# GALOIS MODULES ARISING FROM FALTINGS'S STRICT MODULES

Victor Abrashkin

## 0. Introduction.

Throughout all this paper $O$ is the valuation ring of a complete discrete valuation field $K$ with perfect residue field $k$ of characteristic $p>0$, $\Gamma_K$ — the absolute Galois group of $K$ and $\pi$ — a uniformising element of $K$.

If characteristic of $K$ is 0, denote by $\mathrm{FGr}'(\mathbb{Z}_p)_O$ the category of finite flat commutative group schemes $G$ over $O$ such that the order $|G|$ is a power of $p$. Any such group scheme appears as a kernel of an isogeny of abelian schemes defined over $O$ and reflects important properties of these abelian schemes.

The classification of objects of the category $\mathrm{FGr}'(\mathbb{Z}_p)_O$ was done in [Fo1] under the restriction $e=1$ on the absolute ramification index $e=e(K)$ of the extension $K/\mathbb{Q}_p$ in terms of finite Honda systems (this classification was not complete for $p=2$, for an improved version cf. [Ab1]). Further progress was done in papers [Ab3] for $e\leqslant p-1$ (group schemes killed by $p$), [Co] for $e<p-1$ and, finally, in [Br] for an arbitrary $e$.

Most interesting number theoretic application of the theory of finite flat group schemes come from the study of the structure of the $\Gamma_K$-module $H=G(K_{\mathrm{sep}})$ of geometric points of $G\in\mathrm{FGr}'(\mathbb{Z}_p)_O$. We mention the following three results:

A. *Serre's Conjecture* (proved in [Ra]).

This result describes the action of the inertia subgroup $I_K\subset\Gamma_K$ on the semi-simple envelope of $H$. It is given by characters $\chi: I_K\longrightarrow k^*$ such that for some $N\in\mathbb{N}$, $\chi=\chi_N^a$, where $\chi_N(\tau)=\tau(\pi_N)/\pi_N$, $\pi_N^{p^N-1}=\pi$, and $a=a_0+a_1p+\dots+a_{N-1}p^{N-1}$ with $p$-digits $a_0,a_1,\dots,a_{N-1}\in[0,e]$;

B. *Ramification estimates.*

If $p^M\,\mathrm{id}_G=0$ then the ramification subgroups $\Gamma_K^{(v)}$ of $\Gamma_K$ act trivially on $H$ if $v>e(M-1+1/(p-1))$, cf. [Fo2];

C. *Complete description of* $\Gamma_K$*-module* $H$ (in the case $e=1$, $p>3$ and $pH=0$);

If $e=1$, $p>3$ and $\mathbb{F}_p[\Gamma_K]$-module $H$ satisfies the above Serre's Conjecture and the ramification estimates (i.e. the ramification subgroups $\Gamma_K^{(v)}$ act trivially on $H$ if $v>\dfrac{1}{p-1}$), then there is an $G\in\mathrm{FGr}'(\mathbb{Z}_p)_O$ such that $H=G(K_{\mathrm{sep}})$, cf.[Ab3].

Suppose now that characteristic of $K$ is $p$. In this case a reasonable analogue of the concept of finite flat group scheme would give a way to study kernels of isogenies of Drinfeld modules. This analogue should appear as a finite flat commutative group scheme $G$ with continuous action of a closed subring $O_0 = \mathbb{F}_q[[\pi_0]] \subset O$. The notion of $O_0$-module scheme was not very helpful until Faltings introduced in [Fa] a concept of strict $O_0$-action. The main idea of Faltings's definition can be explained as follows.

Suppose $G = \operatorname{Spec} A$ is a finite flat $O_0$-module over $O$. Present $A$ as a quotient $O[X_1, \dots, X_n]/I$ of a ring of polynomials by an ideal $I$ and define a deformation $A^\flat$ as $O[X_1, \dots, X_n]/(I \cdot I_0)$, where $I_0 = (X_1, \dots, X_n)$. Faltings requires that $O_0$-module structure on $G$, which is given by endomorphisms $[o] : A \longrightarrow A$, $o \in O_0$, should have an extension to an $O_0$-module structure of the deformation $(A, A^\flat)$ of the algebra $A$ and this extension must satisfy the condition of strictness. This means that if $[o]^\flat : A^\flat \longrightarrow A^\flat$ is an extension of $[o]$ for $o \in O_0$, then $[o]^\flat$ must induce multiplication by $o$ on $I_0/I_0^2$ and $I/(I \cdot I_0)$. This definition gives the category $\mathrm{FGr}'(O_0)_O$ of strict $O_0$-modules and its objects have many interesting properties discussed in [Fa].

In this paper we study number theoretic properties of strict $O_0$-modules. In n.1 we present a concept of strict $O_0$-module in a slightly different but an equivalent to the original definition by Faltings way. In n.2 we describe the category of strict $\mathbb{F}_q$-modules over $O$ and apply this in n.3 to the classification of objects of the category $\mathrm{FGr}'(O_0)_O$. This classification does not depend on the ramification index of $K$ over $\operatorname{Frac} O_0 = K_0$ and requires only the study of primitive elements (i.e. the elements $a \in A(G)$ such that $\Delta a = a \otimes 1 + 1 \otimes a$ where $\Delta$ is comultiplication) of the $O$-algebra $A(G)$ of $G \in \mathrm{FGr}'(O_0)_O$. We apply then this classification to prove that any object of $\mathrm{FGr}'(O_0)_O$ can be embedded into a $\pi_0$-divisible group over $O$. This section contains also a comparison of our antiequivalence with parallel results in the theory of finite flat group schemes and $p$-adic representations in the mixed characteristic case from papers [Ab1,3], [Br] and [Fo4]. In n.4 we establish precise analogues of the above properties $A, B$ and $C$ in the category $\mathrm{FGr}'(O_0)_O$.

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## 1. Definition and simplest properties.

Let $O$ be the valuation ring of a complete discrete valuation field $K$ with perfect residue field $k$ of characteristic $p > 0$. All $O$-algebras are assumed usually to be finite, i.e. to be free $O$-modules of finite rank.

1.1. *Deformations of augmented $O$-algebras.*

For an augmented $O$-algebra $A$, we agree to use the following notation: $\varepsilon_A : A \longrightarrow O$ — the morphism of augmentation; $\operatorname{Ker} \varepsilon_A = I_A$ — the augmentation ideal; $A^{loc} = \varprojlim_n A/I_A^n$ — the $I_A$-completion of $A$; $\eta_A : A \longrightarrow A^{loc}$ — a natural projection. Notice, the correspondence $A \mapsto A^{loc}$ is functorial.

The objects of the category $\mathrm{DAlg}_O$ are the triples $\mathcal{A} = (A, A^\flat, i_{\mathcal{A}})$, where $A$ is a finite $O$-algebra, $A^\flat$ is an $O$-algebra such that there is a polynomial ring $O[\bar{X}] = O[X_1, \dots, X_n]$, $n \geqslant 0$, and its ideal $I$ such that $A^{loc} = O[\bar{X}]/I$ and $A^\flat = O[\bar{X}]/(I \centerdot I_0)$ with $I_0 := (X_1, \dots, X_n) \supset I$, and $i_{\mathcal{A}} : A^\flat \longrightarrow A^{loc}$ is a natural epimorphism of $O$-algebras.

A morphism $\bar{f} = (f, f^\flat) : \mathcal{A} \longrightarrow \mathcal{B} = (B, B^\flat, i_{\mathcal{B}})$ in $\mathrm{DAlg}_O$ is given by a morphism of augmented $O$-algebras $f : A \to B$ and an $O$-algebra morphism $f^\flat : A^\flat \longrightarrow B^\flat$ such that $i_{\mathcal{A}} \circ f^{loc} = f^\flat \circ i_{\mathcal{B}}$.

In the category $\mathrm{DAlg}_O$, $\mathcal{O} = (O, O, \mathrm{id}_O)$ is an initial object and any $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$ has a natural augmentation to $\mathcal{O}$, $\varepsilon_\mathcal{A} = (\varepsilon_A, \varepsilon_{A^\flat}) : \mathcal{A} \longrightarrow \mathcal{O}$, where $\mathrm{Ker}\,\varepsilon_{A^\flat} = \mathrm{Ann}(\mathrm{Ker}\, i_\mathcal{A}) := I_{A^\flat}$.

Notice that the $O$-modules $t^*_\mathcal{A} = I_0/I_0^2$ (this module is free) and $N_\mathcal{A} = I/(I \cdot I_0)$ do not depend on the choice of the covering $O[\bar{X}] \longrightarrow A^\flat$. Indeed, the first coincides with $I_{A^\flat}/I^2_{A^\flat}$ and the second — with $\mathrm{Ker}\, i_\mathcal{A}$.

If $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$ and $\mathcal{B} = (B, B^\flat, i_\mathcal{B})$ are objects of $\mathrm{DAlg}_O$ and $f : A \longrightarrow B$ is a morphism of augmented $O$-algebras, then the set of all $f^\flat$ such that $(f, f^\flat) \in \mathrm{Hom}_{\mathrm{DAlg}_O}(\mathcal{A}, \mathcal{B})$ is not empty and has a natural structure of a principal homogeneous space over the group $\mathrm{Hom}_{O\text{-mod}}(t^*_\mathcal{A}, N_\mathcal{B})$.

The deformation $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$ of $A$ will be called minimal if $i_\mathcal{A}$ induces isomorphism of $t^*_\mathcal{A} \otimes k$ onto $I_A/I_A^2 \otimes k$. In other words, $f$ is minimal if minimal systems of generators of $O$-algebras $A^\flat$ and $A^{\mathrm{loc}}$ contain the same number of elements. It is easy to see that if $\mathcal{A}$ is minimal, $\mathcal{A}' = (A, {A'}^\flat, i_{\mathcal{A}'}) \in \mathrm{DAlg}_O$ and $\bar{f} = (\mathrm{id}_A, f^\flat) \in \mathrm{Hom}_{\mathrm{DAlg}_O}(\mathcal{A}, \mathcal{A}')$ then there is an $\bar{g} \in \mathrm{Hom}_{\mathrm{DAlg}_O}(\mathcal{A}', \mathcal{A})$ such that $\bar{f} \circ \bar{g} = \mathrm{id}_\mathcal{A}$. In particular, all minimal deformations of a given $O$-algebra $A$ are isomorphic in $\mathrm{DAlg}_O$.

Let $\mathrm{DAlg}^*_R$ be a quotient category for $\mathrm{DAlg}_O$: it has the same objects but its morphisms are equivalence classes of morphisms from $\mathrm{Hom}_{\mathrm{DAlg}_O}(\mathcal{A}, \mathcal{B})$ arising from the same $O$-algebra morphisms $f : A \longrightarrow B$. Then the forgetful functor $\mathcal{A} = (A, A^\flat, i_\mathcal{A}) \mapsto A$ is an equivalence of $\mathrm{DAlg}^*_O$ and the category of augmented finite $O$-algebras.

