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SEMI-DIRECT PRODUCTS INVOLVING Sp_{2n} OR Spin_n WITH FREE ALGEBRAS OF SYMMETRIC INVARIANTS

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Dedicated to A. Joseph on the occasion of his 75th birthday

ABSTRACT. This is a part of an ongoing project, the goal of which is to classify all semi-direct products $\mathfrak s = \mathfrak g \ltimes V$ such that $\mathfrak g$ is a simple Lie algebra, V is a $\mathfrak g$-module, and $\mathfrak s$ has a free algebra of symmetric invariants. In this paper, we obtain such a classification for the representations of the orthogonal and symplectic algebras.

INTRODUCTION

Let $\Bbbk$ be a field with char $\Bbbk = 0$. Let S be an algebraic group defined over $\Bbbk$ with $\mathfrak s = \mathsf{Lie}\, S$. The invariants of S in the symmetric algebra $\mathcal S(\mathfrak s) = \Bbbk[\mathfrak s^*]$ of $\mathfrak s$ (= the symmetric invariants of $\mathfrak s$ or of S) are denoted by $\Bbbk[\mathfrak s^*]^S$ or $\mathcal S(\mathfrak s)^S$. If S is connected, then we also write $\Bbbk[\mathfrak s^*]^{\mathfrak s}$ or $\mathcal S(\mathfrak s)^{\mathfrak s}$ for them.

Let $\mathfrak g$ be a reductive Lie algebra. Symmetric invariants of $\mathfrak g$ over $\bar\Bbbk$ belong to the classical area of Representation Theory and Invariant Theory, where the most striking and influential results were obtained by Chevalley and Kostant in the 50s and 60s. Then pioneering insights of Kostant and Joseph revealed that the symmetric invariants of certain non-reductive subalgebras of $\mathfrak g$ can explicitly be described and that they are very helpful for understanding representations of $\mathfrak g$ itself, see [J77, J11, K12]. This have opened a brave new world, full of adventures and hidden treasures. Hopefully, we have found (and presented here) some of them.

Although the study of $\mathcal S(\mathfrak s)^S$ is hopeless in general, there are several classes of non-reductive algebras that are still tractable. One of them is obtained via a semi-direct product construction from finite- dimensional representations of reductive groups, which is the main topic of this article, see Section 2 below. Another interesting class of non-reductive algebras consists of truncated biparabolic subalgebras [J07], see also [FL] and references therein. Yet another class consists of the centralisers of nilpotent elements of $\mathfrak g$, see [PPY]. Remarkably, some truncated bi-parabolic subalgebras or centralisers occur also as semi-direct products.

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In [Y17b], the following problem has been proposed:

*To classify the representations V of **simple** algebraic groups G with $\mathsf{Lie}\, G = \mathfrak g$ such that the ring of symmetric invariants of the semi-direct product $\mathfrak s = \mathfrak g \ltimes V$ is polynomial.*

It is easily seen that if $\mathfrak s$ has this property, then $\Bbbk[V^*]^G$ is also a polynomial ring. (But not vice versa!) Therefore, the suitable representations (G, V) are contained in the list of "coregular representations" of simple algebraic groups, see [S78, AG79]. If a generic stabiliser for (G, V) is trivial, then $\Bbbk[\mathfrak s^*]^S \simeq \Bbbk[V^*]^G$. Therefore, it suffices to handle only "coregular representations" with non-trivial generic stabilisers. The latter can be determined with the help of Elashvili's tables [E72]. As it should have been expected, type A is the most difficult case. The solution for just one particular item, $V = m(\mathbb C^n)^* \oplus k\mathbb C^n$ for $G = \mathrm{SL}_n$, occupies the whole paper [Y17b]. This certainly means that obtaining classification in the SL_n-case requires considerable effort. Although the results of [Y17b] are formulated over $\mathbb C$, we notice that they are actually valid over an arbitrary field of characteristic zero. The case of exceptional groups G bas been considered in [PY17b]. The next logical step is to look at the symplectic and orthogonal groups G, which is done in this paper. To a great extent, our classification results rely on the theory developed by the second author in [Y17a].

Let us give a brief outline of the paper. In Sections 1, we gather some properties of the arbitrary coadjoint representations, whereas in Section 2, we stick to the coadjoint representations of semi-direct products and describe our classification techniques. After a brief interlude in Section 3 devoted to an example in type A, we dwell upon the classification of the suitable representations V of the orthogonal (Section 4) and symplectic (Section 5) groups. Our results are summarised in Theorem 2.13 and Tables 1, 2. We are taking a somewhat unusual approach towards a classification and trying to present the essential ideas for **all** pairs (G, V) under consideration. Many pairs can be handled using general theorems presented in Section 2, but some others require lengthy elaborated ad hoc considerations, see e.g. Theorem 4.13. It appears *a posteriori* that, for **all** representations V of $G = \mathrm{Sp}_{2n}$ with polynomial ring $\Bbbk[V^*]^{\mathrm{Sp}_{2n}}$, the algebra of symmetric invariants $\mathcal S(\mathfrak s)^S$ is also polynomial. In most of the $\mathfrak{sp}_{2n}$-cases, we explicitly describe the basic invariants. There is an interesting connection with the invariants of certain centralisers. In particular, if $V = \Bbbk^{2n}$ is the standard (defining) representations of Sp_{2n}, then there is a kind of matryoshka-like structure between the invariants of the semi-direct product and the symmetric invariants of the centraliser of the minimal nilpotent orbit in $\mathfrak{sp}_{2n-2}$.

Notation. Let an algebraic group Q act on an irreducible affine variety X. Then $\Bbbk[X]^Q$ stands for the algebra of Q-invariant regular functions on X and $\Bbbk(X)^Q$ is the field of Q-invariant rational functions. If $\Bbbk[X]^Q$ is finitely generated, then $X/\!\!/Q := \mathsf{Spec}\, \Bbbk[X]^Q$. Whenever $\Bbbk[X]^Q$ is a graded polynomial ring, the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants*. If V is a Q-module and $v \in V$, then $\mathfrak{q}_v = \{\xi \in \mathfrak{q} \mid \xi{\cdot}v = 0\}$ is the *stabiliser* of v in $\mathfrak{q}$ and $Q_v = \{g \in Q \mid g{\cdot}v = v\}$ is the *isotropy group* of v in Q.

Let X be an irreducible variety (e.g. a vector space). We say that a property holds for "generic $x \in X$" if that property holds for all points of an open subset of X. An open subset is said to be *big*, if its complement does not contain divisors.

Write $\mathfrak{heis}_n$, $n \geqslant 0$, for the Heisenberg Lie algebra of dimension $2n{+}1$.

1. Preliminaries on the coadjoint representations

Let Q be a connected algebraic group and $\mathfrak{q} = \mathsf{Lie}\, Q$. The *index* of $\mathfrak{q}$ is

$$\operatorname{ind}\mathfrak{q} = \min_{\gamma\in\mathfrak{q}^*} \dim \mathfrak{q}_\gamma,$$

where $\mathfrak{q}_\gamma$ is the stabiliser of γ in $\mathfrak{q}$. In view of Rosenlicht's theorem [VP89, § 2.3], $\operatorname{ind}\mathfrak{q} = \operatorname{tr.deg} \Bbbk(\mathfrak{q}^*)^Q$. If $\operatorname{ind}\mathfrak{q} = 0$, then $\Bbbk[\mathfrak{q}^*]^Q = \Bbbk$. For a reductive $\mathfrak{g}$, one has $\operatorname{ind}\mathfrak{g} = \operatorname{rk}\mathfrak{g}$. In this case, $(\dim\mathfrak{g} + \operatorname{rk}\mathfrak{g})/2$ is the dimension of a Borel subalgebra of $\mathfrak{g}$. For an arbitrary $\mathfrak{q}$, set $\boldsymbol{b}(\mathfrak{q}) := (\operatorname{ind}\mathfrak{q} + \dim\mathfrak{q})/2$.

One defines the *singular set* $\mathfrak{q}^*_{\mathsf{sing}}$ of $\mathfrak{q}^*$ by

$$\mathfrak{q}^*_{\mathsf{sing}} = \{\gamma \in \mathfrak{q}^* \mid \dim\mathfrak{q}_\gamma > \operatorname{ind}\mathfrak{q}\}.$$

Set also $\mathfrak{q}^*_{\mathsf{reg}} := \mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathsf{sing}}$. Further, $\mathfrak{q}$ is said to have the "codim–2" property (= to satisfy the "codim–2" condition), if $\dim\mathfrak{q}^*_{\mathsf{sing}} \leqslant \dim\mathfrak{q} - 2$. We say that $\mathfrak{q}$ satisfies *the Kostant regularity criterion* (= ***KRC***) if the following properties hold for $\mathcal{S}(\mathfrak{q})^Q$ and $\xi \in \mathfrak{g}^*$:

- $\mathcal{S}(\mathfrak{q})^Q = \Bbbk[f_1, \dots, f_l]$ is a graded polynomial ring (with basic invariants $f_1, \dots, f_l$);
- $\xi \in \mathfrak{q}^*_{\mathsf{reg}}$ if and only if $(df_1)_\xi, \dots, (df_l)_\xi$ are linearly independent.

Every reductive Lie algebra has the "codim–2" property and satisfies ***KRC***.

Observe that $(df)_\xi \in \mathfrak{q}_\xi$ for each $f \in \Bbbk[\mathfrak{q}^*]^Q$.

Theorem 1.1 (cf. [P07b, Theorem 1.2])**.** *If $\mathfrak{q}$ has the "codim–2" property, $\operatorname{tr.deg}\mathcal{S}(\mathfrak{q})^Q = \operatorname{ind}\mathfrak{q} = l$, and there are algebraically independent $f_1, \dots, f_l \in \mathcal{S}(\mathfrak{q})^Q$ such that $\sum_{i=1}^{l} \deg f_i = \boldsymbol{b}(\mathfrak{q})$, then $f_1, \dots, f_l$ freely generate $\mathcal{S}(\mathfrak{q})^Q$ and the **KRC** holds for $\mathfrak{q}$.*

Suppose that Q acts on an affine variety X. Then $f \in \Bbbk[X]$ is a *semi-invariant* of Q if $g{\cdot}f \in \Bbbk f$ for each $g \in Q$. A semi-invariant is said to be *proper* if it is not an invariant. If Q has no non-trivial characters (all 1-dimensional representations of Q are trivial), then it has no proper semi-invariants. In particular, if Q is a semi-direct product of a semisimple and a unipotent group, then all its semi-invariants are invariants. We record a well-known observation:

- if Q has no proper semi-invariants in $\mathcal{S}(\mathfrak{q})$, then $\Bbbk(\mathfrak{q}^*)^Q = \operatorname{Quot}\Bbbk[\mathfrak{q}^*]^Q$ and hence $\operatorname{tr.deg}\Bbbk[\mathfrak{q}^*]^Q = \operatorname{ind}\mathfrak{q}$.

Theorem 1.2 (cf. [JS10, Prop. 5.2]). *Suppose that Q has no proper semi-invariants in $\mathcal{S}(\mathfrak{q})$ and $\mathcal{S}(\mathfrak{q})^Q$ is freely generated by $f_1,\dots,f_l$. Then the differentials $\textsf{d}f_1,\dots,\textsf{d}f_l$ are linearly independent on a big open subset of $\mathfrak{q}^*$.*

For any Lie algebra $\mathfrak{q}$ defined over $\Bbbk$, set $\mathfrak{q}_{\bar{\Bbbk}} := \mathfrak{q}\otimes_{\Bbbk}\bar{\Bbbk}$. Then $\mathcal{S}(\mathfrak{q}_{\bar{\Bbbk}})^{\mathfrak{q}_{\bar{\Bbbk}}} = \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}\otimes_{\Bbbk}\bar{\Bbbk}$. If we extend the field, then a set of the generating invariants over $\Bbbk$ is again a set of the generating invariants over $\bar{\Bbbk}$. In the other direction, having a minimal set $\mathcal{M}$ of homogeneous generators over $\bar{\Bbbk}$, any $\Bbbk$-basis of $\langle\mathcal{M}\rangle_{\bar{\Bbbk}}\cap\mathcal{S}(\mathfrak{q})$ is a minimal set of generators over $\Bbbk$. The properties like "being a polynomial ring" do not change under field extensions. The results in this paper are valid over fields that are not algebraically closed, but in the proofs we may safely assume that $\Bbbk = \bar{\Bbbk}$.

2. On the coadjoint representations of a semi-direct product

For semi-direct products, there are some specific approaches to the symmetric invariants. Our convention is that G is always a connected **reductive** group and $\mathfrak{g} = \operatorname{Lie} G$, whereas a group Q is not necessarily reductive and $\mathfrak{q} = \operatorname{Lie} Q$. In this section, either $\mathfrak{s} = \mathfrak{g}\ltimes V$ or $\mathfrak{s} = \mathfrak{q}\ltimes V$, where V is a finite-dimensional G- or Q-module. Then S is a connected algebraic group with $\operatorname{Lie} S = \mathfrak{s}$. For instance, $S = Q\ltimes\exp(V)$.

The vector space decomposition $\mathfrak{s} = \mathfrak{q}\oplus V$ leads to $\mathfrak{s}^* = \mathfrak{q}^*\oplus V^*$. For $\mathfrak{q} = \mathfrak{g}$, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Each element $x \in V^*$ is considered as a point of $\mathfrak{s}^*$ that is zero on $\mathfrak{q}$. We have $\exp(V){\cdot}x = \operatorname{ad}^*(V){\cdot}x + x$, where each element of $\operatorname{ad}^*(V){\cdot}x$ is zero on V. Note that $\operatorname{ad}^*(V){\cdot}x \subset \operatorname{Ann}(\mathfrak{q}_x) \subset \mathfrak{q}^*$ and $\dim(\operatorname{ad}^*(V){\cdot}x)$ is equal to $\dim(\operatorname{ad}^*(\mathfrak{q}){\cdot}x) = \dim\mathfrak{q} - \dim\mathfrak{q}_x$. Therefore $\operatorname{ad}^*(V){\cdot}x = \operatorname{Ann}(\mathfrak{q}_x)$.

There is a general formula [R78] for the index of $\mathfrak{s} = \mathfrak{q}\ltimes V$:

$$\operatorname{ind}\mathfrak{s} = \dim V - (\dim\mathfrak{q} - \dim\mathfrak{q}_x) + \operatorname{ind}\mathfrak{q}_x \ \text{ with } x\in V^* \text{ generic.} \qquad (2{\cdot}1)$$

The decomposition $\mathfrak{s} = \mathfrak{q}\oplus V$ defines the bi-grading on $\mathcal{S}(\mathfrak{s})$ and it appears that $\mathcal{S}(\mathfrak{s})^S$ is a bi-homogeneous subalgebra, cf. [P07b, Theorem 2.3(i)].

For any $x \in V^*$, the affine space $\mathfrak{q}^*+x$ is $\exp(V)$-stable and Q_x-stable. Further, there is the restriction homomorphism

$$\psi_x : \Bbbk[\mathfrak{s}^*]^S \to \Bbbk[\mathfrak{q}^*+x]^{Q_x\ltimes\exp(V)} \simeq \mathcal{S}(\mathfrak{q}_x)^{Q_x}.$$

The existence of the isomorphism $\Bbbk[\mathfrak{q}^*+x]^{\exp(V)} \simeq \mathcal{S}(\mathfrak{q}_x)$ is proven in [Y17a]. If we choose x as the origin in $\mathfrak{q}^*+x$, then actually $\psi_x(H) \in \mathcal{S}(\mathfrak{q}_x)$ for each $H \in \Bbbk[\mathfrak{s}^*]^{\exp(V)}$, see [Y17a, Prop. 2.7].

Suppose that $Q \lhd \tilde{Q}$ and there is an action of $\tilde{Q}$ on V that extends the Q-action. Set $\tilde{\mathfrak{s}} = \tilde{\mathfrak{q}} \ltimes V$, $\tilde{S} = \tilde{Q} \ltimes \exp(V)$.

