Analytic combinatorics of the transfinite: generalized Mahler partitions and natural Gödel numberings for ordinals

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Abstract

We study the asymptotic of certain count functions which are connected with a certain well-ordered subset of Hardy's orders of infinity. As special cases we find Mahler partitions and its iterated versions as well as their multiplicative counterparts.

1 Introduction

We determine the asympotics of certain count functions for a natural subclass of Hardy's order of infinity. These results find applications in classifying phase transitions for Gödel incompleteness and logical limit laws. The class of functions $\mathbb{E}$ in question is defined as follows. Each member $f \in \mathbb{E}$ is a function from $\mathbb{N} \to \mathbb{N}$. Let $c_0(x) := 0$ be the function constant zero defined for non negative integers. We put $c_0 \in \mathbb{E}$. Now assume that f, g are elements of $\mathbb{E}$. Let $h(x) := x^{f(x)} + g(x)$. Then h is put into $\mathbb{E}$. Now we define $\mathbb{E}$ to be the least set containing c_0 which is closed under this formation rule.

The set $\mathbb{E}$ comes equipped with a natural order $\prec$ of eventual domination which is defined as follows: $f \prec g$ iff there is a $k \in \mathbb{N}$ such that $f(x) < g(x)$ for all $x > k$. Then e.g. Hardy [**?**] has shown that the ordering $\prec$ is a linear ordering on $\mathbb{E}$. Moreover there will not exist any infinite descending chain $f_0 \succ f_1 \succ f_2 \succ \ldots$ of elements in $\mathbb{E}$, i.e. $\prec$ is a well-ordering on $\mathbb{E}$. Moreover the well-orderedness of $\prec$ with respect to arithmetical sets cannot be proved from the Peano axioms for the natural numbers.

There are several canonical complexity measures which can be assigned to members of $\mathbb{E}$. The desired property of such a measure $c : \mathbb{E} \to \mathbb{N}$ is that for any $k \in \mathbb{N}$ and any $f \in \mathbb{E}$ the number of elements in $\{g \prec f : c(g) \leq k\}$ is finite. A canonical choice for c is given by evaluation. We may put $c(g) := g(k)$ for some fixed k. In this case we put

$$M_f^k(n) := \#\{g \prec f : g(k) = n\} \tag{1}$$

and one may ask for the asymptotic of $M_f^k(n)$ as $n \longrightarrow \infty$. For a certain choice of f there is much information on this problem available from the literature about Mahler partitions. Let $f(x) = x^x$ (which is an element from $\mathbb{E}$ since $f(x) = x^{x^{x^{c_0(x)}}+c_0(x)} + c_0(x)$). Then $M_f^k(n)$ is the number of Mahler partitions of n in sums of exponentials with base k. For example it is well known that

$$\ln(M_f^k(n)) \sim \frac{1}{2\ln(k)}(\ln(n))^2$$

as $n \longrightarrow \infty$. In the sequel we stick to the case $k = 2$ when we consider this type of investigation. The case for general k means that one has to replace in the corresponding results $\ln(2)$ by $\ln(k)$. Therefore we drop in the sequel the upper index in m_f^k and assume that this index is equal to 2.

Another natural complexity functions emerges from a natural Gödel numbering of $\mathbb{E}$. The idea is to assign to each object f in $\mathbb{E}$ a unique natural number such that effective operations on $\mathbb{E}$ translate into elementary recursive operations on the corresponding Gödel numbers. To this end note that every non constant zero element f in $\mathbb{E}$ has a unique representation $f = id^{f_1} + \cdots + id^{f_d}$ with $f_1 \succeq \ldots \succeq f_d$. We further let $(p_i)_{i=1}^{\infty}$ denote the enumeration of the primes starting with $p_1 = 2$.

We put $\lceil c_0 \rceil := 1$ and if $f \in \mathbb{E}$ has the representation $f = id^{f_1} + \cdots + id^{f_d}$ with $f_1 \succeq \ldots \succeq f_d$ then we put $\lceil f \rceil := p_1^{\lceil f_1 \rceil} \cdot \ldots \cdot p_d^{\lceil f_d \rceil}$. There are different choices of the Gödel numbering possible but we have chosen one which typically appears in textbooks on recursion theory. We put

$$G_f(n) := \#\{g \prec f : \lceil g \rceil \leq n\}. \tag{2}$$

For $f = id^{id}$ we get a multiplicative analogue of the Mahler partition function as we will see in a minute. Getting non trivial bounds on G_f seems even more difficult then for M_f. But luckily large machinery from analytic combinatorics has already been developed and a seminal paper by Parameswaran allows to obtain weak asymptotics for M_f as well as for G_f. These results are strong enough for the intended proof theoretic applications. It seems that even better bounds are available by applying the saddle point method a la Dumas and Flajolet [5] but we leave this for the experts in the field. Parameswaran's result is as follows.

