ON SCHLÄFLI'S REDUCTION FORMULA

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Introduction

In polyhedral geometry and in particular for the problem of calculating volumes of non-euclidean polytopes, orthoschemes are the most basic objects.

Let X denote the n-dimensional sphere S^n or the n-dimensional hyperbolic space H^n. An orthoscheme in X is a simplex bounded by hyperplanes $H_0, \dots H_n$ such that $H_i \perp H_j$ for $|i-j| > 1$. Hence, a planar orthoscheme is a right-angled triangle, whose area formula can be expressed by the well-known defect formula.

Ludwig Schläfli generalized this formula to spherical orthoschemes of even dimension; Schläfli's reduction formula represents the volume of an even dimensional spherical orthoscheme in terms of the volumes of certain lower dimensional ones (see [S]). This formula can be easily extended to the hyperbolic case by analytic continuation (see [H], p.134 ff). In this paper, we consider a more general class of hyperbolic polytopes, the so-called (complete) orthoschemes of degree d, $0 \leq d \leq 2$, (see [IH] and [K2]). These polytopes arise e.g. as a particular class of fundamental polytopes in the classification problem for hyperbolic Coxeter groups. Our aim is to show that a generalized reduction formula holds for even dimensional complete orthoschemes; we shall see that this reduction law simplifies with increasing degree (of truncation) d.

For the following, it is most convenient to describe a polytope by means of its scheme. The scheme of a polytope $P \subset X$ is a weighted graph (characterizing $P \subset X$ up to congruence) in which the nodes correspond to the bounding hyperplanes of P. Two nodes are joined by an edge if the corresponding hyperplanes are not orthogonal; the weight on an edge equals either the cosine of the dihedral angle between the corresponding hyperplanes, or, for diverging hyperplanes in H^n, the hyperbolic cosine of their distance. A pair of non-adjacent nodes is characterized by the weight zero. To every scheme $\Sigma = (n_1 \dots n_m)$ with nodes $n_1, \dots, n_m$ corresponds a symmetric matrix (a_{ij}) of order m wherein $a_{ii} = 1$ and, for $i \neq j$, a_{ij} equals the negative of the weight between n_i, n_j. A scheme is called elliptic resp. hyperbolic if its matrix is positive definite resp. of index of inertia -1. For more details, see Chapter 1 of this article.

Hence, a spherical orthoscheme R of dimension n is represented by a linear elliptic scheme Σ of order $n+1$. Denote by $f_n(\Sigma)$ or f_n Schläfli's normalized volume function for R given by

$$f_n(\Sigma) = f_n := c_n \mathrm{Vol}_n(R) \quad \text{with} \quad c_n = \frac{2^{n-1}}{\mathrm{Vol}_n(S^n)} = \frac{2^n}{\pi^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right) \quad , \quad f_0 := 1 \,. \quad (1)$$

If the scheme Σ consists of two disjoint components σ_1, σ_2 of orders $n_1+1, n_2+1 \geq 1$, then $f_n(\Sigma) = f_{n_1}(\sigma_1) \cdot f_{n_2}(\sigma_2)$.

With these preliminaries, Schläfli's reduction formula for spherical orthoschemes can be stated as follows:

$$f_{2n}(\Sigma) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{2k}{k} \sum_{\sigma} f_{2n-(2k+1)}(\sigma) \qquad , \qquad \sum f_{-1} := 1 \quad , \qquad (2)$$

where σ runs through all subschemes of Σ of order $2(n-k)$ all of whose components have even order.

For example, we obtain

$$f(012) = f(01) + f(12) - 1 \quad ,$$

$$f(01234) = f(0123) + f(01)f(34) + f(1234) - \{ f(01) + f(12) + f(23) + f(34) \} + 2 \quad .$$

Now, let R_d, $0 \leq d \leq 2$, denote a (complete) orthoscheme of degree d in H^n, i.e., R_d is a d-times truncated orthoscheme bounded by hyperplanes $H_0, \ldots, H_{n+d}$ such that $H_i \perp H_j$ for $j \neq i-1, i, i+1$ (for $d=2$, indices are taken modulo $n+3$). They are described by a linear hyperbolic scheme of order $n+1$ or $n+2$ for $d=0$ or $d=1$, or, by a cyclic hyperbolic scheme of order $n+3$ for $d=2$.

Modifying Schläfli's function for complete orthoschemes $R_d \subset H^n$ with graph Σ_d by

$$F_n(\Sigma_d) := i^n c_n \mathrm{Vol}_n(R_d) \qquad \text{with} \qquad F_0 := 1 \quad , \quad i^2 = -1 \quad , \qquad (3)$$

we show that the following reduction formula holds for $d = 0, 1, 2$:

$$F_{2n}(\Sigma_d) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{2k}{k} \sum_{\sigma} f_{2n-(2k+1)}(\sigma) \qquad , \qquad \sum f_{-1} := 1 \quad , \qquad (4)$$

where σ runs through all *elliptic* subschemes of Σ_d of order $2(n-k)$ all of whose components are of even order.

Since the Schläfli function takes only rational values on the set of spherical Coxeter orthoschemes (all dihedral angles are of the form $\frac{\pi}{p}$, $p \in \mathbf{N}$, $p \geq 2$), the volume of a complete hyperbolic Coxeter orthoscheme of dimension $2n$ is a rational multiple of π^n. The complete Coxeter orthoschemes were classified by Im Hof in 1983 (see [IH]). He showed that they exist only for dimensions ≤ 9; in even dimensions ≥ 4, there are only finitely many examples, whose volumes are determined explicitly in an appendix.

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1. Complete orthoschemes

1.1 Let X^n denote either the n-dimensional euclidean space E^n, the n-sphere S^n or the n-dimensional hyperbolic space H^n. Let S^n be embedded in E^{n+1}, and use for H^n the model in the Lorentz space $E^{1,n}$ of signature $(1,n)$, i.e.: If $E^{1,n}$ denotes the real vector space R^{n+1} equipped with the bilinear form of signature $(1,n)$

$$\langle x,y\rangle = -x_0y_0 + x_1y_1 + \ldots + x_ny_n \quad ,$$

$$\forall x = (x_0,\ldots,x_n) \in R^{n+1} \quad , \quad \forall y = (y_0,\ldots,y_n) \in R^{n+1} \quad ,$$

then H^n can be interpreted as

$$H^n = \{\, x \in E^{1,n} \mid \langle x,x\rangle = -1 \quad , \quad x_0 > 0 \,\} \quad .$$

Or, in the projective model, H^n is the interior of real projective space P^n with respect to the quadric $Q_{1,n} = \{\, [x] \in P^n \mid \langle x,x\rangle = 0 \,\} =: \partial H^n$.