Lemma 2.1. *We have $\mathcal{S}(\mathfrak{s})^{\tilde{S}} \subset \mathcal{S}(\mathfrak{s})^S$ and $H \in \mathcal{S}(\tilde{\mathfrak{s}})^{\tilde{S}}$ lies in $\mathcal{S}(\mathfrak{s})$ if and only if the restriction of H to $\tilde{\mathfrak{q}}^*+x$ lies in $\mathcal{S}(\mathfrak{q}_x)$ for a generic $x \in V^*$.*

Proof. The inclusion $\mathcal{S}(\mathfrak{s})^{\tilde{S}} \subset \mathcal{S}(\mathfrak{s})^S$ is obvious. Now let $\mathfrak{m}$ be a vector space complement of $\mathfrak{q}$ in $\tilde{\mathfrak{q}}$. Then $\mathcal{S}(\tilde{\mathfrak{s}}) = \mathcal{S}(\mathfrak{q}) \otimes \mathcal{S}(\mathfrak{m}) \otimes \mathcal{S}(V)$. If H does not lie in $\mathcal{S}(\mathfrak{q}) \otimes \mathcal{S}(V)$, then $H|_{\tilde{\mathfrak{q}}^*+x}$ does not lie in $\mathcal{S}(\mathfrak{q})$ for any x from a non-empty open subset of V^*.

Finally, suppose that $H \in \mathcal{S}(\mathfrak{s})^{\exp(V)}$. Then $H|_{\tilde{\mathfrak{q}}^*+x}$ lies in $\mathcal{S}(\tilde{\mathfrak{q}}_x)$ by [Y17a, Prop. 2.7]. Clearly, $\mathcal{S}(\mathfrak{q}) \cap \mathcal{S}(\tilde{\mathfrak{q}}_x) = \mathcal{S}(\mathfrak{q}_x)$. □

Proposition 2.2 (Prop. 3.11 in [Y17a])**.** *Let Q be a connected algebraic group acting on a finite-dimensional vector space V. Set $\mathfrak{s} = \mathfrak{q} \ltimes V$. Suppose that Q has no proper semi-invariants in $\Bbbk[\mathfrak{s}^*]^{\exp(V)}$ and $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring in $\operatorname{ind}\mathfrak{s}$ variables. For generic $x \in V^*$, we then have*

- *the restriction map $\psi\colon \Bbbk[\mathfrak{s}^*]^S \to \Bbbk[\mathfrak{q}^*+x]^{Q_x \ltimes \exp(V)} \simeq \mathcal{S}(\mathfrak{q}_x)^{Q_x}$ is onto;*
- *$\mathcal{S}(\mathfrak{q}_x)^{Q_x}$ coincides with $\mathcal{S}(\mathfrak{q}_x)^{\mathfrak{q}_x}$;*
- *$\mathcal{S}(\mathfrak{q}_x)^{Q_x}$ is a polynomial ring in $\operatorname{ind}\mathfrak{q}_x$ variables.*

Note that Q is not assumed to be reductive and Q_x is not assumed to be connected in the above proposition!

Let now V be a G-module. By a classical result of Richardson, there is a non-empty open subset $\Omega \subset V^*$ such that the stabilisers G_x are conjugate in G for all $x \in \Omega$, see e.g. [VP89, Theorem 7.2]. In this situation (any representative of the conjugacy class of) G_x is called a *generic isotropy group*, denoted g.i.g.$(G : V^*)$, and $\mathfrak{g}_x = \operatorname{Lie} G_x$ is a *generic stabiliser* for the G-action on V^*.

If G is semisimple and V is a reducible G-module, say $V = V_1 \oplus V_2$, then there is a trick that allows us to relate the polynomiality property for the symmetric invariants of $\mathfrak{s} = \mathfrak{g} \ltimes V$ to a smaller semi-direct product. The precise statement is as follows.

Proposition 2.3 (cf. [PY17b, Prop. 3.5])**.** *With $\mathfrak{s} = \mathfrak{g} \ltimes (V_1 \oplus V_2)$ as above, let H be a generic isotropy group for $(G : V_1^*)$. If $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring, then so is $\Bbbk[\tilde{\mathfrak{q}}^*]^{\tilde{Q}}$, where $\tilde{Q} = H \ltimes \exp(V_2)$ or $H^\circ \ltimes \exp(V_2)$.*

The above passage from $\mathfrak{s}$ to $\tilde{\mathfrak{q}}$, i.e., from $(G, V_1 \oplus V_2)$ to (H°, V_2) is called a *reduction*, and we denote it by $(G, V_1 \oplus V_2) \longrightarrow (H^\circ, V_2)$ in the diagrams below. This proposition is going to be used as a tool for proving that $\Bbbk[\mathfrak{s}^*]^S$ is not polynomial.

In what follows, the irreducible representations of simple groups are often identified with their highest weights, using the Vinberg–Onishchik numbering of the fundamental weights [VO88]. For instance, if $\varphi_1, \dots, \varphi_n$ are the fundamental weights of a simple algebraic group G, then $V = \varphi_i + 2\varphi_j$ stands for the direct sum of three simple G-modules, with highest weights φ_i (once) and φ_j (twice). A full notation is $V = V_{\varphi_i} + 2V_{\varphi_i}$. Note that adding a trivial 1-dimensional G-module $\Bbbk$ to V does not affect the polynomiality property for $\mathfrak{s}$.

Example 2.4. There is a diagram (tree) of reductions:

$$\begin{array}{ccccccc}
(\mathrm{Spin}_{11}, 2\varphi_1{+}\varphi_5) \searrow & (\mathrm{Spin}_{10}, \varphi_1{+}\varphi_4{+}\varphi_5) \downarrow & \swarrow (\mathrm{Spin}_{10}, \varphi_1{+}2\varphi_4) & & (\mathrm{Spin}_{8}, 2\varphi_1{+}\varphi_3) \swarrow & & \\
(\mathrm{Spin}_{12}, 3\varphi_1{+}\varphi_5) \longrightarrow & (\mathrm{Spin}_9, 2\varphi_4) & \longrightarrow & (\mathrm{Spin}_7, \varphi_1{+}\varphi_3{+}\Bbbk) & \longrightarrow & \boxed{(\mathrm{SL}_4, \varphi_1{+}\varphi_1^*)} .
\end{array}$$

For instance, the first diagonal arrow means that for $G = \mathrm{Spin}_{11}$ and $V_1 = 2\varphi_1$, we have g.i.g.$(G, V_1) = \mathrm{Spin}_9$ and the restriction of $V_2 = \varphi_5$ to $H = \mathrm{Spin}_9$ is the H-module $2\varphi_4$. The terminal item (in the box) does not have the polynomiality property by [Y17b]. Therefore all the items here do not have the polynomiality property by Proposition 2.3.

The action $(G : V)$ is said to be *stable* if the union of closed G-orbits is dense in V. Then g.i.g.$(G : V)$ is necessarily reductive.

We mention the following good situation. Suppose that G is semisimple. If a generic stabiliser for the G-action on V^* is reductive, then the action $(G : V^*)$ is stable [VP89, § 7]. Moreover, S has only trivial characters and no proper semi-invariants.

Example 2.5 (cf. [Y17a, Example 3.6])**.** If G is semisimple, $\mathfrak{g}_x = \mathfrak{sl}_2$ for $x \in V^*$ generic, and $\Bbbk[V^*]^G$ is a polynomial ring, then $\mathcal{S}(\mathfrak{s})^{\mathfrak{s}}$ is a polynomial ring.

We say that $\dim V/\!\!/G$ is the *rank* of the pair (G, V). For (G, V^*) of rank one, we have two general results.

Consider the following assumptions on G and V:

($\diamondsuit$) the action $(G : V^*)$ is stable, $\Bbbk[V^*]^G$ is a polynomial ring, $\Bbbk[\mathfrak{g}_\xi^*]^{G_\xi}$ is a polynomial ring for generic $\xi \in V^*$, and G has no proper semi-invariants in $\Bbbk[V^*]$.

Theorem 2.6 ([PY17b, Theorem 2.3])**.** *Suppose that G and V satisfy condition ($\diamondsuit$) and $V^*/\!\!/G = \mathbb{A}^1$, i.e., $\Bbbk[V^*]^G = \Bbbk[F]$ for some homogeneous F. Let L be a generic isotropy group for $(G : V^*)$. Assume further that $D = \{x \in V^* \mid F(x) = 0\}$ contains an open G-orbit, say $G{\cdot}y$, $\mathrm{ind}\,\mathfrak{g}_y = \mathrm{ind}\,\mathfrak{l} =: \ell$, and $\mathcal{S}(\mathfrak{g}_y)^{G_y}$ is a polynomial ring in ℓ variables with the same degrees of generators as $\mathcal{S}(\mathfrak{l})^L$. Then $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring in $\mathrm{ind}\,\mathfrak{s} = \ell + 1$ variables.*

Lemma 2.7. *Suppose that G is semisimple, $\Bbbk[V^*]^G = \Bbbk[F]$ and a generic isotropy group for $(G : V^*)$, say L, is connected and is either of type $\mathbf{B}_2$ or $\mathbf{G}_2$. Then $\mathfrak{s} = \mathfrak{g} \ltimes V$ has the polynomiality property.*

Proof. Let $x \in V^*$ be generic and $G_x = L$, hence $\mathfrak{g}_x = \mathfrak{l}$. By [Y17a, Lemma 3.5], there are irreducible bi-homogeneous S-invariants H_1 and H_2 such that their restrictions to $\mathfrak{g}+x = \mathfrak{g}^*+x$ yield the basic symmetric invariants of $\mathfrak{l}$ under the isomorphism $\Bbbk[\mathfrak{g}^*+x]^{G_x\ltimes\exp(V)} \simeq \mathcal{S}(\mathfrak{g}_x)^{G_x}$. Furthermore, $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[F,H_1,H_2]$ if and only if H_1 and H_2 are algebraically independent over $\Bbbk[D]^G = \Bbbk$ on $\mathfrak{g}\times D$, where D is the zero set of F. W.l.o.g., we may assume that $\deg_{\mathfrak{g}} H_1 = 2$ and $\deg_{\mathfrak{g}} H_2 = 4$ (if $L = \mathbf{B}_2$) or $\deg_{\mathfrak{g}} H_2 = 6$ (if $L = \mathbf{G}_2$). We may also assume that a non-trivial relation among $H_1|_{\mathfrak{g}\times D}$, $H_2|_{\mathfrak{g}\times D}$ is homogeneous w.r.t. $\mathfrak{g}$ and therefore boils down to $\frac{H_1^\alpha}{H_2} \equiv a \mod (F)$ for $\alpha \in \{2,3\}$, depending on L, and $a \in \Bbbk$. Such a relation means that H_2 is chosen wrongly and has to be replaced by a polynomial $(H_2 - aH_1^\alpha)/F^r$ with the largest possible $r \geqslant 1$. This modification decreases the total degree of H_2 and hence it cannot be performed infinitely many times. □

The following result holds for actions of arbitrary rank.

Lemma 2.8. *Suppose that G is semisimple, $\Bbbk[V^*]^G$ is a polynomial ring and a generic isotropy group for $(G : V^*)$ is a connected group of type A_2. Assume further that, for any G-stable divisor $D \subset V^*$ and a generic point $y \in D$, we have $\dim \mathcal{S}^2(\mathfrak{g}_y)^{G_y} = \dim \mathcal{S}^3(\mathfrak{g}_y)^{G_y} = 1$ and that these unique (up to a scalar) invariants are algebraically independent. Then $\mathfrak{s} = \mathfrak{g} \ltimes V$ has the polynomiality property.*

Proof. The statement readily follows from [Y17a, Lemma 3.5]. □

2.1. **Yet another case of a surjective restriction.** By Proposition 2.2, if $x \in V^*$ is generic, then the restriction homomorphism $\psi_x : \Bbbk[\mathfrak{s}^*]^S \to \Bbbk[\mathfrak{q}^*+x]^{Q_x\ltimes\exp(V)}$ is surjective, whenever $\Bbbk[\mathfrak{s}^*]^S$ is a a polynomial ring and Q has no proper semi-invariants in $\Bbbk[\mathfrak{s}^*]^{\exp(V)}$. On the other hand, ψ_x is surjective for generic $x \in V^*$ if $Q = G$ is reductive and the G-action on V^* is stable [Y17a, Theorem 2.8]. It is likely that the surjectivity holds for a wider class of semi-direct products.

Suppose that $\Bbbk$ is algebraically closed. Take Q and V such that $\dim(Q{\cdot}\xi) = \dim Q - 1$ for generic $\xi \in V^*$. Assume that $\Bbbk[V^*]^Q \neq \Bbbk$. Then $\Bbbk[V^*]^Q = \Bbbk[F]$, where F is a homogeneous polynomial of degree $N \geqslant 1$, $\Bbbk(V^*)^Q = \Bbbk(F)$, and F separates generic Q-orbits on V^*. Hence $\Bbbk\xi \cap Q{\cdot}\xi = \{a x \mid a \in \Bbbk, a^N = 1\}$ for generic $\xi \in V^*$. Let $N_Q(\Bbbk\xi)$ be the normaliser of the line $\Bbbk\xi$. Then $N_Q(\Bbbk\xi) = C_N{\times}Q_\xi$, where $C_N \subset \Bbbk^\times$ is a cyclic group of order N. Let C_N act on V faithfully, then $\tilde{Q} := C_N{\times}Q$ acts on V and $\tilde{Q}_\xi \simeq C_N{\times}Q_\xi$. If $H \in \Bbbk[\mathfrak{s}^*]^Q$ is homogeneous in V, then $\psi_\xi(H)$ is an eigenvector of $C_N \subset \tilde{Q}_\xi$ and the corresponding eigenvalue depends only on $\deg_V H$.

Theorem 2.9 (Generalised surjectivity or the "rank-one argument")**.** *Let Q be a connected algebraic group with $\mathsf{Lie}\, Q = \mathfrak{q}$. Suppose that V is a Q-module such that Q has no proper semi-invariants in $\Bbbk[V^*]$ and $\Bbbk[V^*]^Q = \Bbbk[F]$ with $F \notin \Bbbk$. Set $\mathfrak{s} = \mathfrak{q}{\ltimes}V$, $S = Q{\ltimes}\exp(V)$. Then the* *natural homomorphism*

$$\psi_\xi : \Bbbk[\mathfrak{s}^*]^S \to \Bbbk[\mathfrak{q}^* + \xi]^{Q_\xi \ltimes \exp(V)} \simeq \mathcal{S}(\mathfrak{q}_\xi)^{Q_\xi}$$

is onto for generic $\xi \in V^*$. Moreover, if $h \in \Bbbk[\mathfrak{q}^*{+}\xi]^{Q_\xi \ltimes \exp(V)}$ is a semi-invariant of $N_Q(\Bbbk\xi)$, then there is a homogeneous in V polynomial $H \in \Bbbk[\mathfrak{s}^*]^S$ with $\psi_\xi(H) = h$.

Proof. Let S act on an irreducible variety X. A classical result of Rosenlicht [VP89, § 2.3] implies that the functions $f_1, \dots, f_m \in \Bbbk(X)^S$ generate $\Bbbk(X)^S$ if and only if they separate generic S-orbits on X. Let $U \subset \mathfrak{s}^*$ be a non-empty open subset such that for every two different orbits $S{\cdot}u, S{\cdot}u' \subset U$, there is $\mathbf{f} \in \Bbbk(\mathfrak{s}^*)^S$ separating them, meaning that $\mathbf{f}$ takes finite values at u, u' and $\mathbf{f}(u) \neq \mathbf{f}(u')$. Then $U \cap (\mathfrak{q}^*{+}\xi) \neq \varnothing$ for generic $\xi \in V^*$ and hence generic $Q_\xi \ltimes \exp(V)$-orbits on $\mathfrak{q}^*{+}\xi$ are separated by rational S-invariants for any such ξ. In other words, for every $h \in \Bbbk(\mathfrak{q}^*{+}\xi)^{Q_x \ltimes \exp(V)}$ there is $\tilde{\boldsymbol{r}} \in \Bbbk(\mathfrak{s}^*)^S$ such that $\psi_\xi(\tilde{\boldsymbol{r}}) := \tilde{\boldsymbol{r}}|_{\mathfrak{q}^*+\xi} = h$.

The same principle applies to the group $\Bbbk^{\times}{\times}S$, where $\Bbbk^{\times}$ acts on V by $t{\cdot}v = tv$ for all $t \in \Bbbk^{\times}$, $v \in V$. A rational invariant of $(\Bbbk^{\times}{\times}Q)_\xi \ltimes \exp(V)$ on $\mathfrak{q}^*{+}\xi$ extends to a rational $(\Bbbk^{\times}{\times}S)$-invariant on $\mathfrak{s}^*$.

The absence of proper semi-invariants implies that $\Bbbk(V^*)^Q = \Bbbk(F)$. Hence a generic Q-orbit on V^* is of dimension $\dim V{-}1$. Assume that F is homogeneous and set $N := \deg F$.

