Duality in the Spaces of Solutions of Elliptic Systems

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MPI / 96-4

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December 20, 1995

Abstract

Let P be a determined or overdetermined elliptic differential operator of order p with real analytic coefficients on an open set $X \subset \mathbf{R}^n$. Using Green's functions for the Laplacian P^*P we prove that the dual for the space $sol(\mathcal{D})$ of solutions to the system $Pu = 0$ in a domain $\mathcal{D} \Subset X$ with real analytic boundary can be represented as the space $sol(\overline{\mathcal{D}})$ of solutions on neighborhoods of the closure of $\mathcal{D}$, provided the domain $\mathcal{D}$ possesses some convexity property with respect to the operator P.

AMS subject classification: primary: 35J45; secondary: 35N10.
Key words and phrases: elliptic equations, duality.

*The authors thank Prof. E. Vesentini for many useful discussions.
§Supported by a grant of the Ministry of Science of the Land Brandenburg.

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Introduction

The aim of this paper is to give representations of the strong dual of the space of solutions of a linear elliptic system $Pu = 0$ of partial differential equations on an open subset of $\mathbb{R}^n$. We consider both determined and overdetermined elliptic systems.

Let U be an open subset of the domain $X \subset \mathbb{R}^n$ where the operator P is defined. Denote by $sol(U, P)$ the vector space of all smooth solutions to the equation $Pu = 0$ on U, with the usual Fréchet-Schwartz topology. We will write it simply $sol(U)$ when no confusion can arise.

Denote by $sol(U)'$ the dual space of $sol(U)$, i.e., the space of all continuous linear functionals on $sol(U)$. We tacitly assume that this dual space $sol(U)'$ is endowed with the *strong topology*, i.e., the topology of uniform convergence on every bounded subset of $sol(U)$.

Any successful characterization of the dual space $sol(U)'$ results in the analysis of solutions to $Pu = 0$ (*Golubev series*, etc., see Havin [3], Tarkhanov [14]).

There are a few classical examples of representation of this dual space, such as *Grothendieck duality* and Poincaré duality (see for instance Tarkhanov [15, Ch.5]). The Grothendieck duality is of analytical nature; it has been of particular interest in complex analysis. On the other hand, the Poincaré duality can be stated in an abstract framework.

For determined elliptic operators of the type P^*P we obtain in Section 3 an analogue of the duality result of Grothendieck [2] (cf. Mantovani and Spagnolo [6]). Note that the system $P^*Pu = 0$ is a straightforward generalization of the *Laplace equation.* In this way we obtain what we shall call generalized harmonic functions, or simply *harmonic functions* when no confusion can arise.

Our main result for general elliptic systems is concerned with the case where the coefficients of P are real analytic and U is a relatively compact subdomain of X with real analytic boundary. In this case we prove the following theorem.

Theorem A. *Let the coefficients of the operator P be real analytic on X and $\mathcal{D} \Subset X$ be a domain with real analytic boundary. Suppose that, given any neighborhood U of $\overline{\mathcal{D}}$, there is a neighborhood $U' \subset U$ of $\overline{\mathcal{D}}$ such that $sol(U')$ is dense in $sol(\mathcal{D})$. Then*

$$sol(\mathcal{D})' \overset{top.}{\cong} sol(\overline{\mathcal{D}}).$$

In fact, in Sections 6, 7 below, we will formulate and prove a stronger statement with weaker assumptions on analyticity. Moreover, in these sections we provide also an explicit formula for the pairing.

In fact, there is a trasparent heuristic explanation of this duality. Given any solution $v \in sol(\overline{\mathcal{D}})$, the *Petrovskii Theorem* shows that v is real analytic in a neighborhood of $\overline{\mathcal{D}}$. On the other hand, each $u \in sol(\mathcal{D})$ is real analytic in $\mathcal{D}$, and so u is a hyperfunction there. As the sheaf of hyperfunctions is *flabby*, u can be extended to a hyperfunction in X with a support in the closure of $\mathcal{D}$. Thus, v can be paired with every $u \in sol(\mathcal{D})$.

By *Runge Theorem*, the approximation assumption of Theorem A holds for every determined elliptic operator with real analytic coefficients or in the case where P is an elliptic operator with constant coefficients and $\mathcal{D}$ is convex.

The approximation condition on the couple P and $\mathcal{D}$ in this theorem is to some extent an analogue of the so-called *approximation property* introduced by Grothendieck [2]. In several complex variables a close concept is known as *Runge property* (cf. Hörmander [4]).