1.2 Let $P \subset H^n$ denote a convex polytope bounded by finitely many hyperplanes H_i, $i \in I$, which are characterized by unit normal vectors $e_i \in E^{1,n}$ directed inwards with respect to P, say, i.e. (for basic notations and properties, see [**V1**], §1):

$$H_i = e_i^{\perp} := \{\, x \in H^n \mid \langle x,e_i\rangle = 0 \,\} \quad \text{with} \quad \langle e_i,e_i\rangle = 1 \quad .$$

We always assume that P is acute-angled (i.e., all dihedral angles $\neq \frac{\pi}{2}$ are of measure strictly less than $\frac{\pi}{2}$) and of finite volume. Then, every face $F \subset P$ of dimension k, $1 \le k \le n-1$, is contained in exactly $n-k$ of the bounding hyperplanes of P (see [**A2**], Lemma 1, p.762), and F is itself an acute-angled polytope of finite volume (see [**A2**], Lemma 2, p.762).

The Gram matrix $G(P) := (\,\langle e_i,e_j\rangle\,)_{i,j\in I}$ of the vectors e_i, $i \in I$, associated to P is an indecomposable matrix of signature $(1,n)$ with entries $\langle e_i,e_i\rangle = 1$ and $\langle e_i,e_j\rangle \le 0$ for $i \neq j$, having the following geometrical meanings (see [**V1**], §1):

$$\langle e_i,e_j\rangle = \begin{cases} 0, & \text{if } H_i \perp H_j \quad , \\ -\cos\alpha_{ij}, & \text{if } H_i, H_j \text{ intersect on } P \text{ under the angle } \alpha_{ij} = \angle(H_i,H_j) \quad , \\ -1, & \text{if } H_i, H_j \text{ are parallel} \quad , \\ -\cosh l_{ij}, & \text{if } H_i, H_j \text{ admit a common perpendicular of length } l_{ij} \quad . \end{cases}$$

On the other hand, if $G = (g_{ij})$ is an indecomposable symmetric $m \times m$ matrix of rank $n+1$ with $g_{ii} = 1$ and $g_{ij} \le 0$, for $i \neq j$, then G can be realized as Gram matrix $G(P)$ of an acute-angled polytope $P \subset X^n$ of finite volume in the following way (see [**V1**], §2):

1. If G is positive definite (G is *elliptic*), then $m = n+1$, and G is the Gram matrix of a simplex in S^n uniquely defined up to a motion.
2. If G is positive semidefinite (G is *parabolic*), then $m = n+2$, and G is the Gram matrix of a simplex in E^{n+1} uniquely defined up to a similarity.
3. If G is of signature $(1, n)$ (G is *hyperbolic*), then G is the Gram matrix of a convex polytope with m facets (faces of codimension m) in H^n uniquely defined up to a motion.

In terms of the Gram matrix $G(P)$, the combinatorial structure of an acute-angled polytope $P \subset H^n$ can be described as follows (see **[V1]**, §3):
If P is compact, then the positive definite principal submatrices $G_J := (g_{ij})_{i,j\in J}$ of $G(P)$, $J \subset I$ with $1 \leq |J| \leq n$, are in one-to-one-correspondence with the non-empty faces

$$P^J := P \cap \big(\bigcap_{j\in J} H_j \big) \quad ,$$

and P^J has codimension $|J|$ (see **[V1]**, Theorem 3.1). In particular, a vertex $p \in P$ is characterized by a positive definite principal submatrix of $G(P)$ of order n describing the spherical vertex polytope P_p (intersection of P with the surface of a sufficiently small ball around p) of dimension $n-1$ associated to p.
If P is not compact, but of finite volume, then a point $q \in \partial H^n$ is an infinite vertex of P if and only if for $J_q := \{\, i \in I \mid H_i \ni q \,\}$ the principal submatrix $G_{J_q} = (g_{ij})_{i,j\in J_q}$ is parabolic of rank $n-1$ (or equivalently, the vertex polytope P_q is a euclidean polytope of dimension $n-1$) (see **[V1]**, Theorem 3.2).

1.3 In practice, however, the language of schemes is much more convenient for the geometrical description of certain classes of polytopes (see **[V2]**, §3). A scheme Σ is a weighted graph (see **[V2]**, §2) whose nodes n_i, n_j are joined by an edge with positive weight σ_{ij} or not; the last fact will be indicated by $\sigma_{ij} = 0$. A subscheme of Σ is a subgraph of Σ with each pair of nodes connected by the same weighted edge as in Σ. The number $|\Sigma|$ of nodes is called the order of Σ. To every scheme Σ of order m corresponds a symmetric matrix $A(\Sigma) = (a_{ij})$ of order m with $a_{ii} = 1$ on the diagonal and non-positive entries $a_{ij} = -\sigma_{ij} \leq 0$, $i \neq j$, off it. Σ is called connected if and only if $A(\Sigma)$ is indecomposable. Rank, determinant and character of definiteness of Σ are defined to be the corresponding ones of $A(\Sigma)$. Furthermore, Σ is said to be either elliptic, or parabolic, or hyperbolic if either all its components are elliptic, or - apart from elliptic components - there is at least one parabolic component, or exactly one component is hyperbolic.

Now, the scheme $\Sigma(P)$ of an acute-angled polytope $P \subset X^n$ is the scheme whose matrix $A(\Sigma)$ coincides with the Gram matrix $G(P)$, i.e., whose nodes i correspond to the bounding hyperplanes $H_i = e_i^\perp$ (or equivalently to their normal vectors e_i) of P and whose weights equal $-\langle e_i, e_j\rangle_{X^n}$, $i,j \in I$. The scheme of a face $F \subset P$ is denoted by $\Sigma(F)$ and is called the *face scheme* of F in P; $\Sigma(F)$ is not a subscheme of $\Sigma(P)$, since, in general, it does not inherit the weights of $\Sigma(P)$.