Choose a generic point $\xi \in V^*$ with $F(\xi) \neq 0$ and with $\dim(Q{\cdot}\xi) = \dim V{-}1$. Then $N_Q(\Bbbk\xi) = C_N{\times}Q_\xi$. As above, set $\tilde{Q} := C_N{\times}Q$ and also $\tilde{S} := C_N{\times}S$. We regard $\tilde{Q}$ as a subgroup of $\Bbbk^{\times}{\times}Q$. Now $\tilde{Q}_\xi = (\Bbbk^{\times}{\times}Q)_\xi$.

The group $C_N \subset \tilde{Q}_\xi$ acts on $\Bbbk[\mathfrak{q}^* + \xi]^{Q_\xi \ltimes \exp(V)}$ and this action is diagonalisable. Suppose that $h \in \Bbbk[\mathfrak{q}^* + \xi]^{Q_\xi \ltimes \exp(V)}$ is an eigenvector of C_N. First we show that there is $r \in \Bbbk(\mathfrak{s}^*)^S$ such that $\psi_\xi(r)$ is an eigenvector of $C_N \subset \tilde{Q}_\xi$ with the same weight as h.

Recall that h extends to a rational S-invariant $\tilde{\boldsymbol{r}} \in \Bbbk(\mathfrak{s}^*)^S$. The group C_N is finite, hence $\tilde{\boldsymbol{r}}$ is contained in a finite-dimensional C_N-stable vector space and thereby $\tilde{\boldsymbol{r}}$ is a sum of rational S-invariant C_N-eigenvectors. Since a copy of C_N sitting in $\tilde{Q}$ stabilises ξ, we can replace $\tilde{\boldsymbol{r}}$ with a suitable C_N-semi-invariant component. By a standard argument, this new $\tilde{\boldsymbol{r}}$ is a ratio of two regular $\tilde{S}$-semi-invariants, say $\tilde{\boldsymbol{r}} = q/f$ now. Each bi-homogenous w.r.t. $\mathfrak{s} = \mathfrak{q}{\oplus}V$ component of q (or f) is again a semi-invariant of $\tilde{S}$ of the same weight as q (or f). Let us replace f (and q) with any of its non-zero bi-homogenous components. The resulting rational function r has the same weight as $\tilde{\boldsymbol{r}}$. In particular, r is an S-invariant. Thus, we have found the required rational function. Since r is a semi-invariant of $\Bbbk^{\times}$, it is defined on a non-empty open subset of $\mathfrak{q}^*{\times}Q{\cdot}x$ for each $x \in V^*$ such that $F(x) \neq 0$ and $\dim(Q{\cdot}x) = \dim V{-}1$.

Set $\bar r := \psi_\xi(r) \in \Bbbk(\mathfrak q^*{+}\xi)$. Then $h/\bar r \subset \Bbbk(\mathfrak q^*{+}\xi)^{\tilde Q_\xi \ltimes \exp(V)}$ and therefore extends to a rational $(\Bbbk^{\times}{\times}S)$-invariant on $\mathfrak s^*$. Multiplying the extension by r, we obtain a rational S-invariant R, which is also an eigenvector of $\Bbbk^{\times}$. Let $R = H/P$, where $H, P \in \Bbbk[\mathfrak s^*]$ are relatively prime. Then both H and P are homogenous in V. Note that R is defined on $\mathfrak q^*{+}\xi$, therefore also on $\mathfrak q^*{\times}Q{\cdot}\xi$ and finally on $\mathfrak q \times \Bbbk^{\times}(Q{\cdot}\xi)$, because $R(\eta{+}a\xi) = a^k R(\eta{+}\xi)$ for some $k \in \mathbb Z$ and for all $a \in \Bbbk^{\times}$, $\eta \in \mathfrak q^*$. Hence P is a polynomial in F, more explicitly, $P = F^d$ fore some $d \geqslant 0$. Multiplying R by $\frac{F^d}{F(\xi)^d}$ yields the required pre-image H. □

Remark 2.10. Since $\Bbbk[V^*]^Q = \Bbbk[F]$ and there are no proper Q-semi-invariants in $\Bbbk[V^*]$, $\mathfrak q^*{\times}Q{\cdot}\xi$ is a big open subset of

$$Y_\alpha = \{\mathfrak q^*{+}x \mid F(x) = F(\xi)\} = \{\gamma \in \mathfrak s^* \mid F(\gamma) = \alpha\},$$

where $\alpha = F(\xi)$. For a reductive group G, one knows that any regular G-invariant on a closed G-stable subset $Y \subset X$ of an affine G-variety X extends to a regular G-invariant on X. Assuming that the image of Q in $\mathrm{GL}(V^*)$ is reductive, we could present a different proof of Theorem 2.9, similar to the proof of Theorem 2.8 in [Y17a].

2.2. **Tables and classification tools.** Our goal is to classify the pairs (G, V) such that G is either Spin_n or Sp_{2n} and the semi-direct product $\mathfrak s = \mathfrak g \ltimes V$ has a Free Algebra of symmetric invariants, (FA) for short. We also say that (G, V) is a *positive* (resp. *negative*) case, if the property (FA) **is** (resp. **is not**) satisfied for $\mathfrak s$.

Example 2.11. If G is arbitrary semisimple, then $\mathfrak g \ltimes \mathfrak g^{\mathsf{ab}}$, where $\mathfrak g^{\mathsf{ab}}$ is an Abelian ideal isomorphic to $\mathfrak g$ as a $\mathfrak g$-module, always has (FA) [T71]. Therefore we exclude the adjoint representations from our further consideration.

- If $\Bbbk[\mathfrak s^*]^S$ is a polynomial ring, then so is $\Bbbk[V^*]^G$ [P07b, Section 2 (A)] (cf. [Y17a, Section 3]). For this reason, we only have to examine all representations of G with polynomial rings of invariants.
- Since the algebras $\Bbbk[V]^G$ and $\Bbbk[V^*]^G$ (as well as $\mathcal S(\mathfrak g \ltimes V)^{G\ltimes V}$ and $\mathcal S(\mathfrak g \ltimes V^*)^{G\ltimes V^*}$) are isomorphic, it suffices to keep track of either V or V^*. The same principle applies to the two half-spin representations in type $\mathbf D_{2m}$.

Example 2.12. If a generic stabiliser for $(G{:}V^*)$ is trivial, then $\Bbbk[\mathfrak s^*]^S \simeq \Bbbk[V^*]^G$ [P07a, Theorem 6.4] (cf. [Y17a, Example 3.1]). Therefore all such semi-direct products have (FA).

We are lucky that there is a classification of the representations of the simple algebraic groups with non-trivial generic stabilisers obtained by A.G. Elashvili [E72]. In addition, the two independent classifications in [S78, AG79] provide the list of representations of simple algebraic groups with polynomial rings of invariants. Combining them, we obtain the representations in Tables 1 and 2.

Explanations to the tables. As in [AG79, E72, Y17a, PY17a, PY17b], we use the Vinberg–Onishchik numbering of fundamental weights, see [VO88, Table 1]. In both tables, $\mathfrak h$ is a generic stabiliser for $(G : V)$ and the last column indicates whether (FA) is satisfied for $\mathfrak s$ or not. Naturally, the positive cases are marked with '$+$'. This last column represents the main results of the article. The ring $\Bbbk[V^*]^G$ is always a polynomial ring in $\dim V/\!\!/G$ variables. If the expression for $\dim V/\!\!/G$ is bulky, then it is not included in Table 1. However, one always has $\dim V/\!\!/G = \dim V - \dim G + \dim \mathfrak h$. If $\mathfrak s$ has (FA), then $\operatorname{ind}\mathfrak s = \dim V/\!\!/G + \operatorname{ind}\mathfrak h$ is the total number of the basic invariants in $\Bbbk[\mathfrak s^*]^S$. The symbol $\mathbf{U}_n$ in Table 2 stands for a commutative Lie subalgebra of dimension n that consists of nilpotent elements.

Our classification is summarised in the following

Theorem 2.13. *Let G be either Spin_n or Sp_{2n}, V a finite-dimensional rational G-module, and $\mathfrak s = \mathfrak g \ltimes V$. Then $\Bbbk[\mathfrak s^*]^S$ is a free algebra if and only if one of the following conditions is satisfied:*

- (i) $V = \mathfrak g$;
- (ii) *V or V^* occurs in Tables 1 and 2, and the last column is marked with '$+$'. It is also possible to permute φ_5 and φ_6 for $\mathbf{D}_6$, and take any permutation of $\varphi_1, \varphi_3, \varphi_4$ for $\mathbf{D}_4$.*
- (iii) *$\Bbbk[V]^G$ is a free algebra and* g.i.g.$(G : V)$ *is finite, i.e., (G, V) is contained in the lists of* [S78, AG79], *but is not contained in the tables of* [E72].

• Generic stabilisers for the representations in the tables are taken from [E72]. To verify that the generic isotropy groups are connected, we use Proposition 4.10 and Remark 4.11 in [S78]. In case of reducible representations, this can be combined with the group analogue of [E72, Lemma 2].

• Apart from a generic isotropy group for $(G : V^*)$, we often have to compute the isotropy group G_y, where y is a generic point of a G-stable divisor $D \subset V^*$, cf. Theorem 2.6. Mostly this is done by *ad hoc* methods. Also the following observation is very helpful. Any divisor $D \subset V_1 \oplus V_2$ projects dominantly to at least one factor V_i. Hence it contains a subset of the form $\{x_i\} \times D_{i'}$, where $x_i \in V_i$ is generic, $D_{i'} \subset V_{i'}$ is a divisor, and $\{i, i'\} = \{1, 2\}$.

• Another major ingredient in obtaining the classification is (the presence of) the "codim–2" property for $\mathfrak s$. Some methods for checking the "codim–2" condition are presented in [PY17a, Sect. 4]. Similarly to the Raïs formula, see Eq. (2·1), we also have

$$\dim \mathfrak s_{\gamma+y} = \dim(\mathfrak g_y)_{\bar\gamma} + (\dim V - \dim(G{\cdot}y)),$$

where $y \in V^*$, $\gamma \in \mathfrak g$, and $\bar\gamma = \gamma|_{\mathfrak g_y}$, cf. [Y17a, Eq. (3·1)]. Therefore, $\mathfrak s$ has the "codim–2" property if and only if

(i) $\mathfrak g_x$ with $x \in V^*$ generic has the "codim–2" property and

TABLE 1. The representations of the orthogonal groups with polynomial ring $\Bbbk[V]^G$ and non-trivial generic stabilisers

№	G	V	$\dim V$	$\dim V/\!\!/G$	$\mathfrak h$	$\mathrm{ind}\,\mathfrak s$	(FA)
1	SO_n	$m\varphi_1,\ m<n-1$	mn	$\frac{m(m+1)}{2}$	$\mathfrak{so}_{n-m}$	$\binom{m+1}{2}+[\frac{n-m}{2}]$	$+$
2a	$\mathbf{B}_3$	φ_3	8	1	$\mathbf{G}_2$	3	$+$
2b		$m\varphi_1+m'\varphi_3$ $2\leqslant m+m'\leqslant 3$ $m'>0$	$7m+8m'$		$\mathbf{A}_{4-m-m'}$		$-$, if $(1,1)$
3a	$\mathbf{B}_4$	φ_4	16	1	$\mathbf{B}_3$	4	$+$
3b		$\varphi_1+\varphi_4$	25	3	$\mathbf{G}_2$	5	$+$
3c		$2\varphi_1+\varphi_4$	34	6	$\mathbf{A}_2$	8	$+$
3d		$3\varphi_1+\varphi_4$	43	10	$\mathbf{A}_1$	11	$+$
3e		$2\varphi_4$	32	4	$\mathbf{A}_2$	6	$-$
3f		$\varphi_1+2\varphi_4$	41	8	$\mathbf{A}_1$	9	$+$
4	$\mathbf{B}_5$	$m\varphi_1+\varphi_5$, $0\leqslant m\leqslant 3$	$32+11m$	$1+m+m^2$	$\mathbf{A}_{4-m}$	$5+m^2$	$+$, if $m=0{,}3$ $-$, if $m=1{,}2$
5a	$\mathbf{B}_6$	φ_6	64	2	$\mathbf{A}_2+\mathbf{A}_2$	6	$+$
5b		$\varphi_1+\varphi_6$	77	5	$\mathbf{A}_1+\mathbf{A}_1$	7	$+$
6a	$\mathbf{D}_4$	$\varphi_1+\varphi_3$	16	2	$\mathbf{G}_2$	4	$+$
6b		$m\varphi_1+\varphi_3, m=2{,}3$	$8(m+1)$		$\mathbf{A}_{4-m}$		$+$, if $m=3$
6c		$m\varphi_1+\varphi_3+\varphi_4$ $m=1,2$	$8(m+2)$		$\mathbf{A}_{3-m}$		$+$
7a	$\mathbf{D}_5$	φ_4	16	0	$\mathfrak{so}_7\ltimes V_{\varphi_3}$	3	$+$
7b		$\varphi_1+\varphi_4$	26	2	$\mathbf{B}_3$	5	$+$
7c		$2\varphi_1+\varphi_4$	36	5	$\mathbf{G}_2$	7	$+$
7d		$m\varphi_1+\varphi_4, m=3{,}4$	$16+10m$		$\mathbf{A}_{5-m}$		$+$
7e		$2\varphi_4$	32	1	$\mathbf{G}_2$	3	$+$
7f		$m\varphi_1+2\varphi_4, m=1{,}2$	$32+10m$		$\mathbf{A}_{3-m}$		$+$, if $m=2$
7g		$3\varphi_4$ or $2\varphi_4+\varphi_5$	48	6	$\mathbf{A}_1$	7	$+$
7h		$m\varphi_1+\varphi_4+\varphi_5$ $0\leqslant m\leqslant 2$	$32+10m$	$2+2m+m^2$	$\mathbf{A}_{3-m}$	$5+m+m^2$	$-$, if $m\leqslant 1$ $+$, if $m=2$
8a	$\mathbf{D}_6$	$m\varphi_1+\varphi_5$ $0\leqslant m\leqslant 4$	$32+12m$	$1+m^2$	$\mathbf{A}_{5-m}$	$6-m+m^2$	$+$, if $m=0{,}4$ $-$, if $1\leqslant m\leqslant 3$
8b		$2\varphi_5$	64	7	$3\mathbf{A}_1$	10	$+$
8c		$\varphi_5+\varphi_6$	64	4	$2\mathbf{A}_1$	6	$+$
9a	$\mathbf{D}_7$	φ_6	64	1	$2\mathbf{G}_2$	5	$+$
9b		$m\varphi_1+\varphi_6, m=1{,}2$	$64+14m$		$2\mathbf{A}_{3-m}$		$+$

TABLE 2. The representations of the symplectic group with polynomial ring $\Bbbk[V]^G$ and non-trivial generic stabilisers

№	G	V	$\dim V$	$\dim V/\!\!/G$	$\mathfrak h$	$\operatorname{ind}\mathfrak s$	(FA)
1	$\mathbf{C}_n$	$m\varphi_1$, $m\leqslant 2n-1$	$2mn$	$\binom{m}{2}$	$\mathbf{C}_{n-l}$, $m=2l$ $\mathbf{C}_{n-l}\ltimes\mathfrak{heis}_{n-l}$, $m=2l-1$	$\binom{m}{2}+n-[\frac{m}{2}]$	$+$
2	$\mathbf{C}_n$	φ_2	$2n^2-n-1$	$n-1$	$n\mathbf{A}_1$	$2n-1$	$+$
3	$\mathbf{C}_n$	$\varphi_1+\varphi_2$	$2n^2+n-1$	$n-1$	$\mathbf{U}_n$	$2n-1$	$+$
4	$\mathbf{C}_3$	φ_3	14	1	$\mathbf{A}_2$	3	$+$
5		$\varphi_1+\varphi_3$	20	2	$\mathbf{A}_1$	3	$+$
6		$2\varphi_2$	28	8	$\mathfrak t_1$	9	$+$

(ii) for any divisor $D\subset V$, $\operatorname{ind}\mathfrak g_y+(\dim V-\dim(G{\cdot}y))=\operatorname{ind}\mathfrak s$ holds for all points y of a non-empty open subset $U\subset D$, cf. [Y17a, Eq. (3·2)].

