Restriction of Holomorphic Cohomology of a Shimura Variety to a Smaller Shimura Variety

L. Clozel* and T.N. Venkataramana

*
Université de Paris-Sud
Mathematiques
Bât 425, Orsay-Cedex

France

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-44

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1. Introduction

Suppose $S = S(\Gamma) = \Gamma\backslash X$ is a *Shimura variety* obtained as the quotient of a Hermitian symmetric space $X = G(\mathbf{R})/K_\infty$ - G is a semisimple group over $\mathbf{Q}$, $K_\infty \subset G(\mathbf{R})$ a maximal compact subgroup - by an arithmetic subgroup Γ of $G(\mathbf{Q})$.

Suppose that H is a semisimple $\mathbf{Q}$ -subgroup of G such that $Y = H(\mathbf{R})/K_H$ is also Hermitian symmetric, with $K_H = K_\infty \cap H$ a maximal compact subgroup of $H(\mathbf{R})$ and such that the natural inclusion i of Y in X is holomorphic.

For every covering $S(\Gamma') \to S(\Gamma)$ with $\Gamma' \subset \Gamma$ of finite index the resulting map $i = i(\Gamma') : \Gamma' \cap H\backslash Y \longrightarrow \Gamma'\backslash X$ is , as is well known , a morphism of varieties.

Let C be a correspondence on S which is of the form $z \to g(z)$ on the universal covering X of S with $g \in G(\mathbf{Q})$. We therefore get a finite covering $C : S(\Gamma') \to S(\Gamma)$ for some Γ'.

In this article , we are concerned with the following question . Let ω be a cohomology class on S whose restriction (i.e. pullback via the composite of C with i) to $\Gamma' \cap H\backslash Y$ is zero for all the correspondences C defined above - we will say then that ω *vanishes stably* along H . Then is ω itself zero ?

Membre de l'institut Universitaire de France

This question is hard to answer for an arbitrary class ω , but we can give a criterion purely in terms of the linear algebra of G and H, for *holomorphic* forms on S (which are cuspidal if S is not compact). One of the reasons for the holomorphic case being easier is that the restriction of a holomorphic form to a subvariety is indeed holomorphic whereas even if ω is a harmonic form on S , its restriction to $\Gamma' \cap H \backslash Y$ need not be harmonic in general .

As in [Clo 1] and [Clo 2], we make use of the explicit description - as formulated in [V-Z] - of (g,K)-modules with non-zero cohomology . Indeed, the papers [Clo 1] and [Clo 2] deal with the question of vanishing or not, of cup-products of two holomorphic forms on S , which may be interpreted as the question of the vanishing of the restriction to the diagonal of the tensor-product of these two forms.

The question of the stable vanishing of ω along H (at least in the co-compact case ; when S is not compact, one must restrict oneself to cuspidal cohomology) has a simple description in terms of the Parthasarathy-Vogan-Zuckerman theory : suppose that the infinity type of ω is A_q, associated to the θ-stable parabolic subalgebra q of the Lie-algebra $g = LieG(\mathbf{R}) \otimes \mathbf{C}$. Let p^+ be the holomorphic tangent space of $X = G(\mathbf{R})/K_\infty$, u be the unipotent radical of q , u^+ its intersection with p^+ and R the dimension of u^+. Then, ω is stably non-zero along H if and only if the R-th exterior power of u^+ (a line) lies in the smallest subspace of the R-th exterior power of p^+ which contains the R-th exterior power of $p^+ \cap h$ (here, $h = LieH(\mathbf{R}) \otimes \mathbf{C}$) , and is stable under the adjoint action of K_∞ . We prove this in section 2 , first in the compact case ; the non-compact case is dealt with similarly (for cuspidal holomorphic cohomology) and we will briefly indicate the modifications necessary .

We then use this criterion in the case of the classical hermitian symmetric domains and some naturally embedded sub- hermitian domains. The results are set out in section 3 .

The conjectures of Langlands , Arthur and Kottwitz on the Zeta functions of Shimura varieties impose strong restrictions on the Galois representations occurring in the etale (intersection) cohomology of the Borel-Bailey-Satake Compactification of S. By working out the predictions of these conjectures in the special cases of U(g,h) ($g \leq h$, $g \geq 2$ and $h \neq 2$) and GSp(g) ($g \geq 2$) we will see in section 4 , that the action of the Galois group on the cohomology degree g , of the Shimura varieties corresponding to these two groups is potentially Abelian .

As an application of the calculations of section 3 , we show in section 5 , that given a holomorphic g-form on a Siegel-modular variety (the Shimura variety associated to the group GSp(g), with $g \geq 2$) , its restriction to some product of g modular curves is non-zero. This is shown to imply that the

Mumford-Tate group of the compactly supported cohomology in degree g of the Siegel-modular variety is Abelian (this was implicitly proved in a paper of Weissauer in the case g = 2) , thereby confirming the heuristics of section 4 . As a consequence , we find that the action of the Galois-group on the (image of the compactly supported cohomology in degree g in the) etale intersection cohomology of the associated (Borel-Bailey -Satake) compactification of the Siegel-modular variety is potentially Abelian .

As another application , we show that the Mumford-Tate group of the (compactly supported) cohomology in degree g , of some Shimura varieties attached to U(g,h) ($2 \leq g \leq h$ and $(g,h) \neq (2,2)$) is also Abelian , by restricting the cohomology to an appropriate product of curves . Analogously , we show that the action of the Galois-group on the (image of the compactly supported cohomology in the) etale cohomology in degree g of the associated compactification of this Shimura variety is potentially Abelian .

The second named author would like heartily to thank the first for patiently explaining many of the ideas referred to above , especially the theory of Parthasarathy , Vogan and Zuckerman and the conjectures on the Zeta functions of Shimura varieties . He would also like to thank the Universite de Paris-Sud , Orsay* for its hospitality while part of this work was done and the Institut Universitaire de France and the Commission on Developement and Exchanges of the International Mathematical Union for providing travel support to enable him to visit Paris .

The first named author wants to record here that the results contained in [Clo 1] concerning the *vanishing* of cup-products of holomorphic forms had been proved by Parthasarathy [Par 1] . Thus the new content of [Clo 1] is the *non-vanishing* of certain cup-products . An earlier result (for U(n,1)) can be found in Shimura [Sh 1] .

* and the MPI in Bonn

2. — General criteria

2.1. — Let G be a connected semi-simple group defined over $\mathbb{Q}$; by abuse of notation we will also denote by G the group $G(\mathbb{R})$. Let $K \subset G$ be a maximal compact subgroup. We assume that the symmetric space $X = G/K$ is of Hermitian type. There is then an element c belonging to the center of K such that $Ad(c)$ induces, on the tangent space $\mathfrak{p}_0$ of X at its base point $o = \overline{K}$, multiplication by $i = \sqrt{-1}$. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- \tag{2.1}$$

be the associated decomposition of $\mathfrak{g}$: thus $\mathfrak{p}^+ = \{X \in \mathfrak{p} : Ad(c)X = iX\}$ is the holomorphic tangent space.

Now let $H \subset G$ be a connected reductive subgroup over $\mathbb{Q}$.

We will assume

$$H \cap K \quad \textbf{is a maximal compact subgroup of } H. \tag{2.2}$$

Then the restriction to H of a Cartan involution θ of G is a Cartan involution of $H = H(\mathbb{R})$. We have a corresponding decomposition

$$\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H \tag{2.3}$$

with $\mathfrak{p}_H = \mathfrak{p} \cap \mathfrak{h}$. We further assume

$$\mathfrak{p}_H \quad \textbf{is stable by} \quad Ad(c). \tag{2.4}$$

Then $Ad(c)$ defines a K_H-invariant complex structure on $\mathfrak{p}_{H,0}$. The space $X_H = H/H \cap K$ is Hermitian symmetric. We have a triangular decomposition

$$\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H^+ \oplus \mathfrak{p}_H^- \tag{2.5}$$

compatible with (2.1); finally, the embedding $X_H \hookrightarrow X$ is holomorphic.

Now we assume that $\Gamma \subset G(\mathbb{Q})$ is a neat congruence subgroup. Precisely, we suppose that $\Gamma = G(\mathbb{Q}) \cap K_f$, $K_f \subset G(\mathbb{A}_f)$ being a compact-open subgroup such that $G(\mathbb{Q}) \cap (K \times gK_fg^{-1}) = \{1\}$ for any $g \in G(\mathbb{A}_f)$. We consider the Shimura variety $S(\Gamma) = \Gamma\backslash X$: it is one of the connected components of $S(K_f) = G(\mathbb{Q})\backslash G(\mathbb{A})/K \cdot K_f = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$.

Now assume $K_f^H \subset H(\mathbb{A}_f)$ is compact-open. If $K_f^H \subset K_f$ there is a natural map $S(K_f^K) \underset{j}{\longrightarrow} S(K_f^G)$. By our assumption on K_f, j is finite and unramified. We recall the following fact due to Deligne [Del, Prop. 1.15] :

LEMMA 2.1. — *Given $K_f^H \subset H(\mathbb{A}_f)$, there exists a compact open subgroup $K_f^1 \subset G(\mathbb{A}_f)$, with $K_f^H \subset K_f^1$, such that the natural map $j' : S(K_f^H) \to S(K_f^1)$ is injective.*

In particular, if we take $K_f^H = K_f \cap H(\mathbb{A}_f)$ we get a natural map $j : S(K_f^H) \to S(K_f)$; j is finite and if we replace K_f by a sufficiently small subgroup K_f^1 we get a diagram

$$
\begin{tikzcd}
S(K_f^H) \arrow[r, "j'"'] \arrow[dr, "j"'] & S(K_f^1) \arrow[d, "\pi"] \\
& S(K_f)
\end{tikzcd}
\tag{2.6}
$$

where π is the natural projection and j' is injective. By restriction to the connected component $\Gamma \cap H\backslash H/K_H$ we get a map $j : \Gamma \cap H\backslash X_H \to \Gamma\backslash X$ with analogous properties.

Now assume $g \in G(\mathbb{Q})$. Fix $K_f^H \subset H(\mathbb{A}_f)$ and consider the map $j_g : H(\mathbb{A}) \to G(\mathbb{A})$ given by $j_g(h) = gh$. It is easy to check that j_g yields an injective map $H(\mathbb{Q})\backslash H(\mathbb{A})/K_f^H \to G(\mathbb{Q})\backslash G(\mathbb{A})/K_f^H$; assuming $K_f^H = K_f \cap H(\mathbb{A}) \subset K_f$ (where K_f is compact–open in $G(\mathbb{A}_f)$) we then obtain a natural map $j_g : S(K_f^H) \to S(K_f)$ with our previous notation; this map is unramified and finite. On the connected component of $S(K_f^H)$ given by the orbit of o under $H(\mathbb{R})$, j_g is the natural map $(H \cap g^{-1}\Gamma g)\backslash X_H \to \Gamma\backslash X_G$. In this manner we obtain a family, parametrized by $g \in G(\mathbb{Q})$, of complex subvarieties of $\Gamma\backslash X$, the images of the j_g.

In the remainder of this section we assume that G is anisotropic over $\mathbb{Q}$; $S(\Gamma)$ is then compact. For $r \geq 0$, let $H^{r,0}(S(\Gamma))$ be the holomorphic subspace of $H^r(S(\Gamma), \mathbb{C})$ for the Hodge decomposition. Following Oda [Oda] we will study the restriction map

$$H^{r,0}(S(\Gamma)) \xrightarrow[\text{Res}]{} \prod_{g \in G(\mathbb{Q})} H^{r,0}(S_H(g)) \tag{2.7}$$

where $S_H(g) = (H \cap g^{-1}\Gamma g)\backslash X_H$, and the restriction map is deduced from the family of maps (jg). We want to obtain sufficient conditions for the injectivity of (2.7). We will denote by $R_g : H^{r,0}(S(\Gamma)) \to H^{r,0}(S_H(g))$ the component of (2.7) associated to jg.

PROPOSITION 2.2. — *Assume that $\Lambda^r\mathfrak{p}^+$ is spanned over K by $\Lambda^r\mathfrak{p}_H^+$. Then* (2.7) *is injective.*

Proof : suppose $\omega \in H^{r,0}(S(\Gamma))$ verifies $\operatorname{Res}\omega = 0$.

Identify $\mathfrak{p}^+$ to the holomorphic tangent space at the base point o of $S(\Gamma)$; if $\lambda \in \Lambda^r\mathfrak{p}^+$ than $g \cdot \lambda$ is the translate of λ by g in the holomorphic tangent space at $g \cdot o$. Then $\omega_{g\cdot o}(g \cdot \lambda) = 0$ for all $g \in G(\mathbb{Q})$ and $\lambda \in \Lambda^r\mathfrak{p}_H^+$ by our assumption. Therefore this is true for all $g \in G$. If $k \in K$ we than have $\omega_{gk\cdot o}(gk \cdot \lambda) = \omega_{g\cdot o}(gk \cdot \lambda) = 0$. Therefore $\omega_{g\cdot o}(g\lambda) = 0$ for any $\lambda \in \Lambda^r\mathfrak{p}^+$, $g \in G$. This implies that $\omega = 0$.

2.2. — We now give a more precise version of Proposition 2.2 using representation theory. If the representation–theoretic type of the form ω is fixed, we will obtain a necessary and sufficient condition for the vanishing of $\mathrm{Res}\,\omega$. We still assume G anisotropic.

According to Parthasarathy, Kumaresan and Vogan–Zuckerman, the holomorphic cohomology of $S(\Gamma)$ can be described as follows. Let $\mathfrak{t}_0 \subset \mathfrak{k}_0$ be a Cartan subalgebra. We consider θ–stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}$ [Vo–Z, § 2] : $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{l}$ is the centralizer of an element $X \in i\mathfrak{t}_0$ and $\mathfrak{u}$ is the span of the positive roots of X in $\mathfrak{g}$. Then $\mathfrak{q}$ is stable by θ, whence a decomposition $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{k}) \oplus (\mathfrak{u} \cap \mathfrak{p})$. We assume that $\mathfrak{u} \cap \mathfrak{p} \subset \mathfrak{p}^+$. Let $r = \dim(\mathfrak{u} \cap \mathfrak{p}^+)$. We write $\mathfrak{u}^+ = \mathfrak{u} \cap \mathfrak{p}^+$.

Associated to $\mathfrak{q}$, there is a well-defined irreducible $(\mathfrak{g}, K)$–module $A_\mathfrak{q}$ characterized by the following properties. We assume that a choice of positive roots for $(\mathfrak{k}, \mathfrak{t})$ has been made, compatibly with $\mathfrak{u}$. Let $e(\mathfrak{q})$ be a generator of the line $\Lambda^r \mathfrak{u}^+ \subset \Lambda^r \mathfrak{p}$. Then $e(\mathfrak{q})$ is the highest vector of an irreducible representation $V(\mathfrak{q})$ of K contained in $\Lambda^r \mathfrak{p}_+$. The representation $A_\mathfrak{q}$ is then uniquely characterized by

(2.8) **$A_\mathfrak{q}$ is unitary, with the same infinitesimal character as the trivial representation**

$$\mathrm{Hom}_K(V(\mathfrak{q}), A_\mathfrak{q}) \neq 0 . \tag{2.9}$$

Moreover, $V(\mathfrak{q})$ occurs with multiplicity 1 in $A_\mathfrak{q}$ and $\Lambda^r \mathfrak{p}^+$ and

$$H^{r,0}(\mathfrak{q}, K; A_\mathfrak{q}) = \mathrm{Hom}_K(\Lambda^r \mathfrak{p}^+, A_\mathfrak{q}) \cong \mathbb{C} . \tag{2.10}$$

Cohomology classes (of type $(r, 0)$ and) **of type $A_\mathfrak{q}$** are then obtained as follows. Suppose $\varphi : A_\mathfrak{q} \to C^\infty(\Gamma \backslash G)$ is an intertwining map. We then get a natural map $H^{r,0}(\mathfrak{g}, K; A_\mathfrak{q}) \xrightarrow[\varphi_*]{} H^{r,0}(\mathfrak{g}, K; C^\infty(\Gamma\backslash G)) = H^{r,0}(S(\Gamma))$, the equality being Matsushima's isomorphism. Explicitly, φ_* is obtained as follows : fix a non-zero K–map $\omega : \Lambda^r \mathfrak{p}^+ \to A_\mathfrak{q}$, which necessarily factorizes through the $V(\mathfrak{q})$–component. Define $\omega_\varphi \in H^{r,0}(S(\Gamma))$ by $\omega_\varphi(g \cdot \lambda) = \varphi(\omega(\lambda))(g)$ $(\lambda \in \mathfrak{p}^+ = T_o(X),\ g \in G)$. Then $\omega_\varphi = \varphi_* \omega$, where ω is considered as an element of $H^{r,0}(\mathfrak{g}, K; A_\mathfrak{q})$. The $A_\mathfrak{q}$–component of $H^{r,0}(S(\Gamma))$ is the sum, over a basis $\{\varphi\}$ of $\mathrm{Hom}_{\mathfrak{g},K}(A_\mathfrak{q}, C^\infty(\tau\backslash G))$, of the forms ω_φ.

Write

$$\Lambda^r \mathfrak{p}^+ = V' \oplus V(\mathfrak{q}) \tag{2.11}$$

V' being the orthogonal complement of $V(\mathfrak{q})$ for any invariant scalar product; equivalently, V' is the sum of the K–submodules of $\Lambda^r \mathfrak{p}^+$ non–equivalent to $V(\mathfrak{q})$.

PROPOSITION 2.3. — *The following conditions are equivalent :*

(i) $\mathrm{Res}(\omega) = 0$ *for any* $\omega \in H^{r,0}(S(\tau))$ *of type* $A_\mathfrak{q}$.

(ii) $\Lambda^r \mathfrak{p}_H^+ \subset V' = V(\mathfrak{q})^\perp$

(iii) $\Lambda^r \mathfrak{p}_H^+$ *is orthogonal to the* K*-span of* $e(\mathfrak{q})$.

Proof : suppose ω_φ is associated to $\varphi : A_{\mathfrak{q}} \to C^\infty(\Gamma\backslash G)$. Then (i) says that $\omega_\varphi(g\cdot\lambda) = \varphi(\omega(\lambda))(g) = 0$ for any $g \in G(\mathbb{Q})$, $\lambda \in \Lambda^r\mathfrak{p}_H^+$. Then this is true for any $g \in G$, so $\varphi(\omega(\lambda)) = 0$, whence $\omega(\lambda) = 0$ by injectivity. Since $A_{\mathfrak{q}}$ and $\Lambda^r\mathfrak{p}_H^+$ have a unique component of type $V(\mathfrak{q})$ and ω is a K–map, we may view $\omega : \Lambda^r\mathfrak{p}^+ \to A(\mathfrak{q})$ as the projection $\pi_{\mathfrak{q}}$ onto the second component in (2.11). Our assumption (i) then means that $\pi_{\mathfrak{q}}(\Lambda^r\mathfrak{p}_H^+) = \{0\}$, which is (ii). Since $V(\mathfrak{q})$ is spanned over K by $e(\mathfrak{q})$ this is equivalent to (iii).

The following formulation will be useful :

COROLLARY 2.4. — *The following are equivalent :*
(i) $\mathrm{Res}(\omega) = 0$ *for any* $\omega \in H^{r,0}(S(\Gamma))$ *of type* $A_{\mathfrak{q}}$.
(ii) *The* K*-span of* $\Lambda^r\mathfrak{p}_H^+$ *in* $\Lambda^r\mathfrak{p}^+$ *does not contain* $e(\mathfrak{q})$.

Proof : this follows from condition (iii) above and the fact that $V(\mathfrak{q})$ occurs with multiplicity 1.

We will denote by $E(G,H,r)$ the K–span of $\Lambda^r\mathfrak{p}_H^+$ in $\Lambda^r\mathfrak{p}_+$. By Corollary 2.4, the restriction problem is then reduced to :

PROBLEM 2.5. — *Describe the θ–stable (holomorphic)* $\mathfrak{q}$ *(with* $\dim\mathfrak{u}^+ = r$*) such that* $e(\mathfrak{q}) \in E(G,H,r)$.

We end this section with a useful, negative criterion. Let $T_H \subset K_H$ be a maximal torus. We may assume, up to conjugacy, that $T_H \subset T$ where $T \subset K$ is a maximal torus whose Lie algebra is $\mathfrak{t}_0$ as above.

Let $T_{H,\mathbb{C}}$ be the complexification of T_H. Suppose we are given a one –parameter torus $M \subset T_{H,\mathbb{C}}$. Fix an isomorphism $M \cong \mathbb{G}_m$. Then M acts on H by conjugation, and on $\Lambda^r\mathfrak{p}^+$ via $M \hookrightarrow K_{\mathbb{C}}$ and the adjoint action.