For the space of holomorphic functions in simply connected domains in $\mathbb{C}$ and in (p, q)-circular domains in $\mathbb{C}^2$ a similar result was obtained by Aizenberg and Gindikin [1]. For the spaces of harmonic and holomorphic functions a similar result was recently obtained by Stout [12]. However they constructed isomorphisms different from ours. The advantage of our approach is the fact that it highlights the close connection between the duality of Theorem A and the Grothendieck duality (see Section 3).

1 Preliminaries

Assume that X is an open set in $\mathbf{R}^n$, and $E = X \times \mathbb{C}^k$, $F = X \times \mathbb{C}^l$ are (trivial) vector bundles over X. Sections of E and F of a class $\mathfrak{C}$ on an open set $U \subset X$ can be interpreted as columns of complex valued functions from $\mathfrak{C}(U)$, that is, $\mathfrak{C}(E|_U) \cong [\mathfrak{C}(U)]^k$, and similarly for F.

Throughout the paper we will usually write the letters u, v for sections of E, and f, g for sections of F.

A differential operator P of order $p \geq 1$ and type $E \to F$ can be written in the form $P(x, D) = \sum_{|\alpha| \leq p} P_\alpha(x) D^\alpha$, with suitable $(l \times k)$-matrices $P_\alpha(x)$ of smooth functions on X.

The *principal symbol* $\sigma(P)$ of P is a function on the cotangent bundle of X with values in the space of bundle morphisms $E \to F$. Given any $(x, \xi) \in X \times \mathbf{R}^n$, we have $\sigma(P)(x, \xi) = \sum_{|\alpha|=p} P_\alpha(x) \xi^\alpha$.

We say that P is *elliptic* if the mapping $\sigma(P)(x, \xi) : \mathbf{C}^k \to \mathbf{C}^l$ is injective for every $x \in X$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Hence it follows that $l \geq k$; we say that P is *determined* elliptic if $l = k$, and *overdetermined* elliptic if $l > k$.

Every elliptic operator is hypoelliptic, i.e. all distribution sections satisfying $Pu = 0$ on an open set U of X are infinitely differentiable there. If U is an open subset of X, then we denote by $sol(U, P)$ the vector space of all C^∞ solutions to the equation $Pf = 0$ on U. We will write it simply $sol(U)$ when no confusion can arise.

We endow the space $sol(U)$ with the topology of uniform convergence on compact subsets of U. This topology is generated by the family of seminorms

$$\|u\|_{C(E|_K)} = \sup_{x \in K} |u(x)|,$$

where K runs over all compact subsets of U.

Lemma 1.1 *If $U \subset X$ is open, then the topology in $sol(U)$ coincides with that induced by $C^\infty_{loc}(E|_U)$. In particular, $sol(U)$ is a Fréchet-Schwartz space.*

Proof. By a priori estimates for solutions of elliptic equations, if K' and K'' are compact subsets of U and K' is a subset of the interior of K'', then

$$\sup_{|\alpha| \leq j} \|D^\alpha u\|_{C(E|_{K'})} \leq c \, \|u\|_{C(E|_{K''})} \qquad \text{for all } u \in sol(U), \tag{1.1}$$

with c a constant depending only on K', K'' and j. Hence it follows that the original topology on $sol(U)$ coincides with that induced by $C^\infty_{loc}(E|_U)$. To finish the proof we use the fact that $C^\infty_{loc}(E|_U)$ is a Fréchet-Schwartz space.

□

Throughout this paper we assume that the operator P possesses the following Unique Continuation Property:

$(U)_s$ *given any domain $\mathcal{D} \subset X$, if $u \in sol(\mathcal{D})$ vanishes on a non-empty open subset of $\mathcal{D}$, then $u \equiv 0$ on $\mathcal{D}$.*

Here and in the sequel, by a domain is meant any open connected subset of $\mathbb{R}^n$. This property holds, for instance, if the coefficients of the operator P are real analytic.

It is natural to consider solutions to the system $Pu = 0$ on open sets. However, some problems require to consider solutions on sets $\sigma \subset X$ which are not open. Here we are interested not simply in restrictions of solutions to the given set, but also in the *local solutions* of the system $Pu = 0$ on σ, that is, solutions of the system in some (open) neighborhoods of σ.