1.2. *Deformations of affine group schemes.*

Let $\mathrm{DSch}_O$ be the dual category for $\mathrm{DAlg}_O$. Its objects appear in the form $\mathcal{H} = \mathrm{Spec}\,\mathcal{A} = (H, H^\flat, i_\mathcal{H})$, where $H = \mathrm{Spec}\, A$ and $H^\flat = \mathrm{Spec}\, A^\flat$ are finite flat $O$-schemes, $\mathcal{A} = (A, A^\flat, i_\mathcal{A}) \in \mathrm{DAlg}_O$, and $i_\mathcal{H} : H \to H^\flat$ is a closed embedding of $O$-schemes. This category has direct products: if for $i = 1, 2$, $\mathcal{A}_i = (A_i, A_i^\flat, i_{\mathcal{A}_i})$ with $A_i^{loc} = O[\bar{X}_i]/I$, $A_i^\flat = O[\bar{X}_i]/(I_i \cdot I_{0i})$, then the product $\mathrm{Spec}\,\mathcal{A}_1 \times \mathrm{Spec}\,\mathcal{A}_2$ is given by $\mathrm{Spec}(\mathcal{A}_1 \otimes \mathcal{A}_2)$, where $\mathcal{A}_1 \otimes \mathcal{A}_2 := (A_1 \otimes_O A_2, (A_1 \otimes_O A_2)^\flat, \kappa)$, $(A_1 \otimes_O A_2)^\flat$ is the quotient of $O[\bar{X}_1 \otimes 1, 1 \otimes \bar{X}_2]$ by the product of ideals $I_1 \otimes 1 + 1 \otimes I_2$ and $I_{01} \otimes 1 + 1 \otimes I_{02}$ and $\kappa$ is the natural projection. Notice that for $i = 1, 2$, two projections $\mathrm{pr}_i$ from this product to its components $\mathrm{Spec}\,\mathcal{A}_i$ come from the natural embeddings of $O[\bar{X}_i]$ into $O[\bar{X}_1 \otimes 1, 1 \otimes \bar{X}_2]$.

Let $\mathrm{FGr}_O$ be the category of group objects in $\mathrm{DSch}_R$. If $\mathcal{G} = \mathrm{Spec}\,\mathcal{A} \in \mathrm{FGr}_O$ then its group structure is given via comultiplication $\bar{\Delta} = (\Delta, \Delta^\flat) : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$, counit $\bar{\varepsilon} = (\varepsilon, \varepsilon^\flat) : \mathcal{A} \longrightarrow \mathcal{O}$ and coinversion $\bar{\imath} = (i, i^\flat) : \mathcal{A} \longrightarrow \mathcal{A}$ morphisms, which satisfy usual axioms. The morphisms in $\mathrm{FGr}_O$ are morphisms of group objects. As usually, $\mathrm{FGr}_O$ is an additive category.

Notice that

a) $G = \mathrm{Spec}\, A$ is a finite flat group scheme over $O$ with comultipilication $\Delta$, counit $\varepsilon$ and coinversion $i$;

b) $\bar{\varepsilon} = \varepsilon_\mathcal{A}$, cf. n.1.1.

c) the counit axiom gives for $i = 1, 2$, $\Delta_i^\flat \circ \mathrm{pr}_i = \mathrm{id}_{A^\flat}$ and implies a uniqueness of $\Delta^\flat$ as a lifting of $\Delta$;

d) if $\mathcal{A} = (A, A^\flat, i_\mathcal{A}) \in \mathrm{DAlg}_O$ and $G = \mathrm{Spec}\, A$ is an affine group scheme then there is a unique structure of group object on $\mathrm{Spec}\,\mathcal{A}$ compatible with that of $G$;

e) if $f : G \longrightarrow H$ is a morphism of group schemes and $(f, f^\flat) \in \mathrm{Hom}_{\mathrm{DSch}_O}(\mathcal{G}, \mathcal{H})$

then $(f, f^\flat) \in \mathrm{Hom}_{\mathrm{FGr}_O}(\mathcal{G}, \mathcal{H})$.

The above properties have the following interpretation. Define the quotient category $\mathrm{FGr}_O^*$ as the category consisiting of the objects of the category $\mathrm{FGr}_O$ but where $\mathrm{Hom}_{\mathrm{FGr}_O^*}(\mathcal{G}, \mathcal{H})$ consists of equivalence classes of morphisms from the category $\mathrm{FGr}_O$ which induce the same morphisms of affine group schemes $G \to H$. Then the natural functor $\mathcal{G} \mapsto G$ is an equivalence of categories. We can use this equivalence to define the abelian groups of equivalence classes of short exact sequences $\mathrm{Ext}(\mathcal{G}, \mathcal{H})$ in $\mathrm{FGr}_O$. These groups are functorial by both arguments and there are standard 6-terms exact Hom – Ext-sequences.

1.3. *The categories of strict $R$-modules.*

Suppose $R$ is a ring and $O$ is an $R$-algebra.

Suppose $\mathcal{G}$ is an $R$-module object in the category $\mathrm{DSch}_O$. Then $\mathcal{G}$ is an object of $\mathrm{FGr}_O$ and there is a map $R \longrightarrow \mathrm{End}_{\mathrm{FGr}_O}(\mathcal{G})$ satisfying the usual axioms from the definition of $R$-module. For $r \in R$ and $\mathcal{G} = \mathrm{Spec}\,\mathcal{A}$, denote by $\bar{[r]} = ([r], [r]^\flat)$ the morphism of action of $r$ on $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$. Clearly, $G = \mathrm{Spec}\,A$ is an $R$-module in the category of finite flat schemes over $O$. For any such $G$, the $R$-module structure on the deformation $(G, G^\flat, i) \in \mathrm{FGr}_O$ is given by liftings $[r]^\flat : A^\flat \longrightarrow A^\flat$ of morphisms $[r] : A \longrightarrow A$, $r \in R$. Notice that $[r]^\flat$ are morphisms of augmented algebras. All such liftings are automatically compatible with the group structure on this deformation, i.e. for any $r \in R$, it holds $[r] \circ \Delta^\flat = \Delta \circ ([r] \otimes [r])$. So, the above system gives an $R$-module structure if and only if for any $r_1, r_2 \in R$,

$$[r_1 + r_2]^\flat = \Delta^\flat \circ ([r_1] \otimes [r_2])^\flat, \quad [r_1 r_2]^\flat = [r_1] \circ [r_2]^\flat \tag{1}$$

where $([r_1] \otimes [r_2])^\flat$ is induced by $[r_1]^\flat \otimes [r_2]^\flat$.

We denote by $\mathrm{FGr}(R)_O$ the category of such $R$-module objects $\mathcal{G}$ where $R$-action is strict, i.e. if $\mathcal{G} = \mathrm{Spec}\,\mathcal{A}$ then any $r \in R$ acts on $t^*_\mathcal{A}$ and $N_\mathcal{A}$ via scalar multiplication by $r$. This is a basic definition from Faltings' paper [Fa].

Suppose $\mathcal{G}_1 = (G, G_1^\flat, i_1) \in \mathrm{FGr}_O$ and $\mathcal{G}_2 = (G, G_2^\flat, i_2) \in \mathrm{FGr}_O$ are two deformations of a finite flat group scheme $G$ over $O$. By 1.2, $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic in the category $\mathrm{FGr}_O^*$. Suppose $\mathcal{G}_1$ is equiped with the strict $R$-action. Then there is a unique (strict) $R$-action on $\mathcal{G}_2$ such that any $(\mathrm{id}_G, \phi) \in \mathrm{Hom}_{\mathrm{FGr}_O}(\mathcal{G}_1, \mathcal{G}_2)$ and any $(\mathrm{id}_G, \psi) \in \mathrm{Hom}_{\mathrm{FGr}_O}(\mathcal{G}_2, \mathcal{G}_1)$ are, actually, morphisms in the category $\mathrm{FGr}(R)_O$.

Denote by $\mathrm{FGr}^*(R)_O$ the quotient category of $\mathrm{FGr}(R)_O$ where morphisms are equivalence classes of morphisms $(G, G^\flat, i) \longrightarrow (H, H^\flat, j)$ in the category $\mathrm{FGr}(R)_O$ inducing the same morphism $G \longrightarrow H$. By the above property, all isomorphism classes of objects in $\mathrm{FGr}(R)_O$ appear as $R$-module finite flat schemes $G$ together with a lifting of its $R$-action to some chosen deformation $G^\flat$ which satisfies the above conditions (1).

For example, if $q = p^n$ with $n \in \mathbb{N}$, the objects of the category $\mathrm{FGr}(\mathbb{F}_q)_O$ appear as $\mathrm{Spec}\,\mathcal{A}$, where $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$, $A^{loc} = O[\bar{X}]/I$ and $A^\flat = O[\bar{X}]/(I \cdot I_0)$, and $\mathbb{F}_q$-action is induced by the scalar action of $\mathbb{F}_q$ on $O[\bar{X}]$ (i.e. $[\alpha](\bar{X}) = \alpha\bar{X}$, $\alpha \in \mathbb{F}_q$) and there are generators $j_1, \dots, j_n$ of the ideal $I$ such that $[\alpha]j_i = \alpha j_i$ for all $i = 1, \dots, n$ and $\alpha \in \mathbb{F}_q$.

If $R = \mathbb{F}_q[\pi]$ with an indeterminate $\pi$ and $\mathcal{G} \in \mathrm{FGr}(R)_O$ then $\mathcal{G} \in \mathrm{FGr}(\mathbb{F}_q)_O$ and (in addition to the above assumptions) $R$-action will be determined completely by the action of $\pi$ given by the correspondence

$$\bar{X} \mapsto [\pi](\bar{X}) = \pi\bar{X} + \bar{F}(\bar{X})$$

where $\bar{F}$ is any vector power series with coefficients in $O$ from $I_0^2$ such that $\bar{F}(\alpha\bar{X}) = \alpha\bar{F}(\bar{X})$ for all $\alpha \in \mathbb{F}_q$. This action is strict iff $[\pi]j_i \equiv \pi j_i \operatorname{mod}(I \cdot I_0)$ for the above generators $j_1, \dots, j_n$ of $I$.

1.4. Suppose $i : \mathcal{G}_1 \longrightarrow \mathcal{G}$ is a closed embedding of strict $R$-modules. Then the quotient $\mathcal{G}_2 = \mathcal{G}/\mathcal{G}_1$ has a natural structure of a strict $R$-module scheme and the projection $j : \mathcal{G} \longrightarrow \mathcal{G}_2$ is a morphism in the category $\mathrm{FGr}(R)_O$. Similarly, if $j : \mathcal{G} \longrightarrow \mathcal{G}_2$ is a fully faithful morphism of strict $R$-module schemes then its kernel $\mathcal{G}_1$ is a strict $R$-module and its (closed) embedding $i : \mathcal{G}_1 \longrightarrow \mathcal{G}$ is a morphism from $\mathrm{FGr}(R)_O$.

