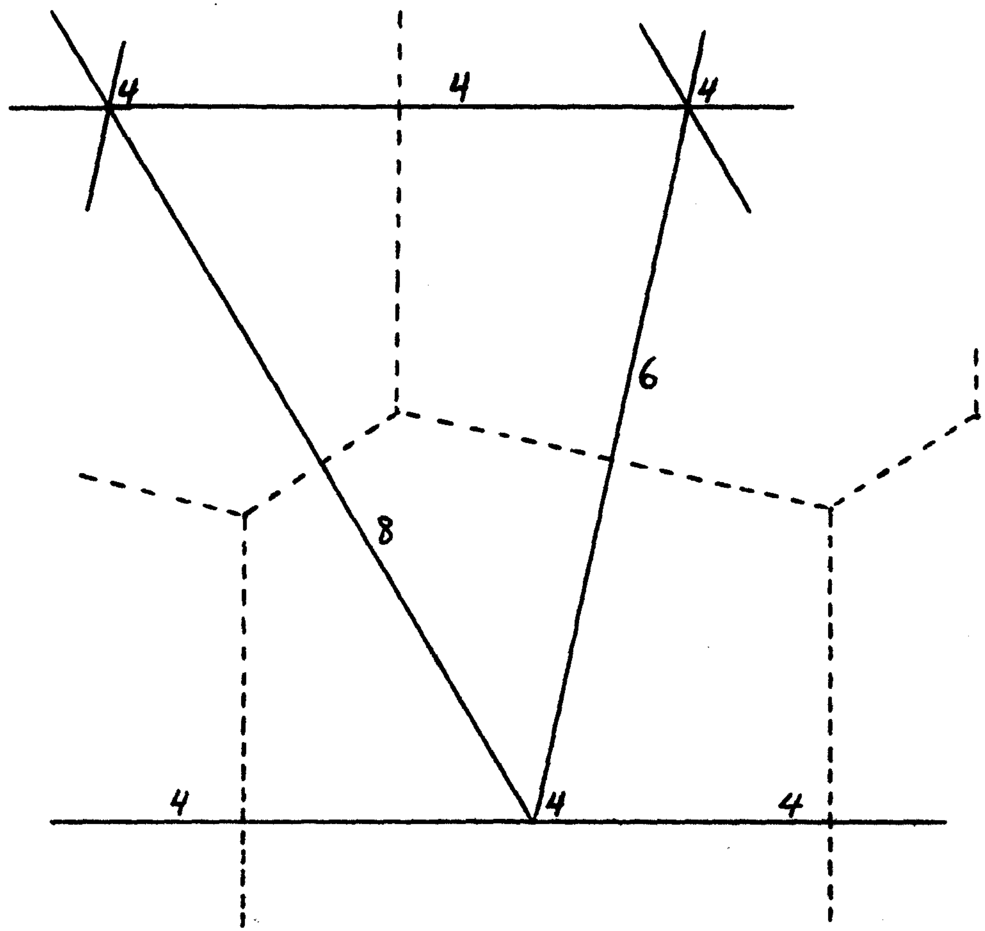


Non-principal cusp

Figure 10. First triangulation for $\mathbb{Q}(\sqrt{-23})$



Principal cusp



Non-principal cusp

Figure 11. Second triangulation for $\mathbb{Q}(\sqrt{-23})$

Bibliography

[1] A. Borel, Commensurability classes and volumes of hyperbolic 3-manifolds. Ann. Scuola Norm. Sup. Pisa 8 (1981) 1-33.

[2] K.-H. Hörsch, Ein Verfahren zur Berechnung von L-Reihen, Diplomarbeit, Bonn 1982.

[3] H. Klingen, Über die Werte der Dedekindschen Zetafunktionen, Math. Ann. 145 (1962) 265-272.

[4] L. Lewin, Polylogarithms and Associated Functions, North Holland, New York-Oxford 1981.

[5] C.L. Siegel, Berechnung von Zetafunktionen an ganzzahligen Stellen, Nachr. Akad. Wiss. Göttingen (1969) 87-102.

[6] W.P. Thurston, The Geometry and Topology of Three-Manifolds, Mimeographed lecture notes, Princeton University, 1979.

[7] M.-F. Vignéras, Arithmétique des Algèbres de Quaternions, Lecture Notes No. 800, Springer-Verlag, Berlin-Heidelberg-New York 1980.