Two acute-angled polytopes $P_1, P_2 \subset H^n$ are said to be *of the same schematic type* if their schemes $\Sigma(P_1), \Sigma(P_2)$ are of the same graphical type (i.e., their underlying graphs as one-dimensional simplicial complexes are simplicial homeomorphic) and if corresponding weights σ_{ij}^1 of $\Sigma(P_1)$ and σ_{ij}^2 of $\Sigma(P_2)$ satisfy:

$$\sigma_{ij}^1 \quad \begin{cases} > \\ = \\ < \end{cases} 1 \quad \Longleftrightarrow \quad \sigma_{ij}^2 \quad \begin{cases} > \\ = \\ < \end{cases} 1 \quad .$$

It follows that polytopes of the same schematic type are of the same combinatorial type (see **1.2** and [**A1**]).

For the schemes of Coxeter polytopes $P_C \subset X^n$ (all dihedral angles are of the form $\frac{\pi}{p}, p \in \mathbb{N}, p \geq 2$) we adopt the usual conventions and - for convenience - use them sometimes even in the non-Coxeter case: If two nodes are related by the weight $\cos\frac{\pi}{p}$, then they are joined by a $(p-2)$-fold line for $p = 3,4$ and by a single line marked p (or $\alpha = \frac{\pi}{p}$) for $p \geq 5$. If two bounding hyperplanes of $P_C \subset X^n$, $X^n \neq S^n$, are parallel, then the corresponding nodes are joined by a line marked ∞; if they are divergent (occuring at most in the hyperbolic case), then their nodes are joined by a dotted line and the weight ≤ -1 is dropped.

The elliptic and parabolic Coxeter schemes were classified by Coxeter in 1934 (see [**C**]). Hyperbolic Coxeter schemes, however, are only partially classified (see e.g. [**V1**], Chapter II).

1.4 The simplest examples of schemes are the linear and cyclic ones. One class of acute-angled hyperbolic polytopes with linear and cyclic schemes is the following (see [**IH**] and [**K2**]): An *n-dimensional complete orthoscheme of degree* d, $0 \leq d \leq 2$, or, for short, an *n-orthoscheme of degree* d is a convex polytope in H^n, $n \geq 2$, denoted by R_d such that its scheme $\Sigma(R_d)$ is connected and linear of length $n+d+1$ for $d = 0,1$ or cyclic of order $n+3$ for $d = 2$.

Hence, orthoschemes of degree d in H^n are bounded by $n+d+1$ hyperplanes $H_0, \ldots, H_{n+d}$ such that

$$H_i \perp H_j \quad \text{for} \quad j \neq i-1, i, i+1, \tag{5}$$

where, for $d = 2$, indices are taken modulo $n + 3$.

Geometrically, orthoschemes of degree d can be described as follows:

For degree $d = 0$, they coincide with the class of (ordinary) orthoschemes introduced by Schläfli (see [S], [BH]): An orthoscheme in X^n $(n \geq 1)$ is an n-simplex bounded by $n+1$ hyperplanes $H_0, \dots, H_n$ such that $H_i \perp H_j$ for $|i-j| > 1$. This is equivalent to say that it has vertices $P_0, \dots, P_n$ numbered in such a way that $\text{span}(P_0, \dots, P_i) \perp \text{span}(P_i, \dots, P_n)$ for $1 \leq i \leq n-1$. The initial and final vertices P_0, P_n of the orthogonal edge-path P_iP_{i+1}, $i = 0, \dots, n-1$, are called principal vertices, since they are distinguished in several ways. E.g. in H^n, at most the principal vertices may be points at infinity (see [BH], Satz 15, p.188).

Using the projective model for H^n (see **1.1**), we can derive orthoschemes of degree $d = 1$ or $d = 2$ from an ordinary one by allowing one or both of its principal vertices (and with them possibly further vertices) to lie outside the quadric $Q_{1,n}$, and then by cutting off the ideal vertices by means of the polar hyperplanes, say H_{n+1} resp. H_{n+2}, corresponding to P_n resp. P_0 (inasmuch as they lie outside $Q_{1,n}$). Hence, orthoschemes of degree $d = 0, 1, 2$ are d-times truncated (or "polarily completed") orthoschemes bounded by hyperplanes $H_0, \dots, H_{n+d}$ with the property (5).

Complete orthoschemes $R_d \subset H^n$, $0 \leq d \leq 2$, have at most $n+3$ non-right dihedral angles (or *essential* angles) $\alpha_1, \dots, \alpha_m$, $m \leq n+3$, and all of them are acute, i.e., $\alpha_i < \frac{\pi}{2}$ for $i = 1, \dots, m$ (see [BH], §4.8, Hilfssatz 2, and the definition). Furthermore, by construction, complete orthoschemes are of finite volume (see [V1], Theorem 4.1). Hence, a face of R_d is also an acute-angled polytope of finite volume (see **1.2**); in fact, it is itself a complete orthoscheme (see [K2], **1.3**). If Σ_d is the scheme of R_d, we denote by $\Sigma_d(l)$ the face scheme of the apex $R_d(l) = R_d \cap H_{l-1} \cap H_l$ in R_d associated to the essential angle $\alpha_l = \angle(H_{l-1}, H_l)$, $1 \leq l \leq m$, of R_d. $\Sigma_d(l)$ is not a subscheme of Σ_d. However, the scheme of a vertex polytope of R_d is a subscheme of Σ_d of order n (see **1.2**) and is therefore a $(n-1)$-orthoscheme with essential angles of the same measure as certain essential angles of R_d. Since the vertex polytope of a face $F \subset R_d$, $2 \leq \dim F \leq n-1$, associated to a vertex $p \in F$ is the (non-empty) intersection of F with the vertex polytope of R_d associated to p, we conclude by iteration that every subscheme of order k of the scheme $\Sigma(F)$, $2 \leq k \leq |\Sigma(F)| - 1$, is the face scheme of order k of a subscheme of Σ_d, and vice versa. In particular, the set of subschemes of order k of $\Sigma_d(l)$ $(2 \leq k \leq n+d-2)$ is identical with the set of face schemes of order k of subschemes of Σ_d describing the apex of the dihedral angle of measure α_l in R_d.