If σ is a closed subset of X, then $sol(\sigma)$ stands for the space of (equivalence classes of) local solutions to $Pu = 0$ on σ. Two such solutions are *equivalent* if there is a neighborhood of σ where they are equal. In $sol(\sigma)$, a sequence $\{u_\nu\}$ is said to converge if there exists a neighborhood $\mathcal{N}$ of σ such that all the solutions are defined at least in $\mathcal{N}$ and converge uniformly on compact subsets of $\mathcal{N}$.

Alternatively, $sol(\sigma)$ can be described as the inductive limit of the spaces $sol(U_\nu)$, where $\{U_\nu\}$ is any decreasing sequence of open sets containing σ such that each neighborhood of σ contains some U_ν and such that each connected component of each U_ν intersects σ. (This latter condition guarantees that the maps $sol(U_\nu) \to sol(\sigma)$ are injective. Then the space $sol(\sigma)$ is necessarily a Hausdorff space.)

Lemma 1.2 *Let the operator P possess the Unique Continuation Property $(U)_s$. Then the space $sol(\sigma)$ is separated, a subset is bounded if and only if it is contained and bounded in some $sol(U_\nu)$, and each closed bounded set is compact.*

Proof. This follows by the same method as in Köthe [5, p.379].

□

2 Green's function

Denote by $E^* = X \times (\mathbb{C}^k)'$ the conjugate bundle of E, and similarly for F. For the operator P, we define the transpose P' as usual, so that P' is a differential operator of type $F^* \to E^*$ and order p on X.

Fix the standard Hermitian structure in the fibers $E_x = \mathbb{C}^k$ $(x \in X)$ of E: $(u,v)_x = \sum_{j=1}^k u_j \overline{v_j}$ for $u, v \in \mathbb{C}^k$. This gives the conjugate linear bundle isomorphism $\star_E : E \to E^*$ by $\langle \star_E v, u\rangle_x = (u,v)_x$ for $u, v \in E_x$.

Using matrix operation conventions, we have $\langle \star_E v, u\rangle_x = v^* u$ for $u \in \mathbb{C}^k$, where v^* is the conjugate matrix: we have $\star_E v = v^*$ under this identification.

The operator $\star_E$ also acts on sections of E via $(\star_E u)(x) = \star_E(u(x))$ for all $x \in X$. Thus, for a class $\mathfrak{C}$ of sections of E we have $\star_E : \mathfrak{C}(E) \to \mathfrak{C}(E^*)$.

The operator $\star_E$ is similar to Hodge's *star operator* on differential forms. We write simply $\star$ when no confusion can arise.

We are now in a position to endow the spaces $C^\infty_{comp}(E)$ and $C^\infty_{comp}(F)$, consisting of infinitely differentiable sections with compact supports of E and F respectively, with (L^2-) pre-Hilbert structures by $(u,v)_X = \int_X \langle \star v, u\rangle_x \, dx$.

Under these structures, the operator P has a formal adjoint operator which is denoted by P^*. This is the differential operator of type $F \to E$ and order p on X given by $P^* g\,(x) = \sum_{|\alpha| \le p} D^\alpha (P_\alpha(x)^* g(x))$ for $g \in C^\infty_{comp}(F)$.

The relation between the transposed operator and its (formal) adjoint becomes clear by using the bundle isomorphism $\star$. Namely, $P^* = \star_E^{-1} P' \star_F$ (see Tarkhanov [14, 4.1.4] for more details).

The operator $\Delta = P^*P$ is usually referred to as the generalized *Laplacian* associated to P. It is easy to see that Δ is an elliptic differential operator of type $E \to E$ and order $2p$ on X.

Throughout the paper we shall even assume that the operator Δ possesses the Unique Continuation Property $(U)_s$. Obviously, this implies that P does so.

If P is the gradient operator in $\mathbf{R}^n$, then $\Delta = P^*P$ is the usual Laplace operator up to a -1 factor. On the other hand, if P is the Cauchy-Riemann operator in $\mathbb{C}^n$, then $\Delta = P^*P$ coincides with the usual Laplace operator on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ up to a $-\frac{1}{4}$ factor.

In the general case, the solutions of the system $\Delta u = 0$ are also said to be generalized *harmonic functions.*

Let $\mathcal{O} \Subset X$ be a domain with C^∞ boundary. Denote by $n(x)$ the unit outward normal vector to the boundary surface $\partial\mathcal{O}$ at a point x. The system of boundary operators $\{(\partial/\partial n)^j\}_{j=0,1,\dots,p-1}$ is known to be a Dirichlet system of order $p-1$ on $\partial\mathcal{O}$.