PROPOSITION 2.6. — *If there exists a* 1*-dimensional torus* $M \subset T_{H,\mathbb{C}}$ *that centralizes* H *but acts by strictly positive weights on* $V(\mathfrak{q})$*, then* $E(G,H,r) \cap V(\mathfrak{q}) = \{0\}$.

Proof : with the notations introduced before Proposition 2.3, we have $\mathrm{Hom}_M(\Lambda^r\mathfrak{p}_H^+, V(\mathfrak{q})) = \{0\}$ by assumption, whence $\mathrm{Hom}_K(\Lambda^r\mathfrak{p}_H^+, V(\mathfrak{q})) = \{0\}$ and $\Lambda^r\mathfrak{p}_H^+ \subset V'$, q.e.d.

We now include another criterion that is easier to check in some cases. Let $E(G,H,r)$ be as before. Given a θ–stable parabolic subalgebra $\mathfrak{q}_H$ of $\mathfrak{h}$ such that $\mathfrak{p}_H^- \cap \mathfrak{u}(\mathfrak{q}_H) = \{0\}$ (a *holomorphic* θ–stable parabolic subalgebra) and $r = \dim(\mathfrak{p}_H^+ \cap \mathfrak{u}(\mathfrak{q}_H))$, let $V_G(\mathfrak{q}_H)$ denote the K–span of the line $\Lambda^r(\mathfrak{p}_H^+ \cap \mathfrak{u}(\mathfrak{q}_H))$ in $\Lambda^r(\mathfrak{p}_G^+)$. Let

$$A(G,H,r) = \sum V_G(\mathfrak{q}_H) \subset E(G,H,r) \tag{2.12}$$

where the sum runs over all holomorphic θ–stable parabolic subalgebras $\mathfrak{q}_H$ relative to our choice of T.

PROPOSITION 2.7. — *The following conditions are equivalent :*
(i) $\mathrm{Res}(\omega) = 0$ *for any* $\omega \in H^{r,0}(S(\Gamma))$ *of type* $A_{\mathfrak{q}}$.
(ii) $A(G,H,r) \subset V' = V(\mathfrak{q})^{\perp}$.

Proof : (i) $\Longrightarrow$ (ii) by Proposition 2.3. Conversely, if $\mathrm{Res}\,\omega \neq 0$, we may, as in the proof of Proposition 2.3, replace ω – seen, say, as a form on X – by a $G(\mathbb{Q})$ translate ω' such that ω'_0 does not vanish on $\Lambda^r\mathfrak{p}_H^+$. Then the form ω' restricts to a non-vanishing holomorphic form on a quotient of H, with must be a sum of forms of type $A(\mathfrak{q}_H)$ in the notation of the paragraph following (2.10), applied to H. Thus $< \omega', e_{\mathfrak{q}_H}) >\neq 0$ for some θ–stable, holomorphic parabolic $\mathfrak{q}_H$. This implis that ω' does not vanish on $A(G,H,r)$ and therefore $A(G,H,r) \supset V(\mathfrak{q})$. Thus (ii) $\Longrightarrow$ (i).

2.3. — Generalizations

The results of § 2.2 generalize quite naturally in two directions : (a) non–trivial systems of coefficients; (b) non–anisotropic groups.

Part (a) is obvious : suppose E is a finite–dimensional representation of $G \underset{\mathbb{Q}}{\times} \mathbb{C}$; E defines a local system $\mathcal{E}$ on $S(\Gamma)$ and on all the varieties considered in the previous arguments. Suppose for simplicity E irreducible. Since $\mathcal{E}$ is locally trivial, $H^{\bullet}(S(\Gamma),\mathcal{E})$ again has a Hodge decomposition and $H^{r,0}(S(\Gamma),\mathcal{E})$ can be decomposed according to certain representations $A_{\mathfrak{q}}(E)$ associated to certain θ–stable parabolic subalgebras [Vo–Z]. The results in § 2.2 then remain true.

In the sequel we will generally neglect coefficient systems, which are irrelevant to our problem.

Consider now the case when G is isotropic over $\mathbb{Q}$. Then $H^{\bullet}(S(\Gamma),\mathbb{C})$ is no longer endowed with a Hodge decomposition. The subspace $H^{\bullet}_{\mathrm{cusp}}(S(\Gamma),\mathbb{C})$ of classes represented by cusp forms, however, inherits of Hodge decomposition (cf. Borel [Bo3], as well as [Bo–Wa, § II.4]). If $\omega \in H^{r,0}_{\mathrm{cusp}}(S(\Gamma))$, we may again restrict ω, via the maps j_g, to $S_H(g)$. The restriction is obviously a holomorphic r–form on this arithmetic quotient. Moreover :

PROPOSITION 2.8. — *If* $\omega \in H^{r,0}_{\mathrm{cusp}}(S(\Gamma))$, $j_g^*\omega \in H^{r,0}_{\mathrm{cusp}}(S_H(g))$.

Proof : we extend an argument in [Cl1].

Since Hodge theory applies to cuspidal cohomology, we may view ω as a holomorphic form on $S(\Gamma)$. Let $\eta = j_g^*\omega$ seen as a holomorphic form. Then $\overline{\partial}\eta = 0$, and therefore η is annihilated by the Hodge Laplacian : in terms of the automorphic functions that are coefficients of η, this translates into the vanishing of the Casimir operator. A finite–dimensionality argument in [Cl1, p. 80] then shows that η is annihilated by an ideal of finite codimension in the center $\mathfrak{z}_H$ of the enveloping algebra for H, if we can show that η (or its coefficients) are square–integrable. We will in fact show :

LEMMA 2.9. — *η is rapidly decreasing on* $H \cap g^{-1}\Gamma g\backslash H$.

Then, by the preceding argument, η is an automorphic form of rapid decrease. By Lemma 5.3 in [Cl1], η is a cusp form, and Lemma 2.8 is proved.

Proof of Lemma 2.9 : if G is the set of $\mathbb{R}$–points of a semi-simple $\mathbb{Q}$–group and f a function on G, left-invariant by an arithmetic subgroup Γ and K–finite on the right ($K \subset G$ maximal compact), the following conditions are equivalent :

(2.13) $\quad f$ **is rapidly decreasing in Siegel domains**

in the usual sense ([Bo–Ja, (1.6)]) and

$$|f(x)| = 0(\|x\|^{-N}) \quad \textbf{for all} \quad N \geq 0, \tag{2.14}$$

x ranging over a Siegel domain $\mathfrak{S}$.

In (2.14), $\| \ \|$ is any norm on G, in the sense of [Bo–Ja]; for the equivalence see Moeglin–Waldspurger [Mo–Wa, p. 20].

Since ω is cuspidal, its coefficients verify (2.13) and therefore (2.14). A norm on G restricts to one on H. We now need only check that the coefficients of η – i.e., the restrictions of the coefficients of ω – verify (2.14), but now on a Siegel domain $\mathfrak{S}_H$ **for** H.

Fix a minimal parabolic $\mathbb{Q}$–subgroup P_H of H, and let U_H be its unipotent radical. Let T_H be a maximal $\mathbb{Q}$–split torus in P_H and consider $B_H = U_H \rtimes T_H \subset P_H$. Then B_H is a "split solvable group" in the sense of [Bo1]. We may similarly define P, T, U for G, and $B_G = U \rtimes T$. By a theorem of Borel and Tits [Bo2, Vol. III, p. 533], there exists $g \in G(\mathbb{Q})$ such that $g\, B_H\, g^{-1} \subset B_G$. Thus we may assume that $B_H \subset B_G$. Then $U_H \subset U$, and we may assume (upon conjugation by an element of U) that $T_H \subset T$. Let Φ^+ be the set of roots of T in U, Φ_H^+ the set of roots of T_H in U_H.

Let M be the centralizer of T in P (or G). Then $M = M^0T$ where M^0 is anisotropic and therefore $M^0 \cap T$ is finite. We similarly define M_H^0. Let A be the connected component of 1 in $T(\mathbb{R})$, $\Omega_U \subset U(\mathbb{R})$ and $\Omega_M \subset M^0(\mathbb{R})$ be compact subsets, and for $t > 0$ define

$$A_t = \{a \in A : a^\alpha > t \quad \forall \alpha \in \Phi\}. \tag{2.15}$$

Fix maximal compact subgroups K_H and K of H and G with $K_H \subset K_G$. A Siegel domain $\mathfrak{S}_G$ of G is then a set

$$\mathfrak{S}_G = \Omega_U \cdot \Omega_M \cdot A_t\, K. \tag{2.16}$$

Similarly a Siegel domain $\mathfrak{S}_H$ of H can be written

$$\mathfrak{S}_H = \Omega_{U_H} \Omega_{M_H}\ A_{H,t} K_H \tag{2.17}$$

where $A_{H,t}$ is of course defined by Φ_H.

Suppose now Ω_G is a compact subset of G. We first prove that

$$|f(x)| = 0(\|x\|^{-N} \quad \forall N \geq 0 \quad \textbf{for } x \textbf{ ranging in } \mathfrak{S}_G \Omega_G, \tag{2.18}$$

f being a cusp form. Indeed, if $x = g\omega$, $g \in \mathfrak{S}_G$, $\omega \in \Omega_G$, write $g = nmak$ according to (2.16) and $h = k\omega$. By the Iwasawa decomposition, $h = n_0 a_0 k_0$, $n_0 \in U(\mathbb{R})$, $a_0 \in A$, $k_0 \in K$. Clearly a_0 is constrained to lie in a fixed compact subset of A. Now

$$x = nmah = nman_0a_0k_0 = n({}^{ma}n_0)maa_0k_0$$

where ${}^uv := uvu^{-1}$. Let $\Omega'_U \subset U(\mathbb{R})$ be a compact subset such that $U(\mathbb{R}) = (\Gamma \cap U(\mathbb{R}))\Omega'_U$. Then $n({}^{ma}n_0) \in (\Gamma \cap U(\mathbb{R}))u$ for $u \in \Omega'_U$, whence $f(x) = f(umaa_0k_0)$. Now we may replace Ω_U by Ω'_U ; $m \in \Omega_M$, a_0 is bounded, and the growth property of f on $K_U\Omega_M A_t K$ implies (2.18).

We now note that if $\sigma \in G(\mathbb{Q})$ is fixed, (2.18) remains true for x ranging in $\sigma\mathfrak{S}_G\Omega_G$ since the left-translate of f is a cusp form. The corresponding estimates are uniform in σ if σ belongs to a finite set. Therefore Lemma 2.9 will follow if we can prove :

LEMMA 2.10. — *Let $\mathfrak{S}_H$ be a Siegel domain* (2.17), *and $W = N_G(T)/Z_G(T)$ be the* ($\mathbb{Q}$)*-Weyl group of* (G,T). *Then there exists a compact subset Ω_G of G and a Siegel domain $\mathfrak{S}_G$ such that*

$$\mathfrak{S}_H \subset \bigsqcup_{\sigma\in W} \sigma\, \mathfrak{S}_G\, \Omega_G \tag{2.20}$$

Indeed, the coefficients f of η then satisfy, by (2.18), the estimate (2.14) on $\mathfrak{S}_H$, whence Lemma 2.9.

Proof of Lemma 2.10 : let $\Sigma \subset \Phi_H^+$ be a fixed subset. Then Ω_{U_H} is contained in a product $\Omega_\Sigma^+\Omega_\Sigma^-$ where $\Omega_\Sigma^+ \subset U_\Sigma^+ = \prod_{\alpha\in\Sigma} U_{H,\alpha}$ and $\Omega_\Sigma^- \subset U_\Sigma^- = \prod_{\alpha\notin\Sigma} U_H$ are compact. If $a \in A_H$ let $\Sigma(a) = \{\alpha \in \Phi_H^+ : a^\alpha \leq 1\}$. Let $\mathfrak{S}_H$ be a Siegel domain (2.17). Then $\mathfrak{S}_H$ is contained in the union over all Σ of the subsets

$$\bigsqcup_a \Omega_\Sigma^-\Omega_\Sigma^+\Omega_{M_H}\, aK_H \tag{2.21}$$

where $a \in A_{H,t}$ verifies the condition $\Sigma(a) = \Sigma$. The eigenvalues of $X \longmapsto a^{-1}Xa$ in $\mathrm{Lie}(U_\Sigma^+)$ are given by $a^{-\alpha}$ ($\alpha \in \Sigma(a)$) and are therefore bounded by t^{-1}. Thus $a^{-1}\Omega_\Sigma^+a$ is contained in a fixed, compact set, that we now denote by Ω_Σ^+. Then (2.21) is contained in

$$\bigsqcup_a \Omega_\Sigma^-\, a\,\Omega_\Sigma^+\, \Omega_{M_H}K_H\,. \tag{2.22}$$

Now fix a, and choose $\sigma \in W$ such that

$$a' = \sigma a \sigma^{-1} \in A_1 = \{a \in A : a^\beta \geq 1\ \forall \beta \in \Phi_G^+\}\,.$$

If x belongs to a root subgroup of U_{Σ}^{-}, x is (strictly) dilated by $Ad(a)$ and $\sigma x \sigma^{-1}$ is therefore dilated by $Ad(a')$. This implies that $\sigma x \sigma^{-1} \in U_G$. Now

$$\sigma(\Omega_{\Sigma}^{-} a) = (\sigma \Omega_{\Sigma}^{-} \sigma^{-1}) a' \sigma \subset \Omega_U' \; A_1 \sigma \tag{2.23}$$

where $\Omega_U' \subset U_G$ is compact. Finally, we see that $\mathfrak{S}_H$ is contained in $\bigsqcup_{\sigma,\Sigma} \sigma^{-1} \Omega_U' A_1 \sigma \Omega_{\Sigma}^{+} \Omega_{MH} K_H$. This is an expression (2.20), whence the lemma.

In conclusion we note that all the arguments in § 2.2 extend to cusp forms in the non-compact case. The differential criteria given there permit one to test when the restriction Res(ω) of a holomorphic cusp form (as a differential form) is non-trivial. If the restriction does not vanish, the associated cohomology class is non-vanishing by results of Borel [Bo3].

3 . Computations in the case of the Classical Hermitian domains

In this section we consider the hermitian symmetric domains of type A,B C and D. The subsections on groups of type A will be denoted (3 .A . *) and so on . We refer to [Clo 2] for the explicit description of these domains .

In the following $M_{m\times n}(\mathbf{C})$ denotes the space of matrices with m-rows and n-columns and with complex entries. $E_{i,j} \in M_{m\times n}(\mathbf{C})$ is the matrix whose entry in the i-th row and j-th column is one and the other entries are zero. The group of non-singular $p\times p$ matrices (resp. of determinant one is denoted GL_p (*resp.* SL_p) . The group of $p \times p$ unitary matrices (resp. of determinant 1) is denoted U(p) (resp. SU(p)). Similarly the group of $p\times p$ real orthogonal matrices (resp. of determinant 1) is denoted O(p) (resp. SO(p)).

If E is a representation of a group then denote by E^* its contragredient . If $e_1,\ e_2,\ \ldots\ e_m$ is a basis of E then its dual basis in E^* is denoted $e_1^*,\ e_2^*,\ \ldots\ e_m^*$. The r-th exterior (resp. symmetric) power of E is denoted $\wedge^r(E)$ (resp. $sym^r(E)$).

We assume from now on, that the reductive group G is almost $\mathbf{Q}$-simple , i.e., has no connected non-central normal subgroups defined over $\mathbf{Q}$. It follows, as is well known , that all the simple factors of the complex Lie algebra g (mod centre) are isomorphic. We assume furthermore that the group $G(\mathbf{R})$ of real points is the product of a compact group and a real **almost simple** non-compact group ; denote the latter by G^{nc}.

(3.A.1).Notation : Let G^{nc} = U(p,q) where p and q are positive integers with $p \le q$. Thus the real rank of G is p. Now ,

$$(3.1)\qquad G^{nc} = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ;\ {}^t\overline{g}\begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} g = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \right\}$$

where $A \in M_{p\times p}(\mathbf{C})$, $B \in M_{p\times q}(\mathbf{C})$, $C \in M_{q\times p}(\mathbf{C})$, $D \in M_{q\times q}(\mathbf{C})$. Let

$$K = \left\{ g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in G^{nc} ;\ A \in U(p) ,\ D \in U(q) \right\}.$$

The complexification $K_{\mathbf{C}}$ of K is the group

$$K_{\mathbf{C}} = \left\{ g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in G(\mathbf{C}) ;\ A \in GL_p ,\ D \in GL_q \right\}.$$

The Cartan involution θ is given by $x \to -{}^t\overline{x}$. Let

$$C = \left\{ g \in K_{\mathbf{C}} \; ; \; g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\; with \;\; A, \; D \; scalar \; matrices \right\}$$

and

$$T = \left\{ g \in K_{\mathbf{C}} \; ; \; g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\; with \;\; A, \; D \; diagonal \; matrices \right\}.$$

Elements of T or of its Lie algebra are denoted $(x_1,...,x_p; \; y_1,...,y_q)$. Now the Lie algebra of $G(\mathbf{C})$ is obviously $M_{(p+q)\times(p+q)}$ and we will view its elements in block form as in (1). Let

$$p^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \;\; with \;\; B \in M_{p\times q} \right\},$$

and

$$p^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \;\; with \;\; C \in M_{q\times p} \right\}.$$

Let $\mathbf{C}^p$ (resp. $\mathbf{C}^q$) be the standard representation of U(p) (resp. of U(q)). Then, as a representation of $K_{\mathbf{C}}$, $p^+ = \mathbf{C}^p \otimes (\mathbf{C}^q)^*$.

Let $e_1, \; . \; . \; . \; e_p$ and $f_1, \; . \; . \; . \; f_q$ be the standard bases of $\mathbf{C}^p$ and $\mathbf{C}^q$ respectively. Fix the Borel-subalgebra b_0 of k to be the one which is upper triangular on $\mathbf{C}^p$ and lower triangular on $\mathbf{C}^q$ with respect to these bases . The roots of T occurring in p^+ are the linear forms $x_i - y_j$ with $1 \leq i \leq p$ and $1 \leq j \leq q$.

(3.A.2) Proposition: *Assume that G^{nc} =U(1,q) and that H is an arbitrary subgroup as in section (1.1). Let A_q , and u=u(q) be as in section (1.1) such that A_q has holomorphic cohomology in degree R with $dim(h \cap p^+) \geq R$ (= $dim(u \cap p^+)$). Then ,*

$$E(G, H, R) \supset V(q).$$

Proof: As a representation of $K_{\mathbf{C}}$,

$$\wedge^R p^+ \;=\; \wedge^R(\mathbf{C}^1 \otimes (\mathbf{C}^q)^*)$$

is irreducible. The proposition now follows from (2.2).

We now classify all the θ-stable parabolic subalgebras q of g which have holomorphic cohomology. As in (2.2) assume that q=q(x). Then , by (2.2) we have $u \cap p^- = 0$. Let x=(a_1,..., a_p ; b_1,..., b_q) be such that its eigenvalues on the Borel subalgebra b_0 are ≥ 0. Therefore

$$a_1 \geq \ldots \geq a_p \;\; and \;\; b_q \geq \ldots \geq b_1.$$

Now p^+ has the $E_{i,p+j}$ as a basis with $1 \leq i \leq p$ and $1 \leq j \leq q$. Moreover $u \cap p^+$ has the $E_{i,p+j}$ as a basis where i and j are such that $a_i - b_j$ is strictly positive. Let $r \leq p$ and $s \leq q$ be defined by the conditions

$$(*)_{r,s}: \quad a_1 \geq \ldots \geq a_r \; > \; a_{r+1} = \; \ldots = a_p =$$

$$= \; b_q = \ldots = b_{s+1} > b_s \geq \ldots \; \geq b_1.$$

Thus q=q(x) with x satisfying $(*)_{r,s}$ for some r and s exhaust the list of all the paraboloic subalgebras with holomorphic cohomology.

The roots of T occurring in $u \cap p^+$ are of the form $x_i - y_j$ with (a) $i \leq r$ and j arbitrary , or (b) $i \geq r+1$ *and* $j \leq s$. Let $\mu = 2\rho(u \cap p^+)$. Then,

$$\mu = \sum_{i=1}^{r}\sum_{j=1}^{q}(x_i - y_j) \; + \; \sum_{i=r+1}^{p}\sum_{j=1}^{s}(x_i - y_j).$$

Thus,

$$\mu = \; (q-s)(x_1 + \ldots + x_r) + s(x_1 + \ldots + x_p) - (p-r)(y_1 + \ldots + y_s) - r(y_1 + \ldots + y_q).$$

Consider the representation of $K_{\mathbf{C}}$

$$(3.2) \quad \otimes^s (\wedge^p \mathbf{C}^p) \otimes Sym^{q-s}(\wedge^r \mathbf{C}^p) \otimes Sym^{p-r}(\wedge^s \mathbf{C}^q)^* \otimes^r (\wedge^q (\mathbf{C}^q)^*) \;.$$

We note that μ is the weight associated to the vector

$$e = \; \otimes^s(e_1 \wedge \ldots \wedge e_p) \otimes^{q-s} (e_1 \wedge \ldots \wedge e_r) \otimes^{p-r} (f_1^* \wedge \ldots \wedge f_s^*) \otimes^r (f_1^* \wedge \ldots \wedge f_q^*).$$

Also note that e is obviously a highest weight vector in this representation space , therefore under $K_{\mathbf{C}}$ it generates an irreducible submodule . This module is isomorphic to A_q .