Notice that for any $\mathcal{G} = (G, G^\flat, i_\mathcal{G}) \in \mathrm{FGr}(R)_O$, the short exact sequence of group schemes

$$0 \longrightarrow G^{\mathrm{loc}} \longrightarrow G \longrightarrow G^{\mathrm{et}} \longrightarrow 0 \qquad (2)$$

(here, $G^{\mathrm{loc}}$ is the maximal local subgroup scheme of $G$ and $G^{\mathrm{et}}$ is the maximal etale quotient of $G$) induces a short exact sequence in $\mathrm{FGr}(R)_O$

$$0 \longrightarrow \mathcal{G}^{\mathrm{loc}} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^{\mathrm{et}} \longrightarrow 0 \qquad (3)$$

where $\mathcal{G}^{\mathrm{loc}} = (G^{\mathrm{loc}}, G^\flat)$, $\mathcal{G}^{\mathrm{et}} = (G^{\mathrm{et}}, \operatorname{Spec} O)$.

Finally, notice that over the valution ring $O_1$ of some unramified extension $K_1$ of $K$, $G^{\mathrm{et}} \otimes O_1$ is constant, i.e. $A(G^{\mathrm{et}}) \otimes O_1 = \mathrm{Map}(H^{\mathrm{et}}, O_1)$ with $H^{\mathrm{et}} = G^{\mathrm{et}}(K_1)$. This also implies the existence of a finite extension $K_1'$ of $K$ such that the short exact sequence (2) (as well as (3)) splits over the valuation ring of $K_1'$.

## 2. Group schemes with strict $\mathbb{F}_q$-action.

As earlier, $O$ is the valuation ring of a complete discrete valuation field $K$ of characteristic $p$ with perfect residue field $k$, $q = p^{N_0}$ with $N_0 \in \mathbb{N}$. In this section we assume that $O$ is an $\mathbb{F}_q$-algebra (then $\mathbb{F}_q \subset k$ and $O \simeq k[[\pi]]$, where $\pi$ is a uniformiser in $K$) and study the full subcategory $\mathrm{FGr}'(\mathbb{F}_q)_O$ of the category $\mathrm{FGr}(\mathbb{F}_q)_O$ consisting of strict finite $\mathbb{F}_q$-modules $G$ over $O$ with etale generic fibre. As it was noticed in n.1, its objects appear as finite flat group schemes $G = \operatorname{Spec} A(G)$ over $O$ with $\mathbb{F}_q$-action such that $A(G^{loc})$ (or even $A(G)$) can be presented in the form $O[\bar{X}]/I$ in such a way that $\mathbb{F}_q$-action on $G$ is induced by a scalar $\mathbb{F}_q$-action on coordinates of $\bar{X}$ and on some system of generators of the ideal $I$.

2.1. *The category* $\mathrm{Mod}_{O,\sigma}$ *and the functor* $\mathrm{Gr}$.

Let $\sigma : O \to O$ be the endomorphism of $q$-th power. For any $O$-module $M$, set $M_\sigma = M \otimes_O O$ where an $O$-module structure on $O$ is given via $\sigma$. Objects of the category $\mathrm{Mod}_{O,\sigma}$ are free $O$-modules $M$ of finite rank with an $O$-linear morphism $\Phi : M_\sigma \to M$ such that its $K$-linear extension $\Phi_K$ is an isomorphism of $K$-vector spaces $M_{\sigma,K} = M_\sigma \otimes_O K$ and $M_K = M \otimes_O K$. Morphisms in $\mathrm{Mod}_{O,\sigma}$ are morphisms of $O$-modules commuting with $\Phi$-action.

For an $M \in \mathrm{Mod}_{O,\sigma}$, introduce the $O$-algebra

$$A[M] = \mathrm{Sym}_O\, M/\{\Phi m - m^{\otimes q} \mid m \in M\}$$

In other words, if $m_1, \dots, m_n$ is an $O$-basis of $M$ and $\Phi m_i = \sum_j o_{ij} m_j$ (where $\det(o_{ij}) \neq 0$), then $A[M] = O[X_1, \dots, X_n]/(\psi_1, \dots, \psi_n)$, where $\psi_i = X_i^q - \sum_j o_{ij} X_j$,

$i = 1, \dots, n$. This $O$-algebra $A[M]$ does not depend on the choice of a basis $m_1, \dots, m_n$ and gives rise to the strict finite flat $\mathbb{F}_q$-module scheme $G[M] = \operatorname{Spec} A[M]$ with comultiplication $\Delta$ such that $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i$ and an $\mathbb{F}_q$-action such that $[\alpha](X_i) = \alpha X_i$, where $\alpha \in \mathbb{F}_q$, $i = 1, \dots, n$. The equality $\Phi(M_{\sigma,K}) = M_K$ implies that the generic fibre of $G[M]$ is etale, and therefore, $G[M] \in \mathrm{FGr}'(\mathbb{F}_q)_O$.

Identify $M$ with its image in $A[M]$ via the natural map $m_i \mapsto X_i$, $i = 1, \dots, n$.

**Proposition 1.**
a) $M = \{m \in A[M] \mid \Delta m = m \otimes 1 + 1 \otimes m, \quad [\alpha]m = \alpha m, \alpha \in \mathbb{F}_q\}$;
b) *the natural inclusions* $M_i \subset A[M_i]$, $i = 1, 2$, *induce the identification*

$$\mathrm{Hom}_{\mathrm{Mod}_{O,\sigma}}(M_1, M_2) = \mathrm{Hom}_{\mathrm{FGr}(\mathbb{F}_q)_O}(G[M_2], G[M_1]).$$

This proposition implies that the correspondence $M \mapsto G[M]$ induces a fully faithful functor $\mathrm{Gr} : \mathrm{Mod}_{O,\sigma} \longrightarrow \mathrm{FGr}'(\mathbb{F}_q)_O$. The rest of this section will be devoted to the proof of the following theorem.

**Theorem 1.** *The functor* Gr *is an antiequivalence of categories.*

Notice that the correspondence $\mathrm{Gr} \otimes K : M_K \mapsto G[M](K_{sep})$ is an antiequivalence of the category of $\sigma$-etale $K$-modules and the category of $\mathbb{F}_q[\Gamma_K]$-modules, cf. [Fo4].

2.2. Later we need the following property of the Hochschild cohomology of group schemes $G[M]$.

**Lemma 1.** *With the above notation suppose that* $a \in A[M]_K$ *is such that* $\delta^+ a := \Delta(a) - a \otimes 1 - 1 \otimes a \in A[M] \otimes A[M]$ *and* $[\alpha]a = \alpha a$ *for all* $\alpha \in \mathbb{F}_q$. *Then there is an* $a' \in A[M]$ *such that* $\delta^+ a = \delta^+ a'$ *and* $a - a' \in M_K$.

*Proof of Lemma 1.* Let $\delta^+_k = \delta^+ \otimes k : A[M]_k \longrightarrow A[M]_k^{\otimes 2}$, where $A[M]_k = A[M] \otimes k$. It will be sufficient to prove that

$$\{b \in A[M]_k \mid \delta^+_k b = 0, \ [\alpha]b = \alpha b, \ \forall \alpha \in \mathbb{F}_q\} = M_k \tag{4}$$

Note that

$$\{X_1^{i_1} \dots X_n^{i_n} \mid 0 \leqslant i_1, \dots, i_n < q\}$$

is a $k$-basis of $A[M]_k$ and

$$\{X_1^{j_1} \dots X_n^{j_n} \otimes X_1^{l_1} \dots X_n^{l_n} \mid 0 \leqslant j_1, \dots, j_n, l_1, \dots, l_n < q\}$$

is a $k$-basis of $A[M]_k^{\otimes 2}$. Now the equality (4) is implied by the following 3 observations:

a) for any $0 \leqslant i_1, \dots, i_n < q$, $\delta^+_k(X_1^{i_1} \dots X_n^{i_n})$ is a linear combination of $X_1^{j_1} \dots X_n^{j_n} \otimes X_1^{l_1} \dots X_n^{l_n}$ with $j_1, \dots, j_n, l_1, \dots, l_n \geqslant 0$ such that $j_1 + l_1 = i_1$, $\dots, j_n + l_n = i_n$;

b) if $\sum\limits_{0 \leqslant i_1, \dots, i_n < q} \alpha_{i_1 \dots i_n} X_1^{i_1} \dots X_n^{i_n} \in \operatorname{Ker} \delta^+_k$ then for each multi index $(i_1, \dots, i_n)$, either $\alpha_{i_1 \dots i_n} = 0$, or $\delta^+_k(X_1^{i_1} \dots X_n^{i_n}) = 0$;

c) for any $0 \leqslant i_1, \dots, i_n < q$, $\delta^+_k(X_1^{i_1} \dots X_n^{i_n}) = 0$ iff
$X_1^{i_1} \dots X_n^{i_n} \in \{X_s^{p^t} \mid 1 \leqslant s \leqslant n, 0 \leqslant t < N_0\}$.

2.3. For $G\in \mathrm{FGr}(\mathbb{F}_q)_O$, let

$$L(G)=\{a\in A(G)\mid \Delta a=a\otimes 1+1\otimes a,\ [\alpha]a=\alpha a, \forall \alpha\in\mathbb{F}_q\}.$$

Clearly, the above lemma implies that if $M\in \mathrm{Mod}_{O,\sigma}$, then $L(G[M])=M$.

The following lemma can be proved directly from the above definition of $L(G)$.

**Lemma 2.**
a) *If $O_1$ is a complete discrete valuation ring containing $O$, then*
*$L(G\otimes O_1)=L(G)\otimes O_1$;*
b) *For any two objects $G_1$ and $G_2$ of $\mathrm{FGr}(\mathbb{F}_q)_O$, it holds*
*$L(G_1\times G_2)=L(G_1)\oplus L(G_2)$.*

The $q$-th power map on $A(G)$ induces $\Phi: L(G)_\sigma\longrightarrow L(G)$ which provides $L(G)$ with the structure of an object of the category $\mathrm{Mod}_{O,\sigma}$. Indeed, if $H=G(K_{\mathrm{sep}})$ with the natural structure of $\mathbb{F}_q[\Gamma_K]$-module, then $L(G)_K=\mathrm{Hom}_{\mathbb{F}_q[\Gamma_K]\text{-mod}}(H,K_{\mathrm{sep}})$ and

$$\mathrm{Im}\,\Phi_K=\mathrm{Hom}_{\mathbb{F}_q[\Gamma_K]\text{-mod}}(H,\sigma(K_{\mathrm{sep}}))\otimes_{\sigma(K)}K=L(G)_K$$

because $\sigma(K_{\mathrm{sep}})\otimes_{\sigma(K)}K=K_{\mathrm{sep}}$.