We formulate the Dirichlet problem for the generalized Laplacian Δ in the following way.

Problem 2.1 *Given a section f of E over $\mathcal{O}$, find a section u of E over $\mathcal{O}$ such that $\Delta u = f$ in $\mathcal{O}$ and $(\partial/\partial n)^j u = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \dots, p-1$.*

As in the classical case, Problem 2.1 is verified to be an elliptic boundary value problem. Moreover, it is formally selfadjoint and possesses at most one solution in reasonable function spaces for u. So, this problem may be treated by standard tools in the scale $\{H^s(E|_\mathcal{O})\}_{s\in\mathbf{R}}$ of Sobolev spaces on $\mathcal{O}$ (see Roitberg [10]).

From this treatment, we briefly sketch the relevant material on Green's function. For more details we refer the reader to Roitberg [10] and Tarkhanov [14, 9.3.8].

It turns out that the inverse of the operator corresponding to Problem 2.1 is integral. Namely, there exists a unique kernel $\mathcal{G}(x,y)$ on $\mathcal{O} \times \mathcal{O}$ such that, for each data $f \in H^{s-2p}(E|_\mathcal{O})$, the function

$$u(x) = \int_\mathcal{O} \mathcal{G}(x,y) f(y)\, dy \quad (x \in \mathcal{O}) \tag{2.1}$$

belongs to $H^s(E|_\mathcal{O})$ and satisfies $\Delta u = f$ in $\mathcal{O}$ and $(\partial/\partial n)^j u = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \dots, p-1$. Such a kernel $\mathcal{G}(x,y)$ is said to be the *Green's function* for Problem 2.1.

We will later give a precise meaning to the integrals in (2.1), specifying to which spaces the Green's function belongs.

The Green's function $\mathcal{G}(\cdot, y)$ is alternatively defined as the solution to the Dirichlet problem with the data $f = \delta_y$, the Dirac *delta-function* supported at $y \in \mathcal{O}$. This data is easily verified to belong to all Sobolev spaces $H^s(\mathcal{O})$ with $s < -\frac{n}{2}$.

Theorem 2.2 *The kernel $\mathcal{G}$ is a C^∞ section of the bundle $E \otimes E^*|_{\overline{\mathcal{O}}\times\overline{\mathcal{O}}}$ away from the diagonal of $\overline{\mathcal{O}} \times \overline{\mathcal{O}}$.*

Proof. See Roitberg [10, 7.4].

□

A discussion of the singularity of $\mathcal{G}(x,y)$ at the diagonal $\{(x,x): x \in \overline{\mathcal{O}}\}$ can be found in Roitberg [10, Th.7.4.3]. For our purposes, it suffices to know that the mapping (2.1), when restricted to $f \in C^\infty_{comp}(E|_{\mathcal{O}})$, is a pseudodifferential operator of type $E|_{\mathcal{O}} \to E|_{\mathcal{O}}$ and order $-2p$. Thus, if f is sufficiently smooth, the integral in (2.1) is actually a usual Lebesgue integral.

Green's formula enables us to prove that the Green's function is a solution of the adjoint boundary value problem in the y variable. To explain this more accurately, denote by I_k the identity $(k \times k)$-matrix.

Theorem 2.3 *Given any $x \in \mathcal{O}$, we have:*

$$\begin{cases} \Delta'(y,D)\mathcal{G}(x,y) = \delta_x(y)\, I_k & \text{for } \ y \in \mathcal{O}, \\ (\partial/\partial n(y))^j \mathcal{G}(x,y) = 0 & \text{for } \ y \in \partial\mathcal{O} \quad (j = 0,1,\ldots,p-1). \end{cases} \tag{2.2}$$

Proof. See Tarkhanov [14, Th.9.3.24].

□

We are now in a position to state the symmetry of Green's function in the variables x and y. This symmetry could be expected from the fact that the Dirichlet problem is (formally) selfadjoint.

Corollary 2.4 *The matrix $\mathcal{G}(x,y)$ is Hermitian, i.e., $\mathcal{G}(x,y)^* = \mathcal{G}(y,x)$ for all $x, y \in \overline{\mathcal{O}}$.*

Proof. Indeed, since the solution to Problem 2.1 is unique, it follows from Theorem 2.3 that

$$\begin{aligned} \mathcal{G}(y,x) &= \star_x \mathcal{G}(x,y) \star_y^{-1} \\ &= \mathcal{G}(x,y)^*, \end{aligned}$$

as desired.