For future reference we note that

$$dim(u \cap p^+) \ = \ rq + (s)(p-r) \ = \ R^+ \ = R.$$

Let $a \leq p$ and $b \leq q$. Let E^a be the $\mathbf{C}$-span of $e_1, \ldots, e_a$ and let F^b be the $\mathbf{C}$-span of $f_1, \ldots, f_b$. Then the restriction of the hermitian form

$$\begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$$

to the subspace $E^a \oplus F^b$ of $\mathbf{C}^p \oplus \mathbf{C}^q$ is non-degenerate . Thus we get an embedding of $U(a,b)$ *in* $U(p,q)$.

From now on , we assume (as we may , thanks to Proposition (3.A.2)) that $q \geq p \geq 2$.

(3.A.4) Proposition: *Let $G^{nc} = U(p,q)$ with p, $q \geq 2$. Let H^{nc}=$U(a,b)$ with $a < p$ and $b < q$ embedded in U(p,q) as above . Let q be a proper θ-stable parabolic subalgebra which contributes to holomorphic cohomology in degree R . Then*

$$E(G,H,R) \cap V(q) = 0.$$

Proof: Let $\xi : \mathbf{G}_m \to T$ be defined, for $t \in \mathbf{G}_m$, by

$$\xi(t) = (1, \ldots, t^{-1})$$

Then the image of ξ centralises H. To prove the proposition we will use the criterion of Proposition (2.6). We therefore compute the weights of $\xi(t)$ on the tensor space (3.2) above . We see that the weights of $\xi(t)$ are of the form t^N where $N \ = \ (p-r) \times A \ + \ r$, with $0 \leq A$. Therefore N is strictly positive unless A is 0 and r is zero. If r is not zero, then by Proposition (2.6) $E(G,H,R) \cap V(q) = 0$.

Similarly, by looking at the embedding

$$\xi'(t) = (1, \ldots, 1, t; 1, \ldots, 1)$$

(where t occurs as the p-th coordinate) whose image also centralises H , we see that, unless s is zero , $E(G,H,R) \cap V(q) = 0$. This completes the proof.

(3.A.5) Proposition: *Let $G^{nc} = U(p,q)$ with $p,\ q \geq 2$. Let $H^{nc}=U(p,b)$ with $b < q$ be embedded in U(p,q) as above . Let $q = q(x) = q(r,s)$ be a proper θ- stable parabolic subalgebra where x satisfies $(*)_{r,s}$ with $dimu^{+} = R \leq dimp_H^{+}$. Then*

$$E(G,H,R) \supset V(q) \ \ if\ and\ only\ if \ \ r = 0.$$

Proof: Consider again, the map ξ defined in the proof of Proposition (3.A.3). Its image still centralises our new H^{nc}. As in the proof of (3.A.3) we see that the eigenvalues of $\xi(t)$ are of the form t^N where $N = (p-r)A + r$ with $0 \leq A$.

If $r \neq 0$ then N is strictly positive and by Proposition (2.6), $E(G,H,R) \cap V(q) = 0$.

If r=0, then $R^{+} = sp \leq dim(p^{+} \cap h) = pb$. and $V = V(q)$ is the $K_{\mathbf{C}}$-stable subspace generated by the vector

$$e_q = \wedge_{i,j}\ e_i \otimes f_j^{*} \ \ (i \leq j\ ,\ \ s+1 \leq j \leq q)\ .$$

Now $p_H^{+} = \mathbf{C}^p \otimes (F^b)^{*}$ is the span of the vectors

$$e_i \otimes f_j\ ,\ \ (1 \leq i \leq p,\ \ 1 \leq j \leq b) \ \ \ (b \geq s)$$

and therefore , $\wedge^{ps} p_H^{+}$ contains the wedge of the vectors

$$e_i \otimes f_j\ ,\ \ (1 \leq i \leq p,\ \ 1 \leq j \leq s)$$

which is precisely e_q .

(3.A.6) Proposition : *Let $H^{nc} = U(a,q)$ be embedded in $G^{nc} = U(p,q)$ as before. Let q=q(x) be a θ-stable parabolic subalgebra associated to x satisfying $(*)_{r,s}$ and $R^{+} = R \leq dimp_H^{+}$. Then*

$$E(G,H,R) \supset V(q)\ if\ and\ only\ if\ s = 0 \ \ .$$

Proof: The proof is entirely similar to that of (3.A.3) and will be omitted.

Notation: Let $p \leq q$, and $G^{nc} = U(p,q)$. Let H^{nc} be the subgroup of G^{nc} which fixes pointwise the C-span of $f_{p+1}, \ldots, f_q$ and leaves stable the C-span of e_i *and* f_i for each i with $1 \leq i \leq p$. Thus $H^{nc} = U(1,1)^p$. We recall that the real rank of G^{nc} is p .

(3.A.7) Proposition : *Let* $\mathbf{q}$ *be a θ-stable parabolic subalgebra of the Lie algebra* $\mathbf{g}$ *such that* $dim(u \cap p^+) = p$. *Let* $U(1,1)^p$ *be embedded in U(p,q) as above . Then ,*

$$E(G,H,p) \supset V(q).$$

Proof : In the notation of (3.A.3) $\mathbf{q}$ =q(p,0) or q(0,p) (if p=q) . For definiteness , assume that q=q(p,0) . Then $u \cap p^+$ is the span of the vectors $e_1 \otimes f_1^*,\ e_2 \otimes f_1^*, \ldots, e_p \otimes f_1^*$, and V($\mathbf{q}$) is the K-span of the vector

$$(3.3) \quad e_{\mathrm{q}} = e_1 \otimes f_1^* \wedge e_2 \otimes f_1^* \wedge \ldots \wedge e_p \otimes f_1^* .$$

Now , E(G,H,p) is the K-span of the vector

$$(3.4) \quad e_1 \otimes f_1^* \wedge e_2 \otimes f_2^* \wedge \ldots \wedge e_p \otimes f_p^*$$

(the line through (**) is $\wedge^p p_H^+$) .

Let $t_2, t_3, \ldots, t_p$ be variables . Now $K_{\mathbf{C}} = GL_p \times GL_q$ and there exists an element $g \in 1 \times GL_q$ which sends the basis $f_2^*, \ldots, f_p^*$ into the vectors $f_2^* + t_2 f_1^*, \ldots, f_p^* + t_p f_1^*$. The g-translate of (3.4) is a polynomial in $t_2, t_3, \ldots, t_p$ with values in E(G,H,p) and the coefficient of the monomial $t_2 \times \ldots \times t_p$ is precisely the vector (3.3) ; this shows that $V(\mathbf{q}) \subset E(G,H,p)$.

More generally, we may split the hermitian space $\mathbf{C}^p \oplus (\mathbf{C}^q)^*$ into a direct sum of hermitian subspaces and consider the restriction of the holomorphic cohomology .

Notation : G = U(p,q) preserves the standard hermitian form

$$h(x,y) = \sum |x_\mu|^2 - \sum |y_\nu|^2$$

on the direct sum $\mathbf{C}^p \oplus (\mathbf{C}^q)^*$. Write $\mathbf{C}^p = \oplus_{i=1}^m E_i$ and $\mathbf{C}^q = \oplus_{i=1}^m F_i$. Let $p_i = dim(E_i) \geq 1$ and $q_i = dim(F_i) \geq 1$. Write $P_i = \sum_{j \leq i} p_j$ and

$Q_i = \sum_{j \leq i} q_j$. Assume that E_i (resp. F_i) is the span of the vectors e_μ with $P_{i-1} + 1 \leq \mu \leq P_i$ (resp. f_ν^* with $Q_{i-1} + 1 \leq \nu \leq Q_i$) .

Let H be the subgroup of G which leaves stable the subspaces $E_i \oplus F_i$ for each i. We therefore have $H = \prod_i U(p_i, q_i)$, $\sum p_i = p$ and $\sum p_i = p$.

(3.A.8) Proposition : *Let $H = \prod_i U(p_i, q_i)$ be embedded in $G = U(p,q)$ as in the preceding paragraph. Let q=q(r,s) be a θ-stable parabolic subalgebra of the Lie-algebra g , which contributes to holomorphic cohomology . Then a cuspidal holomorphic form on $\Gamma \backslash X$ of type A_{q} is stably non-zero along H if and only if either r=0 and $s \leq q_i$ for each i or s=0 and $r \leq p_i$ for each i .*

Proof : (1) We first show that if r=0 and $s \leq q_i$ for all i , then $A(G, H, R) \supset V(\mathbf{q})$ where R is the dimension of $u \cap p^+$. Observe that R=ps . Let P_i and Q_i be as in (3.A.7) . Then , we have $Q_{i-1} + s \leq Q_{i-1} + q_i = Q_i$. Denote by h_i the Lie algebra of $H_i = U(p_i, q_i) \subset U(p,q)$ and q_i the θ-stable parabolic subalgebra of h_i which contributes to holomorphic cohomology in degree $p_i s$. Let $u_i^+ = q_i \cap p_{H_i}^+$; it is the span of the vectors $e_\mu \otimes f_\nu^*$ with $P_{i-1} < \mu \leq P_i$ and $Q_{i-1} < \mu \leq Q_{i-1} + s \leq Q_i$. Now $\wedge^{p_i s}(u_i^+)$ is the line through the vector $\xi_i = \wedge(e_\mu \otimes f_\nu^*)$ where $P_{i-1} < \mu \leq P_i$ and $Q_{i-1} < \nu \leq Q_{i-1} + s \leq Q_i$.

Let $(t_\nu \ ; \ 1 \leq i \leq m \ , \ Q_{i-1} \leq \nu \leq Q_{i-1} + s)$ be variables and let $g \in 1 \times GL_q \subset K_{\mathbf{C}}$ be the element which takes the basis $(q_\nu^* \ ; \ 1 \leq i \leq m \ , \ Q_{i-1} \leq \nu \leq Q_{i-1} + s)$ of $(\mathbf{C}^q)^*$ into the vectors $(q_\nu^* + t_\nu q_{\nu - Q_{i-1}^*} ; 1 \leq i \leq m \ , \ Q_{i-1} \leq \nu \leq Q_{i-1} + s)$. Then the vector $\wedge_{i=1}^m \xi_i$ changes into the A(G,H,R)-valued polynomial

$$\wedge_{i=1}^m \wedge_{\mu,\nu} (e_\mu \otimes f_\nu^* \ + \ t_\nu f_{\nu - Q_i}^*)$$

(where $P_{i-1} < \mu \leq P_i$ and $Q_{i-1} < \nu \leq Q_{i-1} + s \leq Q_i$). The coefficient of the monomial $\prod_{1 \leq i \leq m} \prod_{Q_{i-1} < \nu \leq Q_{i-1}+s} (t_\nu)$ is the vector $\wedge_{i=1}^p \wedge_{\mu,\nu} (e_\mu \otimes f_\nu^*)$ with $P_{i-1} < \mu \leq P_i$ *and* $1 \leq \nu \leq s$. This vector is precisely e_q . Therefore , $e_q \in A(G, H, R)$ and $V(q) \subset A(G, H, R)$.

(2) The case of $s = 0$ and $r \leq p_i$ for each i can be handled similarly.

(3) We will prove that if $A(G, H, R) \supset V(q)$ then

$$(3.5) \qquad rs = 0 \ , \ r \leq p_i \ , \ s \leq q_i \quad \forall i.$$

Observe that in the product $H = U(p_1, q_1) \times U(p_2, q_2) \times \ldots \times U(p_m, q_m)$

the factors may be switched in any order by conjugating by an element k of K (in fact k may be chosen to be a permutation matrix in $K = GL_p \times GL_q$) . The new group H' then has the property that A(G,H',R) = A(G,H,R) . To prove (3.5) , it is therefore enough to show that $rs = 0,\ r \leq p_1,\ s \leq q_1$. By replacing H by the larger group $U(p_1, q_1) \times U(p - p_1, q - q_1)$, we may assume -while proving (3.5)- that m=2 . We will then show that $rs = 0,\ r \leq p_1,\ r \leq p_2\ and\ s \leq q_1,\ s \leq q_2$.

Let $H = U(p_1, q_1) \times U(p_2, q_2)$ and let q_H be a θ-stable parabolic subalgebra of h which contributes to holmorphic cohomology in degree R . We write , in the notation preceding the Proposition , $\mathbf{C}^p = E_1 \oplus E_2$ and $(\mathbf{C}^q)^* = F_1 \oplus F_2$. Let w be the permutation matrix in GL_p such that it takes the basis

$$e_1, \ldots e_{p_1} ;\ e_{p_1+1}, \ \ldots e_p$$

into the elements

$$e_{p_2+1}, \ldots e_p ;\ e_1, \ \ldots e_{p_2}.$$

The conjugate of H by w is $H' = U(p_2, q_2) \times U(p_1, q_1)$ and the conjugate $q_{H'}$ of q_H is also a θ-stable parabolic subalgebra of the Lie algebra h' of H'. In the notation of (A.2) , suppose $q_H = q(r_1, s_1) \oplus q(r_2, s_2)$. Then $q_{H'} = q(r_2, s_2) \oplus q(r_1, s_1)$.

Let $\pi_q : \wedge^R p^+ \to V(q)$ denote the K-equivariant projection map . Then our assumption ensures that $A(G, H, R) \supset V(q)$ ensures that there exists q_H as in the preceding paragraph such that $v_H = \pi_q(e(q_H)) \neq 0$. As π_q is K-equivariant , we also have $v_{H'} = \pi_q(e(q_{H'})) \neq 0$. Since K acts irreducibly on V(q) , e(q) is (upto scalar multiples) the unique vector in V(q) which is invariant under the nilradical n of b (the Borel-subalgebra of k) . Let u(n) be the universal enveloping algebra of n .

Denote by n_H and $n_{H'}$ the intersections of n with h and h' respectively . Let m be the subalgebra of n which is the span of the vectors $E_{a,b}$ with $1 \leq a \leq p_1$ and $1 + p_1 \leq b \leq p$ and the vectors $E_{c+p,d+p}$ with $1 \leq c \leq q_1$ and $1 + q_1 \leq d \leq q$. Similarly let m' be the subalgebra of n which is the span of the vectors $E_{a,b}$ with $1 \leq a \leq p_2$ and $1 + p_2 \leq b \leq p$ and the vectors $E_{c+p,d+p}$ with $1 \leq c \leq q_2$ and $1 + q_2 \leq d \leq q$. We have $n = m \oplus n_H$ and $n = m' \oplus n_{H'}$. By the Poincare-Birkhoff-Witt Theorem we get

$$(3.6) \qquad u(n) = u(m) \otimes u(n_H) \quad and \quad u(n) = u(m') \otimes u(n_{H'}).$$

There exist elements α and β in $u(n)$ such that

$$(3.7) \qquad \alpha(v_H) = e(q)$$

and

$$(3.8) \qquad \beta(v_{H'}) = e(q).$$

As e(q), v_H and $v_{H'}$ are all eigenvectors for the action of T , we may assume that so are α and β . Furthermore , as v_H and $v_{H'}$ are annihilated by n_H and $n_{H'}$ respectively , we may assume from (3.6) that $\alpha \in u(m)$ and $\beta \in u(m')$.

We now compare the T-weights of both sides of the equation (3.7). Recall from (A.2) that the weight of e(q) (q=q(r,s)) is

$$q(x_1 + .. + x_r) + s(x_{r+1} + .. + x_p) - p(y_1 + .. + y_s) - r(y_{s+1} + .. + y_q).$$

The weight of α is of the form

$$\sum_{a \leq p_1 < b} m_{ab}(x_a - x_b) + \sum_{c \leq q_1 < d} n_{cd}(y_d - y_c)$$

where m_{ab}, n_{cd} are non-negative integers.

The weight of $e(q_H)$ is (since $q_H = q(r_1, s_1) \oplus q(r_2, s_2)$) the sum of

$$q_1(x_1 + .. + x_{r_1}) + s_1(x_{r_1+1} + .. + x_{p_1})$$
$$-p_1(y_1 + .. + y_{s_1}) - r_1(y_{s_1+1} + .. + y_{q_1})$$

and

$$q_2(x_{p_1+1} + .. + x_{p_1+r_2}) + s_2(x_{p_1+r_2+1} + .. + x_p)$$

$$-p_2(y_{q_1+1} + .. + y_{q_1+s_2}) - r_2(y_{q_1+s_2+1} + .. + y_q).$$

(Observe that if r=p then $dim(u \cap p^+(= rq+s(p-r)) = pq$, and therefore s can be assumed to be arbitrary .In the following, we use, in order that all the statements make uniform sense also for the case r=p , the convention that if r=p then s=q). Comparing the coefficients of x_p on both sides of (3.7), we obtain $s = s_2 - \sum_{a \leq p_1} m_{ap}$, which shows that $s \leq s_2 \leq q_2$. By symmetry , from (3.8)) we get $s \leq s_1 \leq q_1$.

Comparing the coefficients of y_q on both sides of (3.7), we obtain

$$-r = -r_2 + \sum_{c \leq q_1} n_{cq},$$

which shows that $r \leq r_2 \leq p_2$. By symmetry , from (3.8) we get $r \leq r_1 \leq p_1$.

We now need only show that rs = 0 . We divide the proof into several cases . Observe that $R = qr + (p-r)s = R_1 + R_2$.

Case 1. $r = p_1 = p_2 (< p)$. Then $s_1 = q_1$ *and* $s_2 = q_2$. The the formula for R shows that $R = qr + (p-r)s = p_1 q_1 + p_2 q_2 = rq_1 + rq_2 = rq$. Therefore (p-r)s=0 , and s=0.

Case 2. $r < p_1$,$r = p_2$. Then comparing x_{p_1}-coefficients in (3.7) we get $s_1 \leq s$, and so, $s = s_1$. Then $R = rq + (p-r)s = r_1 q_1 + (p_1 - r_1)s + rq_2$, and $rq_1 + p_1 s - (p_1 - r_1)s = r_1 q_1 = rq_1 + r_1 s$. If $s < q_1$, we similarly get , $r = r_1$ and the last equation in the previous sentence shows that $rs = 0$. If $s = q_1$, then the same equation shows that r = 0 .

Case 3. $r = p_1 < p_2$. The proof is similar to that in Case 2.

Case 4. $r < p_1$ *and* $r < p_2$. Replacing r by s , and eliminating the above 3 cases for s , we may assume that $s < q_1$ *and* $s < q_2$. Comparing x_{p_1}- coefficients in (I) , we get $s = s_1$ and similarly $s = s_2$. Similarly, we get $r = r_1 = r_2$. Then , we have the equation $R = rq + (p-r)s = rq_1 + (p_1 - r)s + rq_2 + (p_2 - r)s$ i.e. rs=0 .

The proposition is now proved in all cases.

Notation: We now assume that G^{nc} is U(p,p) and that H^{nc} is the subgroup GSp_p given by

$$\left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(p,p) \ : \ {}^t g \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix} \right\}$$

Then $K \cap H^{nc} = U(p)$ is embedded in K by the map

$$g \to \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}$$

and the action of U(p) on $p^+ = \mathbf{C}^p \otimes (\mathbf{C}^p)^*$ is isomorphic to the representation $\rho \otimes \rho$ where ρ is the standard representation of U(p) on $\mathbf{C}^p$. Under the embedding of H^{nc} in G^{nc}, $p^+ \cap h$ gets identified with

$$Sym^2(\rho) \ = \ Sym^2(\mathbf{C}^p) \subset \mathbf{C}^p \otimes \mathbf{C}^p.$$

(3.A.9) Proposition : *$G^{nc} = U(p,p)$. Let $H^{nc} = GSp_p$ is embedded in U(p,p) as above. Let q be a θ-stable parabolic subalgebra of g such that it contributes to holomorphic cohomology in degree R . Then*

$$A(G,H,R) \supset V(q)$$

if and only if R=p or R=2p-1 .