2. The volume function of Schläfli

2.1 For $n \geq 1$, let Σ denote the elliptic scheme of order $n+1 \geq 1$ of a spherical n-orthoscheme R. Then, the normalized volume function

$$f_n(\Sigma) = f_n := c_n \mathrm{Vol}_n(R) \quad \text{with} \quad c_n = \frac{2^{n-1}}{\mathrm{Vol}_n(S^n)} = \frac{2^n}{\pi^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right) \quad , \quad f_0 := 1 \quad , \tag{6}$$

is called the function of Schläfli (see Introduction, (1), and **[S]**, Nr.23, p.238). The function f_n is proportional to $\mathrm{Vol}_n(R)$ such that $f_n = 1$ for the orthoscheme with all dihedral angles equal to $\frac{\pi}{2}$. The function of Schläfli satisfies the following factorization property (see **[S]**, Nr.23, p.238, or **[BH]**, Hilfssatz 1, p.213):

LEMMA.

Let Σ denote a linear elliptic scheme of order $n+1$ consisting of disjoint components $\sigma_1, \ldots, \sigma_r$ of orders $n_1+1, \ldots, n_r+1 \geq 1$. Then

$$f_n(\Sigma) = f_{n_1}(\sigma_1) \cdots f_{n_r}(\sigma_r) \quad . \tag{7}$$

For spherical Coxeter orthoschemes (see **1.3**), Schläfli determined explicitly all possible values of f_n (see **[S]**, Nr.30, p.268 ff). Using the standard notations for schemes of spherical Coxeter orthoschemes (see **[V1]**, §5, Table 1), his results read as:

$$f_1(G_2^p) = \frac{2}{p} \quad , \quad p \geq 2 \quad ; \tag{8}$$

$$f_3(F_4) = \frac{1}{72} \quad ; \quad f_3(H_4) = \frac{1}{900} \quad ; \tag{9}$$

$$f_n(A_{n+1}) = \frac{2^{n+1}}{(n+2)!} \quad , \quad n \geq 0 \quad ; \tag{10}$$

$$f_n(B_{n+1}) = \frac{1}{(n+1)!} \quad , \quad n \geq 0 \quad . \tag{11}$$

By means of the trigonometric principle, or equivalently, by interpreting hyperbolic n-space H^n as upper half of the pseudosphere of radius $i = \sqrt{-1}$ in R^{n+1} (see **[BH]**, p.20-21 or p.210), it is obvious how to generalize the notion of Schläfli's function to orthoschemes $R_d \subset H^n$ of degree d, $0 \leq d \leq 2$, and with graph Σ_d (see also **[BH]**, p.212):

The function

$$F_n(\Sigma_d) := i^n c_n \mathrm{Vol}_n(R_d) \quad \text{with} \quad i^2 = -1 \quad , \quad F_0 := 1 \quad , \tag{12}$$

where the constant c_n is defined as in (6), is called the *Schläfli function of the complete orthoscheme R_d*.

Hence, for even dimensions,

$$F_{2n}(\Sigma_d) = (-1)^n \left(\frac{2}{\pi}\right)^n \cdot \prod_{p=1}^{n} (2p-1) \cdot \mathrm{Vol}_{2n}(R_d) \quad , \quad n \geq 1 \quad ,$$

is a real-valued function.

2.2 Let $\mathcal{R}_\kappa$ be the set of compact complete orthoschemes in H^n of combinatorial type κ (see [**A1**], §1). Since every element of $\mathcal{R}_\kappa$ is a polytope with dihedral angles not exceeding $\frac{\pi}{2}$, its congruence class is uniquely determined by its dihedral angles (see [**A1**], §3, Uniqueness Theorem). Hence, Schläfli's volume function $F_n = F_n|\mathcal{R}_\kappa$ restricted on $\mathcal{R}_\kappa$ may be regarded as a function of the dihedral angles. The differential of F_n depending on the dihedral angles can be represented by Schläfli's formula in the following way (see [**K2**], §2, and **1.4**):

THEOREM. (Schläfli's differential formula)

Let F_n , $n \geq 2$, be the Schläfli function on the set $\mathcal{R}_\kappa$ of compact complete orthoschemes in H^n of combinatorial type κ with essential angles $\alpha_1, \ldots, \alpha_{m(\kappa)}$ and with scheme Σ. Denote by $F_{n-2}(k)$ the Schläfli function of the apex of codimension 2 associated to the essential angle α_k of measure $f_1(k) := f_1(\alpha_k)$, $1 \leq k \leq m(\kappa)$. Then

$$dF_n(\Sigma) = \sum_{k=1}^{m(\kappa)} F_{n-2}(k)\, df_1(k) \quad . \tag{13}$$

This formula was established by Schläfli for spherical simplexes, and separately for the more basic orthoschemes. Much later, H. Kneser gave a second, very elegant proof (see [**Kn**]) for both, the spherical and the hyperbolic case. As Schläfli already pointed out (see [S], Nr. 25, p.246 ff, Nr.32, p.272 ff, and [**V1**], Corollary, p.48), the differential formula for orthoschemes can be extended to arbitrary acute-angled polytopes. For $\widetilde{R} \in \mathcal{R}_\kappa$, this can be seen analogously by subdivision into orthoschemes and application of Schläfli's formula for each of the dissecting orthoschemes. Then, collecting all differential expressions in $d\mathrm{Vol}_n(\widetilde{R})$ suitably, one obtains the generalized Schläfli formula for complete hyperbolic orthoschemes in terms of the dihedral angles.