□

3 Grothendieck duality for harmonic functions

In the sequel, we shall denote by $\mathcal{O}$ a fixed relatively compact domain in X with C^∞ boundary $\partial\mathcal{O}$, as in Section 2.

Inspired by the work of Grothendieck [2] who used solution to $\Delta v = 0$ at infinity, we shall consider the manifold with boundary $\hat{\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O}$ as the compactification of $\mathcal{O}$.

We use $\hat{\mathcal{O}}$ instead of $\overline{\mathcal{O}}$ to conceptually distinguish this manifold with boundary from the closed subset $\overline{\mathcal{O}}$ of X.

The topology of $\hat{\mathcal{O}}$ is given by the following neighborhoods bases:

- If $x \in \mathcal{O}$, then we take the usual basis of neighborhoods of x (for example, the family $\{B \cap \mathcal{O}\}$, where B runs over all balls in X centered at x).
- If $x \in \partial\mathcal{O}$, then the basis of neighborhoods of x is defined to be the family $\{B \cap (\mathcal{O} \cup \partial\mathcal{O})\}$, where B runs over all balls in X centered at x.

We shall say that an open set U in $\hat{\mathcal{O}}$ is a neighborhood of infinity if U contains the part $\partial\mathcal{O}$ at infinity of $\hat{\mathcal{O}}$.

We shall also need the concept of a solution to $\Delta u = 0$ in a neighborhood $B \cap (\mathcal{O} \cup \partial\mathcal{O})$ of a point $x \in \partial\mathcal{O}$.

By this, we mean any solution to $\Delta u = 0$ on the $B \cap \mathcal{O}$ (finite part) which is C^∞ up to the $B \cap \partial\mathcal{O}$ (infinite part) and satisfies $(\partial/\partial n)^j u = 0$ on $B \cap \partial\mathcal{O}$ for $j = 0, 1, \ldots, p-1$.

Given an open set $U \subset \hat{\mathcal{O}}$, denote by $sol(U, \Delta)$ the set of all solutions to $\Delta u = 0$ on U.

Lemma 3.1 *Let U be a neighborhood of infinity in $\hat{\mathcal{O}}$. Then $sol(U, \Delta)$ is a closed subspace of $sol(U \cap \mathcal{O}, \Delta)$.*

Proof. Pick a sequence $\{u_\nu\}$ in $sol(U, \Delta)$ converging to a solution u_∞ in $sol(U \cap \mathcal{O}, \Delta)$. We shall have established the lemma if we prove that u_∞ is C^∞ up to the boundary of $\mathcal{O}$ and $(\partial/\partial n)^j u_\infty = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \ldots, p-1$.

To this end, let U' be a sufficiently thin open band close to the boundary in $\mathcal{O}$, so that $\partial\mathcal{O} \subset \partial U'$ and $U' \Subset U$. We can certainly assume that the boundary of U' is of class C^∞.

By the above, the Dirichlet problem for the Laplacian in U' is coercive. Hence for any integer $s \geq p$ there is a constant c such that

$$\|u\|_{H^s(E|_{U'})} \leq c \left(\sum_{j=0}^{p-1} \|(\partial/\partial n)^j u\|^2_{H^{s-j-\frac{1}{2}}(E|_{\partial U'})} \right)^{\frac{1}{2}} \tag{3.1}$$

whenever $u \in H^s(E|_{U'}) \cap sol(U', \Delta)$.

Let us apply this estimate to a solution $u \in sol(U, \Delta)$. Since the normal derivatives of u up to order $p-1$ vanish on the part $\partial\mathcal{O}$ of the boundary of U', we can assert that the norm of u in $H^s(E|_{U'})$ is dominated by Sobolev norms of the normal derivatives of u up to order $p-1$ on the remaining part of the boundary of U'. What is especially important here is that this remaining part $\partial U' \setminus \partial\mathcal{O}$ is a subset in $U \cap \mathcal{O}$. Hence combining the *Sobolev Embedding Theorem* with interior a priori estimates (1.1) yields

$$\sup_{|\alpha| \leq j} \|D^\alpha u\|_{C(E|_{\overline{U'}})} \leq c\, \|u\|_{C(E|_K)} \qquad \text{for all } u \in sol(U, \Delta), \tag{3.2}$$

with K a compact subset of $U \cap \mathcal{O}$, whose interior contains $\partial U' \backslash \partial \mathcal{O}$, and c a constant depending only on $\mathcal{O}'$, K and j.