Proof : We observe that in the notation of (A.7) , we have

$$H'^{nc} = \ SU(1,1)^p \subset H^{nc} = \ Sp_p \subset G^{nc} = SU(p,p) \ .$$

We have seen, from (3.A.7) , that $A(G,H',p) \supset V(q)$. Therefore (3.A.9) follows when R=p.

Suppose now, that R=2p-1 . Then q=q(r,s) with r=s=1. Moreover, $q_H = q_H(k)$ with k=2 . Now, $\wedge^R(u \cap p^+)$ is the line generated by $e(q) = \wedge_{j=1}^p(e_1 \otimes f_j^*) \ \wedge_{k=2}^p(e_k \otimes f_1^*)$ and $\wedge^R(u(q_H \cap p_H^+))$ is the line generated by the vector

$$e(q_H) = \ \wedge_{j=1}^p(e_1 f_j^* + e_j f_1^*) \ \wedge_{k=1}^p (e_2 f_j^* + e_j f_2^*) \ .$$

Let $t_1, \ldots t_p$ be variables and $g \in GL_p \times 1_p \subset K$ be the element which sends the basis $e_1, e_2, \ldots, e_p$ into the vectors $t_1e_1, t_2e_2, \ldots, t_pe_p$. The g-translate of the vector $e(q_H)$ is a polynomial P in the $t_i's$ with values in

$A(G, H, 2p-1)$ whence the coefficient of the monomial $t_1^p t_2 \ldots t_p$ of P is in $A(G, H, 2p-1)$. This coefficient is precisely

$$(3.9) \quad \wedge_{j=1}^{p} (e_1 \otimes f_j^*) \ \wedge_{k=2}^{p} (e_k \otimes f_2^*).$$

Let $h \in 1_p \times GL_p \subset K$ be the element which takes the basis $f_1^*, f_2^*, \ldots, f_p^*$ into the vectors $f_2^*, f_1^*, \ldots, f_p^*$. The h-translate of the vector (3.9) is nothing but -e(q) , and so $e(q) \in A(G, H, 2p-1)$ and $V(q) \subset A(G, H, 2p-1)$.

We will now prove that if $V(q) \subset A(G, H, R)$, then $r \leq 1$, $s \leq 1$. This shows that $R = pr + (p-r)s = p$ *or* $2p-1$.

Let $\mathbf{n}$ be the nil-radical of $\mathbf{b}$ -the Borel-subalgebra as in (A.2). Let $n_H = n \cap k_H$, $n_1 = (gl_p \times 0) \cap n$ and $n_2 = (0 \times gl_p) \cap n$. If l is a Lie-algebra then let u(l) be its universal enveloping algebra. Now both n_1 and n_2 are ideals in $\mathbf{n}$ and $n_1 \oplus n_H = n_2 \oplus n_H = n_H$. Therefore, by the Poincare-Birkhoff-Witt theorem , we have

$$(3.10) \quad u(n) = u(n_1) \otimes u(n_H) = u(n_2) \otimes u(n_H) \ .$$

Let $\pi_q : \wedge^R p^+ \to V(q)$ denote the K-equivariant projection map. By assumption , $A(G, H, R) \supset V(q)$. Therefore there exists a θ-stable parabolic subalgebra q_H which gives holomorphic cohomology in degree R such that $v_H = \pi_q(e(q_H)) \neq 0$. As V(q) is irreducible (and e(q) is the unique vector in V(q) which is invariant under n) , there exists an element $\alpha \in u(n)$ such that $e(q) = \alpha(v_H)$.

Now , elements of n_H kill any vector of type $e(q_H)$ and hence kill v_H . We assume as we may by (3.10) , that

$$(3.11) \quad e(q) = \alpha(v_H) = \beta(v_H) = \delta(v_H)$$

with $\beta \in u(n_1)$ and $\delta \in u(n_2)$. The T-weight of e(q) is

$$\mu = q(x_1 + \ldots + x_r) + s(x_{r+1} + \ldots + x_p) - p(y_1 + \ldots + y_s) - r(y_{s+1} + \ldots + y_q) \ .$$

Suppose that $r \geq 2$. Let $g \in T$ be of the form $g = (t, 1, \ldots, 1)$. The g-translate of both sides of the equation $e(q) = \delta(v_H)$ are polynomials in t and $g.e(q) = t^p e(q)$. Now, g commutes with δ. Moreover ,

$$v_H = \pi_q(\wedge_{j=1}^{p}(e_1 f_j^* + e_j f_1^*) \wedge_{j=2}^{p} (e_2 f_j^* + e_j f_2^*) \wedge \ldots \wedge_{j=k}^{p} (e_k f_j^* + e_j f_k^*)) \quad .$$

The t^p-th coefficient of $g.v_H$ is precisely

$$v_H' = \pi_q(\wedge_{j=1}^{p}(e_1 f_j^*) \wedge_{j=2}^{p} (e_2 f_j^* + e_j f_2^*) \wedge \ldots \wedge_{j=k}^{p} (e_k f_j^* + e_j f_k^*)) \quad .$$

Therefore $e(q) = \delta(v_H')$.

Now let $g' \in T$ be of the form $g' = (1, t, 1, \ldots, 1)$ The g'-translates of both sides of the equation $e(q) = \delta(v_H')$ are polynomials in t and $g'.e(q) = t^p e(q)$, because $r \geq 2$. Now, g' commutes with δ. Moreover ,

$$g'(v_H') = \pi_q(\wedge_{j=1}^{p}(e_1 f_j^*) \wedge_{j=2}^{p} (te_2 f_j^* + e_j f_2^*) \wedge \ldots \wedge_{j=k}^{p} (e_k f_j^* + e_j f_k^*)) \quad .$$

which is a polynomial in t of degree $\leq p-1$, which contradicts the equation

$$t^p e(q) = \delta(g'(v_H')) \quad .$$

Therefore, the assumption that $r \geq 2$ is false and $r \leq 1$. Similarly $s \leq 1$. The proof is complete.

We will now fix our attention on $G^{nc\cdot} = U(p,q)$, $p \leq q$, but, will consider θ-stable *proper* parabolic subalgebras q=q(x) such that $u \cap p^-$ is not necessarily zero. That is, we assume as before, that $a_1 \geq \ldots \geq a_p$ and $b_1 \geq, \ldots \geq b_q$ but not necessarily, that $a_i - b_j \geq 0$. Suppose $\pi = A_q$ is such that $H^p(g, K_\infty, \pi) \neq 0$. Note that p is the real rank of G.

(3.A.10)Proposition : *With the notation above we have*

$$H^p(g, K_\infty, \pi) = H^{p,0}(g, K_\infty, \pi) \oplus H^{0,p}(g, K_\infty, \pi)$$

unless $p = q = 2$.

proof : We write the Hodge decomposition for the cohomology of π (we refer to [VZ] , section (6) for the necessary facts) :

$$H^p(g, K_\infty, \pi) = \oplus H^{R^+ + l, R^- + l}(g, K_\infty, \pi)$$

where the sum is over all $l \geq 0$ with

$$(3.12) \qquad R^+ + l + R^- + l = p$$

and $R^+ = dim(u \cap p^+)$, $R^- = dim(u \cap p^-)$. We may assume that $p \geq 2$. Since $l \geq 0$ this means that $R^+ + R^- \leq p$. Define the integers r, $r_2 \leq p$, $s \leq q$ by the inequalities

$$a_1 = \ldots = a_r = b_1 = \ldots b_s >$$

$$> a_{r+1} \geq \ldots \geq a_{r_2} \geq b_{s+1} \geq \ldots \geq b_q.$$

Now, either r or s is ≥ 1.

We may assume that either $s \neq q$ or $r \neq p$.

The roots of t lying in $u \cap p^+$ *include* the roots

$$x_1 - y_{s+1}, \ \ldots \ x_1 - y_q$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_r - y_{s+1}, \ \ldots \ x_r - y_q$$

whence

$$(3.13) \qquad R^+ \geq (q-s)r \ .$$

If s=0 then $r \geq 1$ and since $q \geq p$, (3.12) and (3.13) show that $R^- = 0 = l$ and therefore, in the Hodge decomposition for π above , only the (p,0) term survives and the proposition follows. We may thus assume that $s \geq 1$.

The roots of t occurring in $u \cap p^-$ *include* the roots

$$y_1 - x_{r+1}, \ \ldots \ y_1 - x_p$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_s - x_{r+1}, \ \ldots \ y_s - x_p$$

whence

$$(3.14) \qquad R^- \geq s(p-r) \geq p - r \ .$$

If r=0 then (3.14) and (3.12) show that $R^- = p$ and $l = 0 = R^+$ and so, in the Hodge decomposition for π above, only the (0,p) term survives and the proposition follows. We may thus assume that $r \geq 1$.

If s=q then $r \neq p$ as we saw before and (3.14) shows that $R^- \geq q$ and by (3.12) it follows that $l = 0 = R^+$ and again the proposition follows. So we may assume that $s \neq q$. Similarly we may assume that $r \neq p$. Now (3.13) , (3.14) and (3.12) show that

$$p \geq R^+ + R^- \geq (q-s)r + s(p-r) ,$$

i.e.

$$(3.15) \quad 0 \geq (q-s-1)r + (s-1)(p-r).$$

Since we may assume that $1 \leq s \leq q-1$ and $1 \leq r \leq p-1$, (3.15) shows that s=q-1 and s=1 , i.e., q=2 ;and by assumption $p \geq 2$. But $p \leq q$ so it follows that $p = q = 2$; the proposition is completely proved.

(3.BD.1) Notation : We now assume that G^{nc} is locally isomorphic to O(2,p). Let $K_\infty = O(2) \times O(p)$ and $K_{\mathbf{C}}$ its complexification. The natural representation $\mathbf{C}^2$ of the group $\mathbf{G}_m = SO(2, \mathbf{C})$ is a sum of two lines $\mathbf{C}^+$ and $\mathbf{C}^-$ on which $t \in \mathbf{G}_m$ acts by t and t^{-1} respectively. Let $\mathbf{C}^p$ be the natural representation of SO(p). We assume that H in G satisfies (2.1) and is so that $H^{nc} = SO(2,k) \subset SO(2,p)$ is the subgroup which leaves the vectors f_{k+1} , . . . , f_p fixed.

(3.BD.2) Proposition : *Let G be an algebraic* $\mathbf{Q}$ *group such that modulo centre,* G^{nc} *is locally isomorphic to SO(2,p) and let H be an algebraic* $\mathbf{Q}$*-subgroup of G so that* $H^{nc} = SO(2,k)$ *is embedded in SO(2,p) as above. Let* Γ *be an arithmetic subgroup of G. If* $m \leq k (= dim(h \cap p^+))$ *then every cuspidal holomorphic m-form on the Shimura variety* $S = S(\Gamma)$ *is stably non-zero along H.*

Proof : We will show that if $m \leq k$, then

$$(3.16) \quad E(G,H,m) = \wedge^m p^+.$$

Then the proposition follows from Proposition (2.2) . Now,

$$h \cap p^+ = \mathbf{C}^+ \otimes (\mathbf{C}f_1 \oplus . . \mathbf{C}f_k).$$

Therefore, E(G,H,m) is the $K_{\mathbf{C}}$-span of $f_{i_1} \wedge \ldots \wedge f_{i_m}$ (with $i_\mu \leq k$) and $\wedge^m p^+$ is the $K_{\mathbf{C}}$-span of $f_{j_1} \wedge \ldots \wedge f_{j_m}$ (with $j_\nu \leq p$). Obviously there is a matrix in SO(p) (in fact, a *permutation* matrix) , which takes $f_1 , \ldots , f_m$ into $f_{j_1} , \ldots , f_{j_m}$. This proves (3.16).

(3.BD.3) Proposition : *Let* $m \geq 5$, $G^{nc} = SO(2,m)$ *and* q *a* θ*-stable parabolic subalgebra of* g *such that* $H^2(g, K, A_{\mathbf{q}}) \neq 0$. *Then*

$$H^2(g, K, A_{\mathbf{q}}) \;=\; H^{1,1}(g, K, A_{\mathbf{q}}) \;\; .$$

Proof : By [VZ] , section 6, the Hodge types of a cohomological representation are of the form $(R^+ + p, R^- + p)$. We will refer to the (R^+, R^-) Hodge-component as the primitive cohomology. If $R^+ + R^- + 2p = 2$, then, we have the following three cases .

(i) the primitive cohomology in degree 0, and q=g ,

(ii)the primitive cohomology is in degree 2 of type (1,1) ,

(iii) the primitive cohomology is of type (2,0) or (0,2) . The dimensions of the holomorphic cohomology are listed in [Clo 2], and are of the form (if m is even) $l-1$ or $l+k-1$ $(k \geq 0)$ where $l \geq 4$ is the absolute rank ; hence the primitive cohomology can never be of type (2,0) or (0,2) . the case when m is odd can be similarly handled . This completes the proof.

(3.C.1) Notation: Let G be a $\mathbf{Q}$-group such that $G^{nc} = GSp_g$. We have seen in section (3.A.4) that $p^+ = Sym^2(\mathbf{C}^g)$. Similarly , it can be shown that $p^- = Sym^2((\mathbf{C}^g)^*)$. Let T denote the subgroup of diagonal elements of G .

Let q=q(x) be a θ-stable parabolic subalgebra with holomorphic cohomology. We may then assume that the diagonal matrix x is of the form x= $(a_1, \ldots , a_g, -a_1, \ldots, -a_g)$ with $a_1 \geq \ldots \geq a_g$ and $a_i + a_j \geq 0$. Then

$$u \cap p^+ \;=\; \oplus \mathbf{C}(e_i \otimes f_j^* \;+\; e_j \otimes f_i^* \;)$$

where the sum is over all i and j such that $x_i + x_j$ is *strictly* positive. Let k be defined by the inequalities

$$a_1 \geq \ldots \geq a_k > 0 = \; a_{k+1} = \; \ldots = a_g.$$

Then, the roots of t occurring in $u \cap p^+$ are

$$2x_1 \; , \; x_1 \; + \; x_2 \; , \; \ldots x_1 + x_g,$$

$$2x_2\ ,\ ..\ x_2+x_g,$$
$$2x_k,\ ..\ x_k+x_g.$$

Moreover,

$$R=R^+=dim(u\cap p^+)=\ k(k+1)/2+\ k(g-k),$$

$$\mu(u\cap p^+)=\ (g+1)(x_1+x_2+..\ +x_k)\ +\ k(x_{k+1}+\ ..+x_g)$$

$$=\ k(x_1+\ .\ +x_g)+\ (g-k+1)(x_1+\ ..\ +x_k),$$

and the $K_{\mathbf{C}}$-span of $\wedge^R(u\cap p^+)$ in $\wedge^R p^+$ is (irreducible and) isomorphic to a subrepresentation of

$$V_{k,g}=\ (\wedge^g\mathbf{C}^g)^k\otimes Sym^{g-k+1}(\wedge^k\mathbf{C}^g).$$

Consider a $\mathbf{Q}$-subgroup H of G such that H^{nc} is the subgroup of G^{nc} which takes the span of $e_1,\ ..\ ,e_h\ ;\ f_1,\ ..\ ,f_h$ into itself and acts trivially on the basis elements $e_{h+1},\ ..\ ,e_g\ ;\ f_{h+1},\ ..\ ,f_g$. Here , $1\leq h\leq g-1$. Let $M=\ \mathbf{G}_m\subset T_{\mathbf{C}}$ be the subgroup which acts trivially on the basis elements $e_1,\ ..\ ,e_{g-1}\ ;\ f_1,\ ..\ ,f_{g-1}$, and acts by t (resp. t^{-1}) on e_g (resp. f_g) for all $t\in\mathbf{G}_m$.

(3.C.2) Proposition : *Let H be a $\mathbf{Q}$-subgroup of G as above so that $H^{nc}=GSp_h$ with $1\leq h\leq g-1$. Then, every cuspidal holomorphic R-form on the Shimura variety S vanishes along H.*

Proof : We assume , as we may , that the holomorphic R-form ω is of type A_q with q as in (3.A.1) and that $1\leq R\leq dim(p^+\cap h)$. We use the criterion of Proposition (2.6) , with M as defined above. The weights of M occurring in $V_{k,g}$ are , by inspection , of the form $t^k\times t^B=t^{k+B}$, , $k+B\geq 1$, whereas M centralises (all of H , and in particular) $p^+\cap h$. Therefore the proposition follows by Proposition (2.6).

(3.C.3) Notation : Let V_i be the span of e_i and f_i for each i. Let Ω be the symplectic form on the sum of all the V_i's which is preserved by G. Its restriction Ω_i to V_i is non-degenerate . Thus we get an embedding of $H^{nc}=\ Sp(\Omega_1)\times..\ \times Sp(\Omega_g)\subset G^{nc}=GSP_g$.

(3.C.4) Proposition : *Let G and H be $\mathbf{Q}$-groups so that H^{nc} is imbedded in $G^{nc} = GSp_g$ as in (3.C.3). Then , a cuspidal holomorphic g-form on the Shimura variety S (associated to a congruence subgroup of G) is stably non-zero along H .*

Proof : Suppose first that R=g. Assume , as one may, that the holomorphic g-form is of type A_q associated to a parabolic q, with holomorphic cohomology. Then (3.C.1) shows that g is of the form k(k+1)/2 + k(g-k) , with $1 \leq k \leq g$. Solving this , we see that k=1 .

Now $\mu(u \cap p^+)$ is the $K_{\mathbf{C}}$-span of the vector $e_1^2 \wedge e_1e_2 \wedge \dots \wedge e_1e_g$ in $\wedge^g(sym^2(\mathbf{C}^g))$. To prove the proposition , it is enough to show , by Corollary (2.4) , that $e_1^2 \wedge e_1e_2 \wedge \dots \wedge e_1e_g$ belongs to E(G,H,g).

Now , the element $e_1^2 \wedge e_2^2 \wedge \dots \wedge e_g^2$ belongs to $\wedge^g(h \cap p^+)$. Let $t_2, \dots, t_g$ be variables. The unipotent matrix which sends the basis $e_1, e_2, \dots, e_g$ to the elements $e_1, e_2 + t_2e_1, \dots, e_g + t_ge_1$ lies in $K_{\mathbf{C}} = GL(\mathbf{C}^g)$ and so

$$e_1^2 \wedge (e_2 + t_2e_1)^2 \wedge \dots \wedge (e_g + t_ge_1)^2$$

may be viewed as an E(G,H,g)-valued polynomial in the t's . Therefore all its coefficients lie in E(G,H,g) and in particular, the coefficient of $t_2 \times \dots \times t_g$, which is precisely $e_1^2 \wedge e_1e_2 \wedge \dots \wedge e_1e_g$, lies in E(G,H,g). The proof is over.

(3.C.5) Notation : More generally, we may consider subgroups which are products of lower dimensional symplectic groups. To be precise, let P_i , $1 \leq i \leq m$ be a partition of the set whose elements are 1 ,2 , . . ., g . Let E_i (resp. F_i) be the span of the vectors e_μ (resp. f_μ) with $\mu \in P_i$. The restriction Ω_i of the form Ω to the direct sum V_i of E_i and F_i is clearly non-degenerate . Let H^{nc} be the product $\Pi_{i=1}^m Sp(\Omega_i)$.

(3.C.6) Proposition : *Let G and H be $\mathbf{Q}$-groups so that H is imbedded in $G = GSp_g$ as in (3.C.5). Then , a cuspidal holomorphic R-form on the Shimura variety S associated to a congruence subgroup of G , is stably non-zero along H if and only if R=g .*

Proof : Suppose that R=g and ω is a holomorphic g-form on S. Then, by replacing H by the *smaller* group $H' = Sp_1^g$ as in (3.A.4), we see that ω is stably non-zero along H' and hence along H.

Assume now that ω is a holomorphic R-form of type A_q which is stably non-zero along H. We will show that R=g. We assume, as we may, by replacing E_i $(i \geq 2)$ (resp F_i $(i \geq 2)$) by their direct sum E'(resp. F') and replacing H by the possibly larger group $Sp(E_1 \oplus F_1) \times Sp(E' \oplus F')$, that m=2 and $H = Sp_a \times Sp_b$ with $a + b = g$.

Let b be the Borel-subalgebra of upper-triangular matrices in $k = gl_g$, n the nil-radical of b and $n_H = n \cap k_H$. Let $m = m_{ab}$ be the span of the matrices E_{ij} with $1 \leq i \leq a \leq j \leq g$. Then $n = n_H \oplus m$.