• Finally, we recall an important class of semi-direct products. Let $\mathfrak g=\mathfrak g_0\oplus\mathfrak g_1$ be a $\mathbb Z_2$-grading of $\mathfrak g$, i.e., $(\mathfrak g,\mathfrak g_0)$ is a symmetric pair. Then the semi-direct product $\mathfrak s=\mathfrak g_0\ltimes\mathfrak g_1^{\mathsf{ab}}$, where $[\mathfrak g_1^{\mathsf{ab}},\mathfrak g_1^{\mathsf{ab}}]=0$, is called the $\mathbb Z_2$-*contraction* of $\mathfrak g$ related to the symmetric pair $(\mathfrak g,\mathfrak g_0)$. Set $l=\operatorname{rk}\mathfrak g$ and let $H_1,\dots,H_l$ be a set of the basic symmetric invariants of $\mathfrak g$. Let $H_i^\bullet$ denote the bi-homogeneous component of H_i that has the highest $\mathfrak g_1$-degree. Then $H_i^\bullet$ is an $\mathfrak s$-invariant in $\mathcal S(\mathfrak s)$ [P07b]. We say that a $\mathbb Z_2$-contraction $\mathfrak s=\mathfrak g_0\ltimes\mathfrak g_1^{\mathsf{ab}}$ is *good* if $\mathcal S(\mathfrak s)^{\mathfrak s}$ is freely generated by the polynomials $H_1^\bullet,\dots,H_l^\bullet$ for some well-chosen generators $\{H_i\}$. Note that $\deg H_i=\deg H_i^\bullet$ for the usual degree.

3. AN EXAMPLE IN TYPE A

The example considered in this section will be needed below in our treatment of $G=\mathrm{SO}_n$. It can also be regarded as a small step towards the classification in type A.

Suppose that $G=\mathrm{SL}_n\subset\mathrm{GL}_n=\tilde G$ and $V=\bigwedge^2\Bbbk^n\oplus(\bigwedge^2\Bbbk^n)^*$. Then $\tilde{\mathfrak s}=\tilde{\mathfrak g}\ltimes V$ is the $\mathbb Z_2$-contraction of $\mathfrak{so}_{2n}$ related to the symmetric pair $(\mathfrak{so}_{2n},\mathfrak{gl}_n)$. By [Y14, Theorem 4.5], this $\mathbb Z_2$-contraction is good and satisfies ***KRC***. Our goal is to describe $\mathcal S(\mathfrak s)^{\mathfrak s}$ using the known description for $\tilde{\mathfrak s}$. Let us denote the basic symmetric invariants of $\tilde{\mathfrak s}$ by $H_1,\dots,H_\ell$, $F_1,\dots,F_r$, where $\deg F_i=2i$ and $\Bbbk[F_1,\dots,F_r]=\Bbbk[V^*]^{\mathrm{GL}_n}$. Then necessary $\ell=[\frac{n+1}{2}]$, $r=[\frac{n}{2}]$.

Proposition 3.1. *If $n=2r$ is even, then $\mathcal S(\mathfrak s)^{\mathfrak s}$ is freely generated by $H_1,\dots,H_r,F_1,\dots,F_{r-1}$, F_r', F_{r+1} with $\deg F_r'=\deg F_{r+1}=r$.*

Proof. Since n is even, the generic isotropy group of the GL_n-action on V^* is $(\mathrm{SL}_2)^r$ and it lies in SL_n. Therefore each H_i lies in $\mathcal S(\mathfrak s)$, see Lemma 2.1. The new generators F_r', F_{r+1} are the pfaffians on $\bigwedge^2 \Bbbk^n$ and $(\bigwedge^2 \Bbbk^n)^*$, respectively. We have

$$\left(\sum_{i=1}^{r} \deg H_i + \sum_{j=1}^{r-1} \deg F_j\right) + 2r = \frac{\dim \mathfrak{s} + \operatorname{ind} \mathfrak{s}}{2}.$$

The generic isotropy groups of $(G\colon \bigwedge^2 \Bbbk^n)$ and $(\tilde{G}\colon \bigwedge^2 \Bbbk^n)$ are the same and $\tilde{\mathfrak{s}}$ has the "codim-2" property by [P07b]. Therefore $\mathfrak{s}$ has the "codim–2" property as well. The polynomials $F_1, \dots, F_{r-1}, F_r', F_{r+1}$ freely generate $\Bbbk[V^*]^G$ [S78, AG79] and the other generators, $H_1, \dots, H_r$, are algebraically independent over $\Bbbk[V^*]$. Therefore Theorem 1.1 applies and provides the result. □

The case of an odd n is much more difficult, because a generic stabiliser for $(G{:}V)$ is not reductive. We conjecture that $\mathcal{S}(\mathfrak{s})^{\mathfrak{s}}$ is still a polynomial ring, but the proof would require a subtle detailed analysis of the generators $H_1, \dots, H_\ell$. Since that case is not used in this paper, we postpone the exploration. Note only that if $n = 3$, then there is an isomorphism $\bigwedge^2 \Bbbk^3 \simeq (\Bbbk^3)^*$. The pair $(\mathrm{SL}_3, \Bbbk^3 \oplus (\Bbbk^3)^*)$ was considered in [Y17b], where it is shown that the corresponding $\mathfrak{s}$ has (FA).

4. The classification for the orthogonal algebra

In this section, $G = \mathrm{Spin}_n$. We classify the finite-dimensional rational representations $(G : V)$ such that g.i.g.$(G : V)$ is infinite and the symmetric invariants of $\mathfrak{s} = \mathfrak{g} \ltimes V$ form a polynomial ring. The answer is given in Table 1.

4.1. **The negative cases in Table 1.** Most of the negative cases (i.e., those having '$-$' in column (FA) in Table 1) are justified by Proposition 2.3 and the reductions of Example 2.4. Another similar diagram is presented below:

(4·1)
$$\begin{array}{ccc} (\mathrm{Spin}_{12}, 2\varphi_1 + \varphi_5) & & \\ & \searrow & \\ (\mathrm{Spin}_{11}, \varphi_1 + \varphi_5) & \longrightarrow & \boxed{(\mathrm{Spin}_{10}, \varphi_4 + \varphi_5)}\,. \end{array}$$

That is, our next step is to show that $(\mathrm{Spin}_{10}, \varphi_4 + \varphi_5)$ does not have (FA). Once this is done, we will know that all the cases in Diagram (4·1) are indeed negative. Afterwards, only one negative case is left, namely $(\mathrm{Spin}_{12}, \varphi_1 + \varphi_5)$.

Theorem 4.1. *The semi-direct product $\mathfrak{s} = \mathfrak{so}_{10} \ltimes (\varphi_4 + \varphi_5)$ does not have* (FA).

Proof. Here $G = \mathrm{Spin}_{10}$ is a subgroup of $\mathrm{Spin}_{11} \subset \mathrm{GL}(V)$ and $V \simeq V^*$ as a Spin_{11}-module. A generic isotropy group in Spin_{11} is SL_5. A generic isotropy group in Spin_{10} is SL_4. There is a divisor $D \subset V$ such that G_y is connected and $\mathfrak{g}_y = \mathfrak{sl}_3 \ltimes \mathfrak{heis}_3$ for a generic point $y \in D$. The stabiliser $\mathfrak{g}_y$ is obtained as an intersection of $\mathfrak{sl}_5$ and a specially chosen $\mathfrak{so}_{10} \subset \mathfrak{so}_{11}$.

Assume that $\mathfrak s$ has (FA). Then $\mathcal S(\mathfrak s)^{\mathfrak s} = \Bbbk[H_1,H_2,H_3,F_1,F_2]$, where $\Bbbk[V^*]^G = \Bbbk[F_1,F_2]$. According to Proposition 2.2, the restrictions $H_i|_{\mathfrak g+x}$ are generators of $\mathcal S(\mathfrak{sl}_4)^{\mathrm{SL}_4}$ for $x \in V^*$ generic. Therefore we may assume that $\deg_{\mathfrak g} H_i = i{+}1$. By Theorem 1.2, there is $y \in D$ with G_y as above such that the differentials $dF_1, dF_2, dH_1, dH_2, dH_3$ are linearly independent on a non-empty open subset of $\mathfrak g + y$ that is stable w.r.t. $G_y \ltimes \exp(V)$.

Take $\xi = \gamma + y$ with $\gamma \in \mathfrak g$ generic. Replacing γ by another point in $\gamma + \mathrm{ad}^*(V)y$ we may safely assume that γ is zero on $\mathrm{Ann}\,(\mathfrak g_y)$. Let $\bar\gamma$ stand for the restriction of γ to $\mathfrak g_y$. Then $\mathfrak s_\xi = (\mathfrak g_y)_{\bar\gamma} \oplus \Bbbk^2 = (\mathfrak t_2 {\oplus} \Bbbk z) \oplus \Bbbk^2$, where $\Bbbk z$ is the centre of $\mathfrak{heis}_3$, $\mathfrak t_2$ is a Cartan subalgebra of $\mathfrak{sl}_3$, and $\Bbbk^2 \subset V$.

We have $(dF_i)_\xi \in \mathfrak s_\xi \cap V = \Bbbk^2$. At the same time $(dH_i)_\xi = \eta_i + u_i$, where $u_i \in V$, $\eta_i \in \mathfrak g$, and η_i is the differential of $H_i|_{\mathfrak g+y}$ at γ. Since γ was chosen to be generic, the elements η_1, η_2, η_3 are linearly independent. Hence the restrictions $\mathbf h_i := H_i|_{\mathfrak g+y}$ are algebraically independent.

It can be easily seen that $\mathrm{ind}\,\mathfrak g_y = 3$ and that $\mathfrak g_y$ satisfies the "codim–2" condition. Since $\deg \mathbf h_i = i{+}1$, we have $\mathcal S(\mathfrak g_y)^{G_y} = \Bbbk[\mathbf h_1, \mathbf h_2, \mathbf h_3]$ by Theorem 1.1. But $z \in \mathcal S(\mathfrak g_y)^{G_y}$ and $\deg z = 1$. A contradiction! □

Theorem 4.2. *The semi-direct product $\mathfrak s = \mathfrak{so}_{12} \ltimes (\varphi_1 + \varphi_5)$ does not have* (FA).

Proof. Here $G = \mathrm{Spin}_{12}$ and a generic isotropy group for the G-action on V_{φ_5} (resp. $V_{\varphi_1} \oplus V_{\varphi_5}$) is SL_6 (resp. SL_5). Let $\boldsymbol f$ be a Spin_{12}-invariant quadratic form on $V_{\varphi_1} \simeq V^*_{\varphi_1}$. Then $D = \{\boldsymbol f = 0\} \times V^*_{\varphi_5}$ is a G-stable divisor in V^*. It can be verified that, for a generic point $y \in D$, one has $\mathfrak g_y = \mathfrak{sl}_4 \ltimes \mathfrak{heis}_4$ and G_y is connected. As in the proof of Theorem 4.1, $\mathcal S(\mathfrak g_y)^{G_y}$ has an element of degree 1, i.e., it is not generated by symmetric invariants of degrees $2, 3, 4, 5$, but it would have been if $\mathfrak s$ had (FA). □

4.2. **The positive cases in Table 1.** We now proceed to the positive cases. Note first that all the instances, where $\mathfrak h$ is of type $\mathbf{A}_1$, are covered by Example 2.5.

Proposition 4.3 (Item 1). *The semi-direct product $\mathfrak s = \mathfrak{so}_n \ltimes m\Bbbk^n$ with $m < n$ has* (FA).

Proof. We have $G \lhd \tilde G$ with $\tilde G = \mathrm{SO}_n {\times} \mathrm{SO}_m$ and $\mathfrak s \lhd \tilde{\mathfrak s}$ for $\tilde{\mathfrak s} = \tilde{\mathfrak g} {\ltimes} V$. The Lie algebra $\tilde{\mathfrak s}$ is the $\mathbb Z_2$-contraction of $\mathfrak{so}_{n+m}$ related to the symmetric subalgebra $\mathfrak{so}_n {\oplus} \mathfrak{so}_m$. Let $x \in V^*$ be generic. Then $\tilde G_x = G_x = \mathrm{SO}_{n-m}$. According to [P07b], $\Bbbk[\tilde{\mathfrak s}^*]^{\tilde{\mathfrak s}} = \Bbbk[V^*]^{\tilde G}[H_1, \dots, H_\ell]$ is a polynomial ring, $\ell = [\frac{n-m}{2}]$. By Lemma 2.1, $H_i \in \mathcal S(\mathfrak s)$ for every i. Next, $\mathfrak s$ has the "codim–2" property if $m = 1$ by [P07b], hence $\mathfrak s$ always has it. The polynomials H_i are algebraically independent over $\Bbbk(V^*)$ and $\Bbbk[V^*]^G$ has $\frac{m(m+1)}{2}$ generators of degree 2. Thereby we have $\mathrm{ind}\,\mathfrak s$ algebraically independent homogeneous invariants with the total sum of degrees being equal to

$$m(m+1) + \sum_{i=1}^{\ell} \deg H_i = m(m+1) + \boldsymbol{b}(\tilde{\mathfrak{s}}) - m(m+1) = \boldsymbol{b}(\tilde{\mathfrak{s}}) = \boldsymbol{b}(\mathfrak{so}_{n+m}) = \boldsymbol{b}(\mathfrak{s}).$$

According to Theorem 1.1, $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[V^*]^G[H_1, \dots, H_\ell]$. □

Theorem 4.4 (Item 9b). *The semi-direct product $\mathfrak{s} = \mathfrak{so}_{14} \ltimes (\varphi_1 + \varphi_6)$ has* (FA)*.*

Proof. Here $G = \mathrm{Spin}_{14}$ and the pair $(\mathrm{Spin}_{14}, V^*_{\varphi_6})$ is of rank one. Let $v \in V^*_{\varphi_6}$ be a generic point. Then $G_v = L \times L$, where L is the connected group of type $\mathbf{G}_2$. By Theorem 2.9, the restriction homomorphism

$$\psi_v : \Bbbk[\mathfrak{s}^*]^S \to \Bbbk[\mathfrak{g}^* \oplus V^*_{\varphi_1} + v]^{G_v \ltimes \exp(V)} \simeq \Bbbk[\mathfrak{g}^*_v \oplus V^*_{\varphi_1}]^{G_v \ltimes \exp(V_{\varphi_1})}$$

is surjective. Further, $V_{\varphi_1} \simeq \Bbbk^{14} = \Bbbk^7 \oplus \Bbbk^7$ as a $L \times L$-module, where each $\Bbbk^7$ is a simplest irreducible $\mathbf{G}_2$-module. Hence $G_v \ltimes \exp(V_{\varphi_1}) = Q \times Q$, where $Q = L \ltimes \exp(\Bbbk^7)$. The group Q has a free algebra of symmetric invariants and $\mathrm{ind}\, \mathfrak{q} = 3$ [PY17b].

There are irreducible tri-homogeneous polynomials $H_1, \dots, H_6 \in \Bbbk[\mathfrak{s}^*]^S$ such that, for a generic point $v \in V^*_{\varphi_6}$, their images $h_i = \psi_v(H_i)$ generate $\mathcal{S}(\mathfrak{q} \times \mathfrak{q})^{Q \times Q}$. Let f be a basic G-invariant in $\Bbbk[V^*_{\varphi_6}]$.

Although the group $G \ltimes \exp(\Bbbk^{14})$ is not reductive, we can argue in the spirit of [Y17a, Section 2] and conclude that $\Bbbk[\mathfrak{s}^*]^S[\frac{1}{f}] = \Bbbk[H_1, \dots, H_6, f, \frac{1}{f}]$. Then the equality

$$\Bbbk[\mathfrak{s}^*]^S = \Bbbk[H_1, \dots, H_6, f]$$

holds if and only if the restrictions of the polynomials $\{H_i\}$ to $V^*_{\varphi_1} \times D$ are algebraically independent, where $D = \{f = 0\} \subset V^*_{\varphi_6}$.

Let $G{\cdot}y \subset D$ be the dense open orbit. Then G_y is connected and $\mathfrak{g}_y = \mathfrak{l} \ltimes \mathfrak{l}^{\mathsf{ab}}$ is the Takiff Lie algebra in type $\mathbf{G}_2$, $\mathfrak{l} = \mathsf{Lie}\, L$. There is only one possible embedding of $\mathfrak{g}_y$ into $\mathfrak{so}_{14}$. Under the non-Abelian $\mathfrak{l}$ the space $\Bbbk^{14}$ decomposes as a sum of two 7-dimensional simple modules. The Abelian ideal $\mathfrak{l}^{\mathsf{ab}}$ takes one copy of $\Bbbk^7$ into another. In other words, $\mathfrak{g}_y \ltimes \Bbbk^{14} = \mathfrak{q} \ltimes \mathfrak{q}^{\mathsf{ab}}$.