The correspondence $L: G\mapsto L(G)$ is the functor from $\mathrm{FGr}'(\mathbb{F}_q)_O$ to $\mathrm{Mod}_{O,\sigma}$. Clearly, the composition $\mathrm{Gr}\circ L$ is equivalent to the identity functor on $\mathrm{Mod}_{O,\sigma}$. So, Gr is an antiequivalence (with the quasi-inverse functor $L$) if any $G\in\mathrm{FGr}'(\mathbb{F}_q)_O$ is isomorphic to $G[M]$ for a suitable $M\in\mathrm{Mod}_{O,\sigma}$. When proving this property we can enlarge (if necessary) the basic ring $O$ by the following lemma.

**Lemma 3.** *If $O'$ is a complete discrete valuation ring containing $O$ and $G\in\mathrm{FGr}'(\mathbb{F}_q)_O$ then $G\simeq G[M]$ with $M\in\mathrm{Mod}_{O,\sigma}$ if and only if $G\otimes O_1\simeq G[M_1]$ for some $M_1\in\mathrm{Mod}_{O_1,\sigma}$.*

*Proof.* Clearly, if $G\simeq G[M]$ with $M\in\mathrm{Mod}_{O,\sigma}$, then $G\otimes O_1\simeq G[M\otimes O_1]$.

If $G\otimes O_1\simeq G[M_1]$ with $M_1\in\mathrm{Mod}_{O,\sigma}$, then $M_1\simeq L(G\otimes O_1)=L(G)\otimes O_1$, therefore, $G\otimes O_1\simeq G[L(G)]\otimes O_1$ and the natural inclusion $G[L(G)]\subset G$ is an isomorphism. Lemma 3 is proved

2.4. *$\mathbb{F}_q$-module schemes of order $q$.*

**Proposition 2.** *Suppose $G$ is an $\mathbb{F}_q$-module scheme of order $q$ such that $G\otimes K$ is etale and $\mathbb{F}_q$ acts on the cotangent space $t^*_G$ via scalar multiplication. Then $G=G[M]$, where $M=Om$ and $\Phi m=\lambda m$ with some $\lambda\in O$, $\lambda\neq 0$ (in particular, $G$ is a strict $\mathbb{F}_q$-module).*

*Proof.* Let $I_G=\mathrm{Ker}\,e_G$ (where $e_G$ is the counit morphism). If $I_G=I_G^2$ then $G$ is etale and over some bigger (unramified over $O$) basic ring $O_1$, $G\otimes O_1$ is constant, i.e. $G\otimes O_1\simeq G[M_1]$, where $M_1=O_1m_1$ and $\Phi(m_1)=m_1$. Therefore, $G\simeq G[M]$ where $M\otimes O_1\simeq M_1$, i.e. $M=Om$, $m=\xi m_1$ with $\xi\in O_1^*$ and $\Phi m=\lambda m$ with $\lambda=\xi^{q-1}\in O\cap O_1^*=O^*$.

Suppose now that $I_G\neq I_G^2$. Consider the decomposition $I_G=\oplus_{1\leqslant n<q}I_{G,n}$, where an $\alpha\in\mathbb{F}_q$ acts on $I_{G,n}$ via multiplication by $\alpha^n$.

Because $t^*_G=I_G/I_G^2\neq 0$, $\mathrm{rk}_O\,I_{G,1}\geqslant 1$ and, therefore, for all $1\leqslant n<q$, $\mathrm{rk}_O\,I_{G,n}\geqslant 1$ (if $x\in I_{G,1}$, $x\neq 0$, then $x^n\neq 0$ and $x^n\in I_{G,n}$). So, all these ranks are equal to 1 and, if $I_{G,1}=xO$, then the equality $I_G=I_{G,1}+I_G^2$ implies for all $1\leqslant n<q$, $I_{G,n}=x^nO$.

Finally, $x^q \in I_{G,1}$ and $x^q = \lambda x$ for some $\lambda \in O$. Clearly, $\lambda \neq 0$ because $G \otimes K$ is etale. This proves that $A(G) = A[M]$ with $M = Om$, $\Phi m = \lambda m$.

Suppose $\Delta x = x \otimes 1 + 1 \otimes x + \delta$. Then $\delta^q = \lambda\delta$ and $\delta \in (x \otimes x)A(G) \otimes A(G)$. This implies easily that $\delta = 0$. It remains to note that by the choice of $x$, $[\alpha]x = \alpha x$ for any $\alpha \in \mathbb{F}_q$, and this action of $\mathbb{F}_q$ is strict.

**Definition.** We shall denote by $\mu_\lambda$ the $\mathbb{F}_q$-module scheme $G[M]$ from the above proposition.

2.5. Suppose $G \in \mathrm{FGr}'(\mathbb{F}_q)_O$ satisfies the following four assumptions:

1) $G = G^{loc}$;

2) there is a short exact sequence in the category $\mathrm{FGr}'(\mathbb{F}_q)_O$

$$0 \longrightarrow \mu_\lambda \xrightarrow{i} G \longrightarrow G_1 \longrightarrow 0$$

where $\lambda \in O, \lambda \neq 0$;

3) the above exact sequence splits over $K$, i.e. there is a $\Gamma_K$-invariant section of the projection of $\mathbb{F}_q[\Gamma_K]$-modules $G(K_{sep}) \longrightarrow G_1(K_{sep})$;

4) there is an $M_1 \in \mathrm{Mod}_{O,\sigma}$ such that $G_1 \simeq G[M_1]$.

**Proposition 3.** *With the above assumptions 1)-4), there is an $M \in \mathrm{Mod}_{O,\sigma}$ such that $G \simeq G[M]$.*

*Proof.* Let $A(\mu_\lambda) = O[u]$ with $u^q = \lambda u$, $\Delta u = u \otimes 1 + 1 \otimes u$, $[\alpha]u = \alpha u$ for $\alpha \in \mathbb{F}_q$. Suppose the closed embedding $i : \mu_\lambda \longrightarrow G$ is given by the $O$-algebra morphism $i^* : A \longrightarrow A(\mu_\lambda)$. Set $A = A(G)$ and $B = A(G_1)$. Then

$$s = \Delta_G \circ (i^* \otimes \mathrm{id}_A) : A \longrightarrow A(\mu_\lambda) \otimes A$$

is the coaction of $\mu_\lambda$ on $A$ and $B = A^{\mu_\lambda} = \{a \in A \mid s(a) = 1 \otimes a\}$.

Choose an $a_0 \in A$ such that $i^*(a_0) = u^{q-1}$ and $[\alpha]a_0 = a_0$ for all $\alpha \in \mathbb{F}_q$. Set

$$s(a_0) = 1 \otimes a_0 + u \otimes a_1 + \cdots + u^{q-1} \otimes a_{q-1}$$

where $a_1, \ldots, a_{q-1} \in A$. Then for any $\alpha \in \mathbb{F}_q$ and $m = 1, \ldots, q-1$, $[\alpha]a_m = \alpha^{-m}a_m$ and the identity

$$(\mathrm{id} \otimes i^*)(s(a_0)) = \Delta(i^* a_0) = \Delta(u^{q-1}) = (u \otimes 1 + 1 \otimes u)^{q-1}$$

implies that $i^*(a_{q-1}) = 1$ and $i^*(a_{q-2}) = -u$. Therefore, $a_{q-1} \in A^*$ (because $A = A^{loc}$) and $A = B[a_{q-2}]$ (because $a_{q-2}$ generates $A$ modulo a nilpotent ideal $I_B A$).

The equality

$$1 \otimes s(a_0) + u \otimes s(a_1) + \cdots + u^{q-1} \otimes s(a_{q-1}) = (\mathrm{id} \otimes s)(s(a_0))$$

$$= (\Delta \otimes \mathrm{id})(s(a_0)) = 1 \otimes 1 \otimes a_0 + (u \otimes 1 + 1 \otimes u) \otimes a_1 + \cdots + (u \otimes 1 + 1 \otimes u)^{q-1} \otimes a_{q-1}$$

implies that $s(a_{q-1}) = 1 \otimes a_{q-1}$ and $s(a_{q-2}) = 1 \otimes a_{q-2} - u \otimes a_{q-1}$.

Therefore, $a_{q-1} \in B \cap A^* = B^*$ and for $\theta = -a_{q-2}/a_{q-1}$, we have $A = B[\theta]$, $s(\theta) = 1 \otimes \theta + u \otimes 1$, $[\alpha]\theta = \alpha\theta$ for $\alpha \in \mathbb{F}_q$, and $\theta^q - \lambda\theta = b \in B$.

Notice also that $\delta^+\theta = \Delta\theta - \theta\otimes 1 - 1\otimes\theta \in B\otimes B$, because this is a $(\mu_\lambda\times\mu_\lambda)$-invariant element of $A\otimes A$.

From condition 3) it follows the existence of $v\in A_K$ such that $i^*(v)=u$, $v^q=\lambda v$, $\Delta v = v\otimes 1 + 1\otimes v$ and $[\alpha]v=\alpha v$ for $\alpha\in\mathbb{F}_q$. Therefore, denoting the $K$-linear extension of $s$ by $s_K$, we have $s_K(v)=1\otimes v + u\otimes 1$. This implies that $\theta - v = b_0 \in B\otimes K$, and $\delta^+b_0=\delta^+\theta$.

Therefore, by the property of Hochshild cohomology from the end of n.2.1, there is a $b_1\in B$ (with the property $[\alpha]b_1=\alpha b_1$, $\forall\alpha\in\mathbb{F}_q$) such that $\delta^+b_1=\delta^+\theta$ and replacing $\theta$ by $\theta-b_1$ we can assume that $\Delta\theta=\theta\otimes 1+1\otimes\theta$. Therefore, $M=M_1+O\theta\in\mathrm{Mod}_{O,\sigma}$, $A=A[M]$ and $G=G[M]$. The proposition is proved.

2.6. Now we can finish the proof of Theorem 1.

Suppose $G\in\mathrm{FGr}'(\mathbb{F}_q)_O$ is of order $q^N$, $N\in\mathbb{N}$. As it was noticed in n.2.3, it will be sufficient to prove that $G\simeq G[M]$ for a suitable $M\in\mathrm{Mod}_{O,\sigma}$. When proving this property we can enlarge (if necessary) the basic ring $O$.

Apply induction on $N$.

The case $N=1$ follows from Proposition 2 in n.2.4.

Suppose that $G\in\mathrm{FGr}'(\mathbb{F}_q)_O$ is of order $q^N$ and any $\mathbb{F}_q$-module scheme of order $q^{N-1}$ appears in the form $G[M_1]$ with $M_1\in\mathrm{Mod}_{O,\sigma}$. By n.1.4 and Lemma 2b), it will be sufficient to consider the following two cases:

a) $G=G^{\mathrm{et}}$;

In this case we can enlarge $O$ to assume that $G$ is a constant group scheme, where the property $G\simeq G[M]$ with $M\in\mathrm{Mod}_{O,\sigma}$ is obviously true.

b) $G=G^{\mathrm{loc}}$.