Then m is an ideal in n and by the Poincare-Birkhoff-Witt Theorem we have

$$(3.17) \qquad u(n) = u(m) \otimes u(n_H)$$

and there exists a θ stable parabolic $q_H \subset h$ (which is holomorphic in degree R) and an $\alpha \in u(n)$ such that

$$(3.18) \qquad e(q) = \alpha\pi_q(e(q_H)) \quad .$$

We assume, as we may by (3.17) , that $\alpha \in u(m)$ and that it is an eigenvector for T. We may write $q_H = q_1 \oplus q_2$ with $q_1 \subset sp_a$ and $q_2 \subset sp_b$ such that $dim u_i^+ = R_i$ for i=1,2 and $u_i^- = 0$. Let v_H denote the image of $e(q_H)$ under the map π_q .

The weight of e(q) under T is given by

$$(3.19) \quad (g+1)x_1 + \ \ldots \ + (g+1)x_k + k(x_{k+1} + \ \ldots \ + kx_g).$$

The weight of α is of the form

$$(3.20) \qquad \sum_{i \leq a \ < \ j} m_{ij}(x_i - x_j)$$

where m_{ij} are non-negative integers.

The weight of v_H is , (for suitable numbers $r \leq a$ and $s \leq b$) of the form

$$(3.21) \quad (a+1)x_1 + \ \ldots \ + (a+1)x_r + r(x_{r+1} + \ \ldots \ + rx_a) \ .$$

$$+(b+1)x_{a+1} + \ \ldots \ + (b+1)x_{a+s} + s(x_{a+s+1} + \ \ldots \ + sx_g) \ .$$

Denote by V(k,g) the representation V(q). Then as $K_H = GL_a \times GL_b$-modules, the inclusion of $V(q_H)$ in V(q) as above implies the inclusion of $V(r,a) \otimes V(s,b)$ in V(k,g) .

Consider the permutation matrix $w_0 \in GL_g = K$ which takes the basis $e_1, \ldots, e_b$; $e_{b+1}, e_{b+2}, \ldots, e_g$ into the vectors $e_{a+1}, \ldots, e_g$; $e_1, e_2, \ldots, e_a$. Conjugation by w_0 takes $H = Sp_a \times Sp_b$ into the group $H' = Sp_b \times Sp_a$ and the w_0 -translate of v_H is a vector $v_{H'}$ which corresponds to the heighest weight vector for $K_{H'}$ and gives a $K_{H'} = w_0 K_H w_0^{-1}$-equivariant inclusion of $V(s,b) \otimes V(r,a)$ in V(k,g) .

As before, we get an element $\beta \in u(m_{ba})$ where $m' = m_{ba}$ is defined as the span of E_{ij} with $i \leq b$ and $j > a$ such that

$$(3.22) \quad e(q) = \beta(v_{H'}) .$$

We now prove that R=g by considering several cases.

Case 1: $r < a$ and $s < b$. Compare the coefficient of x_g on both sides of (3.18). We get $s - \sum m_{ig} = k$ and hence $k < g$ and $s \geq k$. Similarly, we get from (3.22) that $a > r \geq k$. Comparing the weights of x_a in (3.18) we get $r + \sum m_{aj} = k$ and $r \leq k$. Similarly , $s \leq k$. Therefore r=s=k. Then

$$R = gk-k(k-1)/2 = ar-r(r-1)/2+bs-s(s-1)/2 = (a+b)k-k(k-1)$$

i.e., $k(k-1)/2 = 0$ and k=1.

Case 2: $r = a$ and $s < b$. If $a > k$ compare x_a coefficients in (3.18) and get $a+1 \leq k$ -a contradiction. Therefore $a \leq k$. Compare x_g coefficients in (3.18) and get $s \geq k$. Thus $b > k$. Now compare the coefficient of x_g (resp. of x_b) in (3.22) and get $a+1 \geq k$ (resp. $s \leq k$) . Thus $s = k$ and $a+1 = k$ and

$$R = gk - k(k-1)/2 = bs - s(s-1)/2 + a(a+1)/2 = bk$$

and we get $(k-1)k = ak = k(k-1)/2$ and so, k=1.

Case 3 : $a < r$ and $s = b$. By symmetry , we get k=1 , exactly as in Case 2.

Case 4: $r = a$ and $s = b$. If $a > k$ compare the coefficients of x_a in (1) and get $a+1 \leq k$ a contradiction . Hence $a \leq k$ and similarly $b \leq k$. Compare x_g coefficients in (3.18) and get $b+1 \geq k$. Similarly $a+1 \geq k$. If $a = k$ then

$$k(k+1)/2 + bk = gk - k(k-1)/2 = R = k(k+1)/2 + b(b+1)/2$$

i.e., $(b+1)/2 = k \geq b$ and $b \leq 1$; so b=1 and k=1. Similarly if $b = k$ we get $a = 1$ and $k = 1$.

If $a+1 = b+1 = k$ then $R = k(k-1) = 2(k-1)k - k(k-1)/2$ which again shows that k=1.

The proof is complete .

(3.C.7) Notation : In this section we will consider θ-stable parabolic subalgebras q which contribute to cohomology but not necessarily to holomorphic cohomology. Let q=q(x) be associated with the diagonal matrix x with entries $(a_1, \ldots, a_g)$ where , this time, we do not necessarily have $u \cap p^+$ =0 . We may still assume that the a's are in decreasing order . Let $r \leq s$ be defined by the inequalities

$$a_1 \geq \ldots \geq a_r > 0 = a_{r+1} = \ldots = a_s > a_{s+1} \geq \ldots \geq a_g .$$

Then, the roots of t occurring in $u \cap p^+$ *contain* the roots $x_i + x_j$ with $i \leq r$ and $j \leq s$. Therefore

$$(3.23) \qquad R^+ \geq (s(s+1)/2) - (s-r)(s-r+1)/2$$

Similarly , the roots of t occurring in $u \cap p^-$ *contain* the roots $-(x_l + x_m)$ with $r+1 \leq l \leq g$ and $s+1 \leq m \leq g$. Therefore

$$(3.24) \qquad R^- \geq ((g-r)(g-r+1)/2) - (s-r)(s-r+1)/2$$

We are interested in those A_q's which have cohomology of degree g. By [VZ] Section 6 , it follows that

$$(3.25) \qquad g = R^+ + l + R^- + l \quad with \quad l \geq 0.$$

(3.C.8) Proposition : *Let q be a θ -stable proper parabolic subalgebra of g such that the cohomological representation A_q has cohomology in degree g). Then , the cohomology in degree g of A_q is either holomorphic or anti-holomorphic , unless g=2 .*

Proof : If r=0 then *all* roots are non-positive on x and so A_q has anti-holomorphic cohomology in degree g and by (3.25) , has no mixed or holomorphic cohomology of degree g. If s=g, then again all roots are non-negative on x and all the cohomology of A_q in degree g is holomorphic. We may thus assume that $r = u + 1$ and $g = s + v + 1$ with u , $v \geq 0$. Put s=a+r with $a \geq 0$.

In (3.23) , (3.24) and (3.25) we write all inequalities in terms of a,u,v and l and obtain , after simplification ,

$$(u(u+1))/2 \ + \ (v(v+1)/2) \ + \ a(u+v+1) \ + \ l \ \leq 0 \ .$$

This can happen if and only if u,v,a and l are all zero, i.e., $r = s = 1$ and $g = 2$. This proves the proposition.

(3.D.1) Notation : In this section $G^{nc} = SO^*(2n)$ with $n \geq 4$. Thus,

$$G^{nc} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \ g \in SU(n,n) \ ; \ {}^t g \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} g \ = \ \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \right\}.$$

Moreover ,

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \ g \in G^{nc} \ ; \ B = C = 0 \ (and \ D \ = \ {}^t A^{-1}) \right\}.$$

is a maximal compact subgroup of G^{nc} . The Lie-algebra $so^*(2n)$ of G^{nc} is stable under the Cartan-involution $X \to \ -{}^t\overline{X}$ of the Lie-algebra of U(n,n) . Furthermore , the intersection T of G^{nc} with the diagonal matrices in U(n,n) is a θ-stable Cartan-subgroup of $SO^*(2n)$. We also have

$$p^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \ ; \ B ={}^t B \right\} \ ,$$

$$p^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \ ; \ C ={}^t C \right\} \ .$$

As a representation of $K_{\mathbf{C}} = GL_n = GL(\mathbf{C}^n)$, $p^+ = \wedge^2(\mathbf{C}^n)$. Fix $X \in iLie(T)$,

$$X = (a_1, \ . \ . \ . \ , \ a_n, \ -a_1, \ . \ . \ . \ , \ -a_n)$$

with $a_1 \geq a_2 \geq . \ . \ . \ \geq a_n$.

We determine the parabolic θ-stable subalgebras q which contribute to holomorphic cohomology . As $u \cap p^- = 0$, we must have $a_i + a_j \geq 0$(for all i, j with $i \neq j$) . Thus ,

$$a_1 \geq a_2 \geq . \ . \ . \ \geq a_{n-1} \geq +a_n, -a_n.$$

If $a_{n-1} + a_n > 0$, then $a_i + a_j > 0$ for all i and j and therefore $q = p^+ + k$. We assume then that $a_{n-1} + a_n = 0(\geq 2a_n)$.

Case 1 : $a_n = 0$. Let $k(\leq n-1)$ be defined by the inequalities

$$a_1 \geq . \ . \ . \ \geq a_k > 0 = a_{k+1} = \ . \ . \ . = a_{n-1} = a_n \ .$$

Then the roots in p^+ which are positive on X are

$$x_1 + x_2, x_1 + x_3, \ . \ . \ . \ , x_1 + x_n$$

$$. \ .$$

$$x_k + x_{k+1}, x_k + x_{k+2}, \ . \ . \ . \ , x_k + x_n \ .$$

Then $\mu(q) = (n-1)(x_1 + . \ . \ . + x_k) + k(x_1 + \ . \ . \ . \ + x_n)$ and so , the representation V(q) -which we denote by $W_{k,n}$ in order to keep track of k - may be thought of as a subrepresentation of

$$(\wedge^n \mathbf{C}^n)^k \otimes Sym^{n-1-k}(\wedge^k \mathbf{C}^n) \ .$$

Case 2 : $a_n < 0$. Let $k \leq n-2$ be defined by the inequalities

$$a_1 \geq . \ . \ . \ \geq a_k > a_{k+1} = \ . \ . \ . = a_{n-1} = -a_n > 0 \ .$$

The roots in p^+ which are positive on X are :

$$x_1 + x_2, x_1 + x_3, \ . \ . \ . \ , x_1 + x_n$$

$$. \ .$$

$$x_k + x_{k+1}, x_k + x_{k+2}, \ldots, x_k + x_n ,$$

and

$$x_{k+1} + x_{k+2}, x_{k+1} + x_{k+3}, \ldots, x_{k+1} + x_{n-1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_{n-2} + x_{n-1} \quad .$$

Consequently

$$\mu(q) = (n-1)(x_1 + \ldots + x_k) + (n-2)(x_{k+1} + \ldots + x_{n-1}) + kx_n$$

and so , the representation V(q) -which we denote by $V_{k,n}$ - may be thought of as a subrepresentation of

$$(\wedge^n \mathbf{C}^n)^k \otimes Sym^{n-2-k}(\wedge^k \mathbf{C}^n) \otimes (\wedge^k \mathbf{C}^n) \quad .$$

We now consider restriction from U(n,n) to $SO^*(2n)$.

(3.D.2) Proposition : *Let G be a $\mathbf{Q}$-group , H a $\mathbf{Q}$-subgroup such that $G^{nc} = U(n,n) \supset H^{nc} = SO^*(2n)$ embedded as in (3.D.1) . Then every holomorphic cuspidal form on $\Gamma\backslash G/K$ (Γ is an arithmetic subgroup of $G(\mathbf{Q})$) vanishes along H .*

Proof : We will check that for every $r,\ s\ \geq 0\ r+s\ >\ 0$ ($q = q(r,s)$ is defined as in (3.A.3) , and $V_{r,s}(q) = V(q(r,s))$)

$$A(G,H,R) \cap V_{r,s}(q) = 0 \ .$$

Then , the proposition follows from (2.7) .

We assume that $r \geq 1$. Let b be the Borel-subalgebra of $k_{\mathbf{C}} = u(n)_{\mathbf{C}} \oplus u(n)_{\mathbf{C}} \subset g_{\mathbf{C}}$, where the first factor is the space of upper-triangular matrices and the second is the space of lower triangular matrices . Then $b_H = b \cap h$ is a Borel-subalgebra of k_H . Now $k_2 = 0 \oplus u(n)_{\mathbf{C}}$ is an ideal in $k_{\mathbf{C}}$ and therefore $b \cap k_2 = b_2$ is an ideal in b . Thus , by the Poincare-Birkhoff-Witt Theorem , we see that

$$(3.26) \quad u(b) = u(b_2) \otimes u(b_H) \quad .$$

Suppose $V(q_H) = V_{k,n}$ or $V(q_H) = W_{k,n}$ occurs as a sub-representation of the K_H-module V(q) . Write

$$R = dim(u(q) \cap p^+) = dim(u(q_H) \cap p_H^+)$$
$$v_H = \wedge^R(p_H^+ \cap u(q_H)) , \quad v_G = \wedge^R(p^+ \cap u(q)) .$$

As v_G is the unique highest weight vector in V(q) , there exists a $\alpha \in u(b)$ such that

$$(3.27) \quad v_G = \alpha(v_H).$$

Using (3.26) and the fact that v_H is a highest weight vector for $u(b_H)$, we may assume that $\alpha \in u(b_2)$. Now $K \subset U(n,n)$ acts on $\mathbf{C}^n \oplus \mathbf{C}^n$. In the notation of (3.A.1) ,

$$v_G = (e_1 f_1^* \wedge \ldots \wedge e_1 f_n^*)) \wedge \ldots \wedge (e_r f_1^* \wedge \ldots \wedge e_r f_n^*) \wedge$$
$$\wedge((e_{r+1} f_1^* \wedge \ldots \wedge e_{r+1} f_s^*)) \wedge \ldots \wedge (e_n f_1^* \wedge \ldots \wedge e_n f_s^*)$$

Now α is in the tensor space generated by f_j and f_j^* and

$$v_H = (e_1 f_2^* - e_2 f_1^*) \wedge \ldots \wedge (e_1 f_n^* - e_n f_1^*) \wedge (etc)$$

the other terms (marked etc) in v_H do not involve e_1 . The e_1-degrees of v_G , α and v_H are respectively n (because $r \geq 1$) , 0 , n-1 , which makes the equation (3.27) impossible . Therefore $r = 0$. Similarly $s = 0$. This completes the proof .

(3.D.3) Notation : We now choose $G^{nc} = SO^*(2n)$ and for $m \leq n-1$ consider $H^{nc} = SO^*(2m)$ embedded as the subgroup of $G^{nc} \subset GL_{2n}$ of elements g such that $g_{ij} = \delta_{ij}$ for $i \leq n-m$ or $j > n+m$.

(3.D.4) Proposition : *Let G and H be* $\mathbf{Q}$*-groups such that* $G^{nc} = SO^*(2n) \supset H^{nc} = SO^*(2m)$ *embedded as above . Then , every cuspidal holomorphic form on* $\Gamma\backslash G / K$ ($\Gamma \subset G(\mathbf{Q})$ *an arithmetic subgroup) vanishes along H .*

Proof : Let

$$S = \left\{\theta(t) \in T_{\mathbf{C}} \ : \ \theta(t)(e_1) = te_1 \ , \ \theta(t)f_n = t^{-1}f_n \ and \ \theta(t)(v) = v\right\}$$

where v is in the span of e_i with $i \neq 1$ and f_j with $j \neq n$.

Thus S is a G_m and centralises H . Its weights on $V_{k,n}$ and $W_{k,n}$ are of the form t^{k+m} where m is non-negative . By (2.6) our proposition follows .

4. Review of a conjecture on the Zeta-functions of Shimura Varieties

(4.1) Notation : To each non-archimedean local field F_v we associate the group $W_{F_v} \times SU(2, \mathbf{R})$ which we call the Langlands group $\mathcal{L}_{F_v}$ of the local field F_v. If F_v is archimedean , the Langlands group is taken to be the Weil group of F_v. Given a reductive algebraic group G over F_v, the semi-direct product ${}^L G = \hat{G} \times W_{F_v}$ is the *Langlands dual* . We recall the

Local Langlands Conjecture . There is a partition of the set $\Pi(G(F_v))$ of equivalence classes of irreducible admissible representations of $G(F_v)$ into finite subsets , called *L-packets* , such that there is a natural bijection between L-packets $\Pi = \Pi(\phi')$ and the set of $\hat{G}$-conjugacy classes of continuous homomorphisms ϕ' (with image of each element being semi-simple) of $\mathcal{L}_{F_v}$ into ${}^L G(\mathbf{C})$ which is compatible with the natural maps to W_{F_v} .

We assume the existence of the Langlands group $\mathcal{L}_F$, associated to a number-field F. This is a conjectural extension of the Weil group W_F by a compact group . This group is supposed to satisfy , among others , the following conditions .

The isomorphism classes of continuous n-dimensional representations ϕ' of the Langlands group are in natural bijection with the set of equivalence classes of cuspidal representations π of $GL_n(\mathbf{A}_F)$. In particular , the Abelianisations of the Weil group and the Langlands group are the same .

Given now a number field F and a place v of F , there exists a special conjugacy class of embeddings $i_v : \mathcal{L}_{F_v} \to \mathcal{L}_F$. Let ϕ be an irreducible representation of $\mathcal{L}_F$. Its restriction ϕ_v to $\mathcal{L}_{F_v}$ has a local factor $L(s, \phi_v)$ as in [Tate (Corvallis)] and the bijection between ϕ and π is such that the corresponding local factors $L(s, \phi_v)$ and $L(s, \pi_v)$ are the same for almost all the places .

A **Langlands Parameter** is a continuous homomorphism

$$\phi' : \mathcal{L}_F \to^L G$$

such that the image of every element is semi-simple and such that ϕ' commutes with the projections to W_F. An **Arthur Parameter** is a continuous homomorphism

$$\phi : \mathcal{L}_F \times SL_2(\mathbf{C}) \to^L G$$

such that the restriction to $SL_2(\mathbf{C})$ is holomorphic , the restriction to $\mathcal{L}_F$ is a Langlands Parameter , and the image of $\mathcal{L}_F$ is bounded modulo the center of $\hat{G}$. Given an Arthur parameter ϕ , define the Langlands parameter

$$\phi'(w) := \phi(w , \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}$$

where $|w| : \mathcal{L}_F \to \mathbf{R}_+^*$ is the pullback of the absolute-value map of W_F.

Fix a place v ; the restriction ϕ_v of ϕ to $\mathcal{L}_{F_v} \times SL(2, \mathbf{C})$ is conjectured to correspond to an Arthur- packet $\Pi(\phi_v)$ which contains the L-packet $\Pi(\phi_v')$ corresponding to ϕ_v'. For almost all finite v, $\Pi(\phi_v)$ should contain a unique unramified representation π_v^0 . Define the global A-packet $\Pi(\phi) = \otimes\Pi(\phi_v)$ which is the set of restricted tensor products $\otimes\pi_v$ where $\pi_v \in \Pi(\phi_v)$ for all v and $\pi_v = \pi_v^0$ for almost all v .

If F is a local field , the Arthur parameter $\phi_{G\ ,\ triv}$ of the *trivial* representation of G is given as follows.

$$\phi_{G\ ,\ triv} : W_F \times SL_2(\mathbf{C}) \to^L G$$

is trivial on the Weil group of F and is a map which is non-trivial on $SL_2(\mathbf{C})$ and takes the upper triangular unipotent group in $SL_2(\mathbf{C})$ to the one parameter subgroup generated by a regular unipotent element in ${}^LG$.

Suppose now, that the local field is $\mathbf{R}$ and that G is as in (2.1). An Arthur parameter

$$\phi : W_{\mathbf{R}} \times SL_2(\mathbf{C}) \to^L G$$

is called **cohomological** if the associated Arthur-packet consists of cohomological representations. Write $W_{\mathbf{R}} = \mathbf{C}^* \cup \mathbf{C}^*\sigma_\infty$. Upto equivalence , the cohomological Arthur-parameters are indexed by parabolic subgroups P (containing a fixed Borel Subgroup B) , whose Lie algebra is of the form q(x) and contains b_0 (as defined in(2.1)). Let T denote the maximal torus in K and (hence in G) whose Lie algebra is the centraliser of x . Let M be the Levi-part of P which contains T . Let $\hat{M}$ denote the dual group of M which embeds naturally into $\hat{G}$.