By [PY, Example 4.1], Theorem 2.2 of the same paper [PY] applies to $\mathfrak{q}$ and guarantees us that the symmetric invariants of $\mathfrak{q} \ltimes \mathfrak{q}^{\mathsf{ab}}$ form a polynomial ring in 6 generators, where the degrees of the basic invariants are the same as in the case of $\mathfrak{q} \oplus \mathfrak{q}$.

It remains to observe that the proof of [PY17b, Theorem 2.3] can be repeated for the semi-direct product $(G \ltimes \exp(V_{\varphi_1})) \ltimes \exp(V_{\varphi_6})$ producing a suitable modification of the elements $H_1, \dots, H_6$, cf. Theorem 2.6. □

Corollary 4.5 (Item 5a.). *The reduction*

$$(\mathrm{Spin}_{14}, \varphi_1 + \varphi_6) \longrightarrow (\mathrm{Spin}_{13}, \varphi_6)$$

shows that also $(\mathrm{Spin}_{13}, \varphi_6)$ *has* (FA)*, see Proposition* 2.3*.*

Theorem 4.6. *The semi-direct product* $\mathfrak s = \mathfrak{so}_{14} \ltimes (2\varphi_1 + \varphi_6)$ *has* (FA)*.*

Proof. Here $G = \mathrm{Spin}_{14}$ and the proof follows the same lines as the proof of Theorem 4.4. We split the group S as $(G \ltimes \exp(2V_{\varphi_1})) \ltimes \exp(V_{\varphi_6})$. Now $Q = L \ltimes \exp(2\Bbbk^7)$ and again $G_v \ltimes \exp(2V_{\varphi_1}) = Q \times Q$. By [PY17b], $\mathfrak q$ has (FA) and the "codim–2" property. Here $\mathrm{ind}\,\mathfrak q = 4$ and we have eight polynomials $H_i \in \Bbbk[\mathfrak s^*]^S$ such that their restrictions to $\mathfrak g \oplus (2V^*_{\varphi_1}) + v$ generate $\mathcal S(\mathfrak q \oplus \mathfrak q)^{Q\times Q}$. These polynomials are tri-homogeneous w.r.t. the decomposition $\mathfrak s = \mathfrak g \oplus 2V_{\varphi_1} \oplus V_{\varphi_6}$. Again $\mathfrak g_v \ltimes (V_{\varphi_1} \oplus V_{\varphi_1}) = \mathfrak q \ltimes \mathfrak q^{\mathsf{ab}}$, [PY, Theorem 2.2] applies to $\mathfrak q$ and assures that the symmetric invariants of $\mathfrak q \ltimes \mathfrak q^{\mathsf{ab}}$ form a polynomial ring in 8 generators, where the degrees of the basic invariants are the same as in the case of $\mathfrak q \oplus \mathfrak q$. □

Corollary 4.7. *The reductions*

$$(\mathrm{Spin}_{14}, 2\varphi_1 + \varphi_6) \longrightarrow (\mathrm{Spin}_{13}, \varphi_1 + \varphi_6 + \Bbbk) \longrightarrow (\mathrm{Spin}_{12}, \varphi_5 + \varphi_6 + \Bbbk)$$

show that the pairs $(\mathrm{Spin}_{13}, \varphi_1 + \varphi_6)$ *and* $(\mathrm{Spin}_{12}, \varphi_5 + \varphi_6)$ *also have* (FA)*, see Proposition* 2.3*.*

Many representations in types $\mathbf{D}_4$, $\mathbf{B}_4$, and $\mathbf{B}_3$ are covered by reductions from $\mathbf{D}_5$. Among the type $\mathbf{D}_5$ cases, the following one is easy to handle.

Example 4.8 (Item7a). The pair $(\mathbf{D}_5, \varphi_4)$ is of rank zero and therefore the open Spin_{10}-orbit in $\Bbbk^{10}$ is big. The existence of the isomorphism $\Bbbk[\mathfrak g + x]^{G_x \ltimes \exp(V)} \simeq \mathcal S(\mathfrak g_x)^{G_x}$ [Y17a] shows that $\mathcal S(\mathfrak s)^{\mathfrak s} \simeq \mathcal S(\mathfrak h)^H$, where H is the isotropy group of an element in the open orbit and $\mathfrak h = \mathsf{Lie}\, H$. In order to be more explicit, H is connected and $\mathfrak h = \mathfrak{so}_7 \ltimes \Bbbk^8$, where $\mathfrak{so}_7$ acts on $\Bbbk^8$ via the spin-representation. The algebra $\mathcal S(\mathfrak h)^{\mathfrak h}$ is free by [Y17a, Example 3.8]. By a coincidence, the semi-direct product encoded by $(\mathbf{D}_5, \varphi_4)$ is also a truncated maximal parabolic subalgebra $\mathfrak p$ of $\mathbf{E}_6$. The symmetric invariants of $\mathfrak p$ are studied in [FL] and by a computer aided calculation it is shown there that $\mathcal{S}(\mathfrak p)^{\mathfrak p}$ is a polynomial ring with three generators of degrees 6, 8, and 18.

Below we list the 'top' pairs that have to be treated individually. They are divided into two classes, in the first class $\dim V/\!\!/G = 1$ and in the second $\dim V/\!\!/G > 1$.

$$(4{\cdot}2)\qquad \begin{cases} \text{Rank one pairs: } (\mathbf{B}_5, \varphi_5), (\mathbf{D}_5, 2\varphi_4), (\mathbf{D}_6, \varphi_5), (\mathbf{D}_7, \varphi_6); \\ \text{higher rank pairs: } (\mathbf{D}_5, \varphi_1 + \varphi_4), (\mathbf{D}_5, 2\varphi_1 + \varphi_4), (\mathbf{D}_5, 3\varphi_1 + \varphi_4), (\mathbf{D}_6, 2\varphi_5). \end{cases}$$

Theorem 4.9. *The rank one pairs listed in* (4·2) *have* (FA)*.*

Proof. In case of $(\mathbf{D}_5, 2\varphi_4)$ a generic stabiliser is of type $\mathbf{G}_2$. This pair is covered by Lemma 2.7. For the other three pairs, many conditions of Theorem 2.6 are satisfied. For each pair, there is an open orbit $G{\cdot}y \subset D$, where D stands for the zero set of the generator $F \in \Bbbk[V^*]^G$. It remains to inspect the symmetric invariants of G_y.

A generic isotropy group for $(\mathsf{B}_5, \varphi_5)$ is SL_5, G_y is connected, and $\mathfrak{g}_y$ is a $\mathbb{Z}_2$-contraction of $\mathfrak{sl}_5$, the semi-direct product $\mathfrak{so}_5 \ltimes V_{\varphi_1^2}$, which is a good $\mathbb{Z}_2$-contraction [P07b].

A generic isotropy group for $(\mathsf{D}_6, \varphi_5)$ is SL_6, G_y is connected, and $\mathfrak{g}_y$ is a $\mathbb{Z}_2$-contraction of $\mathfrak{sl}_6$, the semi-direct product $\mathfrak{sp}_6 \ltimes V_{\varphi_2}$, which is a good $\mathbb{Z}_2$-contraction [Y14, Theorem 4.5].

A generic isotropy group for $(\mathsf{D}_7, \varphi_6)$ is $L \times L$, where L is the connected group of type G_2, G_y is connected, and $\mathfrak{g}_y$ is the Takiff algebra $\mathfrak{l} \ltimes \mathfrak{l}^{\mathsf{ab}}$, where $\mathfrak{l} = \mathsf{Lie}\, L$. The basic symmetric invariants of $\mathfrak{g}_y$ have the same degrees as in the case of $\mathfrak{l} \oplus \mathfrak{l}$ [T71]. □

Example 4.10 (Item 7d). For the pair $(\mathsf{D}_5, 3\varphi_1 + \varphi_4)$, a generic isotropy group is connected and is of type A_2. Let $D \subset V^*$ be a G-invariant divisor. Then there are at least two copies of $\Bbbk^{10}$ in V^* such that the projection of D on each of them is surjective. For a generic $y \in D$, $G_y = (\mathrm{Spin}_8)_{\tilde{y}}$, where $\tilde{y}$ is a generic point of a Spin_8-invariant divisor $\tilde{D} \subset V_{\varphi_1} \oplus V_{\varphi_3} \oplus V_{\varphi_4}$ (here highest weights of Spin_8 are meant). Continuing the computation one obtains that $G_y = L_v$, where L is the connected group of type G_2 and v is a highest weight vector in $\Bbbk^7$. The group L_v has a free algebra of symmetric invariants generated in degrees 2 and 3, see [PY17b, Lemma 3.9]. Therefore Lemma 2.8 applies.

The remaining three higher rank pairs listed in (4·2) require elaborate arguments. For all of them, Theorem 2.9 will be the starting point. Note that the pair $(\mathrm{SO}_n, \Bbbk^n)$ is of rank one. We let $(\ ,\ )$ denote a non-degenerate SO_n-invariant scalar product on $\Bbbk^n$.

Theorem 4.11 (Item 7b.). *The semi-direct product $\mathfrak{s} = \mathfrak{so}_{10} \ltimes (\varphi_1 + \varphi_4)$ has* (FA)*.*

Proof. Here $G = \mathrm{Spin}_{10}$ and we use the reduction

$$(\mathrm{Spin}_{10}, \varphi_1{+}\varphi_4) \to (\mathrm{Spin}_9, \varphi_4) \tag{4·3}$$

in the increasing direction, starting from the smaller representation and its invariants. By Theorem 2.9, the restriction homomorphism

$$\psi_v \colon \Bbbk[\mathfrak{s}^*]^{\mathfrak{s}} \to \Bbbk[\mathfrak{g}^* \oplus V_{\varphi_4}^* + v]^{G_v \ltimes \exp(V)} \simeq \Bbbk[\mathfrak{g}_v^* \oplus V_{\varphi_4}^*]^{G_v \ltimes \exp(V_{\varphi_4})}$$

is surjective for generic $v \in V_{\varphi_1}^*$. Here $G_v = \mathrm{Spin}_9$. The group $Q = G_v \ltimes \exp(V_{\varphi_4})$ has a free algebra of symmetric invariants [P07b, Theorem 4.7]. More explicitly, $\mathcal{S}(\mathfrak{q})^Q$ is generated by $(\ ,\ )$ on $\Bbbk^{16}$ and three bi-homogeneous polynomials $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ of bi-degrees $(2,4)$, $(4,4)$, $(6,6)$. Note that each generator is unique up to a non-zero scalar. Whenever $(\xi,\xi) \neq 0$ for $\xi \in V_{\varphi_4}^*$, we have $\mathbf{h}_i|_{\mathfrak{so}_9+\xi} = \Delta_{2i}$, where each Δ_{2i} is a basic symmetric invariant of $\mathfrak{so}_7 = (\mathfrak{so}_9)_\xi$. The generators Δ_{2i} are now fixed and they do not depend on the choice of ξ.

Take $H_i \in \Bbbk[\mathfrak{s}^*]^{\mathfrak{s}}$ with $\psi_v(H_i) = \mathbf{h}_i$. Without loss of generality, we may assume that H_i is homogeneous w.r.t. to $\mathfrak{g}$ and V_{φ_4}. The uniqueness of the basic symmetric $\mathfrak{q}$-invariants, allows us to take a suitable tri-homogeneous component of each H_i, see Theorem 2.9.

Now assume that each H_i is irreducible. Whenever $(\xi,\xi)\ne 0$ for $\xi\in\Bbbk^{16}$ and $(\eta,\eta)\ne 0$ for $\eta\in\Bbbk^{10}$, we have $H_i|_{\mathfrak g+x}=a_x\Delta_{2i}$, where $x=\eta+\xi$ and $a_x\in\Bbbk^{\times}$.

According to [Y17a, Lemma 3.5(ii)], we have $\mathcal S(\mathfrak s)^{\mathfrak s}=\Bbbk[V^*]^G[H_1,H_2,H_3]$ if and only if the restrictions $H_i|_{\mathfrak g\times D}$ are algebraically independent over $\Bbbk[D]^G$ for each G-invariant divisor $D\subset V^*$.

If D contains a point $av+\xi$ with $\xi\in\Bbbk^{16}$ and $a\ne 0$, a relation among $H_i|_{\mathfrak g\times D}$ leads to a relation among the restrictions of $\mathbf h_i$ to $\mathfrak{so}_9\times\tilde D$ for some Spin_9-invariant divisor $\tilde D\subset\Bbbk^{16}$. Moreover, this new relation is over $\Bbbk[\{v\}\times D]^{G_v}=\Bbbk$. Since the polynomials $\mathbf h_i$ freely generate $\mathcal S(\mathfrak q)^{\mathfrak q}$ over $\Bbbk[V^*_{\varphi_4}]^{\mathrm{Spin}_9}$, nothing of this sort can happen. Therefore there is a unique suspicious divisor, namely, the divisor $D=\tilde D\times\Bbbk^{16}$, where $\tilde D=\{u\in\Bbbk^{10}\mid (u,u)=0\}$.

Since each H_i is irreducible, it is non-zero on $\mathfrak g\times D$. Therefore there is a point $\xi\in\Bbbk^{16}$ such that $(\xi,\xi)\ne 0$ and $H_i|_{\mathfrak g\times\tilde D\times\{\xi\}}\ne 0$ for all i. Here $G_\xi=\mathrm{Spin}_7\ltimes\exp(\Bbbk^8)$ and $\Bbbk^{10}\subset V^*$ decomposes as $\Bbbk\oplus\Bbbk^8\oplus\Bbbk$ under G_ξ. The Abelian ideal $\Bbbk^8$ of $\mathfrak g_\xi$ takes $\Bbbk$ to $\Bbbk^8$ and then $\Bbbk^8$ to another copy of $\Bbbk$. Note that the vectors in each copy of $\Bbbk$ are isotropic. Take $u\ne 0$ in the first copy and $u'\ne 0$ in the second copy of $\Bbbk$. Set $\eta_t=u+tu'$, $x_t=\eta_t+\xi$ for $t\in\Bbbk$, $y=u+\xi$. Then $G_{x_t}=G_y\simeq\mathrm{Spin}_7$.

We have $(\eta_t,\eta_t)\ne 0$ for $t\ne 0$ and hence $H_i|_{\mathfrak g+x_t}=a_t\Delta_{2i}\ne 0$, whenever $t\ne 0$. Here $a_t\Delta_{2i}\in\mathcal S(\mathfrak g_{x_t})=\mathcal S(\mathfrak g_y)$. Clearly $H_i|_{\mathfrak g\times\{y\}}=\lim\limits_{t\to 0}a_t\Delta_{2i}$ and it is either zero or a non-zero scalar multiple of Δ_{2i}. If the second possibility takes place for all i, when the restrictions of H_i to $\mathfrak g\times D$ are algebraically independent over $\Bbbk[D]$. Thus, it remains to prove that $H_i|_{\mathfrak g\times\{y\}}\ne 0$ for all i.

Assume that $H_i|_{\mathfrak g\times\{y\}}=0$. Then H_i vanishes on $\mathfrak g\times G_\xi{\cdot}u\times\{\xi\}$ and also on $\mathfrak g\times G_\xi{\cdot}\Bbbk u\times\{\xi\}$, since H_i is tri-homogeneous. The subset $G_\xi{\cdot}\Bbbk u$ is dense in $\tilde D$ (it equals $\tilde D\setminus\{0\}$), hence H_i vanishes on $\mathfrak g\times\tilde D\times\{\xi\}$, too. However, this contradicts the choice of ξ. □

Theorem 4.12 (Item 7c.). *If $\mathfrak s$ is given by $(\mathbf D_5, 2\varphi_1+\varphi_4)$, then $\Bbbk[\mathfrak s^*]^S=\Bbbk[V^*]^G[H_2,H_6]$ is a polynomial ring and the multi-degrees of H_i are $(2,2,2,4)$, $(6,4,4,8)$.*

Proof. For this pair, the chain of reductions is

$$(\mathrm{Spin}_{10},2\varphi_1{+}\varphi_4)\to(\mathrm{Spin}_9,\varphi_1{+}\varphi_4{+}\Bbbk)\to(\mathrm{Spin}_8,\varphi_3{+}\varphi_4{+}\Bbbk)\to(\mathrm{Spin}_7,\varphi_3{+}\Bbbk)\qquad(4{\cdot}4)$$

and again we are tracing the chain from the smaller groups to the larger.