In this case enlarge $O$ to be able to assume that $G\otimes K$ is constant, i.e. $H=G(K_{\mathrm{sep}})$ has a trivial $\Gamma_K$-action. Choose a 1-dimensional $\mathbb{F}_q$-subspace $H_2$ in $H$. It gives rise to a short exact sequence of $\mathbb{F}_q$-module group schemes

$$0\longrightarrow G_2\longrightarrow G\longrightarrow G_1\longrightarrow 0$$

where $G_2(K_{\mathrm{sep}})=H_2$.

Because $G$ is a strict $\mathbb{F}_q$-module, $\mathbb{F}_q$ acts on $t_G^*$ and, therefore, on $t_{G_2}^*$, via scalar multiplication. By Proposition 2, $G_2\simeq\mu_\lambda$, $\lambda\in O,\lambda\neq 0$, belongs to $\mathrm{FGr}'(\mathbb{F}_q)_O$ and, therefore, $G_1\in\mathrm{FGr}'(\mathbb{F}_q)_O$, cf. n.1.4. By induction, we have $G_1\simeq G[M_1]$, $M_1\in\mathrm{Mod}_{O,\sigma}$.

Finally, $G$ satisfies the properties 1)-4) from n.2.5 and by Proposition 3, $G\simeq G[M]$ for $M\in\mathrm{Mod}_{O,\sigma}$. The Theorem is proved.

## 3. Group schemes with strict $O_0$-action.

In this section $R=O_0$ is the valuation ring of a closed subfield $K_0$ of $K$ with finite residue field $\mathbb{F}_q$, $q=p^n$, $n\in\mathbb{N}$. Fix a choice of uniformising element $\pi_0$ in $K_0$. So, $O_0=\mathbb{F}_q[[\pi_0]]$.

3.1. Denote by $\mathrm{FGr}'(O_0)_O$ a full subcategory in $\mathrm{FGr}(O_0)_O$ consisting of objects killed by some power of $[\pi_0]$. This is an analogue of the classical category of $p$-torsion finite flat group schemes over the valuation ring of a complete discrete valuation field of mixed characteristic.

Let $\mathrm{Mod}(O_0)_O$ be the category, consisting of $M \in \mathrm{Mod}_{O,\sigma}$ with $O_0$-module structure given for $o \in O_0$ by endomorphisms $[o] \in \mathrm{End}_{\mathrm{Mod}_{O,\sigma}}(M)$ such that for all $m \in M$, $[\alpha]m = \alpha m$ if $\alpha \in \mathbb{F}_q \subset k$ and $[\pi_0^N]m = 0$ for sufficiently large $N$. Morphisms in the category $\mathrm{Mod}(O_0)_O$ are morphisms of the category $\mathrm{Mod}_{O,\sigma}$ which commute with all endomorphisms $[o]$, $o \in O_0$. In other words, any $M \in \mathrm{Mod}(O_0)_O$ is a free $O$-module $M$ of finite rank equiped with

a) an $O$-linear map $\Phi : M_\sigma \longrightarrow M$ such that $\Phi \otimes K$ is isomorphism;

b) an $O$-linear nilpotent endomorphism $[\pi_0] : M \longrightarrow M$ commuting with $\Phi$.

We shall denote by $\mathrm{Mod}'(O_0)_O$ a full subcategory of $\mathrm{Mod}(O_0)_O$ consisting of objects $M$ such that

c) for any $m \in M$, $[\pi_0]m \equiv \pi_0 m \bmod \Phi(M_\sigma)$.

The category $\mathrm{Mod}(O_0)_O$ is not generally abelian. But similarly to the category $\mathrm{FGr}'(O_0)_O$, one can work with short exact sequences in $\mathrm{Mod}(O_0)_O$. The sequence of objects and morphisms in $\mathrm{Mod}(O_0)_O$

$$0 \longrightarrow M_1 \xrightarrow{i} M \xrightarrow{j} M_2 \longrightarrow 0$$

is exact if and only if $i$ is a pure morphism of $O$-modules (i.e. $M/i(M_1)$ has no $O$-torsion) and $j$ is epimorphism of $O$-modules with kernel $i(M_1)$. This allows to define $O$-modules of equivalence classes of short exact sequences which satisfy the usual functorial properties.

3.2. Let $i : \mathrm{FGr}(O_0)_O \longrightarrow \mathrm{FGr}(\mathbb{F}_q)_O$ be the forgetful functor.

**Proposition 4.** *If $\mathcal{G} \in \mathrm{FGr}'(O_0)_O$, then $i(\mathcal{G}) \in \mathrm{FGr}'(\mathbb{F}_q)_O$.*

*Proof.* Indeed, if $\mathcal{G} = (G, G^\flat)$ then $O_0$ acts via a scalar multiplication on $t^*_{G^\flat}$. This implies that $t^*_G$ is killed by some power of $\pi_0$ and, therefore, $t^*_{G\otimes K} = t^*_G \otimes K = 0$, i.e. $G \otimes K$ is etale. The proposition is proved.

Let $L : \mathrm{FGr}'(\mathbb{F}_q)_O \longrightarrow \mathrm{Mod}_{O,\sigma}$ be the functor from n.2. For any $\mathcal{G} \in \mathrm{FGr}'(O_0)_O$, $L(i(\mathcal{G}))$ has a natural structure of an object of the category $\mathrm{Mod}(O_0)_O$. We denote by $L$ the functor from $\mathrm{FGr}'(O_0)_O$ to $\mathrm{Mod}(O_0)_O$ induced by the above correspondence $\mathcal{G} \mapsto L(i(\mathcal{G}))$.

**Theorem 2.** *The functor $L$ induces antiequivalence of the categories $\mathrm{FGr}'(O_0)_O$ and $\mathrm{Mod}'(O_0)_O$.*

*Proof.* Let $\mathcal{G} \in \mathrm{FGr}'(O_0)_O$ and $M = L(\mathcal{G}) \in \mathrm{Mod}(O_0)_O$. Considering $M$ as an object of $\mathrm{Mod}_{O,\sigma}$ we can recover the $\mathbb{F}_q$-module scheme $i(\mathcal{G})$ in the form $G[M]$. This reduces the proof of our theorem to the following statement:

**Proposition 5.** *$\mathcal{G}$ is a strict $O_0$-module iff $M$ satisfies the condition* c) from n.3.1.

Let $\mathcal{G}^{loc}$ be the maximal local subobject in $\mathcal{G}$ and $M^{loc} = L(\mathcal{G}^{loc})$. Then the natural projection $M \longrightarrow M^{loc}$ induces isomorphism of $(O_0, O)$-modules $M/\Phi(M_\sigma)$ and $M^{loc}/\Phi(M^{loc}_\sigma)$. So, it will be sufficient to prove the above proposition for a local $\mathcal{G}$. Under this assumption choose a vector column $\bar{m} = (m_1, \dots, m_n)^t$ such that its coordinates create an $O$-basis of $M = L(\mathcal{G})$ and set $\Phi\bar{m} = C\bar{m}$ and $[\pi_0]\bar{m} = D\bar{m}$, where $C, D \in \mathbb{M}_n(O)$ are $n \times n$-matrices with coefficients from $O$.

**Lemma 4.** *The above matrices $C$ and $D$ give a structure of an object of the category $\mathrm{Mod}'(O_0)_O$ on $M$ if and only if*

1) $\det C \neq 0$*;*

2) *$D$ is nilpotent and $\sigma(D)C = CD$;*

3) *there is a $B \in \mathbb{M}_n(O)$ such that $D - \pi_0 E_n = BC$.*

*Proof.* Indeed, 1) means that $\Phi(M_\sigma)$ is a lattice in the $O$-module $M$.

Property 2) means that $M$ is $[\pi_0]$-torsion and the morphisms $\Phi$ and $[\pi_0]$ commute one with another. Indeed, $(\Phi \circ [\pi_0])(\bar{m}) = [\pi_0](\Phi\bar{m}) = CD\bar{m}$ and $([\pi] \circ \Phi)(\bar{m}) = \sigma(D)C\bar{m}$.

Notice that the coordinates of the vector $C\bar{m}$ generate the $O$-module $\Phi(M_\sigma)$. So, the coordinates of $[\pi_0](\bar{m}) - \pi_0\bar{m}$ belong to $\Phi(M_\sigma)$ if and only if there is an $B \in \mathbb{M}_n(O)$ such that $D - \pi_0 E_n = BC$.

The lemma is proved.

*Remark.* a) $M = M^{loc}$ if and only if $C$ is $\sigma$-nilpotent, i.e. there is an $N \in \mathbb{N}$ such that $\sigma^N(C) \ldots \sigma(C)C = 0$;
b) the above properties 1) and 3) imply that $\sigma(D) - \pi_0 E_n = CB$.

With the above notation and agreements let $\mathcal{G} = (G, G^\flat)$, where $A(G) = O[\bar{X}]/I$ and $A(G^\flat) = O[\bar{X}]/(I \cdot I_0)$, where $\bar{X} = (X_1, \ldots, X_n)$ and the ideal $I$ is generated by the coordinates of the vector $\bar{X}^q - C\bar{X}$.

Then $A(G)$ has an $O$-basis $\{X_1^{a_1} \ldots X_n^{a_n} \mid 0 \leqslant a_1, \ldots, a_n < q\}$ and this basis can be completed to an $O$-basis of $A(G^\flat)$ by joining the elements of the set $\{X_i^q \mid 1 \leqslant i \leqslant n\}$.

The action of $O_0$ is given by $[\pi_0]\bar{X} = D\bar{X}$ on $A(G)$. This action is strict if and only if it can be extended to $A(G^\flat)$ via the relation

$$[\pi_0]\bar{X} = D\bar{X} + B(\bar{X}^q - C\bar{X})$$

with some $B \in \mathbb{M}_n(O)$ in such a way that it induces the scalar multiplication by $\pi_0$ on $t^*_{A(G^\flat)}$ and $N_{A(G^\flat)}$.

The first condition is equivalent to the equality $D - BC = \pi_0 E_n$.

The second can be analysed as follows,

$$[\pi_0]^\flat(\bar{X}^q - C\bar{X}) \equiv ([\pi_0]^\flat\bar{X})^q - C([\pi_0]^\flat\bar{X}) \equiv (\sigma(D) - CB)(\bar{X}^q - C\bar{X})$$

and is equivalent to the matrix equality $\sigma D - CB = \pi_0 E_n$.

So, Proposition 5 and Theorem 2 follow from Lemma 4 and the above Remark b).

3.3. Clearly, the antiequivalence $L$ transforms short exact sequences in $\mathrm{FGr}'(O_0)_O$ to short exact sequences in the category $\mathrm{Mod}'(O_0)_{O,\sigma}$. In particular, we have the following property.