Let δ_M denote half the sum of positive roots of of T occurring in M. Put $\hat{\delta}_P = \delta_G - \delta_M$. We may think of $\hat{\delta}_P$ as a weight of T or as a co-weight of $\hat{T}$. Now, there exists an element of the Weyl group of T in G (or in M) such that conjugation by it acts as the inverse on T because T is anisotropic. Let

w_M be an element of the Weyl group of $\hat{T}$ in $\hat{M}$ which does the same for $\hat{T}$. We now describe the Arthur Parameter corresponding to P by the formulae (see [Art 1])

$$\xi_P(z) = (z/\bar{z})^{\hat{\delta}_P}$$

$$\xi_P(\sigma_\infty) = w_G w_M^{-1} \times \sigma_\infty$$

$$\phi_P = \xi_P \circ \phi_{M\ ,\ triv}.$$

We return to the notation of (2.1). Let L^2_d denote the **discrete** part of the right-regular representation of $G(\mathbf{A})$ on the space $L^2(\omega)$ of functions on the quotient $G(\mathbf{Q})Z(\mathbf{A})\backslash G(\mathbf{A})$ which are square-integrable modulo the centre of $G(\mathbf{A})$ and which transform according to a fixed unitary character ω under $Z(\mathbf{A})$. It is well known that $L^2_d(\omega)$ is a direct sum of irreducible representations π , of $G(\mathbf{A})$, each occurring with a finite multiplicity which we denote by $m(\pi)$. We will call these π *discrete representations* . Of course, if $G(\mathbf{Q})Z(\mathbf{A})\backslash G(\mathbf{A})$ is compact , then $L^2_d(\omega)$ is equal to $L^2(\omega)$.

Arthur conjectures that there exists a *partition* of the set $\mathcal{E}$ of (equivalence classes) of discrete representations , into subsets $\mathcal{E}\cap(\Pi)$, where Π is a global A-packet corresponding to an A-parameter ϕ as above. The A-parameter satisfies some additional properties in this case ;for a discussion of these , we refer the reader to [Bl-Ro],section (3.4) .

We assume from now on , that G is as in (2.1) . More precisely , we assume as in [De 1] , that we have a an algebraic group G defined over $\mathbf{Q}$ and a homomorphism $h\ :\ R_{\mathbf{C}/\mathbf{R}}(G_m) \to G$ defined over $\mathbf{R}$ which satisfies the axioms of [De 1] . We will assume that the image of the restriction μ of h to the "first" factor $\mathbf{G}_m$ (in the identification of the complex points of the group $R_{\mathbf{C}/\mathbf{R}}$ as a product of two copies of $\mathbf{G}_m$) lies in a maximal torus T of G which is defined over $\mathbf{Q}$. Let Γ_μ denote the subgroup of the Galois-group of $\mathbf{Q}$ which fixes the conjugacy class of μ . This is an open subgroup of finite index and corresponds to a finite extension E (called the reflex field of (G,h))

.

Let K_f be an open compact subgroup of $G(\mathbf{A}_f)$. Form the quotient $S = S(K_f) = G(\mathbf{Q}Z(\mathbf{A})\backslash G(\mathbf{A})/K_f$. Each of its connected components is of the form $S = S(\Gamma)$ for some arithmetic subgroup Γ of $G(\mathbf{Q})$. Let $S(\widehat{K_f})$ denote the Bailey-Borel-Satake compactification of S . This is a projective variety, which is not smooth in general, and is defined over the reflex field E . By Zucker's conjecture (Theorem of Looienga, Saper-Stern) , the intersection-cohomology $IH^*(\widehat{S(K_f)})$ with middle perversity , of $\widehat{S(K_f)}$, is naturally isomorphic to the L^2 -cohomology $H_2^*(S(K_f))$ of the space $S(K_f)$. By the Matshushima formula , we then have

$$IH^*(\widehat{S(K_f)}, \mathbf{C}) = \oplus m(\pi) H^*(g, K, \pi_\infty) \otimes \pi_f^{K_f}$$

where π runs through the discrete representations , $\pi_f^{K_f}$ denotes the space of vectors in π_f which are fixed under K_f .

The intersection cohomology IH^i is defined over $\mathbf{Q}$ and it has $\mathbf{Q}_l$ analogues in etale cohomology , and these satisfy the Weil (purity) conjectures ([Bei-Ber-De]) . The etale cohomology comes equipped with an action of the Galois-group Γ_E of the number-field E over which all these $\widehat{S(K_f)}$ are defined

.

Consider the direct limit, as the open subgroups K_f of the group $G(\mathbf{A}_f)$ become smaller and smaller , of the i-th L_d^2- cohomology groups of the Shimura varieties Sh (G, K_f) with coefficients in $\overline{Q}_l$. This limit admits an action by the group $G(\mathbf{A}_f)$, and decomposes as a direct sum of irreducible representations π_f (with finite multiplicity) under it:

$$lim \quad H_2^i(S(K_f), \mathbf{Q}_l) = \oplus H^i(\pi_f) \otimes \pi_f,$$

and $H^i(\pi_f)$ is finite dimensional . Moreover , the representations are defined over a finite extension of $\mathbf{Q}_l$ and hence so is the space $H^i(\pi_f)$. The action of Γ_E commutes with the action of $G(\mathbf{A}_f)$ on this direct limit and therefore we get a representation $\rho^i(\pi_f)$ of Γ_E on $H^i(\pi_f)$. Let ϕ be the Arthur parameter corresponding to a representation π whose finite part is π_f and whose infinite part is cohomological .

We return to the homomorphism μ which gives dually , a homomorphism $\hat{\mu}$ of $\hat{T}$ into $\mathbf{G}_m$ .. After a conjugation by an element of the Weyl-group of $\hat{G}, \hat{T}$ this yields a representation (r, V) of the semi-direct product $\hat{G} \times W_E$. If ϕ is as in the above paragraph , we may form the representation $r \circ \phi_E$ of $\mathcal{L}_E \times SL_2(\mathbf{C})$, where ϕ_E is the restriction to $\mathcal{L}_E \times SL_2(\mathbf{C})$ of ϕ .

Denote (by an abuse of notation) by $\mathbf{G}_m$ the group of diagonal matrices in SL_2 . Restrict $r \circ \phi$ to $\mathbf{G}_m$. For each integer i , denote by V^i the subspace of V on which $t \in \mathbf{G}_m$ acts by the character t^{i-d} where d is the dimension of the Shimura variety . We note that the image of $\mathcal{L}_E$ commutes with that of $\mathbf{G}_m$ and so , the image of $\mathcal{L}_E$ under the representation $r \circ \phi'_E$ (ϕ'_E is as in (4.1)) also commutes with $\mathbf{G}_m$. Therefore V^i is left stable by the image of $\mathcal{L}_E$ under $r \circ \phi'_E$. We denote this representation of $\mathcal{L}_E$ also by V^i .

(4.2) Conjecture on the Zeta-Function of $Sh(G, K_f)$: The representation $\rho^i(\pi_f)$ is a *subrepresentation* of $\theta^d \otimes V^i$. Here d is the complex dimension of the Shimura variety and θ^d is the d-th power of the cyclotomic character .

(We note that $\rho^i(\pi_f)$ is a representation of Γ_E and therefore of $\mathcal{L}_E$). For a much more precise statement of the conjecture see (5.2) of [Bl-Ro].

We now collect together some consequences of the conjecture (4.2) . Note that in the consequences below , the assumptions on the A-parameters need be verified only at the Archimedean places.

(1) Suppose that the centraliser of the image of SL_2 under the map ϕ_∞ has **Abelian** Lie algebra. Then, for each $\pi \in \Pi(\phi)$ and for each i , the representation $\rho^i(\pi_f)$ is **potentially Abelian** , i.e., the image of an open subgroup of Γ_E under $\rho^i(\pi_f)$ is Abelian .

(2) Suppose that for some i , the space

$$V^i = \left\{ v \in V \; ; \; r \circ \phi(1 \, , \, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) \, v = t^{i-d} v \right\}$$

is **one-dimensional** , where d is the complex dimension of the Shimura variety. Then the image of $\rho^i(\pi_f)$ is Abelian .

(4.3.A) Suppose that $G^{nc} = GU(p,q)$ $p \leq q$. We now assume that q(x) is a θ -stable parabolic subalgebra as in (3.A) and that P is the subgroup of the complexification $GL(p+q, \mathbf{C})$ of GU(p,q) with Lie-algebra q(x) . Assume also that the representation A_q contributes to holomorphic cohomology in dimension p (=real rank of G). Then, by the calculations of (3.A.4), x may be taken to be the diagonal matrix with entries $(a_1 \, , \, a \, , \, . \, . \, . \, , \, a \,)$, with $a_1 > a$ and a occurs $p + q - 1$ times . The semisimple part M_{ss} of the Levi-subgroup of P containing T is the special linear group of the span W of $e_2, \, . \, , \, e_p$ and $f_1, \, . \, . \, f_q$. The homomorphism ξ_P defined above takes SL_2 into M_{ss} and SL_2 acts irreducibly on W. Therefore, the centraliser of SL_2 in

$\hat{G} = GL_{p+q}$ is the diagonal subgroup , which act as scalars on W . Now the conclusion of (1) of (4.2) applies.

(4.3.C) Assume that $G^{nc} = GSp_g$. Let q(x) be a θ-stable parabolic subalgebra as in (3.C) and P the subgroup with Lie algebra q(x) of the complexification of G^{nc}. Then, as we have seen, x may be assumed to be the diagonal $2g \times 2g$ matrix

$$x = (a_1, 0, \ldots, 0, -a_1, 0, \ldots, 0)$$

and the semisimple part of the Levi subgroup M of P containing T is GSp_{g-1} which acts on the span of

$$e_2, , e_g, f_2, \ldots, f_g .$$

The homomorphism ξ takes SL_2 into the subgroup $\hat{M} = GSpin(2g-1)$ of $\hat{G} = GSpin(2g+1)$ and sends a nontrivial upper-triangular unipotent element of SL_2 into a regular unipotent element of $\hat{M}$.

We wish to show that the centraliser of the image of SL_2 in $\hat{G}$ has Abelian Lie algebra. It is enough to prove that the centraliser (of the image of SL_2) in SO(2g+1) under the composite with ξ , of the covering map p of Spin(2g+1) into SO(2g+1) , has Abelian Lie-algebra. Now $p \circ \xi : SL_2 \to GL_{2g+1}$ is , as is easily shown, the direct sum of an irreducible representation of SL_2 of dimension 2g-1, together with the trivial 2-dimensional representation . Therefore, the connected component of identity of the centraliser in SO(2g+1) of SL_2, is $SO(2) \times 1_{2g-1}$ which is clearly Abelian. The conclusion of (1) of (4.2) applies.

Now assume that $(p, q) \geq (2, 2)$ in the GU-case and that $g \geq 3$ in the GSp case . Then by (3.A.10) and (3.C.8) we know that when the degree r_G is equal to the rank (and except for the trivial representation) , the only parameters corresponding to a cohomological representation at ∞ are of type (4.3.A) or (4.3.C) . So , for the associated Shimura varieties, the Galois action on the r_G-th etale cohomology groups is semisimple and potentially Abelian. We will show in the next section , that these predictions are true.

5. — Arithmetic applications : the case of $Sp(g)$

5.1. — We will now apply the results of § § 1, 3 to the Galois representations occurring in the cohomology of the variety of moduli $\mathcal{A}_g$ of Abelian varieties of dimension g.

Thus let $G = GSp(g)/\mathbb{Q}$ be the group of similitudes of the symplectic form on $\mathbb{Q}^{2g}$ of matrix $\begin{pmatrix} 0 & -1g \\ 1g & 0 \end{pmatrix}$; let

$$(5.1) \qquad \begin{aligned} h : \mathbb{C}^\times &\longrightarrow G(\mathbb{R}) \\ z = x + iy &\longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix} . \end{aligned}$$

If $K_f \subset G(\mathbb{A}_f)$, we have an associated variety $S(h, K_f) = S(K_f)$ over $\mathbb{Q}$, with $\mathbb{C}$–points $S(K_f)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/KK_f$, $K = \mathbb{R}^\times U(g)$. For L a finite extension of $\mathbb{Q}_\ell$, we are interested in the cohomology space $H^g_{\text{ét}}(S(K_f) \times \overline{\mathbb{Q}}, L)$ and the action on it of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Note that (except for cohomology contributed by the trivial representation, and therefore composed of Tate classes) this is the first degree where there should be non–zero cohomology.

Denote by $H^g_!(S(K_f))$ the image in $H^g(S(K_f))$ of the cohomology with compact supports H^g_c, in various cohomology theories.

Over $\mathbb{C}$, $H^g_!(S(K_f), \mathbb{C})$ a priori carries a mixed Hodge structure according to Deligne [De3, De4]. Let $IH^\bullet$ denote intersection cohomology with midde perversity [Go–M]. The canonical map $H^\bullet_c(S(K_f)) \to IH^\bullet(S(K_f))$ quotients through $H^\bullet_c(S(K_f)) \to H^\bullet(S(K_f))$, as follows from Poincaré duality for intersection cohomology; in our case it is also a consequence of obvious properties of L^2–cohomology and of the Zucker conjecture stating then $H^\bullet_{(2)}(S(K_f), \mathbb{C}) \cong IH^\bullet(S(K_f), \mathbb{C})$ [Lo, Sa–St]. Thus we get a map :

$$(5.2) \qquad H^\bullet_!(S(K_f)) \longrightarrow IH^\bullet(S(K_f))$$

of $\mathbb{Q}$–mixed Hodge structures. A simple result of Harder and one of us ([Ha]; [Clo3 : Prop. 3.18]) shows that this map is injective. It follows that $H^i_!(S(K_f), \mathbb{C})$ carries a pure $\mathbb{Q}$–Hodge structure of weight i in all degrees (this injectivity property is particular to Shimura varieties and does not follow from the axiomatic properties of intersection cohomology).

Note that the purity of $H^i_!$ can also be seen directly, as was pointed to us by L. Illusie and T. Saito. Indeed, $H^i(S(K_f))$ has a weight filtration whose weights belong to the interval $[i, 2i]$, since $S(K_f)$ is a quotient of a smooth variety with only finite quotient singularities. Analogously, the weights of H^{2d-i} are larger than $2d - i$. By Poincaré duality, H^i_c has a weight filtration with weights $\leq i$. The consequence is that $H^i_!$ is a mixed Hodge structure with **pure** weight i, i.e., a pure Hodge structure of weight i.

When g is even, $H^g_!(S(K_f), \mathbb{C})$ contains a "trivial" part. We describe this using representation theory. Let $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A})/Z_G(\mathbb{R})) = \bigoplus \pi$, where π runs over a complete set of summands of the discrete part of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z_G(\mathbb{R}))$ as a representation of

$G(\mathbb{R})$. Then, according to Borel and Casselman, /A

$$H^g_{(2)}(S(K_f), \mathbb{C}) = \bigoplus_{\pi} H^g(\mathfrak{g}, K; \pi_\infty) \otimes \pi_f^{K_f}$$

where $\mathfrak{g} = \mathrm{Lie}(G \times \mathbb{R})$ and $K \subset G(\mathbb{R})$ is maximal compact. By strong approximation, $\pi_\infty \cong \mathbb{C}$ if, and only if, π is an Abelian character. The part of $H^g_{(2)}$ corresponding to these representations is composed of classes of type $(g/2,\ g/2)$ in the Hodge decomposition. Moreover, it is the image in $H^g_{(2)}$ of a projector composed of Hecke correspondences, which may be defined over $\mathbb{Q}$. In particular, its intersection with $H^g_!$, if it is non-zero, is in any case a direct summand of $H^g_!$ for any of the cohomology theories we will consider[1] We will denote by $\widetilde{H}^g_!(S(K_f))$ the complementary subspace, both in complex and in ℓ-adic étale cohomology.

LEMMA 5.1. — *Assume $g \geq 3$. Then $\widetilde{H}^g_!(S(K_f), \mathbb{C})$ is purely of Hodge type $((g,0);\ (0,g))$.*

Proof. This follows from Proposition 3C8.

We will prove two, obviously related, results :

THEOREM 5.2. — *The Mumford–Tate group of the Hodge structure $(\widetilde{H}^g_!(S(K_f), \mathbb{Q})\,;$ $\widetilde{H}^g_!(S(K_f), \mathbb{C}) = \widetilde{H}^{g,0}_! \oplus \widetilde{H}^{0,g}_!)$ is Abelian.*

For the second result note that if ℓ is a prime, $\widetilde{H}^g_!(S(K_f), \mathbb{Q}_\ell)$ carries a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We will say that this representation is potentially Abelian if there exist finite extensions L of $\mathbb{Q}_\ell$ and F of $\mathbb{Q}$ such that the associated representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ on $\widetilde{H}^g_!(S(K_f), L)$ is an extension of Abelian characters.

THEOREM 5.3. — *The representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on $\widetilde{H}^g_!(S(K_f), \mathbb{Q}_\ell)$ is potentially Abelian.*

Before we proceed, we will state a conjecture which would allow one to give a proof of Theorem 5.3 analogous to the proof we will give for Theorem 5.2.

CONJECTURE 5.4. — *For sufficiently large ℓ, $\widetilde{H}^g_!(S(K_f), \mathbb{Q}_\ell)$ is a Hodge–Tate representation of* $\mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ *with Hodge–Tate type $((g,0);\ (0,g))$.*

N.B. By this we mean that if $V = \widetilde{H}^g_!(S(K_f), \mathbb{Q}_\ell)$ and $\mathbb{C}_\ell$ is a completion of $\overline{\mathbb{Q}}_\ell$, the only irreducible summands of $V \otimes \mathbb{C}_\ell$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$-module are isomorphic to $\mathbb{C}_\ell(0)$ or $\mathbb{C}_\ell(-g)$.

(1) The part of $H^g_{(2)}$ contributed by the Abelian characters is composed of Chern classes on the different components on $S(K_f)$ (cf. e.g. Parthasarathy [Par1]). Probably the intersection with $H^g_!$ vanishes but this is irrelevant to us.

5.2. — Before we give the proof of Theorem 3.2, we make a few remarks an Hodge structures. First note that $\widetilde{H}_!^g(S(K_f),\mathbb{Q})$ carries a polarized Hodge structure (by the corresponding fact for intersection cohomology), so its Mumford–Tate group is reductive. In the sequel we will have to consider L–Hodge structures, $L \subset \mathbb{C}$ being a number field. If L is real, the usual theory applies. In particular the category of polarized Hodge structures is semi–simple.

We will have to consider parts of the cohomology of $S(K_f)$ and related varieties which are only defined over CM–fields L. A complex Hodge structure of weight w on a vector space H of finite dimension over L (L a CM–field embedded in $\mathbb{C}$) is simply a Hodge decomposition $H \bigotimes_L \mathbb{C} = \bigoplus_{p+q=w} H^{pq}$. The usual notions apply; in particular we can define the Mumford–Tate group. If $L_0 \subset L$ is the maximal totally real subfield, a Hodge structure H_0 over L_0 defines a complex Hodge structure over L by $H = H_0 \bigotimes_{L_0} L$. Conversely, if H is a complex Hodge structure over L, let $\overline{H} = H \bigotimes_{\sigma : L \to L} L$ where σ is complex conjugation. Then $\overline{H} \otimes \mathbb{C} \cong H$, the isomorphism sending H^{pq} to $\overline{H}^{qp}$. Then $H \oplus \overline{H} = H^o \bigotimes_{L_0} L$, H^o being the Hodge structure "obtained by restriction of scalars" : $H^o = H$, seen as a L_0–vector space, and $H^0 \bigotimes_{L_0} \mathbb{C} = (H \otimes \mathbb{C}) \oplus (\overline{H} \otimes \mathbb{C})$. Note that H^0 is a true (real) Hodge structure.

Suppose X is a smooth variety, and $H = H^w(X, L)$. Then H carries a complex Hodge structure over L, which is obtained by extension of scalars from the (real) Hodge structure of $H_0 = H^w(X, L_0)$. It follows that the Mumford–Tate group of H is reductive. Therefore H, as an L–Hodge structure, is semi–simple. The same applies to intersection cohomology, and to $\widetilde{H}_!^g(S(K_f))$.

We now come to the proof of Theorem $\cancel{3}$.2. Let /(5.2)

$$H = \{(x_1, \ldots, x_g) \in GL(2)^g : \det x_1 = \det x_2 = \cdots = \det x_g\}\,. \tag{5.4}$$

Thus H is a group over $\mathbb{Z}$, with a natural embedding into $GSp(g)$ coming from the identification of $GL(2)$ with the group of symplectic similitudes of the form $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. For each $x \in G(\mathbb{Q})$, we have a finite map :

$$j_x : S(H, K_H(x)) \longrightarrow S(G, K_f)\,. \tag{5.5}$$

Consider the corresponding maps

$$j_x^* : \widetilde{H}_!^g(S(K_f), \mathbb{C}) \longrightarrow H_!^g(S(H, K_H(x)); \mathbb{C})\,. \tag{5.6}$$

LEMMA 5.5. — *The map* $\mathrm{Res} = \prod_{x \in G(\mathbb{Q})} j_x^*$ *is injective.*

Proof. By the remarks at the beginning of § 5.1, $H_!^g(S(H, K_H(x)); \mathbb{C})$ carries a pure Hodge structure of weight g; the restriction map is, according to Deligne [De3], a map of mixed (i.e., pure) Hodge structures. Therefore it suffices to show that if $\alpha \in \widetilde{H}_!^g(S(K_f), \mathbb{C})$ is of type $(g,0)$ and $\operatorname{Res} \alpha = 0$, then $\alpha = 0$.