Theorem 1 (Parameswaran [8]). *Suppose that the following conditions hold.*

1. *$L(u)$ and $P(u)$ are functions on the non negative reals such that $\int_0^R L(u)du$ and $\int_0^R P(u)du$ exist in the Lebesgue sense for every positive R.*
2. *$\exp(s\int_0^\infty \frac{e^{-su}}{1-e^{-su}}L(u)du) = s\int_0^\infty P(u)e^{-su}du$ for all positive s,*
3. *$\langle M, M^* \rangle$ form a pair of conjugate slowly varying functions,*
4. *M is non decreasing,*
5. *$\int_0^u \frac{L(t)}{t}dt \sim M(u)$ as $u \to \infty$, and*
6. *$P(u)$ is non decreasing.*

Then $\log P(u) \sim \frac{1}{M^*(u)}$ *as* $u \to \infty$.

We now state our main results. For a compact presentation we use the following notations. We put

$$\ln_{d+1}(x) := \ln(\ln_d(x)$$

where $\ln_1(x) = \ln(x)$. Moreover we put $id(x) := x$ and let

$$id_{d+1}(k)(x) := x^{id_d(k)(x)}$$

where $id_1(k)(x) = x^k$. In addition we put

$$\exp_{d+1}(x) := \exp(\exp_d(x))$$

where $\exp_1(x) = \exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.

Theorem 2. *1. If $f = id_1(k)$ then there exist explicitly calculable constants C_1, C_2 such that*

$$M_f(x) \quad \sim \quad C_1 \cdot x^{k-1} \tag{3}$$

$$G_f(x) \quad \sim \quad C_1 \cdot (\frac{\ln(x)}{\ln(\ln(x))})^k \tag{4}$$

2. If $f = id_2(k)$ then there exist explicitly calculable constants C_3, C_4 such that

$$\ln(M_f(x)) \quad \sim \quad C_3 \cdot (\ln(x))^{k+1} \tag{5}$$

$$\ln(G_f(x)) \quad \sim \quad C_4 \cdot \ln(\ln(x)) \cdot (\frac{\ln\ln(x)}{\ln(\ln(\ln(x))})^k \tag{6}$$

3. If $f = id_d(k)$ and $d \geq 2$ then with the same constants C_3, C_4 as in the previous item

$$\ln_{d-1}(M_f(x)) \quad \sim \quad C_3 \cdot (\ln_{d-1}(x))^{k+1} \tag{7}$$

$$\ln_{d-1}(G_f(x)) \quad \sim \quad C_4 \cdot \ln_d(x) \cdot (\frac{\ln_d(x)}{\ln(\ln_d(x))})^k \tag{8}$$

Proof. We prove the results for the generalized Mahler norms. The asymptotic (3) is well known. Indeed we may consider $\{g \prec id^k\}$ as a generalized additive number system generated from the additive primes id^l for $0 \leq l < k$. By Theorem 2.48 in Burris [4] we therefore obtain

$$M_{id^k}(x) \sim \frac{1}{(k-1)!} \frac{1}{\prod_{l<k} 2^l} x^{k-1} \tag{9}$$

and assertion (3) follows.

Let us now prove assertion (5). By remark 2.32 and Theorem 2.48 in Burris [4] we obtain

$$m(x) := \#\{g \prec id^k : g(2) \leq x\} \sim \frac{1}{k!} \frac{1}{\prod_{l<k} 2^l} x^k \tag{10}$$

Let

$$n(u) = \sum_{2^l \leq u} M_{id^k}(l) = m(\frac{\ln(u)}{\ln(2)}).$$

Then

$$n(u) \sim \frac{1}{k!(\prod_{l<k} 2^l)(\ln(2))^k)}(\ln(u)^k =: L(u).$$

Let

$$C := \frac{1}{k!(\prod_{l<k} 2^l)(\ln(2)^k)}.$$

Let

$$M(u) := \int_a^u \frac{L(t)}{t} dt$$

where $a >$ is arbitrary but fixed. Then by de l'Hospital's rule

$$M(u) \sim \frac{C}{k+1}(ln(u)^{k+1}).$$

Let

$$P(u) := \sum_{l \leq u} M_{id^{id^k}}(l).$$

By Theorem 1 of Parameswaran we obtain

$$\ln(P(u)) \sim \frac{C}{k+1}(ln(u)^{k+1}).$$

Moreover this yields $\ln(M_{id^{id^k}}(u)) \sim \frac{C}{k+1}(ln(u)^{k+1})$ as indicated on the last page of Parameswaran. So we may put $C_3 := \frac{C}{k+1}$.