We can now return to the sequence $\{u_\nu\}$. It follows from (3.2) that, given any multi-index α, the sequence of derivatives $\{D^\alpha u_\nu\}$ is a Cauchy sequence in $C(E|_{\overline{U'}})$. Therefore $\{u_\nu\}$ converges to a section $u \in C^\infty(E|_{\overline{U'}})$ uniformly on $\overline{U'}$ and together with all derivatives.

Obviously, $u_\infty = u$ in U'. This shows at once that u_∞ is C^∞ up to the boundary of $\mathcal{O}$ and $(\partial/\partial n)^j u_\infty = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \ldots, p-1$, as desired.

□

In the case where U is an open subset of $\hat{\mathcal{O}}$ containing $\partial\mathcal{O}$ we endow $sol(U, \Delta)$ with the topology induced by $sol(U \cap \mathcal{O}, \Delta)$. Then Lemmas 1.1 and 3.1 show that $sol(U, \Delta)$ is a Fréchet-Schwartz space. (For the moment we shall say nothing about a topology on $sol(U, \Delta)$ in the general case.)

We now invoke the construction of the *inductive limit* of a sequence of Fréchet spaces in order to define the space $sol(\sigma, \Delta)$ also for those closed sets σ in $\hat{\mathcal{O}}$ which are "approximable" by open subsets of $\hat{\mathcal{O}}$ containing $\partial\mathcal{O}$. These are nothing but the close subsets of $\hat{\mathcal{O}}$ containing the "infinitely far" surface $\partial\mathcal{O}$.

Next we fix a *Green operator* G_P for the differential operator P. By definition, G_P is a bidifferential operator of type $(F^*, E) \to \Lambda^{n-1}T^*(X)$ (where $\Lambda^{n-1}T^*(X)$ is the bundle of exterior differential forms of degree-$(n-1)$ on X) and order $p-1$, such that $dG_P(\star g, u) = ((Pu, g)_x - (u, P^* g)_x)\, dx$ pointwise on X, for all smooth sections g of F and u of E.

We immediately obtain:

Lemma 3.2 *A Green operator for the Laplacian Δ is given by*

$$G_\Delta(\star v, u) = G_P(\star P v, u) - \overline{G_P(\star P u, v)}. \qquad (3.3)$$

Having disposed of these preliminary steps, we fix now an open subset U of $\mathcal{O}$ and turn to describing the dual space for $sol(U, \Delta)$.

Given any solution $v \in sol(\hat{\mathcal{O}} \setminus U, \Delta)$, we define a linear functional $\mathcal{F}_v$ on $sol(U, \Delta)$ as follows.

There is an open set $\mathcal{N}_v \Subset U$ with piecewise smooth boundary such that v is still defined and satisfies $\Delta v = 0$ in a neighborhood of $\mathcal{O} \setminus \mathcal{N}_v$. Put

$$\langle \mathcal{F}_v, u \rangle = \int_{\partial \mathcal{N}_v} G_\Delta(\star v, u) \qquad (u \in sol(U, \Delta)). \qquad (3.4)$$

It follows from Stokes' formula that the value $\langle \mathcal{F}_v, u \rangle$ is independent of the particular choice of $\mathcal{N}_v$ with the properties previously mentioned.

Lemma 3.3 *The functional $\mathcal{F}_v$ defined by (3.4) is a continuous linear functional on the space $sol(U, \Delta)$.*

Proof. Use estimate (1.1) with $K' = \partial\mathcal{N}_v$ and $j = 2p - 1$.

□

The following result is related to the work of Grothendieck [2] where the concept of solution to $\Delta v = 0$ regular at the point of infinity of the one-point compactification of $\mathcal{O}$ was used.

Theorem 3.4 *Let the operator P^*P possess the Unique Continuation Property $(U)_s$ on X. Then for each open set $U \subset \mathcal{O}$, the correspondence $v \mapsto \mathcal{F}_v$ induces a topological isomorphism*

$$sol(U,\Delta)' \overset{top.}{\cong} sol(\hat{\mathcal{O}} \setminus U, \Delta).$$

Proof. Pick a continuous linear functional $\mathcal{F}$ on $sol(U,\Delta)$. Since $sol(U,\Delta)$ is a subspace of $C_{loc}(E|_U)$, the space of continuous sections of E over U, this functional can be extended, by the Hahn-Banach Theorem, to an E^*-valued measure m with compact support in U. Set $K = supp\, m$.