By L^2-Hodge theory, α is represented by a form ω of type $(g,0)$ on $S(K_f)$; ω is moreover square-integrable. The variety $S(H, K_H(x))$ has a finite covering which is a product of modular curves $C_1, \ldots, C_g$. Thus we get a finite map factoring through j_x :

$$j : C_1 \times \cdots \times C_g \longrightarrow S(K_f) := X \tag{5.7}$$

Consider a smooth compactification $\overline{X}$ of X. By a basic result of Freitag and Pommerening [Fr–Po], ω extends to a smooth differential form on $\overline{X}$. Let $C = C_1 \times \cdots \times C_g$. Since j is finite, we obtain by normalization a compactification $\overline{\overline{C}}$ of C and a diagram

$$\begin{tikzcd} \overline{\overline{C}} \arrow[r, "\overline{j}"'] & \overline{X} \\ C \arrow[u, hook] \arrow[r, "j"'] & X \arrow[u, hook] \end{tikzcd} \tag{5.8}$$

where $\overline{j}$ is finite. By using the embedded resolution of singularities for $\overline{\overline{C}} \subset \overline{X}$ we obtain a new diagram (5.8) with $\overline{X}$ and $\overline{\overline{C}}$ smooth. Then the extension $\overline{\omega}$ of ω to $\overline{X}$ gives by restriction a form $(\overline{j})^*\overline{\omega}$ on $\overline{\overline{C}}$; on the interior C it coincides with $j^*\omega$.

Let $\overline{C} = \overline{C}_1 \times \cdots \times \overline{C}_g$ be the obvious smooth compactification of C. By the birational invariance of genera $\Gamma(\overline{C}, \underline{\Omega}^g) \cong \Gamma(\overline{\overline{C}}, \underline{\Omega}^g)$. Thus $j^*\omega$ extends to a holomorphic form θ on $\overline{C}$. The cohomology class γ of θ is of type $(g,0)$; it is in the image of the map $H_c^g(C) \to H^g(\overline{C})$; its restriction β to C is equal to $j^*\alpha$. Note that for each factor C_i of C – a modular curve – the map $H^1(\overline{C}_i) \to H^1(C_i)$ is injective. From the Künneth decomposition it follows that $H^{g,0}(\overline{C}) \subset \otimes^g H^1(\overline{C}_i)$, and therefore that $H^{g,0}(\overline{C}) \to H^g(C)$ is injective.

Suppose now that $\operatorname{Res} \alpha = 0$. Then $\beta = 0$, whence $\gamma = 0$, whence $\theta = 0$ by Hodge theory on $\overline{C}$. This implies that $(\overline{j})^*\omega = 0$ as a holomorphic form on $\overline{\overline{C}}$ or, equivalently, on C. Thus $j^*\omega = 0$ as a holomorphic form on C. Then $j_x^*\omega = 0$ for all x, and the arguments of § 2 show that $\omega = 0$ as a form and, therefore, as a cohomology class. This proves Lemma 5.5.

In order to prove Theorem 3.2, it now suffices to consider the image of $\widetilde{H}_!^g(S(K_f))$ in $H_!^g(C_1 \times \cdots \times C_g)$. By the foregoing arguments this injects into $H_{(2)}^g(C_1 \times \cdots \times C_g)$, which is the cohomology of the L^2-spectrum of $H_1(\mathbb{Q})\backslash H_1(\mathbb{A})$, where $H_1 = GL(2)^g$. Since we are only interested in classes of type $(g,0)$ or $(0,g)$, the image is contained in

$H^g_{cusp}(C_1 \times \cdots C_g)$, which is described over $\mathbb{C}$, in the manner of (5.3), as

$$H^g_{cusp}(C_1 \times \cdots C_g) = \bigoplus_{\pi_1,\ldots,\pi_g} H^1(\pi_{1,\infty}) \otimes \cdots \otimes H^1(\pi_{g,\infty}) \otimes \otimes \pi_{1,f}^{K_1} \otimes \cdots \otimes \pi_{g,f}^{K_g}, \tag{5.9}$$

where we have written $H^1(\pi_\infty)$ for $(\mathfrak{g}, K)$-cohomology and where $K_1 \times \cdots K_g$ is a congruence subgroup of $H_1(\mathbb{A}_f)$. Note that $H^g_{cusp}(C_1 \times \cdots C_g) \subset H^g(\overline{C}_1 \times \cdots \overline{C}_g)$ is a sub-Hodge structure, for example by the Drinfeld–Manin principle. Moreover the Hecke algebra acts irreducibly on $\pi_1^{K_1} \otimes \cdots \otimes \pi_g^{K_g}$. Constructing an associated projector (given by a linear combination over a CM-field of Hecke operators), we see that the image is a sum of complex Hodge structures over a sufficiently large CM field L, of the type

$$N \subset M_1 \otimes \cdots M_g = M \tag{5.10}$$

$M_i \subset H^1_{csup}(C_i)$ being a 2-dimensional complex Hodge structure over L; each M_i is of type $\{(1,0);(0,1)\}$, and N is purely of type $((g,0);(0,g))$.

We will now give two proofs of Theorem 5.2, based on different arguments, which we feel have independent interest.

We will denote by G_X the Mumford–Tate group of a L-complex Hodge structure X, for L a sufficiently large CM-field. There is a natural homomorphism $\nu = \nu_X : \mathbb{G}_m \to G_X$ defined over L, as is G_X. The Mumford–Tate group is defined, a priori, as a subgroup of $GL(X) \times \mathbb{G}_m$, but the projection on the first component is an isomorphism, and we will when convenient consider G_X as a subgroup of $GL(X)$. (As a good reference for Mumford–Tate groups, see [DMOS, § I.3]).

By general principles G_M is the image, by the tensor product, of the group $G_0 = G_{M_1 \oplus \cdots \oplus M_g}$; moreover the natural maps $G_0 \to G_{M_i}$ are surjective. Since $G_{M_i} \subset GL(2)/L$ is reductive and G_{M_i} contains the image of ν, $G_{M_i} = GL(2)$ if it is not Abelian.

We first reduce to the case where all G_{M_i} are isomorphic to $GL(2)$. Assume, for example, G_{M_1} Abelian. Then G_{M_1} acts by two characters on M_1; since their restriction to $\mathbb{G}_m$ via ν are $x \longmapsto (x^{-1}, 1)$ they are not isomorphic. This implies that M_1 splits into two L-Hodge structures. Then we see that $M_2 \otimes \cdots \otimes M_g$ verifies analogous conditions, with the number of variables reduced.

We may now assume that $G_{M_i} = GL(2)$ for all i; we consider $G_0 \subset G_{M_1} \times \cdots \times G_{M_g} = GL(2)^g$. (We will now simply argue with the complex points of the Mumford–Tate groups). Consider its derived group $G_{0,der} \subset SL(2)^g$.

We will apply a variant of Goursat's lemma.

LEMMA 5.6. — *Suppose $A \subset SL(2)^g$ is a connected semi-simple subgroup such that the image of A by each projection is equal to $SL(2)$. Then A is, modulo permutation of $\{1, \ldots, g\}$, of the form*

$$\{(\varphi_1(x_1), \ldots, \varphi_{g_1}(x_1), \varphi_{g_1+1}(x_2), \ldots, \varphi_{g_1+g_2}(x_2), \ldots, \varphi_{g-g_{r+1}}(x_r), \ldots, \varphi_g(x_r))\}$$

where $(x_j) \in SL(2)^r$ and the φ_i are automorphisms of $SL(2)$.

Proof. Write $SL(2)^g = SL(2) \times SL(2)^{g-1}$ and let B be the image of A by the map $SL(2)^g \to SL(2)^{g-1}$. Then by induction B is of the form indicated. Up to automorphisms of the factors we may write

$$B = \{(x_1, \ldots, x_1,\ x_2, \ldots, \ldots, x_r, \ldots x_r)\} \cong SL(2)^r . \tag{5.11}$$

If $\mathrm{Ker}(A \to B)$ is finite it must be 1 since $SL(2)$ is simply connected. Thus $A \cong B$. The first projection applied to each $SL(2)$–factor of B must be trivial or the identity. Since the factors commute it is the identity for exactly one factor. The lemma follows.

Suppose then $\mathrm{Ker}(A \to B) = C$ infinite.

The map $C \to SL(2)$ given by the first projection is injective. Since C is clearly semi–simple this map must be an isomorphism. Thus A is clearly semi–simple this map must be an isomorphism. Thus A contains $SL(2)$ embedded into $Sl(2)^r$ in the first component. This implies that $A = C \times B$, q.e.d.

Write then

(5.12)

$$G_{0,der} = \{\varphi_1(x_1), \ldots, \varphi_{g_1}(x_1), \varphi_{g_1+1}(x_2), \ldots \varphi_{g_1+g_2}(x_2), \ldots, \varphi_{g-g_r+1}(x_r), \ldots, \varphi_g(x_r)\}$$

where $g = g_1 + \cdots + g_r$, $x_i \in SL(2)$ and the φ_j are automorphisms of $SL(2)$. In the tensor product representation, $G_{0,der}$ leaves $N \subset M$ invariant. We will denote by e_i, f_i the vectors of type $(1,0), (0,1)$ in each M_i.

Suppose first $g_1 < g$. Then for $x \in SL(2)$,

$$(\varphi_1(x), \ldots, \varphi_{g_1}(x), 1, \ldots)(\otimes e_i) = \varphi_1(x)e_1 \otimes \cdots \otimes \varphi_{g_1}(x)e_{g_1} \otimes e_{g_1+1} \otimes \cdots \otimes e_g .$$

Since this must be a linear combination of $e = \otimes e_i$ and $f = \otimes f_i$, we must needs have $\varphi_1(x)e_i \in \mathbb{C}e_i$ for $i \leq g_1$. This is impossible.

Therefore $g_1 = g$, and the same argument implies that $\varphi_i(x)e_i \in \mathbb{C}e_i$ for all i or $\varphi_i(x)\ell_i \in \mathbb{C}f_i$ for all i, which is again impossible.

The conclusion is that G_{M_i} is abelian for all i, except if $g = 1$. If we return to the reduction to the case $G_{M_i} \cong GL(2)$, we see that our argument is complete except if all G_{M_i} but one are Abelian. However, in this case, M is a direct sum of 2^{g-1} summands of type $((1,0)\oplus(0,1))\otimes T$ where T is of type (p,q) with $p+q \leq g-1$. Each of these summands is irreducible, and its intersection with N would necessarily be one–dimensional, which is impossible (for $g \geq 2$). This concludes the first proof of Theorem 5.2.

The second proof is based on the following lemma :

Lemma 5.7. — *Let X, Y be $\mathbb{Q}$–Hodge structures whose Mumford–Tate group is reductive. Suppose X, Y are pure of weights a, b with $a, b > 0$. Let $Z \subset X \otimes Y$ be a $\mathbb{Q}$–Hodge structure of type $\{(p,0)\,;\,(0,p)\}$ with $p = a + b$. Then*

(i) *The Mumford–Tate group G_Z is Abelian.*

(ii) *If X, Y are irreducible, G_X and G_Y are also Abelian.*
(iii) *X is of type $\{(a,0)\,;\,(0,a)\}$ and Y of type $\{(b,0)\,,\,(0,b)\}$.*

We apply this lemma, arguing inductively, to an embedding $N \subset M_1 \otimes \cdots M_g$ where N is an irreducible summand over $\mathbb{Q}$ of $\widetilde{H}^g_!(S(K_f))$ and M_i are irreducible summands of $H^1(\overline{C}_i)$ over $\mathbb{Q}$. The conclusion is that G_N, and the G_{M_i}, are Abelian.

We now prove Lemma 5.7. We now denote by G_X the Mumford–Tate group of a $\mathbb{Q}$–Hodge structure X; as above we view G_X as a subgroup of $GL(X)$. Let $G = G_{X \oplus Y}$: we then have natural surjections $G \to G_X$, $G \to G_Y$ and $G \to \cancel{G_Z}$. Let $\mathfrak{g}_X$, $\mathfrak{g}_Y$, $\mathfrak{g}_Z$, $\mathfrak{g}_{X\oplus Y} = \mathfrak{g}$ denote the Lie algebras. Then $\mathfrak{g}_Z \subset \mathrm{End}(Z) \cong Z \otimes Z^*$. As such $\mathfrak{g}_Z$ has a Hodge decomposition, of types $(0,0), (p,-p)$ and $(-p,p)$. / G_Z

The Hodge type of $\mathfrak{g}_X$ are of type $(i-k, j-\ell)$ with $a = i+j = k+\ell$. If this is of type $(p,-p)$ we get $a+b = p = i-k \le i \le a$, an impossibility. Thus $\mathfrak{g}_X^{(-p,p)} = \mathfrak{g}_X^{(p,-p)} = \{0\}$ and the same is true for $\mathfrak{g}_Y$. Consequently $\mathfrak{g}^{(p,-p)} = \mathfrak{g}^{(-p,p)} = 0$, whence $\mathfrak{g}_Z^{(p,-p)} = \mathfrak{g}_Z^{(-p,p)} = 0$.

Thus $\mathfrak{g}_Z$ if of type $(0,0)$ which implies that G_Z commutes with $\nu(\mathbb{C}^\times)$. Now denote by M_Z the Mumford–Tate group of Z, **seen as a subgroup** of $GL(Z) \times \mathbb{G}_m$: then [, Prop. 3.4] M_Z is the smallest subgroup of $GL(Z) \times \mathbb{G}_m$, defined over $\mathbb{Q}$, and whose set of complex points contains $\mu(\mathbb{C}^*)$ where

$$\mu(z) = (\nu(z), z^{-1}) \quad (z \in \mathbb{C}^\times)\,. \tag{5.13}$$

Since $M_Z \subset G_Z \times \mathbb{C}^\times$, M_Z commutes with μ; therefore $\mu(\mathbb{C}^\times)$ is contained in the center of M_Z : this is a $\mathbb{Q}$–subgroup of M_Z verifying the defining condition of M_Z, and we conclude that M_Z is Abelian; so is G_Z. This proves (i).

As to (iii), note that if X, Y are irreducible, we may find a number field L and absolutely irreducible summands X_1 and Y_1 of $X \otimes L$ and $Y \otimes L$ such that X, Y are sums of conjugates of X_1 and Y_1 in the obvious sense. Extending scalars to L, we are reduced to the case when $Z \subset X \otimes Y$ and X, Y are absolutely irreducible. Since the Mumford–Tate group of Z is Abelian, we may then replace Z (perhaps after a further extension of scalars) by a one–dimensional sub–Hodge structure contained in $X \otimes Y$. We then have a morphism of Hodge structures over L :

$$X^* \otimes Z \longrightarrow Y \tag{5.14}$$

which is an isomorphism by irreducibility. Moreover Z is now of type $(p,0)$ or $(0,p)$: assume the latter. If (i,j) is a type of X and (k,ℓ) a type of Y we now have

$$-i = k\,, \quad -j + p = \ell \tag{5.15}$$

which implies $i = k = 0$ and $j = a$, $\ell = \cancel{p}$. This, after an obvious argument of Galois descent, implies (iii); moreover the weight homomorphisms $\mathbb{G}_m \to GL(X)$ and $\mathbb{G}_m \to GL(Y)$ are now scalar, and the argument given in the proof of (i) implies (ii). / b

Proof of Theorem 5.3

We now consider the restriction maps

$$j^* : (H_!^g(S(K_f)), \mathbb{Q}_\ell) \longrightarrow H_!^g(C_1 \times \cdots \times C_g, \mathbb{Q}_\ell)$$

in étale cohomology. These maps can be defined over number fields, and therefore commute with the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for some sufficiently large number field F. The previous proof shows that the image is contained in the subspace of $H_!^g(C_1 \times \cdots \times C_g, \cancel{d})$ contributed by modular forms yielding Hodge structures of CM $\mathbb{Q}_\ell$
type. The associated Galois representation then arise from factors of the Jacobians of the curves C_i that are of CM type. According to Shimura–Taniyama and Weil the associated Galois representations are potentially Abelian ([Bor], [Pi]; cf. Serre [Se2]).

5.3. — We end this paragraph with a few remarks. First we note that the argument in the proof of Theorem 5.12 could be applied directly to the Galois representations – hence proving Theorem 5.3 – if one could use Hodge–Tate theory. This would however require information on the Hodge–Tate nature of $\widetilde{H}_!^g(S(K_f)), \mathbb{Q}_\ell)$, i.e., Conjecture 5.4. In turn this conjecture would follow if a suitable comparison theorem between ℓ–adic and De Rham cohomology applied to $\widetilde{H}_!^g$, since De Rham cohomology is computed (over $\mathbb{C}$) by Lemma 5.1. However, the theory of Hodge–Tate for open varieties does not seem to be sufficiently developed for this proof.

For $g = 2$, Weissauer [Weis] has been able to prove Theorem 5.3 directly; Theorem 5.2 is implicit in his paper[1]. We could have obtained Theorem 5.3, in the even case (g even), by reduction to his result, using the analogue of Lemma 5.5 to restrict classes in $\widetilde{H}_!^g$ to products of Siegel threefolds, to which his results apply. It is then also possible to treat the case that g is odd, but at the cost of complications. Our present proof is more direct. Finally, it is also possible to treat the case of coefficient systems : we leave this to the interested reader.

We also note that the previous proof implies that classes in $\widetilde{H}_!^g(S(K_f))$ restrict to **cuspidal** cohomology classes on products of modular curves. On the other hand, there is no **a priori** reason for such classes on $S(K_f)$ to be cuspidal (they are only square–integrable). In fact, Weissauer [Weis] shows the existence, for $g = 2$, of non–cuspidal holomorphic classes.

We note that this phenomenon is more general. Suppose $F_1, \ldots, F_r$ are totally real fields of degrees $g_1, \ldots, g_r$ over $\mathbb{Q}$, with $g_1 + \cdots + g_r = g$. Let $G_i = \mathrm{Res}_{F_i/\mathbb{Q}} GL(2)$. Then G_i has a natural homomorphism "norm of determinant" $\nu_i : G_i \to \mathbb{G}_m$ over $\mathbb{Q}$. Let H be the group

$$H = \{(x_1, \ldots, x_g) : x_i \in G_i,\ \nu_i(x) = \cdots = \nu_r(x_r)\}. \tag{5.16}$$

Then there is a natural map $H \to G$. A corollary of the previous arguments is :

[1] See also Blasius–Rogawski [Bl–Ro].

PROPOSITION 5.8. — *($g \geq 3$) A class $\omega \in \widetilde{H}^g_!(S(K_f))$ restricts to a cuspidal class in $H^g_!(S(H, K_H(x)))$ for any map $j_x : S(H, K_H(x)) \to S(K_f)$.*

Proof. We may assume ω of type $(g, 0)$. The arguments of § 5.2 then show that the restriction of ω yields a class of type $(g, 0)$, in the image of H^g_c, in the g-th cohomology of a product $C_1 \times \cdots \times C_r$ of Shimura varieties associated to G_i. But according to Harder [Ha2] all such classes are cuspidal. More precisely, if $C = C_1 \times \cdots C_r$, Harder shows [Ha2, p. 65] that

$$H^g_!(C) = H^g_{\text{cusp}}(C) \oplus H^g_{!,res}(C), \tag{5.17}$$

the second factor coming from Eisenstein series associated to Grössencharakterer of the F_i. The F_i being totally real all such classes are Tate classes.

If $r = 1$, and we therefore consider an embedding $GL(2, F) \to GSp(g)$ with $[F : \mathbb{Q}] = g$, it is an exercise on Siegel domains to check directly Proposition 5.8 from the growth estimates on ω coming from its square–integrability (use Lemmas I.4.1 and I.4.11 of Moeglin–Waldspurger [Mo–We]. For $r > 1$ we have not tried to give a direct proof; for **cuspidal classes** of course the result follows from Proposition 2.8.