Let us now prove assertion 7 by induction on d. Put

$$m(x) := \#\{g \prec M_{id_d(k)} : g(2) \leq x\}. \tag{11}$$

Let

$$n(u) = \sum_{2^l \leq u} M_{id_d(k)}(l) = m(\frac{\ln(u)}{\ln(2)}) := L(u).$$

The induction hypothesis yields

$$\log_{d-1}(m(x)) \sim C \cdot (\log_{d-1}(x))^{k+1}. \tag{12}$$

for $C = C_3$. Let

$$M(u) := \int_a^u \frac{L(t)}{t} dt$$

for some arbitrary fixed $a > 0$ Let

$$P(u) := \sum_{l \leq u} M_{id_d(k)}(l).$$

By Theorem 1 of Parameswaran we obtain

$$\ln(P(u)) \sim M(u).$$

We claim that

$$\ln_d(P(u)) \sim C \cdot (\ln_d(x))^{k+1}.$$

Proof: Pick

$$\varepsilon > 0.$$

Then (12) yields

$$m(u) \leq \exp_{d-1}((1+\frac{\varepsilon}{2})C(\ln_{d-1}(u))^k$$

for large enough u. Hence

$$L(u) \leq \exp_{d-1}((1+\frac{\varepsilon}{2})C(\ln_{d-1}(\frac{\ln(u)}{\ln(2)}))^{k+1}.$$

Thus

$$M(u) \leq \int_a^u \frac{\exp_{d-1}((1+\frac{\varepsilon}{2})C(\ln_{d-1}(\frac{\ln(u)}{\ln(2)}))^{k+1}}{u} du.$$

Put

$$N(u) := \exp_{d-1}((1+\varepsilon)C(\ln_{d-1}(\ln(u)))^{k+1}.$$

Then de l'Hospital's rule yields

$$M(u) = o(N(u))$$

as $u \to \infty$. In particular we obtain that $M(u) \leq N(u)$ for large enough u. Therefore

$$\ln(P(u)) \sim M(u) \leq N(u)$$

for large enough u. Hence

$$P(u) \leq \exp_d((1+\varepsilon)C(\ln_d(u))^{k+1}.$$

By a similar argument we obtain

$$P(u) \geq \exp_d((1-\varepsilon)C(\ln_d(u))^{k+1}.$$

Thus

$$\log_d(P(u)) \sim C(\ln_d(u))^{k+1}.$$

Further

$$\ln_d(M_{id_{d+1}(k)}) \sim C(\ln_d(u))^{k+1}(u))$$

as indicated on the last page of Parameswaran. $\square$

The intended proof-theoretic applications are as follows. Let small Greek letters range over elements of $\mathbb{E}$. With ω we denote the identity function id. Elements in $\mathbb{E}$ of the form $\alpha+\omega^{c_0}$ are called successors. Non zero elements of $\mathbb{E}$ which are not successors are called limits. For a limit $\lambda \in \mathbb{E}$ let $\lambda[x]$ be the x-th element of the canonical fundamental sequence for λ. This means that if $\lambda = \omega^{\alpha_1}+\cdots+\omega^{\alpha_n}$ where $\alpha_1 \succeq \ldots \succeq \alpha_n$ and $\alpha_n = \gamma+\omega^{c_0}$ then $\lambda[x] := \omega^{\alpha_1}+\cdots+\omega^{\gamma}\cdot x$ and that if $\lambda = \omega^{\alpha_1}+\cdots+\omega^{\alpha_n}$ where $\alpha_1 \succeq \ldots \succeq \alpha_n$ and α_n is a limit then $\lambda[x] := \omega^{\alpha_1}+\cdots+\omega^{\alpha_n[x]}$. Then we have that for all $\beta \prec \lambda$ there is an x such that $\beta \prec \lambda[x]$ so that $\lambda[x]$ converges to λ as $x \to \infty$. It is convenient to introduce a top element for the elements of $\mathbb{E}$. We call this ε_0 and we write $\alpha \prec \varepsilon_0$ in place of $\alpha \in \mathbb{E}$. For ε_0 the fundamental sequence is defined via $\varepsilon_0[x] := \omega_x = id_x$. We then can define F_α for $\alpha \preceq \varepsilon_0$ as follows by recursion on $\prec$.