Moreover, the Theorem remains true for the Schläfli function on the set of acute-angled polytopes of fixed schematic type (see **1.3**) in H^n , $n \geq 2$, which are non-compact but of finite volume (for $n = 3$, we have to cut off a horospherical neighborhood around each vertex at infinity before evaluating Schläfli's function on the apex edges). By dissection, it suffices again to consider asymptotic orthoschemes, and among them only those of dimension $n \geq 3$. Since an orthoscheme in H^n has at most the two principal vertices at infinity (see **1.4**), there are only d-asymptotic orthoschemes, $0 \leq d \leq 2$, and their congruence classes are described by n essential angles $\alpha_1, \ldots, \alpha_n$ ($n - d$ of them form a system of independent parameters). Furthermore, each d-asymptotic orthoscheme may be interpreted as limiting polytope of a sequence of compact orthoschemes. Hence, by analyticity of the volume function, Schläfli's differential formula (13) holds for a family of d-asymptotic orthoschemes in H^n. We formulate this result only partially in the following sense:

COROLLARY.

Let F_n, $n \geq 4$, be the Schläfli function on the set complete orthoschemes of schematic type ς and of finite volume in H^n with essential angles $\alpha_1, \ldots, \alpha_{m(\varsigma)}$ and with scheme Σ_ς. Denote by $F_{n-2}(k)$ the Schläfli function of the apex of codimension 2 associated to the essential angle α_k of measure $f_1(k) := f_1(\alpha_k)$, $1 \leq k \leq m(\varsigma)$. Then

$$dF_n(\Sigma_\varsigma) = \sum_{k=1}^{m(\varsigma)} F_{n-2}(k)\, df_1(k) \quad . \tag{14}$$

3. The reduction formula

3.1 Generalizing Schläfli's method for spherical orthoschemes, we prove the following reduction formula for even dimensional hyperbolic orthoschemes of degree d, $0 \leq d \leq 2$, in terms of the modified Schläfli function (for complete orthoschemes of dimension four, this is already proved in **[K1]**, §4, by a different method):

THEOREM. (Reduction formula)

Denote by $R_d \subset H^{2n}$, $0 \leq d \leq 2$, $n \geq 1$, a $2n$-dimensional orthoscheme of degree d with scheme Σ_d. Then

$$F_{2n}(\Sigma_d) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{2k}{k} \sum_{\sigma} f_{2n-(2k+1)}(\sigma) \quad , \quad \sum f_{-1} := 1 \quad , \tag{15}$$

where σ runs through all elliptic subschemes of order $2(n-k)$ of Σ_d all of whose components are of even order.

Proof: For $0 \le d \le 2$ and $n \ge 1$, let R_d be a compact orthoscheme of degree d and dimension $2n$. Proving the formula (15) we proceed by induction on the dimension.

For $n = 1$, we have by definition (see (8) and (12)) that $F_2(\Sigma_d) = -\frac{2}{\pi}\,\mathrm{Vol}_2(R_d)$ and $f_1(\alpha) := f_1(\sigma) = \frac{2}{\pi}\,\alpha$, if σ is of weight $\cos\alpha$. By varying the degree d of R_d, we obtain the following cases (see **1.4**):

0. For $d = 0$, $R_0 = R_0(\alpha_1, \alpha_2)$ is a right-angled triangle with essential angles $0 \le \alpha_1, \alpha_2 < \frac{\pi}{2}$ and with area $\mathrm{Vol}_2(R_0) = \frac{\pi}{2} - (\alpha_1 + \alpha_2)$. On the other hand, its scheme Σ_0 is given by (using the notation of **1.3**)

$$\circ \overset{\alpha_1}{\text{———}} \circ \overset{\alpha_2}{\text{———}} \circ \quad .$$

   Thus, the formula (15) reads:

$$F_2(\Sigma_0) = f_1(\alpha_1) + f_1(\alpha_2) - 1 \quad ,$$

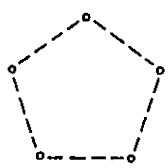
   which is the above area formula for a 2-orthoscheme in terms of the Schläfli functions.

1. For $d = 1$, $R_1 = R_1(\alpha)$ is a Lambert quadrilateral (quadrilateral with three right angles and one acute angle α, $0 \le \alpha < \frac{\pi}{2}$) of area $\mathrm{Vol}_2(R_1) = \frac{\pi}{2} - \alpha$. Since its scheme Σ_1 is of the form (see **1.3**)

$$\circ \cdots\cdot \circ \overset{\alpha}{\text{———}} \circ \cdots\cdot \circ \quad ,$$

   the formula (15) evaluates as $F_2(\Sigma_1) = f_1(\alpha) - 1$.

2. For $d = 2$, R_2 is a totally right-angled pentagon of constant area $\mathrm{Vol}_2(R_2) = \frac{\pi}{2}$. The scheme Σ_2 of R_2 is given by a cycle (see **1.3**)



   Obviously, Σ_2 contains no elliptic subscheme of order > 1. Hence, the formula (15) yields $F_2(\Sigma_2) = -1$ as required.

Now, assume that the assertion holds for complete orthoschemes of even dimension $< 2n$. To show that the volume of a $2n$-dimensional orthoscheme R_d of degree d, $0 \le$

$d \le 2$, satiesfies the formula (15), we interprete the essential angles $\alpha_1, \dots, \alpha_m$ of R_d as independent variables (keeping all notations as before) and differentiate (15) with respect to the measure $f_1(l) := f_1(\alpha_l)$, $1 \le l \le m$. Schläfli's differential formula for hyperbolic complete orthoschemes (see **2.2**, Corollary) has the following effect on the left hand side of (15):

$$\frac{\partial}{\partial f_1(l)} F_{2n}(\Sigma_d) = F_{2n-2}(l) \quad , \tag{16}$$

where, by respecting the dependence among $\alpha_1, \dots, \alpha_m$, $F_{2n-2}(l)$ denotes the normalized volume of the apex $R_d(l)$ corresponding to the angle α_l of measure $f_1(l)$ (see **1.4**). Since $R_d(l)$ is itself a complete orthoscheme of dimension $2(n-1)$ (see **1.4**), we have by induction for the Schläfli function of its scheme $\Sigma_d(l)$:

$$F_{2n-2}(\Sigma_d(l)) = \sum_{k=0}^{n-1} (-1)^k a_k \sum_{\sigma'(l)} f_{2n-2-(2k+1)}(\sigma'(l)) \text{, where } a_k := \frac{1}{k+1}\binom{2k}{k}, \tag{17}$$