**Proposition 6.** *Suppose $\mathcal{G}_1, \mathcal{G}_2 \in \mathrm{FGr}'(O_0)_O$ and $f \in \mathrm{Hom}_{\mathrm{FGr}'(O_0)_O}(\mathcal{G}_1, \mathcal{G}_2)$. Then*
a) *$f$ is a closed embedding if and only if $L(f)$ is surjective;*
b) *$f$ is fully faithful if and only if $L(f)$ is a pure embedding.*

Suppose for $n \in \mathbb{N}$, $\mathcal{G}^{(n)} \in \mathrm{FGr}'(O_0)_O$, $i_n : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n+1)}$ is a closed immersion, $j_n : \mathcal{G}^{(n+1)} \longrightarrow \mathcal{G}^{(n)}$ is a fully faithful morphism. For $m > n$, set

$i_{nm} = i_n \circ \dots \circ i_{m-1}$ and $j_{mn} = j_{m-1} \circ \dots \circ j_n$. Then (following the original deinition of Tate) $\{\mathcal{G}^{(n)}, i_n, j_n\}$ is a $\pi_0$-divisible group in the category $\mathrm{FGr}'(O_0)_O$ if for any $m > n$, $(\mathcal{G}^{(n)}, i_{nm}) = \mathrm{Ker}[\pi_0^m] \,\mathrm{id}_{\mathcal{G}^{(m)}}$ and $[\pi_0^{m-n}] \,\mathrm{id}_{\mathcal{G}^{(m)}} = j_{mn} \circ i_{nm}$.

Similar definitions can be done for the category $\mathrm{Mod}'(O_0)_O$, where a $\pi_0$-divisible group appears as the collection $\{M^{(n)}, i_n, j_n\}$ where $i_n$ is a pure embedding of underlying $O$-modules $M^{(n)} \longrightarrow M^{(n+1)}$ and $j_n : M^{(n+1)} \longrightarrow M^{(n)}$ is a surjection of $O$-modules.

3.4. Clearly, the functor $L$ from Theorem 2 transforms $\pi_0$-divisible groups in the category $\mathrm{FGr}'(O_0)_O$ to $\pi_0$-divisible groups in the category $\mathrm{Mod}'(O_0)_O$.

**Theorem 3.** *For any object $\mathcal{G}$ of the category $\mathrm{FGr}'(O_0)_O$, there is a $\pi_0$-divisible group $\{\mathcal{H}^{(n)}, i_n, j_n\}_{n\geqslant 1}$ and a closed embedding $\mathcal{G} \longrightarrow \mathcal{H}^{(N)}$, where $N \in \mathbb{N}$ is such that $[\pi_0^N] \,\mathrm{id}_{\mathcal{G}} = 0$.*

*Remark.* The statement of the above theorem is equivalent to the existence of a $\pi_0$-divisible group $\{\mathcal{H}'^{(n)}, i_n, j_n\}_{n\geqslant 1}$ and a fully faithful morphism $\mathcal{H}'^{(N)} \longrightarrow \mathcal{G}$.

*Proof.* The antiequivalence $L$ allows us to prove the dual version of this theorem in the category $\mathrm{Mod}'(O_0)_O$.

Let $M \in \mathrm{Mod}'(O_0)_O$ and let $N \in \mathbb{N}$ be such that $[\pi_0^N]M = 0$. Use induction on $N$.

Suppose first, that $N = 1$.

Then $M$ is a free $O$-module with $\sigma$-linear $\Phi : M_\sigma \longrightarrow M$ such that $\pi_0 M \subset \Phi(M_\sigma) \subset M$. Choose an $O$-basis $m_1, \dots, m_n$ in $M$ and take a vector column $\bar{m} = (m_1, \dots, m_n)^t$. Then $\Phi\bar{m} = C\bar{m}$, where $C \in \mathbb{M}_n(O)$ is a divisor of $\pi_0 E_n$. Let $\widetilde{C} \in \mathbb{M}_n(O)$ be such that $C\widetilde{C} = \pi_0 E_n$.

For $l \in \mathbb{N}$, introduce free $O$-modules $M^{(l)}$ with free generators

$$\{m_{1i}^{(k)}, m_{2i}^{(k)} \mid 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant l\}$$

Let $\bar{m}_1^{(0)} = \bar{m}_2^{(0)} = \bar{0}$ and for $k \geqslant 1$, let $\bar{m}_1^{(k)} = (m_{11}^{(k)}, \dots, m_{1n}^{(k)})^t$ and $\bar{m}_2^{(k)} = (m_{21}^{(k)}, \dots, m_{2n}^{(k)})^t$. Define $O$-linear morphisms $\Phi : M_\sigma^{(l)} \longrightarrow M^{(l)}$ and $[\pi_0] : M \longrightarrow M$ by setting for $1 \leqslant k \leqslant l$,

$$\Phi\bar{m}_1^{(k)} = \widetilde{C}\bar{m}_1^{(k)} + \bar{m}_2^{(k-1)}, \quad \Phi\bar{m}_2^{(k)} = C\bar{m}_2^{(k)} + \bar{m}_1^{(k)}$$

and $[\pi_o]\bar{m}_1^{(k)} = \bar{m}_1^{(k-1)}$, $[\pi_0]\bar{m}_2^{(k)} = \bar{m}_2^{(k-1)}$.

It is easy to see that we defined a structure of objects of the category $\mathrm{Mod}'(O_0)_O$ on all $M^{(l)}$, $l \in \mathbb{N}$, and the system $\{M^{(l)}\}_{l\geqslant 1}$ together with the natural inclusions $i_n : M^{(n} \longrightarrow M^{(n+1)}$ and projections $j_l : M^{(l+1)} \longrightarrow M^{(l)}$ gives a $\pi_0$-divisible group in the category $\mathrm{Mod}'(O_0)_O$. Clearly, the correspondences $\bar{m}_1^{(1)} \mapsto \bar{0}$ and $\bar{m}_2^{(1)} \mapsto \bar{m}$ give an epimorphic map from $M^{(1)}$ to $M$. The case $N = 1$ has been considered.

Suppose $N > 1$ and Theorem 3 has been proved for all $M' \in \mathrm{Mod}'(O_0)_O$ such that $[\pi_0]^{N-1}(M') = 0$.

Let $M_1 = \mathrm{Ker}[\pi_0]^{N-1} \,\mathrm{id}_M$ and $M_2 = [\pi_0]^{N-1}(M) \subset M$. Then $M_1$ and $M_2$ have natural structures of objects of the category $\mathrm{Mod}'(O_0)_O$ and in this category we have a natural short exact sequence

$$\varepsilon : \quad 0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

(Notice that, generally, the embedding $M_2 \subset M$ is not pure). By induction, there is a $\pi_0$-divisible group $\{T^{(n)}, i_n, j_n\}_{n\geqslant 1}$ and pure embeddings $\alpha : M_1 \longrightarrow T^{(N-1)}$, $\beta : M_2 \longrightarrow T^{(1)}$. Consider the short exact sequences

$$\alpha_*\varepsilon : \quad 0 \longrightarrow T^{(N-1)} \longrightarrow \alpha_* M \xrightarrow{j_\alpha} M_2 \longrightarrow 0$$

and

$$\beta^*\eta_N : \quad 0 \longrightarrow T^{(N-1)} \longrightarrow \beta^* T^{(N)} \xrightarrow{j_\beta} M_2 \longrightarrow 0$$

where the second sequence is obtained via $\beta$ from the standard short exact sequence

$$\eta_N : \quad 0 \longrightarrow T^{(N-1)} \xrightarrow{i_N} T^{(N)} \xrightarrow{j_{N1}} T^{(1)} \longrightarrow 0$$

Notice that there is a pure embedding $M \longrightarrow \alpha_* M$. Notice also that in the both sequences $\alpha_*\varepsilon$ and $\beta^*\eta_N$, the epimorphic maps $j_\alpha$ and $j_\beta$ are induced by multiplication by $[\pi_0^{N-1}]$. In other words, the endomorphism $[\pi_0^{N-1}]$ on $\alpha_* M$ and, resp., on $\beta^* T^{(N)}$ is a composition of $j_\alpha$ and, resp., of $j_\beta$ with the natural inclusion of $M_2$ into $\alpha_* M$ and, resp., into $\beta^* T^{(N)}$ (which is induced by the embedding $M_2 \subset T^{(1)}$).

Let

$$0 \longrightarrow T^{(N-1)} \longrightarrow \widetilde{M} \longrightarrow M_2 \longrightarrow 0$$

be the difference of $\alpha_*\varepsilon$ and $\beta^*\eta_N$ considered as elements of the group $\mathrm{Ext}_{\mathrm{Mod}'(O_0)_O}(M_2, T^{(N-1)})$. It is easy to see that $[\pi_0^{N-1}](\widetilde{M}) = 0$ and there is an $\widetilde{M_1} \in \mathrm{Mod}'(O_0)_O$, a surjective map $\tilde{j}$ and a pure embedding $\tilde{\imath}$ such that

$$\beta^* T^{(N)} \oplus \widetilde{M} \xrightarrow{\tilde{j}} \widetilde{M_1} \xleftarrow{\tilde{\imath}} \alpha_*(M)$$

By inductive assumption, there is a pure embedding of $\beta^* T^{(N)} \oplus \widetilde{M}$ into a $\pi_0$-divisible group in the category $\mathrm{Mod}'(O_0)_O$. This implies easily the existence of such an embedding for $\alpha_* M$ and, therefore, for $M$.

Theorem 3 is proved.

3.5. *Relation to the mixed characteristic case.*

In nn.3.5.1-3.5.3 below we need a full subcategory $\mathrm{FGr}'_1(O_0)_O$ in $\mathrm{FGr}'(O_))_O$. It consists of $\mathcal{G} = (G, G^\flat)$ such that $[\pi_0]\,\mathrm{id}_G = 0$. Then the functor $L$ induces an antiequivalence of this category and the full subcategory $\mathrm{Mod}'_1(O_0)_O$ of $\mathrm{Mod}'(O_0)_O$ consisting of free $O$-modules $M$ such that $\pi_0 M \subset \Phi(M_\sigma) \subset M$. As usually, $e = e(K/K_0)$ is the ramification index of $K = \mathrm{Frac}\, O$ over $K_0 = \mathrm{Frac}\, O_0$.

3.5.1. Suppose $e(K/K_0) = 1$.

Introduce the category $\mathrm{SH}_1(\mathbb{F}_q)_O$ with objects $(M^0, M^1, \varphi_0, \varphi_1)$, where $M^0$ is an $O$-module of finite length such that $\pi_0 M = 0$, $\varphi_0 : M^0 \longrightarrow M^0$ is a $\sigma$-linear morphism, $M^1 = \mathrm{Ker}\, \varphi_0$, $\varphi_1 : M^1 \longrightarrow M^0$ is a $\sigma$-linear morphism and $\varphi_0(M^0) + \varphi_1(M^1) = M$. This is an analogue of Fontaine's category of filtered modules.