$$\begin{aligned} F_0(x) &:= x+1, \\ F_{\alpha+1}(x) &:= F_\alpha^{(x)}(x) \text{ where the upper index denotes number of iterations,} \\ F_\lambda(x) &:= F_{\lambda[x]}(x) \text{ where } \lambda \text{ is a limit.} \end{aligned}$$

Let c be a complexity measure for the elements of $\mathbb{E}$. Let $\mathrm{SWO}(\beta, f, c)$ be the statement

$$(\forall K)(\exists L)(\forall \alpha_0,\ldots,\alpha_L \prec \beta)\left((\forall i \le L)[c(\alpha_i) \le K + f_\alpha(i)] \to (\exists i < L)[\alpha_i \preceq \alpha_{i+1}]\right).$$

Theorem 3. *Let*

$$f_\alpha(i) := \exp_{F_\alpha^{-1}(i)}\left(\sqrt{\ln_{F_\alpha^{-1}(i)}(i)}\right).$$

Then the following phase transition result holds for $\mathrm{SWO}(\varepsilon_0, f, M)$.

1. *If* $\alpha \prec \varepsilon_0$ *then*
$$\mathrm{PA} \vdash \mathrm{SWO}(\varepsilon_0, f_\alpha, M).$$
2. *If* $\alpha = \varepsilon_0$ *then*
$$\mathrm{PA} \nvdash \mathrm{SWO}(\varepsilon_0, f_\alpha, M).$$

Let $\mathrm{I}\Sigma_d$ be the fragment of PA where the induction scheme is restricted to formulas with at most d quantifiers.

Theorem 4. *Let* $d \ge 1$. *Let*

$$f_\alpha(i) := \exp_d\left(\sqrt[F_\alpha^{-1}(i)]{\ln_d(i)}\right).$$

Then the following phase transition result holds for $\mathrm{SWO}(\omega_{d+1}, f, M)$.

1. *If* $\alpha \prec \omega_{d+1}$ *then*
$$\mathrm{I}\Sigma_d \vdash \mathrm{SWO}(\omega_{d+1}, f_\alpha, M).$$
2. *If* $\alpha = \omega_{d+1}$ *then*
$$\mathrm{I}\Sigma_d \nvdash \mathrm{SWO}(\omega_{d+1}, f_\alpha, M).$$

In the multiplicative situation the following phase transition results are obtained.

Theorem 5. *Let*

$$f_\alpha(i) := \exp\left(\exp_{F_\alpha^{-1}(i)}(\sqrt{\ln_{F_\alpha^{-1}(i)}(i)})\right).$$

Then the following phase transition result holds for $\mathrm{SWO}(\varepsilon_0, f, \lceil\cdot\rceil)$.

1. *If* $\alpha \prec \varepsilon_0$ *then*
$$\mathrm{PA} \vdash \mathrm{SWO}(\varepsilon_0, f_\alpha, \lceil\cdot\rceil).$$

2. *If* $\alpha = \varepsilon_0$ *then*
$$\mathrm{PA} \nvdash \mathrm{SWO}(\varepsilon_0, f_\alpha, \lceil\cdot\rceil).$$

Let $\mathrm{I}\Sigma_d$ be the fragment of PA where the induction scheme is restricted to formulas with at most d quantifiers.

Theorem 6. *Let* $d \geq 1$. *Let*

$$f_\alpha(i) := \exp\left(\exp_d(\sqrt[F_\alpha^{-1}(i)]{\ln_d(i)})\right).$$

Then the following phase transition result holds for $\mathrm{SWO}(\omega_{d+1}, f, \lceil\cdot\rceil)$.

1. *If* $\alpha \prec \omega_{d+1}$ *then*
$$\mathrm{I}\Sigma_d \vdash \mathrm{SWO}(\omega_{d+1}, f_\alpha, \lceil\cdot\rceil).$$

2. *If* $\alpha = \omega_{d+1}$ *then*
$$\mathrm{I}\Sigma_d \nvdash \mathrm{SWO}(\omega_{d+1}, f_\alpha, \lceil\cdot\rceil).$$

In a sequel paper we will exploit our investigations to prove (joint project with A.R. Woods) logical limit laws for segments of $\mathbb{E}$. We plan to investigate further properties of M_f and G_f with J.P. Bell.

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