and where $\sigma'(l)$ runs through all elliptic subschemes of order $2(n-1-k)$ of $\Sigma_d(l)$ all of whose components are of even order. The differentiation of the right hand side of (15) with respect to $f_1(l)$ leads together with Schläfli's differential formula for spherical orthoschemes to

$$\frac{\partial}{\partial f_1(l)} \sum_{k=0}^{n} (-1)^k a_k \sum_{\sigma} f_{2n-(2k+1)}(\sigma) = \sum_{k=0}^{n-1} (-1)^k a_k \sum_{\sigma''(l)} f_{2n-2-(2k+1)}(\sigma''(l)) \quad , \tag{18}$$

where $\sigma''(l)$ corresponds to the apex of the angle α_l in the orthoscheme with scheme σ in the summation (15). Hence, each $\sigma''(l)$ is an elliptic scheme of even order $|\sigma|-2$. Moreover, all components $\sigma_1, \dots, \sigma_r$, $r \ge 1$, of $\sigma''(l)$ are of even order: In fact, the differentiation of $f_{2n-(2k+1)}(\sigma)$ with respect to $f_1(l)$ affects only one, say σ_ν, of the components of σ all of them being of even order; its associated spherical orthoscheme R_ν has therefore no essential angle equal to $\frac{\pi}{2}$, and the same holds for each of its faces of codimension 2 (see [BH], §4.2, Satz 3, and §4.3, Satz 1). Hence, the apex orthoscheme of R_ν associated to the angle α_l is described by a connected elliptic scheme of even order $|\sigma_\nu| - 2$.

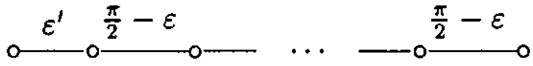
Now, every subscheme $\sigma'(l)$ of order k, $0 \le k \le 2n-2$, in the summation (17) occurs as a subscheme $\sigma''(l)$ of order k in the summation (18), and vice versa (see **1.4**). Hence, by induction hypothesis, we proved (15) up to the value of

$$\frac{(-1)^n}{n+1}\binom{2n}{n} \cdot \sum f_{-1} \quad .$$

It remains to show that this constant of integration, written in the form $(-1)^n\, a_n$ (see (15)), is given by

$$(-1)^n\, a_n = \frac{(-1)^n}{n+1}\binom{2n}{n} \quad . \tag{19}$$

We check (19) first for the case $d = 0$ considering the following scheme of order $2n+1$

$\Sigma_{2n}(\varepsilon,\varepsilon')$:  , $\varepsilon, \varepsilon' > 0$,

with

$\Sigma_k(\varepsilon)$:  , $k = 1, \ldots, 2n-1$,

among the subschemes of order $k+1$. The determinant of these schemes satisfy the following recursion formulas (see [S], Nr. 27, p.257):

$$\det\Sigma_{2n}(\varepsilon,\varepsilon') = \det\Sigma_{2n-1}(\varepsilon) - \cos^2\varepsilon' \det\Sigma_{2n-2}(\varepsilon) \tag{20}$$

$$\det\Sigma_k(\varepsilon) = \det\Sigma_{k-1}(\varepsilon) - \sin^2\varepsilon \det\Sigma_{k-2}(\varepsilon)\,,\ k = 2,\ldots,2n-1\,,\ \det\Sigma_0 := 1\,. \tag{21}$$

From (21) we derive that (see [S], Nr. 28, p.265)

$$\det\Sigma_k(\varepsilon) = \frac{1}{2^k}\,\frac{\left(1+\sqrt{1-4\sin^2\varepsilon}\right)^k - \left(1-\sqrt{1-4\sin^2\varepsilon}\right)^k}{\sqrt{1-4\sin^2\varepsilon}} \quad , \quad k \geq 1 \quad ,$$

i.e., $\det\Sigma_k(\varepsilon) > 0$, and by (21), $\det\Sigma_k(\varepsilon) < \det\Sigma_{k-1}(\varepsilon)$ for $\varepsilon < \frac{\pi}{6}$ and $k \geq 1$. Furthermore, we can represent

$$\frac{\det\Sigma_k(\varepsilon)}{\det\Sigma_{k-1}(\varepsilon)} \quad , \quad k = 1,\ldots,2n-1 \quad ,$$

as a finite continued fraction with k partial quotients (see [S], Nr.27, p.258) using the classical notation of Pringsheim:

$$\frac{\det\Sigma_k(\varepsilon)}{\det\Sigma_{k-1}(\varepsilon)} = 1 - \cfrac{\sin^2\varepsilon}{1 - \cfrac{\sin^2\varepsilon}{\ddots - \cfrac{\sin^2\varepsilon}{1-\sin^2\varepsilon}}} = 1 - \underbrace{\frac{\sin^2\varepsilon|}{|1} - \cdots - \frac{\sin^2\varepsilon|}{|1}}_{k \text{ times}} \quad .$$

Hence, by (20), we can choose ε' in terms of $\varepsilon \in (0, \frac{\pi}{6})$ in such a way that $\det\Sigma_{2n}(\varepsilon,\varepsilon') < 0$, $\det\Sigma_{2n-1}(\varepsilon,\varepsilon') > 0$, and that $\varepsilon'(\varepsilon) \longrightarrow 0$ for $\varepsilon \to 0$; e.g., let $\varepsilon' \in (0, \frac{\pi}{2})$ with

$$\cos^2\varepsilon' = 1 - \underbrace{\frac{\sin^2\varepsilon|}{|1} - \cdots - \frac{\sin^2\varepsilon|}{|1}}_{2n-2 \text{ times}} - \frac{\sin^2\varepsilon|}{|2} \quad , \quad 0 < \varepsilon < \frac{\pi}{6} \quad !$$

Thus, $\Sigma_{2n}(\varepsilon,\varepsilon')$ describes a compact hyperbolic $2n$-orthoscheme $R_{2n}(\varepsilon)$ (see **1.2**) which, for $\varepsilon \to 0$, converges to a degenerate orthoscheme R_e with angles $0, \frac{\pi}{2}, \ldots, \frac{\pi}{2}$ and with scheme

$$\Sigma_e \quad : \quad \circ\overset{\infty}{-\!\!-\!\!-}\circ \quad \circ \quad \circ \quad \cdots \quad \circ \quad \circ \quad .$$