By [Y17a, Prop. 3.10], the symmetric invariants of $\mathrm{Spin}_7\ltimes\exp(V_{\varphi_3})$ are freely generated by the following three polynomials: the scalar product $(\ ,\ )$ on $V_{\varphi_3}\simeq\Bbbk^8$, $\mathbf h_2$, and $\mathbf h_6$. Here the bi-degrees the last two are $(2,2)$, $(6,4)$. We are lucky that all three generators are unique (up to a scalar) and $\mathfrak{so}_7\ltimes V_{\varphi_3}$ has the "codim–2" property. One can easily deduce that all items in (4·4) have the "codim–2" property. A generic isotropy group for $(\mathrm{Spin}_7:V_{\varphi_3})$, say L, is the connected simple group of type $\mathbf G_2$. Take $u\in V_{\varphi_3}$ with $(u,u)\ne 0$.

Then $(\mathfrak{so}_7)_u = \mathfrak{l} = \mathrm{Lie}\, L$. Let $h_2, h_6 \in \mathcal{S}((\mathfrak{so}_7)_u)$ be the restrictions of $\mathbf{h}_2$, $\mathbf{h}_6$ to $\mathfrak{so}_7{+}u$. Then h_2 and h_6 generate $\mathcal{S}(\mathfrak{l})^L$. We have $\dim \mathcal{S}^2(\mathfrak{l})^L = 1$, the generator of degree 2 is unique (up to a non-zero scalar). In the space $\mathcal{S}^6(\mathfrak{l})^L = \Bbbk h_2^3 \oplus \Bbbk h_6$, the generator h_6 is characterised by the property that it is the restriction of an invariant of $\mathrm{Spin}_7 \ltimes \exp(V_{\varphi_3})$ of bi-degree $(6,4)$. This property does not depend on the choice of u.

Consider next $\mathfrak{s}_2 := \mathfrak{so}_8 \ltimes (V_1 \oplus V_2)$, where $V_1 = V_{\varphi_3}$, $V_2 = V_{\varphi_4}$. Choose $v \in V_1^*$ with $(v,v) \ne 0$. By Theorem 2.9, there are $\hat{h}_2, \hat{h}_6 \in \mathcal{S}(\mathfrak{s}_2)^{\mathfrak{s}_2}$ such that $\hat{h}_i|_{\mathfrak{so}^8 \oplus V_2^* + v} = \mathbf{h}_i$. One can safely replace $\hat{h}_2$ by its component of degrees 2 in $\mathfrak{so}_8$, 2 in V_2 and replace $\hat{h}_6$ by its component of degrees 6 in $\mathfrak{so}_8$, 4 in V_1. The uniqueness of generators in the case of $\mathrm{Spin}_7 \ltimes \exp(V_{\varphi_3})$ allows also to take tri-homogeneous components. Suppose now that each $\hat{h}_i$ is irreducible. Set $a_i = \deg_{V_2} \hat{h}_i$. Choose $v_2 \in V_2^*$ with $(v_2, v_2) \ne 0$. The restriction $\hat{h}_2|_{\mathfrak{so}_8 \oplus V_2 + v_2}$ is an invariant of bi-degree $(2, a_2)$ and either $a_2 = 2$ or this restriction is divisible by the invariant of bi-degree $(0,2)$. In the last case, $\hat{h}_2$ is divisible by a generator of $\Bbbk[V_2]^{\mathrm{SO}_8}$. A contradiction. Since $\hat{h}_6|_{\mathfrak{so}_8 + v + v_2} = h_6$ and since in addition $\hat{h}_6$ is irreducible, the restriction $\hat{h}_6|_{\mathfrak{so}_8 \oplus V_1^* + v_2}$ is an invariant of bi-degree $(6,4)$, i.e., $a_6 = 4$. Making use of Theorem 1.1, we conclude that $\Bbbk[\mathfrak{s}_2^*]^{\mathfrak{s}_2} = \Bbbk[V_1^* \oplus V_2^*]^{\mathrm{Spin}_8}[\hat{h}_2, \hat{h}_6]$.

The Spin_9-actions on $V_{\varphi_1} = \Bbbk^9$ and $V_{\varphi_4} = \Bbbk^{16}$ are of rank one. By [E72], g.i.g.$(\mathrm{Spin}_9{:}V_{\varphi_4}) = \mathrm{Spin}_7$, and $\Bbbk^9|_{\mathrm{Spin}_7}$ is the Spin_7-module $V_{\varphi_3} \oplus \Bbbk$. The restriction homomorphism $\Bbbk[V_{\varphi_1}^* \oplus V_{\varphi_4}^*]^{\mathrm{Spin}_9} \to \Bbbk[V_{\varphi_3}^* \oplus \Bbbk]^{\mathrm{Spin}_7}$ is onto. Using Theorem 2.9 and the reductions

$$(4{\cdot}5)\qquad \begin{array}{ccccc} & \nearrow & (\mathrm{Spin}_8, \varphi_3{+}\varphi_4) & \searrow & \\ (\mathrm{Spin}_9, \varphi_1{+}\varphi_4) & & & & (L, 0) \\ & \searrow & (\mathrm{Spin}_7, \varphi_3{+}\Bbbk) & \nearrow & \end{array}$$

we prove that there are algebraically independent over $\Bbbk[V_{\varphi_1}^* \oplus V_{\varphi_4}^*]$ symmetric invariants of tri-degrees $(2,2,4)$, $(6,4,8)$ w.r.t. $\mathfrak{so}_9 \oplus \Bbbk^9 \oplus \Bbbk^{16}$. They generate the ring of symmetric invariants related to $(\mathrm{Spin}_9, \varphi_1{+}\varphi_4)$ over $\Bbbk[V_{\varphi_1}^* \oplus V_{\varphi_9}^*]^{\mathrm{Spin}_9}$ by Theorem 1.1.

One can make a reduction step from $\mathfrak{s}$ to $(\mathrm{Spin}_9, V_{\varphi_1} \oplus V_{\varphi_9})$ using either of the two copies of V_{φ_1}. This allows one to find algebraically independent over $\Bbbk[V^*]$ polynomials $H_2, H_6 \in \Bbbk[\mathfrak{s}^*]^{\mathfrak{s}}$ of multi-degrees $(2,2,2,4)$ and $(6,4,4,8)$, respectively. The basic invariants on V^* are of degrees $2,2,2,3,3$. Thus, the total sum of degrees is

$$10 + 22 + 12 = 44 \ \text{ and } \ \dim \mathfrak{s} + \mathrm{ind}\, \mathfrak{s} = 45 + 20 + 16 + 7 = 88.$$

Therefore, by Theorem 1.1, we have $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[V^*]^G[H_2, H_6]$. □

The case of $(\mathsf{D}_6, 2\varphi_5)$ is very complicated. We begin by introducing some notation and stating a few facts related to this pair. First, $V_{\varphi_5} \simeq V_{\varphi_5}^*$ as a G-module. Second, the representation of G on V_{φ_5} is of rank one and $\Bbbk[V_{\varphi_5}^*]^G = \Bbbk[F]$, where F is a homogeneous polynomial of degree 4. It would be convenient to write $V = V_1 \oplus V_2$, where each V_i

is a copy of V_{φ_5} and let F stand for the generator of $\Bbbk[V_1^*]^G$. Further, there is a natural action of SL_2 on V. We suppose that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot V_2 = 0$ and that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot V_1 = V_2$ for $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2$. The ring $\Bbbk[V^*]^G$ has 7 generators:

$$F = F_{(4,0)}, F_{(3,1)}, F_{(2,2)}, F_{(1,3)}, F_{(0,4)}, F_{(1,1)}, F_{(3,3)}.$$

Here $F_{(\alpha,\beta)}$ stands for a particular G-invariant in $\mathcal{S}^\alpha(V_1)\mathcal{S}^\beta(V_2)$. It is assumed that the first five polynomials build an irreducible SL_2-module and that the last two are SL_2-invariants.

We let SL_2 act on $\mathfrak{g}$ trivially and thus obtain an action of SL_2 on $\Bbbk[\mathfrak{s}^*]^S$. Note that if $H \in \Bbbk[\mathfrak{s}^*]$ and $\deg_{V_1} H > \deg_{V_2} H$, then $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot H \neq 0$.

Let $v \in V_1^*$ be a generic point and

$$\psi_v \colon \Bbbk[\mathfrak{s}^*]^S \to \Bbbk[\mathfrak{g} \oplus V_2^* + v]^{G_v \ltimes \exp(V)} \simeq \mathcal{S}(\mathfrak{g}_v \ltimes V_2)^{G_v \ltimes \exp(V_2)}$$

be the corresponding restriction homomorphism. Here $G_v = \mathrm{SL}_6$ and

$$V_2 = \bigwedge^2 \Bbbk^6 \oplus (\bigwedge^2 \Bbbk^6)^* \oplus 2\Bbbk$$

as a G_v-module. Set $\mathfrak{q} = \mathfrak{g}_v \ltimes V_2$. By Proposition 3.1, $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a polynomial ring and $\Bbbk[\mathfrak{q}^*]^{\mathfrak{q}} = \Bbbk[V_2^*]^{\mathrm{SL}_6}[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]$, where the generators $\mathbf{h}_i$ are of bi-degrees $(2,4)$, $(2,6)$, $(2,8)$.

Let $N_G(\Bbbk v)$ be the normaliser of the line $\Bbbk v$. Then $N_G(\Bbbk v) = C_4 \times G_v$, where $C_4 = \langle \zeta \rangle$ is a cyclic group of order 4. It is not difficult to see that $\mathrm{Ad}\,(\zeta)A = -A^t$ for each $A \in \mathfrak{g}_v$ and that $\zeta \cdot \mathbf{h}_k = (-1)^k \mathbf{h}_k$ for each $k \in \{1,2,3\}$. The element ζ^2 multiplies $\psi_v(F_{(1,3)})$ and $\psi_v(F_{(1,1)})$ by -1, the product $\psi_v(F_{(1,3)})\psi_v(F_{(1,1)})$ is a C_4-invariant.

There is a subgroup $C_4 \subset N_G(\Bbbk v) \times \mathrm{GL}(V_1^*)$, which stabilisers v. This means that if $H \in \Bbbk[\mathfrak{s}^*]^S$ is homogeneous in V_1, then $\psi_v(H)$ is an eigenvector of $C_4 \subset N_G(\Bbbk v)$ and the corresponding eigenvalue depends only on $\deg_{V_1} H$.

Theorem 4.13 (Item 8b). *If $\mathfrak{s}$ is given by the pair $(\mathbf{D}_6, 2\varphi_5)$, then $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[V^*]^G[H_1, H_2, H_3]$ is a polynomial ring and the tri-degrees of H_i are $(2,4,4)$, $(2,6,6)$, $(2,8,8)$.*

Proof. According to Theorem 2.9, there are homogeneous in V_1 elements $H_1, H_2, H_3 \in \mathcal{S}(\mathfrak{s})^{\mathfrak{s}}$ such that $\psi_v(H_i) = \mathbf{h}_i$. There is no harm in assuming that these polynomials are tri-homogeneous. Suppose that $b_i = \deg_{V_1} H_i$ is the minimal possible. Set $a_i = \deg_{V_2} H_i$. Then $a_1 = 4$, $a_2 = 6$, $a_3 = 8$. The eigenvalues of ζ on $\mathbf{h}_i$ indicate that $a_i \equiv b_i \pmod 4$ for each i.

Suppose for the moment that $a_i = b_i$ for all i. It is not difficult to see that $\mathfrak{s}$ satisfies the "codim–2" condition. The elements $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ are algebraically independent over

$\Bbbk(V_2)$, hence H_1, H_2, H_3 are algebraically independent over $\Bbbk(V^*)$. Thus, we have ten algebraically independent homogeneous invariants. The total sum of their degrees is

$$2+6+20+10+14+18=70 \text{ and } \dim\mathfrak{s}+\operatorname{ind}\mathfrak{s}=66+64+10=140.$$

Thereby $\Bbbk[\mathfrak{s}^*]^S=\Bbbk[V^*]^G[H_1,H_2,H_3]$ by Theorem 1.1. It remains to show that the assumption is correct.

For a generic $v'\in V_2^*$, $\mathfrak{g}_{v'}\ltimes V_1\simeq\mathfrak{q}$ and each $H_i|_{\mathfrak{g}\oplus V_1^*+v'}$ is a symmetric invariant of $\mathfrak{g}_{v'}\ltimes V_1$ of degree 2 in $\mathfrak{g}_{v'}$. Since the restrictions $H_i|_{\mathfrak{g}+v+v'}$ are the basic symmetric invariants of $G_{v+v'}=(\mathrm{SL}_2)^3$, the restrictions of H_i to $\mathfrak{g}\oplus V_1^*+v'$ are algebraically independent over $\Bbbk[V_1^*]$. Thereby $\sum b_i\geqslant 18$ and $b_i\geqslant 4$ for each i. Moreover, if $b_1=4$, then $b_2\geqslant 6$.

Set $\tilde H_i := \begin{pmatrix}0&0\\1&0\end{pmatrix}\cdot H_i$. If $b_i>a_i$, then $\tilde H_i\neq 0$. We have $\psi_v(\tilde H_i)\in\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ and $\deg_{\mathfrak{g}_v}\psi_v(\tilde H_i)=2$. Therefore $\psi_v(\tilde H_i)$ is a linear combination of $\mathbf{h}_j$ with coefficients from $\Bbbk[V_2^*]^{\mathrm{SL}_6}$. Moreover, each coefficient is an eigenvector of ζ. The first element, H_1, can be handled easily.

Assume that $\tilde H_1\neq 0$. Then $\psi_v(\tilde H_1)=\mathbf{f}\,\mathbf{h}_1$ with non-zero $\mathbf{f}\in V_2^{G_v}$ and this $\mathbf{f}$ is an eigenvector of ζ. Since $\deg_{V_1}\tilde H_1\equiv 3 \pmod 4$, $\mathbf{f}=\psi_v(F_{(3,1)})$ (up to a non-zero scalar). Since $\psi_{av}(\frac{\tilde H_1}{F_{3,1}})=a^{b_1-4}\mathbf{h}_1$ for each $a\in\Bbbk^\times$ and since $F_{(3,1)}$ and F are coprime, we have $\frac{\tilde H_1}{F_{3,1}}\in\Bbbk[\mathfrak{s}^*]^S$. Also $\psi_v(\frac{\tilde H_1}{F_{3,1}})=\mathbf{h}_1$. Clearly $\deg_{V_1}\frac{\tilde H_1}{F_{3,1}}=b_1-4<b_1$. A contradiction with the choice of H_1. We have established that $\psi_v(\tilde H_1)=0$. Hence $b_1=4$ and H_1 is an SL_2-invariant.

Certain further precautions are needed. It may happen that H_2 (or H_3) does not lie in a simple SL_2-module. In that case we replace H_2 (or H_3) by a suitable (and suitably normalised) component of the same tri-degree, which lies in a simple SL_2-module and which restricts to $\mathbf{h}_2+p$ with $p\in\mathcal{S}^2(V_2)\mathbf{h}_1$ (or to $\mathbf{h}_3+p$ with $p\in\mathcal{S}^4(V_2)\mathbf{h}_1\oplus\mathcal{S}^2(V_2)\mathbf{h}_2$) on $\mathfrak{g}\oplus V_2^*+v$. One may say that $\mathbf{h}_2$ was (or $\mathbf{h}_2$ and $\mathbf{h}_3$ were) changed as well, so that the conditions $\psi_v(H_i)=\mathbf{h}_i$ are not violated. We also normalise F in such a way that $F(v)=1$. Some other normalisations are done below without mentioning.

Assume that $\tilde H_2\neq 0$ and that $\psi_v(\tilde H_2)\in\mathcal{S}^3(V_1)\mathbf{h}_1$. Then $\tilde H_2\in\Bbbk(V^*)H_1$ and so does H_2, which is equal to $\begin{pmatrix}0&1\\0&0\end{pmatrix}\cdot\tilde H_2$ up to a non-zero scalar. A contradiction, here $\psi_v(H_2)\neq\mathbf{h}_2$. Knowing that H_2 is an SL_2-invariant, we can use a similar argument in order to prove that $\psi_v(\tilde H_3)\notin\mathcal{S}^5(V_1)\mathbf{h}_1\oplus\mathcal{S}^3(V_1)\mathbf{h}_2$ in case $\tilde H_3\neq 0$.

We will see below that if $b_i>a_i$, then $\tilde H_i=\mathbf{H}_i+\frac{F_{(3,1)}H_i}{F}$, where $\mathbf{H}_2\in\Bbbk(V^*)^GH_1$ and $\mathbf{H}_3\in\Bbbk(V^*)^GH_1\oplus\Bbbk(V^*)^GH_2$. Recall that F and $F_{(3,1)}$ are coprime. In case $\mathbf{H}_i\in\Bbbk[\mathfrak{s}^*]$, we can replace H_i with $\frac{H_i}{F}\in\Bbbk[\mathfrak{s}^*]$ decreasing $\deg_{V_1}H_i$ by 4. The main difficulties lie with non-regular $\mathbf{H}_i$.