Consider the functor $\mathrm{SH} : \mathrm{Mod}'_1(O_0)_O \longrightarrow \mathrm{SH}_1(\mathbb{F}_q)$ defined by the correspondence $M \mapsto (M^0, M^1, \varphi_0, \varphi_1)$, where $M^0 = M \bmod \pi_0 M$, $\varphi_0 = \Phi \bmod \pi_0$ and $\varphi_1$ is induced by $\dfrac{1}{\pi_0}\Phi$. The functor SH is an equivalence of categories if $q > 2$ and is "very close" to an equivalence if $q = 2$. This shows that strict $O_0$-modules have similar description as conventional group schemes if $e=1$. Actually, one can develop the Dieudonne theory in the context of strict modules and realise the approach from [Ab1] to recover directly an analogue of Fontaine's classification of group schemes over Witt vectors.

3.5.2. Suppose $e \leqslant q-1$.

Define the category $\mathrm{SH}_1(\mathbb{F}_q)_O$ of the collections $(M, M^0, M^1, \varphi_0, \varphi_1)$, where $M$ is an $O$-module of finite length killed by $\pi_0$, $M^0 = \mathrm{Ker}\,\pi|_M$, $\varphi_0 : M^0 \longrightarrow M$ is a $\sigma$-linear map, $M^1 = \mathrm{Ker}\,\varphi_0$, $\varphi_1 : M^1 \longrightarrow M$ is a $\sigma$-linear map, and $\varphi_0(M^0)\otimes_k O + \varphi_1(M^1)\otimes_k O = M$. This category is an analogue of the category $\mathrm{SH}_O$ from [Ab3]. Consider the functor $\mathrm{SH} : \mathrm{Mod}_1'(O_0)_O \longrightarrow \mathrm{SH}(\mathbb{F}_q)_O$ defined by the correspondence $M \mapsto (\bar{M}, \bar{M}^0, \bar{M}^1, \varphi_0, \varphi_1)$, where $\bar{M}$ is the $O$-submodule in $\dfrac{\pi}{\pi_0} O \otimes M \bmod \pi M$ generated by the images of the elements of the sets $\{\dfrac{\pi}{\pi_0}\Phi(m \mid m \in M\}$ and $\{\dfrac{1}{\pi_0}\Phi(m) \mid m \in M,\ \Phi(m)=0\}$. Then $\bar{M}^0 = \mathrm{Ker}\,\pi|_{\bar{M}} = M \bmod \pi M$, $\varphi_0 : \bar{M}^0 \longrightarrow M$ is induced by $\dfrac{\pi}{\pi_0}\Phi$, $\bar{M}^1 = \mathrm{Ker}\,\varphi_0$ and $\varphi_1 : \bar{M}^1 \longrightarrow M$ is induced by $\dfrac{1}{\pi_0}\Phi$.

If $e < q-1$ then SH is an equivalence of categories and, if $e = q-1$ then SH is "very close" to an equivalence of categories. Again, the methods from [Ab3] can be used to obtain directly the classification of objects from $\mathrm{FGr}_1'(O_0)_O$ via objects of the category $\mathrm{SH}(\mathbb{F}_q)_O$. (The details will appear in W.Gibbons's thesis.)

Notice that when working with equal characteristic case the category $\mathrm{SH}(\mathbb{F}_q)_O$ can be replaced by a simpler category $\mathrm{SH}'(\mathbb{F}_q)_O$ consisting of the triples $(\bar{M}, \bar{M}^0, \varphi)$, where $\bar{M}$ is an $O$-module of finite length killed by $\pi_0$, $\bar{M}^0 = \mathrm{Ker}\,\pi|_{\bar{M}}$ and $\varphi_0 : \bar{M}^0 \longrightarrow \bar{M}$ is a $\sigma$-linear morphism such that $\varphi(\bar{M}^0)\otimes O = \bar{M}$. The functor $\mathrm{SH}' : \mathrm{FGr}_1'(O_0)_O \longrightarrow \mathrm{SH}'(\mathbb{F}_q)_O$ is defined by the correspondence $M \mapsto (\bar{M}, \bar{M}^0, \varphi_0)$, where $\bar{M}$ is an $O$-submodule in $\dfrac{1}{\pi_0} O \otimes M \bmod \pi M$ generated by elements of the form $\dfrac{1}{\pi_0}\Phi m$ with $m \in M$, $\bar{M}^0 = \mathrm{Ker}\,\pi|_M = M \bmod \pi M$ and $\varphi_0$ is induced by $\dfrac{1}{\pi_0}\Phi$ (this map is still $\sigma$-linear because of the equal characteristic situation).

3.5.3. Suppose that $e(K/K_0)$ is arbitrary.

Introduce the category $\mathrm{BR}_1(\mathbb{F}_q)_O$. Its objects are the triples $(M, M^1, \varphi)$, where $M$ is an $O$-module of finite length such that $\pi_0 M = 0$, $M^1$ is an $O_1$-submodule in $M \otimes_O O_1$, where $O_1 = O[\pi_1]$ with $\pi_1^p = \pi$, and $\varphi : M^1 \longrightarrow M$ is a $\sigma$-linear map ($\sigma$ is still the $q$-th power map) such that $\varphi(M^1) = M$. This category is an equicharacteristic version of Breuil's category $\mathrm{Mod}_{/S_1}$ appeared in his classification of period $p$ group schemes in the mixed characteristic case, cf. [Br].

Again there is a natural functor from $\mathrm{Mod}_1'(O_0)_O$ to $\mathrm{BR}_1(\mathbb{F}_q)_O$ defined by the correspondence $M \mapsto (\bar{M}, \bar{M}^1, \varphi)$, where $\bar{M} = M \bmod \pi_0$ and $\varphi$ is induced by $\dfrac{1}{\pi_0}\Phi$. But one can't expect that Breuil's method works in the equal characteristic case, because it is based very heavily on crystalline technique.

3.5.4. Again $e(K/K_0)$ is arbitrary.

Introduce an equal characteristic analogue of the concept of $p$-etale $\varphi$-module of $q$-heighjt 1 over $S$ from [Fo4].

First, introduce 2nd copy $\underline{O}_0 = \mathbb{F}_q[[\underline{\pi}_0]]$ of $O_0$ (actually, $\underline{\pi}_0$ is an analogue of $[\pi_0]$). Then $\underline{O}_0$ will be considered as an analogue of $\mathbb{Z}_p$ and $\widetilde{\underline{O}}_0 = k[[\underline{\pi}_0]]$ is an analogue of Witt vectors $W(k)$. Then consider $S = \widetilde{\underline{O}}_0[[\pi]]$, where $k[[\pi]] = O$ appears as a subring of $S$. Introduce $\sigma : S \longrightarrow S$ such that $\sigma|_{\underline{O}_0} = \mathrm{id}$ and $\sigma|_O$ is (as earlier) the $q$th power map.

Suppose that $M$ is an $S$-module of finite type together with an $\underline{O}_0$-linear map $\Phi : M_\sigma := M \otimes_{(O,\sigma)} O \longrightarrow M$. Then $M$ is called a $\underline{\pi}_0$-etale $\varphi$-module of $q$-height 1 (with $q = \underline{\pi}_0 - \pi_0$), if $\mathrm{Coker}\,\Phi$ is killed by multiplication by $\underline{\pi}_0 - \pi_0$.

With this notation Theorem 2 establishes a description of the category of finite flat strict $O_0$-modules over $O$ in terms of $\underline{\pi}_0$-torsion $\underline{\pi}_0$-etale $\varphi$-modules of $q$-height 1.

**4. Properties of arising Galois modules.**

In this section $\mathcal{G} = (G, G^\flat) \in \mathrm{FGr}'(O_0)_O$, $H = G(K_{\mathrm{sep}})$ is $O_0[\Gamma_K]$-module of geometric points of $G$ and $e = e(K/K_0)$ — is the ramification index of $K$ over $K_0$. We also set $\Gamma_K = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ and denote by $I_K$ the inertia subgroup of $\Gamma_K$.

4.1. *Characters of the semisimple envelope of $H$.*

Suppose $\bar{k}$ is an algebraic closure of $k$ and the character $\chi : I_K \longrightarrow \bar{k}^*$ appears with a nonzero multiplicity in the semisimple envelope of the $O_0[\Gamma_K]$-module $H$. An analogue of the Serre Conjecture for $H$ can be stated as follows.

**Theorem 4.** *For the above character $\chi$, there are $a, N \in \mathbb{N}\backslash p\mathbb{N}$ such that $\chi = \chi_N^a$, where $a = a_0 + a_1 q + \cdots + a_{N-1} q^{N-1}$ with $0 \leqslant a_i \leqslant e$ and $\chi_N : I_K \longrightarrow \bar{k}^*$ is such that for any $\tau \in I_K$, $\chi_N(\tau) = \tau(\pi_N)/\pi_N$, where $\pi_N \in K_{\mathrm{sep}}$ and $\pi_N^{q^N-1} = \pi$.*

*Proof.* This can be deduced in the same way as it has been obtained in the case of usual group schemes in [Ra]. First we can assume that $k = \bar{k}$ and $e < q - 1$. Then any simple object of the category $\mathrm{Mod}'(O_0)_O$ appears in the form $M = \oplus_{0 \leqslant i < n} O m_i$, where $\Phi m_0 = \pi^{a_0} m_1, \ldots, \Phi m_{N-1} = \pi^{a_{N-1}} m_0$ and $[\pi_0] m_0 = \cdots = [\pi_0] m_{N-1} = 0$ with $0 \leqslant a_i \leqslant e$, $0 \leqslant i < N$.

Then the corresponding Galois module consists of $K_{\mathrm{sep}}$-points of the $O$-algebra $O[T_0, \ldots, T_{N-1}]$, where $T_0^q = \pi^{a_0} T_1, \ldots, T_{N-1}^q = \pi^{a_{N-1}} T_0$. It can be naturally identified with the $O_0[\Gamma_K]$-module $\{\alpha \pi_N^a \mid \alpha \in \mathbb{F}_{q^N}\}$, where $a = a_0 + a_1 q + \cdots + a_{N-1} q^{N-1}$. Clearly, $I_K$ acts on it via the conjugacy class of characters $\{\sigma^i \chi_N^a \mid 0 \leqslant i < N\}$.

*Remark.* Following Raynaud's method one can deduce from the above description of simple objects in $\mathrm{Mod}'(O_0)_O$ that if $e < q - 1$ then the functor $G \mapsto G(K_{\mathrm{sep}})$ is a fully faithful functor from $\mathrm{Mod}'(O_))_O$ to the category of $O_0[\Gamma_K]$-modules.