Since $\det\Sigma_e = 0$, the $n+1$ normal vectors in $E^{1,n}$ associated to the bounding hyperplanes of R_e are linear dependent. Therefore, we have in the limit a decrease of dimension, i.e., $\mathrm{Vol}_{2n}(R_e) = 0$. Geometrically, this can be seen by observing that, for $\varepsilon \to 0$, the vertex orthoscheme of $R_{2n}(\varepsilon)$, described by $\Sigma_{2n-1}(\varepsilon)$, converges to a totally rectangular spherical orthoscheme with edge lenghts $\frac{\pi}{2}$ (see [**BH**], Folgerung, p.82); this implies that $R_{2n}(\varepsilon)$ has a triangular face of area $\Delta(\varepsilon) = \frac{\pi}{2} - (h(\varepsilon) + \varepsilon') > 0$, where h is a continuous positive function with $h(\varepsilon) \to \frac{\pi}{2}$ for $\varepsilon \to 0$. Hence, we have $\Delta(\varepsilon) \to 0$ for $\varepsilon \to 0$ implying that at least two vertices of the limiting polytope coincide. Hence, (15) yields:

$$0 = F_{2n}(R_e) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{2k}{k} \sum_{\sigma} f_{2n-(2k+1)}(\sigma) + (-1)^n a_n \quad ,$$

where σ runs through all elliptic subschemes of Σ_e of order $2(n-k)$ all of whose components are of even order. This condition implies the following identity (for a tricky proof, see [**S**], p.255-256, and **2.1**):

$$0 = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{2k}{k} \binom{n+k}{2k} + (-1)^n a_n \quad .$$

Since (see [**S**], p.256)

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{2k}{k} \binom{n+k}{2k} = 0 \quad ,$$

we deduce (19).

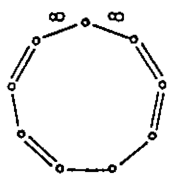
For $d > 0$, we remark first that there are orthoschemes $R_d \subset H^n$, $n \geq 3$, of degree d of different schematic type; although the graphical type and the order of the associated schemes Σ_d are constant, the weights or, more precisely, the number of subschemes of one character of definiteness may differ. Geometrically, this means that orthoschemes of degree d of different schematic type are bounded by the same number of hyperplanes, but in view of their mutual position in H^n, the number of (finite or infinite) vertices may differ. However, the sets of orthoschemes of degree d of different schematic type are not disjoint. Indeed, their intersections consist of the asymptotic complete orthoschemes (i.e., the respective polytopes of transition are described by schemes with parabolic subschemes of rank $n-1$ (see **1.4**)).

Secondly, we observe that a doubly asymptotic orthoscheme of degree 0 (both principal vertices are points at infinity) can be interpreted as asymptotic limiting case of an orthoscheme of degree d (the polar hyperplanes associated to the principal vertices touch the absolute quadric and truncate therefore without effect). Hence, by comparison in the appropriate asymptotic limiting cases, one immediately sees that the constant of integration always equals

$$\frac{(-1)^n}{n+1}\binom{2n}{n} \quad .$$

Q.E.D.

3.2 The Reduction formula (15) can now be applied to all complete Coxeter orthoschemes (for dimensions $2n \geq 4$, see Appendix). We want to give one example in detail: Consider the following cyclic scheme Σ of order 9 describing a 6-dimensional non-compact Coxeter orthoscheme R of degree 2:



Determining the volume $\mathrm{Vol}_6(R)$ according to (15), we have to pick out all different elliptic subschemes σ of Σ of even order $6-2k$, $k = 0, 1, 2$, together with their multiplicities $\mu(\sigma)$, and evaluate $f_{6-(2k+1)}$ on σ (see **2.1**, Lemma and (8)-(11)):

k	$\mu(\sigma)$	σ	$f_{6-(2k+1)}(\sigma)$
0	1	o═o o—o═o—o	$\frac{1}{144}$
	1	o═o o—o—o═o	$\frac{1}{48}$
	1	o—o o═o o═o	$\frac{1}{6}$
1	1	o—o═o—o	$\frac{1}{72}$
	2	o—o—o═o	$\frac{1}{24}$
	1	o—o o—o	$\frac{4}{9}$
	5	o—o o═o	$\frac{1}{3}$
	4	o═o o═o	$\frac{1}{4}$
2	3	o—o	$\frac{2}{3}$
	4	o═o	$\frac{1}{2}$

Hence, according to (15), we obtain:

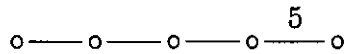
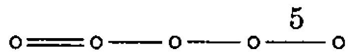
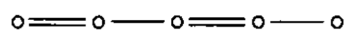
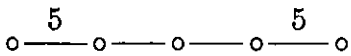
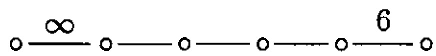
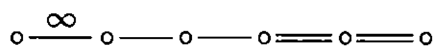
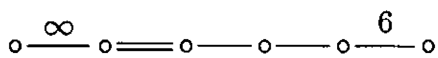
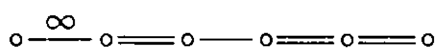
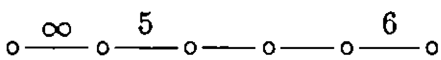
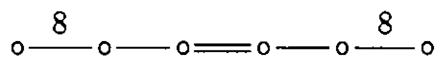
$$\begin{aligned} F_6(\Sigma) = (-1)^3\, c_6 \operatorname{Vol}_6(R) &= \sum_{k=0}^{3} \frac{(-1)^k}{k+1} \binom{2k}{k} \sum f_{6-(2k+1)} \\ &= \sum f_5 - \sum f_3 + 2 \cdot \sum f_1 - 5 \cdot \sum f_{-1} \\ &= \frac{7}{36} - \frac{77}{24} + 2 \cdot 4 - 5 \\ &= -\frac{1}{72} \quad . \end{aligned}$$