Modification for H_2. Assume that $b_2 > 6$. Then $\psi_v(\tilde{H}_2) = \mathbf{f}_3\mathbf{h}_1 + \mathbf{f}_{(3,1)}\mathbf{h}_2$ with $\mathbf{f}_3 \in \mathcal{S}^3(V_2)^{G_v}$, $\mathbf{f}_{(3,1)} \in V_2^{G_v}$ and $\mathbf{f}_{(3,1)} \neq 0$. Both coefficients are eigenvectors of ζ. We have $\mathbf{f}_{(3,1)} = \psi_v(F_{(3,1)})$ and $\mathbf{f}_3$ is the image of

$$c_1F_{(1,3)} + F'_{(5,3)} + c_2F^3_{(3,1)},$$

where $c_1, c_2 \in \Bbbk$ and $F'_{(5,3)}$ is some G-invariant in $\mathcal{S}^5(V_1)\mathcal{S}^3(V_2)$. Set $\delta := \frac{b_2-6}{4}$ and

$$\mathbf{H}_2 := (c_1F^\delta F_{(1,3)} + F^{\delta-1}F'_{(5,3)} + c_2F^{\delta-2}F^3_{(3,1)})H_1.$$

Then $\psi_{av}(\tilde{H}_2 - \mathbf{H}_2) = a^{b_2-1}\mathbf{f}_{(3,1)}\mathbf{h}_2$ for all $a \in \Bbbk^\times$. If $\mathbf{H}_2 \notin \Bbbk[\mathfrak{s}^*]$, then $\delta = 1$ and $c_2 \neq 0$. Here

$$\tilde{H}_2 - c_1FF_{(1,3)}H_1 - F'_{(5,3)}H_1 - c_2\frac{F^3_{(3,1)}H_1}{F} = \frac{F_{(3,1)}H_2}{F}$$

and

$$\frac{F_{(3,1)}H_2 + c_2F^3_{(3,1)}H_1}{F} \in \Bbbk[\mathfrak{s}^*].$$

Since F and $F_{(3,1)}$ are coprime, we have

$$\hat{H}_2 = \frac{H_2 + c_2F^2_{(3,1)}H_1}{F} \in \Bbbk[\mathfrak{s}^*].$$

In this case we replace $\mathbf{h}_2$ with $\mathbf{h}_2 + c_2\mathbf{f}^2_{(3,1)}\mathbf{h}_1$ and H_2 with $\hat{H}_2$. This does not violate the property $\zeta^2{\cdot}\mathbf{h}_2 = -\mathbf{h}_2$. Now $\deg_{V_1} H_2 = \deg_{V_2} H_2 = 6$. If $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}{\cdot}H_2 \neq 0$, then this is an invariant of tri-degree $(2,7,5)$ and hence lies in $\Bbbk(V^*)H_1$. But then also $H_2 \in \Bbbk(V^*)H_1$. This new contradiction shows that H_2 is an SL_2-invariant.

Modification for H_3. Now we know that $b_2 = 6$ and therefore $b_3 \geqslant 8$. Assume that $b_3 > 8$. Then

$$\psi_v(\tilde{H}_3) = \mathbf{f}'_5\mathbf{h}_1 + \mathbf{f}'_3\mathbf{h}_2 + \mathbf{f}_{(3,1)}\mathbf{h}_3$$

with $\mathbf{f}'_k \in \mathcal{S}^k(V_2)^{G_v}$, $\mathbf{f}_{(3,1)} \in V_2^{G_v}$. All three coefficients are eigenvectors of ζ. Studying the eigenvalues one concludes that $\mathbf{f}_{(3,1)} = \psi_v(F_{(3,1)})$, $\mathbf{f}'_3$ is the image of $s_1F_{(1,3)} + F'_{(5,3)} + s_2F^3_{(3,1)}$, where $F'_{(5,3)} \in \mathcal{S}^5(V_1)\mathcal{S}^3(V_2)$, $s_i \in \Bbbk$, and finally $\mathbf{f}'_5$ is the image of a rather complicated expression $\sum\limits_{j=0}^{3} F'_{(4j+3,5)}$. Set $\nu := \frac{b_3-8}{4}$ and

$$\mathbf{H}_3 := (\sum_{j=0}^{3} F'_{(4j+3,5)}F^{\nu-j})H_1 + (s_1F_{(1,3)}F^\nu + F'_{(5,3)}F^{\nu-1} + s_2F^3_{(3,1)}F^{\nu-2})H_2.$$

As above, $\tilde{H}_3 - \mathbf{H}_3 = \frac{F_{(3,1)}H_3}{F}$. If $\mathbf{H}_3 \notin \Bbbk[\mathfrak{s}^*]$, then $\nu = 2$ or $\nu = 1$.

Suppose that $\nu = 2$ and that $F'_{(15,5)} \neq 0$. Then $F'_{(15,5)} = F^5_{(3,1)}$ (up to a non-zero scalar) and

$$\frac{F_{(3,1)}H_3}{F} + \frac{F^5_{(3,1)}H_1}{F} \in \Bbbk[\mathfrak{s}^*] \ \text{ leading to } \ \frac{H_3 + F^4_{(3,1)}H_1}{F} \in \Bbbk[\mathfrak{s}^*].$$

Modifying $\mathbf{h}_3$ and H_3 accordingly, we obtain a new H_3 with $\deg_{V_1} H_3 \leqslant 12$.

Suppose now that $\nu = 1$. If $F'_{(15,5)} \neq 0$, then we obtain $\frac{F^5_{(3,1)}H_1}{F} \in \Bbbk[\mathfrak{s}^*]$, which cannot be the case. Thereby $F'_{(15,5)} = 0$ and

$$\frac{F_{(3,1)}H_3}{F} + \frac{F'_{(11,5)}H_1}{F} + s_2\frac{F^3_{(3,1)}H_2}{F} \in \Bbbk[\mathfrak{s}^*].$$

Since $2{\times}5 = 10 < 11$ and since $\psi_v(F_{(4,0)}) = 1$, the polynomial $F'_{(11,5)}$ is divisible by $F_{(3,1)}$, say $F'_{(11,5)} = F_{(3,1)}\mathbf{F}$. Now

$$\frac{H_3 + \mathbf{F}H_1 + s_2F^2_{(3,1)}H_2}{F} \in \Bbbk[\mathfrak{s}^*].$$

This allows us to replace H_3, modifying $\mathbf{h}_3$ at the same time, by a polynomial of tri-degree $(2, 8, 8)$ keeping the property $\psi_v(H_3) = \mathbf{h}_3$. $\square$

Corollary 4.14. *Suppose that* $\tilde{\mathfrak{s}} = \tilde{\mathfrak{g}}\ltimes V$ *is given by the pair* $(\mathrm{Spin}_{12}{\times}\mathrm{SL}_2, V_{\varphi_5}{\otimes}\Bbbk^2)$*. Then* $\tilde{\mathfrak{s}}$ *has* (FA) *and* $\Bbbk[\tilde{\mathfrak{s}}^*]^{\tilde{\mathfrak{s}}} = \Bbbk[V^*]^{\tilde{G}}[H_1, H_2, H_3]$*, where the bi-degrees of* H_i *are* $(2, 8)$, $(2, 12)$, $(2, 16)$.

Proof. Let $\mathfrak{s} = \mathfrak{g}\ltimes V$ and $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[V^*]^G[H_1, H_2, H_3]$ be as in Theorem 4.13. Then $\tilde{\mathfrak{g}}_x = \mathfrak{g}_x$ for generic $x \in V^*$. Hence $\Bbbk[\tilde{\mathfrak{s}}^*]^{\tilde{S}} \subset \Bbbk[\mathfrak{s}^*]$ by Lemma 2.1. According to the proof of Theorem 4.13, H_1 and H_2 are SL_2-invariants, i.e., they are $\tilde{S}$-invariants, and also $\tilde{H}_3 \notin \mathcal{S}^8(V)H_1 \oplus \mathcal{S}^4(V)H_2$ if $\tilde{H}_3 \neq 0$. At the same time the tri-degree of $\tilde{H}_3$ is $(2, 7, 9)$ if $\tilde{H}_3 \neq 0$. Combining these two observations, we see that $\tilde{H}_3 = 0$, H_3 is an SL_2-invariant, and $\Bbbk[\tilde{\mathfrak{s}}^*]^{\tilde{\mathfrak{s}}} = \Bbbk[V^*]^{\tilde{G}}[H_1, H_2, H_3]$. Since $\Bbbk[V^*]^{\tilde{G}}$ is a polynomial ring, the result follows. $\square$

Proposition 4.15. *All the remaining cases marked with '+' in Table 1 are indeed positive.*

Proof. Making further use of Proposition 2.3, we see that all the remaining cases are covered by reductions from G of type $\mathbf{D}_5$, see Diagrams (4·3), (4·4), and also

$$\begin{array}{ccc} (\mathrm{Spin}_{10}, 3\varphi_1{+}\varphi_4) \longrightarrow & (\mathrm{Spin}_9, 2\varphi_1{+}\varphi_4{+}2\Bbbk) & \\ & \downarrow & \\ & (\mathrm{Spin}_8, \varphi_1{+}\varphi_3{+}\varphi_4{+}\Bbbk) & \longrightarrow (\mathrm{Spin}_7, 2\varphi_3), \end{array}$$

where the initial pair is positive by Example 4.10. $\square$

5. The classification for the symplectic algebra

In this section, $G = \mathrm{Sp}_{2n}$. We classify the finite-dimensional rational representations $(G : V)$ such that g.i.g.$(G : V)$ is infinite and the symmetric invariants of $\mathfrak{s} = \mathfrak{g} \ltimes V$ form a polynomial ring. The answer is given in Table 2. Surprisingly, all the possible candidates for $\mathfrak{s} = \mathfrak{g} \ltimes V$ do have (FA).

Let $e \in \mathfrak{g}$ be a nilpotent element and $\mathfrak{g}_e \subset \mathfrak{g}$ its centraliser. Then $\mathfrak{g}_e$ has (FA) by [PPY]. This does not seem to be relevant to our current task, but it is.

The nilpotent element e can be included into an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$ and this gives rise to the decomposition $\mathfrak{g} = \Bbbk f \oplus e^{\perp}$, where $e^{\perp}$ is the subspace orthogonal to e w.r.t. the Killing form of $\mathfrak{g}$. Let $\Delta_k \in \mathcal{S}(\mathfrak{sp}_{2n})$ be the sum of the principal k-minors. We write the highest f-component of Δ_k as ${}^e\!\Delta_k f^d$. Then $\{{}^e\!\Delta_k \mid k \text{ even}, 2 \leqslant k \leqslant 2n\}$ is a set of the basic symmetric invariants of $\mathfrak{g}_e$ [PPY, Theorem 4.4].

Let now e be a minimal nilpotent element. Then $\mathfrak{g}_e = \mathfrak{sp}_{2n-2} \ltimes \mathfrak{heis}_{n-1}$. Restricting $H \in \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ to the hyperplane in $\mathfrak{g}_e^*$, where $e = 0$, we obtain a symmetric invariant of $\mathfrak{s} := \mathfrak{sp}_{2n-2} \ltimes \Bbbk^{2n-2}$.

Let H_i be the restriction of ${}^e\!\Delta_{2i+2}$ to the hyperplane $e = 0$.

Lemma 5.1. *The algebra of symmetric invariants of $\mathfrak{s} = \mathfrak{sp}_{2n-2} \ltimes \Bbbk^{2n-2}$ is freely generated by the polynomials H_i as above with $1 \leqslant i \leqslant n{-}1$.*

Proof. Set $n' = n{-}1$. The group $G' = \mathrm{Sp}_{2n'}$ acts on $V^* \simeq V = \Bbbk^{2n'}$ with an open orbit, which consists of all non-zero vectors of V^*. Therefore $\mathcal{S}(\mathfrak{s})^{\mathfrak{s}} \simeq \mathcal{S}(\mathfrak{h})^H$, where

$$H = (\mathrm{Sp}_{2n'})_v = \mathrm{Sp}_{2n'-2} \ltimes \exp(\mathfrak{heis}_{n'-1})$$

and $v \in V$ is non-zero. By a coincidence, $\mathfrak{h} = \mathfrak{g}'_{e'}$, where $e' \in \mathfrak{g}'$ is a minimal nilpotent element. We have to show that $\psi_v(H_i)$ form a set of the basic symmetric invariants of $\mathfrak{h}$ for the usual restriction $\psi_v \colon \Bbbk[\mathfrak{s}]^{\mathfrak{s}} \to \Bbbk[(\mathfrak{g}')^* + v]^{G' \ltimes \exp(V)} \simeq \mathcal{S}(\mathfrak{h})^{\mathfrak{h}}$.

Note that the f-degree of each Δ_k with even k is one, see [PPY] and the matrix description of elements of $f + \mathfrak{g}_e$ presented in Figure 1. Further, ${}^e\!\Delta_{2i+2}$ is a sum $e\Delta'_{2i} + H_i$,

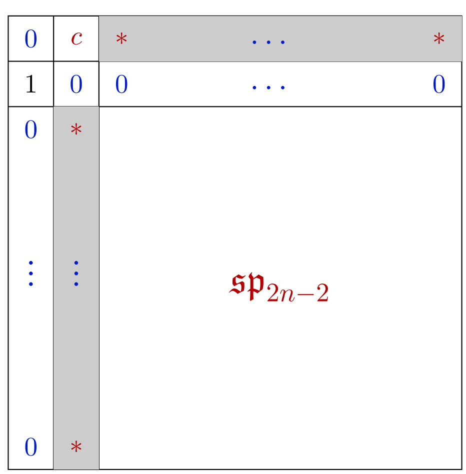

Fig. 1. Elements of $f + \mathfrak{g}_e$.

where $\Delta'_{2i} \in \mathcal{S}(\mathfrak{g}')$. Choosing $v = (1, 0, \ldots, 0)^t$, one readily sees that $\psi_v(H_i) = {}^{e'}\!\Delta'_{2i}$. This concludes the proof. □

Remark 5.2. We have a nice matryoshka-like structure. Starting from $\mathfrak{g}_e$ with $\mathfrak{g} = \mathfrak{sp}_{2n+2}$ and restricting the symmetric invariants to the hyperplane $e = 0$ one obtains the symmetric invariants of the semi-direct product $\mathfrak{sp}_{2n}\ltimes\Bbbk^{2n}$. By passing to the stabiliser of a generic point $x \in V^*$ with $V = \Bbbk^{2n}$, one comes back to $(\mathfrak{sp}_{2n'})_{e'}$ with $n' = n{-}1$. And so on.

Suppose now that $e \in \mathfrak{g}$ is given by the partition $(2^m, 1^{2n})$, $\mathfrak{g} = \mathfrak{sp}_{2m+2n}$. Then $\mathfrak{g}_e = (\mathfrak{so}_m{\oplus}\mathfrak{sp}_{2n})\ltimes(\Bbbk^m{\otimes}\Bbbk^{2n} \oplus \mathcal{S}^2\Bbbk^m)$ and the nilpotent radical of $\mathfrak{g}_e$ is two-step nilpotent. Suppose that m is odd. Set $Y := \mathrm{Ann}\,(\mathcal{S}^2\Bbbk^m) \subset \mathfrak{g}_e^*$ and let $\tilde{H}_i$ be the restriction to Y of ${}^e\!\Delta_k$ with $k = 3m{+}2i{-}1$.