4.2. *Ramification estimates.*

These estimates are given in Theorem 5 below and are completely similar to the known estimates in the case of conventional group schemes, cf. [Fo2]. The proof is based on the knowledge of "equations" of the strict module $G$ and is done below by the methods of the paper [Ab4]. Notice that the methods from [Fo3] also can be adjusted to obtain the same estimates.

**Theorem 5.** *If $H$ is killed by $[\pi_0^N]$ then the ramification subgroups $\Gamma_K^{(v)}$ act trivially on $H$ for $v > e\left(N + \dfrac{1}{q-1}\right) - 1$.*

*Proof.* We can assume that there is a $\pi_0$-divisible group $\{\mathcal{G}^{(i)}\}_{i\geqslant 1}$ of a height $h$ in $\mathrm{FGr}'(O_0)_O$ such that $\mathcal{G}^{(N)} = \mathcal{G}$.

4.2.1. By the above assumption, $L(\mathcal{G})$ is a free $O$-module of rank $hN$ and we can choose its $O$-basis in the form

$$m_1, \dots, m_h, [\pi_0]m_1, \dots, [\pi_0]m_h, \dots, [\pi_0]^{N-1}m_1, \dots, [\pi_0^{N-1}]m_h$$

Introduce vector-columns

$$\bar{m}_1 = (m_1, \dots, m_h)^t, \dots, \bar{m}_N = [\pi_0^{N-1}]\bar{m}_1 = ([\pi_0^{N-1}]m_1, \dots, [\pi_0^{N-1}]m_h)^t$$

Then, in the obvious notation, the structure of $M = L(\mathcal{G})$ can be given in the following form

$$\frac{1}{\pi_0}C_1\Phi\bar{m}_N + \frac{1}{\pi_0}C_2\Phi\bar{m}_{N-1} + \dots + \frac{1}{\pi_0}C_N\Phi\bar{m}_1 = \bar{m}_N - \frac{1}{\pi_0}\bar{m}_{N-1}$$
$$\frac{1}{\pi_0}C_1\Phi\bar{m}_{N-1} + \dots + \frac{1}{\pi_0}C_{N-1}\Phi\bar{m}_1 = \bar{m}_{N-1} - \frac{1}{\pi_0}\bar{m}_{N-2}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{1}{\pi_0}C_1\Phi\bar{m}_1 = \bar{m}_1$$

where all $C_i \in \mathrm{GL}(h, O)$ and $\det C_1 \neq 0$.

Consider vector columns $\bar{X}_i = (X_{i1}, \dots, X_{ih})^t$ of independent variables $X_{ij}$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant h$. Then the algebra $A(G)$ appears as a quotient of $O[\bar{X}_1, \dots, \bar{X}_N]$ by the ideal generated by equations

$$\sum_{1\leqslant i\leqslant s} C_i\bar{X}_{s+1-i}^q = \pi_0\bar{X}_s - \bar{X}_{s-1} \tag{5}$$

where $1 \leqslant s \leqslant N$ and by definition $\bar{X}_0 = \bar{0}$.

Consider the points of $G(K_{\mathrm{sep}})$ as solutions $\bar{a} = (\bar{a}_1, \dots, \bar{a}_N)$ of the system (5). The following lemma can be easily proved by induction on $N$.

**Lemma 5.** *If $\bar{a} = (\bar{a}_1, \dots, \bar{a}_N)$ and $\bar{a}' = (\bar{a}'_1, \dots, \bar{a}'_N)$ are solutions of* (4.1) *such that $\bar{a} \equiv \bar{a}' \bmod \pi_0^{\frac{1}{q-1}} m_{\mathrm{sep}}$, where $m_{\mathrm{sep}}$ is the maximal ideal of the valuation ring of $K_{\mathrm{sep}}$, then $\bar{a} = \bar{a}'$.*

4.2.2. Suppose $\alpha \in \mathbb{Q}_{>0}$ has the $p$-adic valuation 0. Then $\alpha = \dfrac{m}{q^M - 1}$ with suitable $m, M \in \mathbb{N}$, $(m, p) = 1$. For any such $\alpha$ there is an extension $K_\alpha$ of $K$ with $[K_\alpha : K] = q^M$ and the Herbrand function

$$\varphi_{K_\alpha/K}(x) = \begin{cases} x, & \text{for } 0 \leqslant x \leqslant \alpha \\ \alpha + \dfrac{x-\alpha}{q^M}, & \text{for } x \geqslant \alpha. \end{cases}$$

Notice that $\varphi_{K_\alpha/K}$ has only one edge point in $x=\alpha$.

Explicit construction of $K_\alpha$ can be found in [Ab4], where the field $K_\alpha$ appears as an extension of $K$ of degree $q^M$ in $L_\alpha = K(\pi_M)(T)$, where $\pi_M^{q^M-1}=\pi$ and $T^{q^M}-T=\pi_M^{-m}$. Clearly, $K_\alpha$ is totaly ramified over $K$ and, therefore, there is a field isomorphism

$$h_\alpha : K \longrightarrow K_\alpha$$

From the above construction of $K_\alpha$, it follows easily that $h_\alpha$ can be chosen in such a way that for any $a\in m_K$ ($m_K$ is the maximal ideal in $O$),

$$a = h_\alpha(a)^{q^M} + \tilde{a},$$

with $\tilde{a}\in K_\alpha$ such that $v_K(\tilde{a}) \geqslant v_K(a)+\alpha$ ($v_K$ is the normalized valuation in $K$).

4.2.3. Denote by the same symbol an extension of $h_\alpha$ to an isomorphism of $K_{\mathrm{sep}}$ onto $K_{\alpha,\mathrm{sep}} = K_{\mathrm{sep}}$. Clearly, $\bar{X}=(\bar{X}_1,\dots,\bar{X}_s)\mapsto h_\alpha(\bar{X})=(h_\alpha(\bar{X}_1),\dots,h_\alpha(\bar{X}_s))$ is a one-one correspondence between solutions of the system (5) and solutions $\bar{Y}=(\bar{Y}_1,\dots,\bar{Y}_s)$ of the similar system

$$\sum_{1\leqslant i\leqslant s} h_\alpha(C_i)\bar{Y}^q_{s+1-i} = h_\alpha(\pi_0)\bar{Y}_s - \bar{Y}_{s-1} \tag{6}$$

where $1\leqslant s\leqslant N$ and by definition $\bar{Y}_0=\bar{0}$.

**Lemma 6.** *If $\alpha > e\left(N+\dfrac{1}{q-1}\right)-1$, then for any solution $\bar{X}^{(0)}$ of (5) there is a unique solution $\bar{Y}^{(0)}$ of (6) such that $\bar{X}^{(0)}\equiv \bar{Y}^{(0)q^M} \bmod \pi_0^{\frac{1}{q-1}}m_{\mathrm{sep}}$*

*Proof of lemma.* the correspondence $\bar{Y}\mapsto\bar{Z}=\bar{Y}^{q^M}-\bar{X}^{(0)}$ establishes a one-one correspondence between solutions $\bar{Y}$ of (5) and solutions $\bar{Z}=(\bar{Z}_1,\dots,\bar{Z}_s)$ of the system of equations

$$\sum_{1\leqslant i\leqslant s} C_i\bar{Z}^q_{s+1-i} = \pi_0\bar{Z}_s - \bar{Z}_{s-1} + \bar{F}_s \tag{7}$$

where $1\leqslant s\leqslant N$, $\bar{Z}_0=0$ and

$$\bar{F}_s = \tilde{\pi}_0\bar{Y}_s^{q^M} - \sum_{1\leqslant i\leqslant s}\tilde{C}_i\bar{Y}^{q^{M+1}}_{s+1-i} \in \pi_0^{1+\frac{1}{q-1}}m_{\mathrm{sep}}$$

because $v_K(\tilde{\pi}_0), v_K(\tilde{C}_i)\geqslant 1+\alpha > e\left(1+\dfrac{1}{q-1}\right)$.

Now induction on $s$ shows that the system (7) has a unique solution $\bar{Z}$ with coordinates in $\pi_0^{\frac{1}{q-1}}m_{\mathrm{sep}}$.

Lemma is proved.

With the above notation and assumptions we have the following corollary.

**Corollary.** *Suppose $E$, resp. $E_\alpha$, is obtained by joining to $K$, resp. $K_\alpha$, all coordinates of all solutions of the system of equations (4.1), resp. (4.2), in $K_{\mathrm{sep}}$. Then $EK_\alpha = E_\alpha$.*

4.2.4. For any finite extension $F\subset L$ in $K_{\mathrm{sep}}$, let $v(L/F)$ be the minimal rational number such that the ramification groups $\Gamma_F^{(v)}$ act trivially on $L$ for $v>v(L/F)$.

**Proposition 7.** *With the above notation there is the following inequality*

$$v(E/K) \leqslant e\left(N + \frac{1}{q-1}\right) - 1$$

*Proof.* Suppose this inequality does not hold. Then there is a rational number $\alpha$ satisfying the assumptions from the beginning of n.4.2.2 and the inequalities

$$v(E/K) > \alpha > e\left(N + \frac{1}{q-1}\right) - 1.$$

Notice that $E_\alpha = EK_\alpha$ implies that

$$v(E_\alpha/K) = \max\{v(E/K), v(K_\alpha/K)\} = v(E/K)$$

On the other hand, looking at the maximal edge points of Herbrand functions from the identity $\varphi_{E_\alpha/K} = \varphi_{E_\alpha/K_\alpha} \circ \varphi_{K_\alpha/K}$, we obtain that

$$v(E_\alpha/K) = \max\left\{v(K_\alpha/K), \varphi_{K_\alpha/K}(v(E_\alpha/K_\alpha))\right\} =$$
$$\max\left\{\alpha, \frac{v(E/K)-\alpha}{q} + \alpha\right\} < v(E/K)$$

because $v(E/K) = v(E_\alpha/K_\alpha)$. Contradiction.

Theorem 5 is proved.

4.3. As it was noticed in n.3.5.1, if $e = 1$ then killed by $[\pi_0]$ strict $O_0$-modules behave very similarly to group schemes of period $p$ over Witt vectors. For this reason, one can apply directly methods from [Ab2] to prove the following result.

**Theorem 6.** *Suppose $H$ is an $\mathbb{F}_q[\Gamma_K]$-module such that*
a) *the action of inertia subgroup of $\Gamma_K$ on the semisimple envelope of $H$ is given by characters, which satisfy Serre's Conjecture, cf. Theorem 4;*
b) *the ramification subgroups $\Gamma_K^{(v)}$ act trivially on $H$ if $v > \dfrac{1}{q-1}$ (i.e. the ramification estimate from Theorem 5 holds for $H$).*

*Then there is an $\mathcal{G} = (G, G^\flat) \in \mathrm{FGr}'(O_0)_O$ such that $H \simeq G(K_{\mathrm{sep}})$.*

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Dept.of Maths., Univ. of Durham, Sci. Laboratories, Durham, DH1 3LE, UK
*E-mail address*: victor.abrashkin@ dur.ac.uk