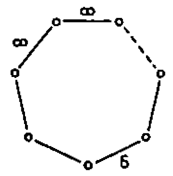
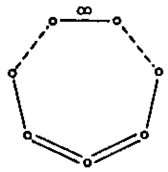
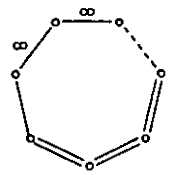
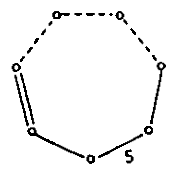
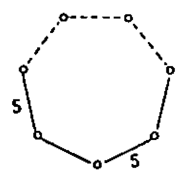
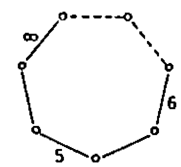
Since $c_6 = \dfrac{120}{\pi^3}$, we get:

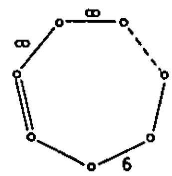
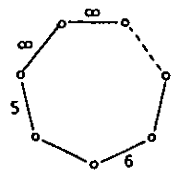
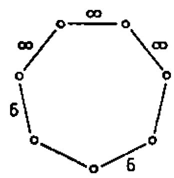
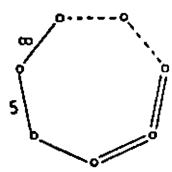
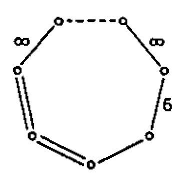
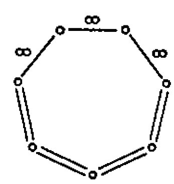
$$\operatorname{Vol}_6(R) = -\frac{1}{c_6} \cdot F_6(\Sigma) = \frac{\pi^3}{8'640} \simeq 0.0036 \quad .$$

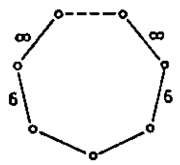
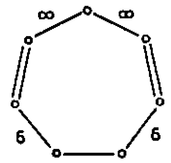
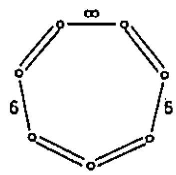
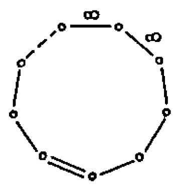
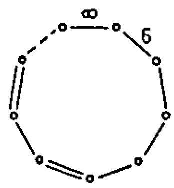
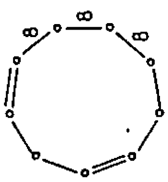
Appendix

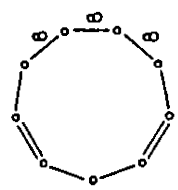
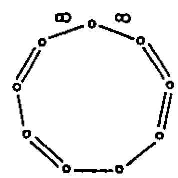
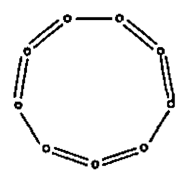
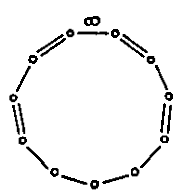
The complete Coxeter orthoschemes $R_{d,C}$ of degree d, $0 \leq d \leq 2$, were classified by Im Hof in 1983 (see **[IH]**). They exist only up to dimension 9; in each dimension ≥ 4, there are finitely many examples. We now list the graphs Σ_d and volumes $\mathrm{Vol}_{2n}(R_{d,C})$ of all complete Coxeter orthoschemes of degree d and of even dimension $2n \geq 4$:

$2n$	d	Σ_d	$\mathrm{Vol}_{2n}(R_{d,C})$
4	0		$\frac{\pi^2}{10'800} \simeq 0.0009$
			$\frac{17\pi^2}{21'600} \simeq 0.0078$
			$\frac{\pi^2}{864} \simeq 0.0114$
			$\frac{13\pi^2}{5'400} \simeq 0.0238$
4	1		$\frac{\pi^2}{540} \simeq 0.0183$
			$\frac{\pi^2}{288} \simeq 0.0343$
			$\frac{41\pi^2}{10800} \simeq 0.0375$
			$\frac{17\pi^2}{4'320} \simeq 0.0388$
			$\frac{5\pi^2}{864} \simeq 0.0571$
			$\frac{\pi^2}{288} \simeq 0.0343$
			$\frac{\pi^2}{144} \simeq 0.0685$
			$\frac{61\pi^2}{900} \simeq 0.6689$
			$\frac{11\pi^2}{1'728} \simeq 0.0628$

$2n$	d	Σ_d	$\mathrm{Vol}_{2n}(R_{d,C})$
4	2		$\frac{\pi^2}{108} \simeq 0.0914$
			$\frac{\pi^2}{108} \simeq 0.0914$
			$\frac{\pi^2}{72} \simeq 0.1371$
			$\frac{\pi^2}{120} \simeq 0.0823$
			$\frac{\pi^2}{90} \simeq 0.1097$
			$\frac{7\pi^2}{540} \simeq 0.1279$

$2n$	d	Σ_d	$\mathrm{Vol}_{2n}(R_{d,C})$
4	2		$\frac{\pi^2}{72} \simeq 0.1371$
			$\frac{\pi^2}{60} \simeq 0.1645$
			$\frac{\pi^2}{54} \simeq 0.1828$
			$\frac{\pi^2}{90} \simeq 0.1097$
			$\frac{\pi^2}{72} \simeq 0.1371$
			$\frac{\pi^2}{48} \simeq 0.02056$

$2n$	d	Σ_d	$\mathrm{Vol}_{2n}(R_{d,C})$
4	2		$\frac{\pi^2}{108} \simeq 0.0914$
			$\frac{\pi^2}{48} \simeq 0.2056$
			$\frac{5\pi^2}{216} \simeq 0.2285$
6	1		$\frac{11\pi^3}{86'400} \simeq 0.0039$
			$\frac{\pi^3}{86'400} \simeq 0.0004$
6	2		$\frac{\pi^3}{43'200} \simeq 0.0007$
			$\frac{7\pi^3}{129'600} \simeq 0.0017$
			$\frac{\pi^3}{17'280} \simeq 0.0018$

$2n$	d	Σ_d	$\mathrm{Vol}_{2n}(R_{d,C})$
6	2		$\frac{389\pi^3}{25'290} \simeq 0.4653$
			$\frac{\pi^3}{8'640} \simeq 0.0036$
			$\frac{\pi^3}{3'240} \simeq 0.0096$
8	2		$\frac{569\pi^4}{43'545'600} \simeq 0.0013$

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