Let $\mathcal{N} \Subset U$ be any open set with piecewise smooth boundary such that $K \subset \mathcal{N}$. For each solution $u \in sol(U,\Delta)$, we have, by Green's formula,

$$u(x) = -\int_{\partial\mathcal{N}} G_\Delta(\mathcal{G}(x,y), u(y)) \qquad (x \in \mathcal{N}).$$

(Here $\mathcal{G}(x,y)$ is the Green's function of the Dirichlet problem for the Laplacian in $\mathcal{O}$, as in Section 2.) Therefore

$$\begin{aligned} \langle \mathcal{F}, u\rangle &= \int_U \langle dm, u\rangle_x \\ &= \int_{\partial\mathcal{N}} G_\Delta(\star v, u), \end{aligned}$$

where $v(y) = -\star_y^{-1} \int_U \langle dm, \mathcal{G}(\cdot,y)\rangle_x$.

Now we look more closely at the properties of this function v called the "Fantappiè indicatrix" of $\mathcal{F}$. Since $\Delta'(y,D)\mathcal{G}(x,y) = \delta_x(y)\, I_k$, we deduce that $\Delta v = 0$ away from K.

Moreover, Theorems 2.2 and 2.3 show that v is C^∞ up to the boundary of $\mathcal{O}$ and satisfies $(\partial/\partial n)^j v = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \ldots, p-1$.

From what has already been proved, it follows that $v \in sol(\hat{\mathcal{O}} \setminus U, \Delta)$ and $\mathcal{F} = \mathcal{F}_v$. Our next claim is that such a v is unique.

To this end, we let $v \in sol(\hat{\mathcal{O}} \setminus U, \Delta)$ satisfy

$$\int_{\partial\mathcal{N}_v} G_\Delta(\star v, u) = 0 \qquad \text{for all } u \in sol(U,\Delta), \tag{3.5}$$

where $\mathcal{N}_v \Subset U$ is an open set with piecewise smooth boundary, such that v is still defined and satisfies $\Delta v = 0$ in a neighborhood of $\mathcal{O} \setminus \mathcal{N}_v$.

We represent v in the complement of $\mathcal{N}_v$ by Green's formula. This is possible because of $(\partial/\partial n)^j v = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \dots, p-1$. We get

$$v(y) = -\star_y^{-1} \int_{\partial\mathcal{N}_v} G_\Delta(\star v(x), \mathcal{G}(x,y)) \qquad \text{for } y \in \mathcal{O} \setminus \overline{\mathcal{N}_v}.$$

For any fixed $y \in \mathcal{O} \setminus U$, we have $\mathcal{G}(\cdot, y) \in sol(U, \Delta)$, and so $v(y) = 0$ by condition (3.5). Since the operator P^*P possesses the Unique Continuation Property $(U)_s$, $v \equiv 0$ if $\overline{U} \subset \mathcal{O}$. To complete the proof in the case where $\overline{U}$ is not contained in $\mathcal{O}$, we use the *Runge Theorem* for solutions of the equation $\Delta u = 0$ (cf. Tarkhanov [14, 5.1.6]).

There exists an open set $\mathcal{N} \Subset U$ with the following properties:

- $\mathcal{N}_v \Subset \mathcal{N}$, and
- the complement of $\mathcal{N}$ has no compact connected components in U.

(The second property can always be achieved by adding all compact connected components of $U \setminus \mathcal{N}$ to $\mathcal{N}$.)

Fix $y \in \mathcal{O} \setminus \mathcal{N}$. Then each column of the matrix $\mathcal{G}(\cdot, y)$ is in $sol(\mathcal{N}, \Delta)$. According to the *Runge Theorem*, it can be approximated uniformly on compact subsets of $\mathcal{O}$ by solutions in $sol(U, \Delta)$. Let $\{u_\nu\}$ be a resulting sequence for $\mathcal{G}(\cdot, y)$, so that the columns of u_ν belong to $sol(\mathcal{N}, \Delta)$ and $u_\nu \to \mathcal{G}(\cdot, y)$ uniformly on compact subsets of $\mathcal{O}$.

Applying (1.1) we can assert that the derivatives up to order $p-1$ of u_ν also converge to the corresponding derivatives of $\mathcal{G}(\cdot, y)$ uniformly on compact subsets of $\mathcal{N}$. Therefore,

$$\begin{aligned} v(y) &= -\lim_{\nu\to\infty} \int_{\partial\mathcal{N}_v} G_\Delta(\star v, u_\nu) \\ &= -\lim_{\nu\to\infty} 0 \\ &= 0. \end{aligned}$$

Thus, $v = 0$ in $\mathcal{O} \setminus \mathcal{N}$, i.e., v is the zero element of $sol(\hat{\mathcal{O}} \setminus U, \Delta)$.