6. — The case of unitary or orthogonal groups

6.1. — In this section we consider a group of unitary similitudes $G = GU(Q)$ associated to a quadratic imaginary extension F of $\mathbb{Q}$ and an Hermitian form Q on $F^n = V$. Write $V = V_h \oplus V_a$, where V_h is a sum of hyperbolic planes and $Q_a = Q|_{V_a}$ is anisotropic. Let $2p = \dim V_h$, $m = \dim V_a$. As is well-known, $Q_a \otimes \mathbb{R}$ is definite if $m \neq 2$. The group $G(\mathbb{R})$ then has signature $(p+m, p)$. If $m = 2$ and $Q_a \otimes \mathbb{R}$ is indefinite, $G(\mathbb{R})$ has signature $(p+1, p+1)$.

In the first case Q can be written in a basis of V with the matrix

$$J = \left(\begin{array}{cc|ccc} & -1_p & & & 0 \\ 1_p & & & & \\ \hline & & \iota\xi_1 & & \\ & & & \ddots & \\ & 0 & & & \iota\xi_m \end{array}\right)$$

with $\xi_i \in \mathbb{Q}^\times$, $\xi_i < 0$ and $\iota \in F$ an element such that $\tau = -\iota$. (We assume $\iota^{-1} Q_a \otimes \mathbb{R}$ **negative**). The group G contains the subgroup H of $GSp(r) \times GU(Q_a)$ defined by the equality of the ratios of similitude. Consider $h : \mathbb{C}^\times \to G(\mathbb{R})$

$$Z = x + iy \longmapsto \left(\begin{array}{cc|ccc} x & y & 0 & & \\ -y & x & & & \\ \hline & & z & & \\ & 0 & & \ddots & \\ & & & & z \end{array}\right).$$

Then h factors through $H(\mathbb{R}) \subset G(\mathbb{R})$. The centralizer K_∞ of h in $G(\mathbb{R})$ is isomorphic to the subgroup $G(U(p) \times U(p+m))$ of $GU(p) \times GU(p+m)$ defined by the equality of the similitude ratios, if $m \neq 0$. (If $m = 0$ this is a subgroup of index 2 of K_∞).

In the second case, we take h given (for an appropriate basis of V_a) by

$$Z \longmapsto \left(\begin{array}{cc|cc} x & y & & \\ -y & x & & \\ \hline & & z & \\ & & & \bar{z} \end{array}\right).$$

The centralizer of h then has a subgroup of index 2 isomorphic to $G(U(p+1) \times U(p+1))$.

The parameter h defines a family of Shimura varieties $S(h, K_f) = S(K_f)$ ($K_f \subset G(\mathbb{A}_f)$), defined over a reflex field E. We will not explicitate the field E; we recall only that for $m = 0$ (even quasi-split case) it is equal to $\mathbb{Q}$; for $m = 1$ (odd quasi-split case) it is equal to F.

Let $q = p + m$. Thus $p \leq q$. We will assume $p \geq 2$ and $q > 2$. Define $H_!^\bullet(S(K_f))$ as in § 5.1. Thus $H_!^i(S(K_f), \mathbb{C})$ carries a pure Hodge structure of weight i. Since it injects into $IH^i(S(K_f), \mathbb{C})$, we can apply the arguments of the previous paragraph :

LEMMA 6.1. — *Let $\widetilde{H}^p_!(S(K_f),\mathbb{C})$ denote the complement in $H^q_!(S(K_f),\mathbb{C})$ of the (possible) contribution of the trivial representation. Then $\widetilde{H}^p_!(S(K_f),\mathbb{C})$ is purely of Hodge type $((p,0)\,;\,(0,p))$.*

Proof. This follows from Proposition 3.A.6 by the arguments given in § 5.1.

Our purpose in this section is to prove :

THEOREM 6.2. — *Assume $p \geq 2$, $q > 2$. Then the representation of* $\mathrm{Gal}(\overline{E}/E)$ *on $\widetilde{H}^p_!(S(K_f),\mathbb{Q}_\ell)$ is potentially Abelian.*

THEOREM 6.3. — *$(p \geq 2$, $q > 2)$. The Hodge structure on $\widetilde{H}^p_!(S(K_f),\mathbb{C})$ has Abelian Mumford–Tate group.*

We will prove these theorems by reduction to the case of the group $GSp(p)$. Until the end of this section, we assume $m \neq 2$. Let $H \subset G$ be the $\mathbb{Q}$–subgroup described above. Then H has a natural morphism toward $M = GSp(g)/\mathbb{Q}$, whose kernel is $\mathbb{R}$–anisotropic. There is a family of associated morphisms $S(H,{K_f}^H) \rightarrow S(M,{K_f}^M)$ (with obvious notations) which are finite coverings defined over number fields. Therefore the assertions of Theorem 5.2 and 5.3 are true for H – using the results of Weissauer [Weis] and Blasius–Rogawski [Bl–Ro] if $p = 2$, and § 5 for $p > 2$.

For $x \in G(\mathbb{Q})$, consider as usual

$$j_x : S(H, K_H(x)) \longrightarrow S(G, K_f) = S(K_f) \tag{6.1}$$

and its effect on $\widetilde{H}^q_!(S(K_f),\mathbb{C})$. We must prove :

LEMMA 6.4. — *The map $\mathrm{Res} = \prod_{x\in G(\mathbb{Q})} j_x^*$ is injective.*

Proof. As in §5 we must show that if $\alpha \in \widetilde{H}^p_!(S(K_f),\mathbb{C})$ is of type $(p,0)$ and $\mathrm{Res}\,\alpha = 0$, then $\alpha = 0$. Let ω an L^2, holomorphic, representative of α. Fix $j = j_x$, and let $\beta = j^*\alpha$, and $\eta = j^*\omega$, a form of type $(p,0)$ on $S({K_f}^H)$ where ${K_f}^H = K_H(x)$. As noticed above $S({K_f}^H)$ is a finite cover of a Shimura variety for $GSp(p)$; since p$\geq$ 2 the congruence property holds for $GSp(p)$ and therefore it is (over $\mathbb{C}$ at least) a Shimura variety of $GSp(p)$. We can therefore apply the geometric arguments of § 5.

Set $X = S({K_f}^H)$ and let $\overline{X}$ be a smooth compactification of X. Then η extends to a smooth differential form $\overline{\eta}$ on $\overline{X}$. We would like to show that η (or $\overline{\eta}$) vanishes if the cohomology class β vanishes; β is the cohomology class associated to η.

Since the map $H^p(\overline{X}) \rightarrow H^p(X)$ has no reason to be injective even on classes of type $(p,0)$, we cannot argue directly on $\overline{X}$.

Consider however the natural morphism $SL(2)^p \rightarrow H$. The effect on Shimura varieties (**over** $\mathbb{C}$! these do not correspond to natural maps of moduli problems) give maps

$$j_1 : C_1 \times \cdots \times C_p \longrightarrow X$$

We can now consider $j_1^*\eta = \theta$, a form on $C_1 \times \cdots \times C_p$, and the associated cohomology class $\gamma = j_1^*\beta$. The argument in § 5.2 shows that θ extends to a form $\overline{\theta}$ on $\overline{C}_1 \times \cdots \times \overline{C}_p$, of type $(p,0)$. Since $\gamma = 0$, $\overline{\theta} = 0$ (again, see § 5.2, proof of Lemma 5.5) whence $\eta = 0$. By (the proof of) Proposition 3.A.5 and Proposition 3.C.4, the assumption that $\theta = 0$ for all j and j_1 implies that η, and then ω, vanish. Therefore $\alpha = 0$, q.e.d.

Proof of Theorems 5.2 and 6.3. They now follow from Lemma 6.4 and the fact that the analogous theorems hold for H.

We now simply sketch the proof when $m = 2$ and $Q_a \otimes \mathbb{R}$ is indefinite. In this case we must consider a group H which is a subgroup of $GSp(r) \times GU(Q_a)$, the group $GU(Q_a)$ being of type $(1,1)$ at infinity.

We first show that the restriction map from $\widetilde{H}^p_!(S(K_f), \mathbb{C})$ to $\prod_x H^q_!(S(H, K_H(x)), \mathbb{C})$ is injective, with the usual notations. This is a geometric problem, so we can as well consider $Sp(r) \times SU(Q_a) \subset G$. We are led to consider holomorphic p-forms on varieties $X_1 \times C$, where X_1 is a Shimura variety for $Sp(r)$ and C a Shimura curve coming from $SU(Q_a)$.

LEMMA 6.5. — *If X_1 is a quasi-projective variety, and C a projective variety, over $\mathbb{C}$, then*

$$H^0(X_1 \times C, \underline{\Omega}^p) = \bigoplus_{p_1+p_2=p} H^0(X_1, \underline{\Omega}^{p_1}) \otimes H^0(C, \underline{\Omega}^{p_2}) \tag{6.2}$$

This is clear. We may apply this to $X_1 \times C$ and to $\overline{X}_1 \times C$ where $\overline{X}_1$ is a smooth compactification of X_1. Using the result of Freitag–Pommerenke as an § 5.2, we then deduce that restriction is injective if it is so infinitesimally. For this we need :

LEMMA 6.6. — *Let G_1 be unitary a group of type $(r+1, r+1)$ at infinity, and $H_1 \subset G_1/\mathbb{Q}$ a subgroup of type $U(r,r) \times U(1,1)$ at infinity. If ω is a holomorphic $(r+1)$-form on a Shimura variety for G_1 then ω does not vanish stably along H_1. Moreover its restrictions are of type $((r,0)\,;\,(1,0))$ on the factors associated to $U(r,r) \times U(1,1)$.*

This follows from Proposition 3.A.8, except for the type of the restriction. For clarity we retrace the proof, using the notations of § 3A. It suffices to show that the vector $v = e_1 \otimes f_1 \wedge \cdots \wedge e_1 \otimes f_{r+1}$ is in the $\mathbb{C}K$-span of $w = e_1 \otimes f_1 \wedge \cdots \wedge e_1 \otimes f_r \wedge e_{r+1} \otimes f_{r+1}$, K being $U(r+1) \times U(r+1)$ acting in the obvious manner on $\mathbb{C}^{r+1} \otimes \mathbb{C}^{r+1}$. We consider a linear transformation $e_{r+1} \longmapsto e_1 + te_{r+1}$, $e_i \longmapsto e_i$ $(i \leq r)$. The constant term of the vector $w(t)$ obtained from w is then equal to v, q.e.d. Further, the first degrees where $U(r,r)$ can have holomorphic cohomology are r, $2r-1$. If $r > 2$ the last assertion follows; for $r = 1$ it is obvious. Assume $r = 2$: thus $K = U(3) \times U(3)$, $v = e_1 \otimes f_1 \wedge e_1 \otimes f_2 \wedge e_1 \otimes f_3$. The space $\Lambda^3 \mathfrak{p}^+_{H_1}$ is spanned by the four vectors obtained from $e_1 \otimes f_1 \wedge e_2 \otimes f_1 \wedge e_2 \otimes f_2$ by permutations of the indices in $\{1,2\}$. On the other hand, $K \cdot v$ is the 3-dimensional space spanned by v, $e_2 \otimes f_1 \wedge e_2 \otimes f_2 \wedge e_2 \otimes f_3$ and $e_3 \otimes f_1 \wedge e_3 \otimes f_2 \wedge e_2 \otimes f_3$. For the natural scalar product, one checks easily that $K \cdot v$ is orthogonal to $\Lambda^3 \mathfrak{p}^+_{H_1}$. Thus v is not in the $\mathbb{C}K$-span of $\Lambda^3 \mathfrak{p}^+_{H_1}$. This completes the proof.

The injectivity of restriction now follows from Lemmas 6.5 and 6.6.

We now finish the proof of Theorem 6.3. We have obtained an injective map of $\widetilde{H}^p_!(S(K_f),\mathbb{C})$ into a product of spaces of the form $H^p_!(S(H,K_H),\mathbb{C})$; H is a subgroup of $GSp(r)\times GU(Q_a)=H'$ and $S(H,K_H)$ is covered by a product of varieties $X_1\times C$ associated to the two factors; finally we have an injection of $\widetilde{H}^p_!(S(K_f),\mathbb{C})$ into a product of spaces of the form $H^g_!(X_1\times C)$, whose image falls in $H^{r,0}_!(X_1)\otimes H^{1,0}(C)$ by Lemma 6.6. Since $H^r_!(X_1)$ and $H^1(C)$ are pure Hodge structures of weights r and 1 respectively, Lemma 5.7 (i) implies that the Mumford–Tate group of (each absolutely irreducible factor of) $H^g_!(S(K_f),\mathbb{C})$ is Abelian. Moreover, Theorem 6.1 now follows as in §5, since by Lemma 5.7 all irreducible Hodge sub–structures $Z\subset H^r_!(X_1)$, $T\subset H^1(C)$ such that $\mathrm{Im}(H^p_!(S(K_f),\mathbb{C})$ meets $Z\otimes T$ are Abelian : this implies that the associated Galois representations are potentially Abelian by the results recalled in § 5. (N.B. : since $r>1$, it follows from § 5 that the Galois representations on $H^r_!(X_1)$ are Abelian : note however that we do need a further argument (Lemma 5.7) to control the contributions of $H^1(C)$, since the whole space $H^1(C)$ is **not** Abelian.

6.2. — We now consider the case of a unitary group G over $\mathbb{Q}$ of type $(2,2)$ at the infinite prime.

Assume first that G is quasi–split : it contains a copy H of $Sp(2)$ as in § 6.1. Let $\mathfrak{g}=LieG(\mathbb{R})$.

Lemma 6.7. — *There are 6 Vogan–Zurkerman modules* $A_{\mathfrak{q}}$ *such that* $H^2(\mathfrak{g},K_\infty;A_{\mathfrak{q}})\neq 0$ *: the trivial representation, two modules whose* H^2 *is primitive of type* $(2,0)$ *or* $(0,2)$*, and one module having primitive cohomology of type* $(1,1)$.

The proof is easy. Write $\mathfrak{q}=\mathfrak{q}(x)$ with $x=(a_1,a_2,b_1,b_2)$ as in § 3A. We get two distinct types of holomorphic $A_{\mathfrak{q}}$'s associated to (a,a,a,b_2) with $a>b_2$, and (a_1,b,b,b) with $a_1>b$. Analogously, there are two types of antiholomorphic modules. Finally, $x=(a_1,a_2,a_1,a_2)$ yields a module of type $(1,1)$, unique up to isomorphism. Denote by A_1, A_2 the holomorphic representations associated to (a,a,a,b_2) and (a_1,b,b,b) respectively.

Now the space $\widetilde{H}^2(S(K_f),\mathbb{C})$ carries a Hodge structure of type $\{(2,0),(0,2),(1,1)\}$. Consider the restriction maps to the spaces $S(H,K_H),\mathbb{C})$ with the notations of § 6.1. We get a map $\widetilde{H}^2(S(K_f),\mathbb{C})\xrightarrow{\mathrm{Res}}\prod_x \widetilde{H}^2(S(H,K_H(x)),\mathbb{C})$. The spaces on the right are again pure Hodge structures, and it follows from the results of Weissauer [Weis] and Blasius–Rogawski that each of these Hodge structures is, over $\mathbb{Q}$, the direct sum of its part of type $\{(2,0),(0,2)\}$ and of its part of type $(1,1)$: these components are separated by Hecke operators.

We may then consider the restriction map Res, composed with projection on the $(2,0),(0,2)$ part. Its kernel contains (over $\mathbb{C}$) $\widetilde{H}^2(S(K_f),\mathbb{C})^{11}$ and is a $\mathbb{Q}$–Hodge structure. Moreover (over $\mathbb{C}$) Res is injective, by Proposition 3.A.9, on $\widetilde{H}^2(S(K_f),\mathbb{C})^{(2,0)+(0,2)}$. Consequently $\widetilde{H}^2(S(K_f),\mathbb{C})^{11}$ splits as a Hodge structure over $\mathbb{Q}$. Moreover the previous arguments imply that the part of type $\{(2,0),(0,2)\}$ of $\widetilde{H}^2$ is Abelian, since this is known for $Sp(2)$. We have proved :

THEOREM 6.8. — *Suppose G is a quasi-split unitary group of (absolute)* rank 4 *over $\mathbb{Q}$: thus $G(\mathbb{R}) \cong U(2,2)$. Then $\widetilde{H}^2(S(K_f),\mathbb{C})$ is a direct sum, as a Hodge structure :*

$$\widetilde{H}^2 = \widetilde{H}^{11} \oplus \widetilde{H}^{(2,0)+(0,2)}$$

over $\mathbb{Q}$. Both Hodge structures are Abelian. The corresponding results hold for the Galois representations.

In fact, $\widetilde{H}^{11}$ is a Hodge structure of type $(1,1)$, thus associated to a scalar representation of $\mathbb{G}_m$: it splits into 1–dimensional subspaces over $\mathbb{Q}$. Note that moreover, because of the Lefschetz $(1-1)$–theorem, all rational classes in $\widetilde{H}^{11}$ should be represented by cycles : this would follow if one could show that their images in the cohomology of a suitable smooth compactification of $S(K_f)$ do not vanish. We have not pursued this problem.

We now consider the anisotropic case. By taking a 3–dimensional subspace of the 4–dimensional Hermitian space, we obtain an embedding $H \hookrightarrow G$, where H is a unitary group of type $(2,1)$ at infinity. Upon restriction to (varieties associated to) H, the classes of type A_1 vanish while restriction is injective for classes of type A_2 (Propositions 3A5, 3A6).

The consequence is that classes of type A_1 or A_2 belong to different Hodge structures (over $\mathbb{Q}$). If one could show that classes associated to the representation of type $(1,1)$ lie in a disjoint Hodge structure, the argument shetched above would again show that they are Tate classes. This will require further work on the trace formula.

6.3. — Orthogonal groups.

Suppose G is a group over $\mathbb{Q}$ such that $G(\mathbb{R})$ is isogenous to $SO(2,m)$. We will assume $m \geq 5$. For $K \subset G(\mathbb{A}_f)$ compact–open, we consider the intersection cohomology space $IH^2(S_K(\mathbb{C}),\mathbb{C})$.

LEMMA 6.9. — $IH^2(S_K(\mathbb{C}),\mathbb{C})$ *is purely of type* $(1,1)$.

This is Proposition 3BD3.

Suppose now G **anisotropic** over $\mathbb{Q}$.

THEOREM 6.10. — *(G anisotropic over $\mathbb{Q}$, $G(\mathbb{R})$ isogenous to $SO(2,m)$)*

(i) *$H^2(S_K,\mathbb{C})$ is purely of type $(1,1)$, and consequently spanned by algebraic classes*

(ii) *If E is a field of definition for S_K, the action of* $\mathrm{Gal}(\overline{E}/E')$ *on $H^2(S_K,\mathbb{Q}_\ell)$ is, for an finite extension E' of E, an extension of* 1*–dimensional representations isomorphic to the Tate character.*

This is clear : (i) follows from the Lemma. By the Lefschetz $(1-1)$–theorem, $H^2(S_K,\mathbb{C})$ is spanned by the classes of cycles, which may be defined over a finite extension E'. Then (ii) follows.

7. — Summation

An infernal conjecture about the cohomology of Shimura varieties is the following – we limit ourselves to varieties over $\mathbb{Q}$.

PROBLEM 7.1. — *Let G be a reductive group over $\mathbb{Q}$, with $G_{der} \times \overline{\mathbb{Q}}$ simple and $G(\mathbb{R})$ Hermitian of real rank r. Suppose the absolute rank of $G > 1$. Then is $H^r(S_K)$ Abelian (as a Hodge structure, or a Galois representation)?* (potentially Abelian as)

Here H^r should be interpreted as IH^r (intersection cohomology), or $H^r_!$ according to our choices in this paper.

Numerous cases of this conjecture are known, the first mention of the problem being seemingly due to Oda [Oda] : see in particular Kumar–Ramakrishnan [KuR], Blasius–Rogawski [Bl–Ro], Weissauer [Weis]. We now review what is known :

7A. — Unitary groups

For H^1 the result is due to [KuR] and [Bl–Ro]. For $r \geq 2$ and G a true unitary group (over $\mathbb{Q}$) it is proved here, except for non-split $U(2,2)$ (see § 6.1, 6.2). This leaves the case where G is a "fake" unitary group, associated to a Hermitian space over a simple central algebra D with an involution of the second kind. If D is a division algebra (and with a few restrictions) the result follows from Theorem 3.3 of [Clo4] and its proof when $G = U(D)$ – the unitary group associated to a 1–dimensional Hermitian module over D.

7B. — Orthogonal groups

The simplest result is Theorem 6.10. It would extend, correctly rephrased, to isotropic groups and $H^2_!$ if one could show that these classes extend injectively to a suitable smooth compactification.

7.C. — Symplectic groups

Here the result is given by Theorems 5.2 and 5.3. We have not treated the case of anisotropic groups (associated to quaternionian forms, see Deligne [Del]) since we consider only groups over $\mathbb{Q}$; however the method of § 5 extends to these cases, at least for $g > 2$ (where one can use Lemma 5.1).

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