Lemma 5.3. *For $1 \leqslant i \leqslant \left(n - \frac{m-1}{2}\right)$, we have $\tilde{H}_i \in \mathcal{S}(\mathfrak{s})^{\mathfrak{s}}$, where $\mathfrak{s} = \mathfrak{sp}_{2n}\ltimes m\Bbbk^{2n}$.*

Proof. By the construction, each $\tilde{H}_i$ is $\mathfrak{g}_e$-invariant. Note that $\mathfrak{g}_e$ acts on Y as the semi-direct product $(\mathfrak{so}_m{\oplus}\mathfrak{sp}_{2n})\ltimes\Bbbk^m{\otimes}\Bbbk^{2n}$. For each even k with $k \geqslant 2m$, the f-degree of Δ_k is m [PPY]. For the corresponding $\tilde{H}_i$, this means that $\tilde{H}_i \in \mathcal{S}(\mathfrak{s})$, see also Figure 2, where $C \in \mathcal{S}^2\Bbbk^m$. □

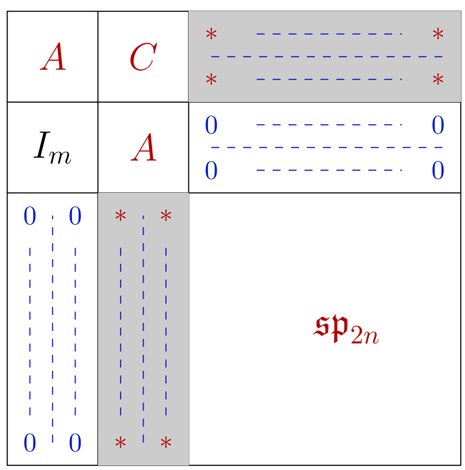

Fig. 2. Elements of $f + \mathfrak{g}_e \subset \mathfrak{sp}_{2m+2n}$.

Theorem 5.4. *All semi-direct products associated with pairs listed in Table 2 have* (FA)*.*

Proof. We begin with Item 1.

Suppose that m is even. Set $\tilde{G} := \mathrm{Sp}_{2n}{\times}\mathrm{Sp}_m$ and $\tilde{S} := \tilde{G}\ltimes\exp(V)$. Then $G \lhd \tilde{G}$ and $S \lhd \tilde{S}$. The Lie algebra $\tilde{\mathfrak{s}} = \mathsf{Lie}\,\tilde{S}$ is the $\mathbb{Z}_2$-contraction of $\mathfrak{sp}_{2n+m}$ related to the symmetric pair $(\mathfrak{sp}_{2n+m}, \mathfrak{sp}_{2n}{\oplus}\mathfrak{sp}_m)$. Let $\Delta_k \in \mathcal{S}(\mathfrak{sp}_{2n+m})$ be the sum of the principal k-minors and let $\Delta_k^\bullet$ be the highest V-component of Δ_k. The elements $\Delta_k^\bullet$ with even k, $2m < k \leqslant 2n{+}m$, belong to a set of the algebraically independent generators of $\mathcal{S}(\tilde{\mathfrak{s}})^{\tilde{\mathfrak{s}}}$, see [Y14, Theorem 4.5]. For a generic point $x \in V^*$, their restrictions $\Delta_k^\bullet|_{\tilde{\mathfrak{g}}+x}$ form a generating set for the symmetric invariants of $(\mathfrak{sp}_{2n})_x = \mathfrak{sp}_{2n-m}$. Hence $\Delta_k^\bullet \in \mathcal{S}(\mathfrak{s})^S$ by Lemma 2.1. According to [Y17a, Lemma 3.5(ii)], these elements $\Delta_k^\bullet$ (freely) generate $\mathcal{S}(\mathfrak{s})^{\mathfrak{s}}$ over $\Bbbk[V^*]^G$ if and only if their restrictions to $\mathfrak{g} \times D$ are algebraically independent over $\Bbbk[D]^G$ for any G-invariant divisor $D \subset V^*$.

In case $\mathrm{Sp}_m{\cdot}D$ is open in V^*, the restrictions of the elements $\Delta_k^\bullet$ to $\mathfrak{g}+y$ are algebraically independent for a generic point $y \in D$. If $\mathrm{Sp}_m{\cdot}D$ is not open in V^*, then D is $\tilde{G}$-invariant and the restrictions of $\Delta_k^\bullet$ to $\mathfrak{g} \times D$ are algebraically independent over $\Bbbk[D]^{\tilde{G}}$ by [Y17a, Lemma 3.5(ii)] applied to $\tilde{\mathfrak{s}}$. If there is a non-trivial relation among these restrictions and not all the coefficients are $\tilde{G}$-invariant, then one can apply an element of $\tilde{G}$ to the relation and by taking a suitable linear combination obtain a smaller non-trivial one. Thus, a minimal non-trivial relation among the restrictions must have $\tilde{G}$-invariant coefficients. Hence the restrictions of $\Delta_k^\bullet$ to $\mathfrak{g} \times D$ are also algebraically independent over $\Bbbk[D]^G$.

Suppose now that m is odd. Consider the standard embedding $\mathfrak{sl}_{2n} \subset \mathfrak{sl}_{2n} \times \mathfrak{sl}_m \subset \mathfrak{sl}_{2n+m}$. The defining representation of Sp_{2n} in $\Bbbk^{2n}$ is self-dual. Therefore we can embed $V \simeq V^*$ into $m\Bbbk^{2n} \oplus m(\Bbbk^{2n})^*$ diagonally. This gives rise to $\mathfrak{s}^* = \mathfrak{g} \oplus V \subset \mathfrak{sl}_{2n+m}$. Let $\Delta_k \in \mathcal{S}(\mathfrak{sl}_{2n+m})$ be the sum of the principal k-minors and $\Delta_k^\bullet$ the highest V-component of the restriction $\Delta_k|_{\mathfrak{s}^*}$. Note that in case $m = 1$, we have $\Delta_k^\bullet = -H_i$, where H_i is the same as in Lemma 5.1 and $k = 2i{+}1$. For $m \geqslant 3$, $\Delta_k^\bullet$ is equal to $\pm\tilde{H}_i$, where $\tilde{H}_i$ is the same as in Lemma 5.3 and $k = 2m{+}2i{-}1$. Suppose that $m \geqslant 3$.

Fix a G-stable decomposition $V = V_1 \oplus V_2$ with $V_1 = \Bbbk^{2n}$. Then there is the corresponding decomposition $V^* = V_1^* \oplus V_2^*$. Choose a generic $v \in V_2^*$ and consider the usual restriction homomorphism

$$\psi_v\colon \Bbbk[\mathfrak{s}^*]^S \to \Bbbk[\mathfrak{g} \oplus V_1^* {+} v]^{G_v \ltimes \exp(V)} \simeq \mathcal{S}(\mathfrak{g}_v \ltimes V_1)^{G_v \ltimes \exp(V_1)}.$$

Here $G_v = \mathrm{Sp}_{2n-m+1}$. Setting $n' := n - \frac{m-1}{2}$, we obtain $\mathfrak{g}_v \ltimes V_1 = (\mathfrak{sp}_{2n'} \ltimes \Bbbk^{2n'}) \oplus \Bbbk^{m-1}$. If $k = 2m{+}2i{-}1$, then the restriction of $\Delta_k^\bullet$ to $\mathfrak{g} \oplus V_1^* + v$ is equal to cH_i, where $c \in \Bbbk^\times$ and H_i is the same symmetric invariant of $\mathfrak{sp}_{2n'} \ltimes \Bbbk^{2n'}$ as in Lemma 5.1.

The ring $\Bbbk[V^*]^G$ is freely generated by $\binom{m}{2}$ polynomials F_j of degree 2. We may (and will) assume that the first $m{-}1$ elements F_j lie in $V_1 \otimes V_2$ and that the remaining ones (freely) generate $\Bbbk[V_2^*]^G$. Then $\psi_v(F_j) \in \Bbbk$ for $j \geqslant m$ and $\langle \psi_v(F_j) \mid 1 \leqslant j \leqslant m{-}1 \rangle_{\Bbbk}$ is the Abelian direct summand $\Bbbk^{m-1}$ of $\mathfrak{g}_v \ltimes V_1$. We see that $F_1, \ldots, F_{m-1}, \Delta_{2m+1}^\bullet, \ldots, \Delta_{2n+m}^\bullet$ are algebraically independent over $\Bbbk[V_2^*]$. Hence

$$\left\{ F_j \mid 1 \leqslant j \leqslant \binom{m}{2} \right\} \cup \{ \Delta_k^\bullet \mid k \text{ odd}, 2m < k \leqslant 2n{+}m \}$$

is a set of algebraically independent homogeneous invariants. Our goal is to prove that this is a generating set.

There is a big open subset $U \subset V^*$ such that G_v is a generic isotropy group for $(G{:}V^*)$ for each $v \in U$. Here $G_v = (\mathrm{Sp}_{2n'})_e$ with $2n' = 2n{-}m{+}1$ and $e \in \mathfrak{sp}_{2n'}$ being a minimal nilpotent element. The algebra $\mathfrak{g}_v$ has the "codim–2" property by [PPY] and hence $\mathfrak{s}$ has the "codim–2" property as well.

Finally we calculate the sum of the degrees of the proposed generators. There are $\binom{m}{2}$ invariants of degree 2, the minors $\Delta_k^\bullet$ are of degrees $2m{+}1$, $2m{+}3$, …, $m{+}2n$. Summing up

$$2\binom{m}{2} + \frac{1}{2}\left(n - \frac{m-1}{2}\right)(2n+3m+1) = \frac{1}{2}\mathrm{ind}\,\mathfrak{s} + n^2 + \frac{n}{2} + nm = \frac{\mathrm{ind}\,\mathfrak{s} + \dim\mathfrak{s}}{2}.$$

Applying Theorem 1.1, we can conclude that $\mathcal{S}(\mathfrak{s})^{\mathfrak{s}}$ is freely generated by the polynomials F_j and $\Delta_k^\bullet$.

Item 2 is a $\mathbb{Z}_2$-contraction of SL_{2n}, and this contraction is good, see [Y14, Theorem 4.5].

Item 4 can be covered by Theorem 2.6 (or Lemma 2.8), this pair (G, V^*) is of rank one. There is an open orbit $G{\cdot}y \subset D$, where D stands for the zero set of the generator $F \in \Bbbk[V^*]^G$. A generic isotropy group for $(G{:}V^*)$ is SL_3, G_y is connected, and $\mathfrak{g}_y$ is equal to $\mathfrak{sl}_2 \ltimes \mathcal{S}^4\Bbbk^2$, see [I70]. This $\mathfrak{g}_y$ is a good $\mathbb{Z}_2$-contraction of $\mathfrak{sl}_3$ [P07b].

Item 5 is covered by Example 2.5.

Item 6 is treated in [PY17a, Appendix A], there it is shown that this pair has (FA).

The final challenge is to describe the symmetric invariants for item 3. A certain similarity with item 2 will help. Now $V = V_1 {\oplus} V_2$ with $V_1 = \Bbbk^{2n}$, $V_2 = V_{\varphi_2}$. Set $\mathfrak{s}_2 := \mathfrak{g}{\ltimes}V_2$ (this is the semi-direct product in line 2). According to [Y14], $\Bbbk[\mathfrak{s}_2^*]^{\mathfrak{s}_2} = \Bbbk[V_2^*]^G[\mathbf{h}_1, \dots, \mathbf{h}_n]$, where each $\mathbf{h}_i$ is bi-homogeneous and $\deg_{\mathfrak{g}} \mathbf{h}_i = 2$. In other words, $\mathbf{h}_i \in (\mathcal{S}^2(\mathfrak{g}){\otimes}\mathcal{S}(V_2))^G$. In $\mathcal{S}^2(V_1)$, there is a unique copy of $\mathfrak{g}$, which gives rise to embeddings $\iota\colon \mathcal{S}^2(\mathfrak{g}) \to \mathfrak{g}{\otimes}\mathcal{S}^2(V_1)$ and

$$\tilde{\iota}\colon (\mathcal{S}^2(\mathfrak{g}){\otimes}\mathcal{S}(V_2))^G \to (\mathfrak{g}{\otimes}\mathcal{S}^2(V_1){\otimes}\mathcal{S}(V_2))^G.$$

Set $H_i := \tilde{\iota}(\mathbf{h}_i)$.

Each H_i is a G-invariant by the construction. Next we check that it is also a V-invariant. Take a generic point $v \in V_2^*$. Then $\mathfrak{g}_v$ is a direct sum of n copies of $\mathfrak{sl}_2$ and under $\mathfrak{g}_v$ the space V_1 decomposes into a direct sum of n copies of $\Bbbk^2$. The restriction of $\mathbf{h}_i$ to $\mathfrak{g} + v$ is an element of $\mathcal{S}^2(\mathfrak{g}_v)^{\mathfrak{g}_v} \subset \mathcal{S}^2(\mathfrak{g}) \subset \mathfrak{g}{\otimes}\mathfrak{g}$. If we regard this restriction as a bi-linear function on $\mathfrak{g}{\otimes}\mathfrak{g}$, then its value on (A, B) for $A, B \in \mathfrak{g}$ can be calculated as follows. From each matrix we cut the $\mathfrak{sl}_2$ pieces A_j, B_j, $1 \leqslant j \leqslant n$, corresponding to the $\mathfrak{sl}_2$ summands of $\mathfrak{g}_v$ and take a linear combination $\sum \alpha_{i,j}\mathrm{tr}\,(A_jB_j)$. With a slight abuse of notation we set $\mathbf{h}_i(A, B, v) := \sum \alpha_{i,j}\mathrm{tr}\,(A_jB_j)$.

The restriction of H_i to $\mathfrak g \oplus V_1^* + v$ is an element of $(\mathfrak g \otimes \mathcal S^2(V_1))^{\mathfrak g_v}$. Take $\xi \in V_1^*$. Let $B(\xi) \in \mathfrak g$ be the projection of ξ^2 to $\mathfrak g \subset \mathcal S^2(V_1)$. Then

$$H_i(A+\xi+v) = \mathbf{h}_i(A, B(\xi), v).$$

Write $\xi = \xi_1 + \ldots + \xi_n$, where each ξ_j lies in its $\mathfrak g_v$-stable copy of $\Bbbk^2$. Then $\xi_j \otimes \xi_k$ with $j \ne k$ is orthogonal to $\mathfrak g_v \subset \mathfrak g \subset \mathcal S^2(V_1)$. Furthermore, $\mathrm{tr}\,(A_j B(\xi)_j) = \det(\xi_j | A_j \xi_j)$. Therefore

$$H_i(A+\xi+v) = \sum \alpha_{i,j} \det(\xi_j | A_j \xi_j).$$

We see that $H_i|_{\mathfrak g\oplus V_1^*+v}$ lies in $\mathcal S(\mathfrak g_v \ltimes V_1)$ and therefore is a V_2-invariant [Y17a]. Moreover, this restriction is a V_1-invariant by [Y17b]. Since these assertions hold for a generic vector $v \in V_2^*$, each H_i is a V-invariant. From the case of $\mathfrak s_2$, we know that the matrix $(\alpha_{i,j})$ is non-degenerate. Hence the invariants H_i are algebraically independent over $\Bbbk(V_2^*)$. Note that $\Bbbk[V_2^*]^G = \Bbbk[V^*]^G$. Further, $\deg H_i = \deg \mathbf{h}_i + 1$. If we sum over all (suggested) generators, then the result is $(\dim \mathfrak s_2 + \mathrm{ind}\,\mathfrak s_2)/2 + n$ and this is exactly $(\dim \mathfrak s + \mathrm{ind}\,\mathfrak s)/2$.

In order to use Theorem 1.1, it remains to prove that $\mathfrak s$ has the "codim–2" property. Let $D \subset V_2^*$ be a G-invariant divisor and let $y \in D$ be a generic point. If $G_y \ne (\mathrm{SL}_2)^n$, then $G_y = (\mathrm{SL}_2)^{n-2} \times (\mathrm{SL}_2 \ltimes \exp(\mathcal S^2\Bbbk^2))$. In particular, $\dim(G{\cdot}y) = \dim V_2 - (n-1)$. If $\mathfrak q$ is the Lie algebra of $Q = \mathrm{SL}_2 \ltimes \exp(\mathcal S^2\Bbbk^2)$, then $\mathfrak q = \mathfrak{sl}_2 \ltimes \mathfrak{sl}_2^{\mathsf{ab}}$. We have

$$G_y \ltimes \exp(V_1) = (\mathrm{SL}_2 \ltimes \exp(\Bbbk^2))^{n-2} \times (Q \ltimes \exp(\Bbbk^4))$$

and $\mathfrak q \ltimes \Bbbk^4 = \mathfrak{sl}_2 \ltimes ((\Bbbk^2 \oplus \mathcal S^2\Bbbk^2) \oplus \Bbbk^2)$ with the unique non-zero commutator $[\Bbbk^2, \mathcal S^2\Bbbk^2] = \Bbbk^2$. An easy computation shows that $\mathrm{ind}\,(\mathfrak q \ltimes \Bbbk^4) = 2$. Thereby $\mathrm{ind}\,(\mathfrak g_y \ltimes V_1) = n$ and hence $\mathfrak g \oplus V_1^* \times D \cap \mathfrak s^*_{\mathsf{reg}} \ne \varnothing$, cf. [Y17a, Eq. (3·2)]. The Lie algebra $\mathfrak s$ does have the "codim–2" property. □

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