We have proved that the correspondence $v \mapsto \mathcal{F}_v$ induces the isomorphism of vector spaces

$$sol(\hat{\mathcal{O}} \setminus U, \Delta) \xrightarrow{\cong} sol(U, \Delta)'.$$

We are now going to invoke an operator-theoretic argument to conclude that this algebraic isomorphism is in fact a topological one.

To this end, we note that the spaces $sol(\hat{\mathcal{O}} \setminus U, \Delta)$ and $sol(U, \Delta)'$ are both spaces of type *DFS*. (For $sol(\hat{\mathcal{O}} \setminus U, \Delta)$, see the proof of Theorem 1.5.5 in Morimoto [7, p.13]. For $sol(U, \Delta)'$, see Lemma 1.1 above.) As the *Closed Graph Theorem* is correct for linear maps between spaces of type *DFS* (see Corollary A.6.4 in Morimoto [7, p.254]), to see that $v \mapsto \mathcal{F}_v$ is a topological isomorphism, it suffices to show that it is continuous. This latter conclusion, however, is obvious from the way the inductive limit topology is defined, and the construction of $\mathcal{F}_v$. This completes the proof.

□

One may conjecture that Theorem 3.4 is still true for *arbitrary* open sets U in $\widehat{\mathcal{O}}$. But we have not been able to do this.

4 A corollary

In this section we derive the following consequence of Theorem 3.4.

Corollary 4.1 *Let $\mathcal{D} \Subset \mathcal{O}$ be a domain with real analytic boundary. Assume that the operator Δ satisfies the Unique Continuation Property $(U)_s$ on X and its coefficients are real analytic in a neighborhood of the boundary of D. Then it follows that*

$$sol(\mathcal{D},\Delta)' \overset{top.}{\cong} sol(\overline{\mathcal{D}},\Delta). \tag{4.1}$$

Before proving this corollary, we briefly discuss a result of Morrey and Nirenberg [8] to be used in the proof.

Theorem 4.2 *Let Δ be a determined strongly elliptic differential operator of order $2p$ with real analytic coefficients on X. Assume that u is a solution to $\Delta u = 0$ in a domain $\mathcal{D} \subset X$. If u vanishes up to order $p-1$ on an open real analytic portion $\mathcal{S}$ of the boundary of $\mathcal{D}$, then for each point $x_0 \in \mathcal{S}$ there is a neighborhood $\mathcal{N}(x_0)$ on X depending only on the operator Δ and the domain near x_0, such that u may be extended as a solution of $\Delta u = 0$ from $\mathcal{N}(x_0) \cap \mathcal{D}$ to the whole neighborhood $\mathcal{N}(x_0)$.*

Proof. See Morrey and Nirenberg [8].

□

The important point to note here is that the neighborhood $\mathcal{N}(x_0)$ in Theorem 4.2 is independent of the particular solution u.

In fact, Morrey and Nirenberg [8] proved the existence of $\mathcal{N}(x_0)$ by showing that there is a real $r > 0$ such that, for any $u \in sol(\mathcal{D},\Delta)$ vanishing up to order $p-1$ on S, the *Taylor series* of u at x_0 converges in the ball $B(x_0,r)$. Thus, the solution u holomorphically extends to a neighborhood $\widetilde{\mathcal{N}}_{x_0}$ of x_0 in $\mathbb{C}^n$).

We are going to apply this corollary in the case where $\Delta = P^*P$ is the generalized Laplacian. To this end, we have to verify that the Laplacian is strongly elliptic (this notion becomes clear below).

Lemma 4.3 *If P is an elliptic differential operator of order p, then the operator $\Delta = P^*P$ is strongly elliptic of order $2p$.*

Proof. What is to be proved is that, given any non-zero vector $z \in \mathbb{C}^k$, we have

$$Re\ v^*\sigma(\Delta)(x,\xi)v \neq 0 \quad \text{for all } (x,\xi) \in X \times (\mathbb{R}^n \setminus \{0\}).$$

Suppose the lemma were false. Then there is a non-zero vector $z \in \mathbb{C}^k$ such that $Re\ v^*\sigma(\Delta)(x,\xi)v = 0$ for some $(x,\xi) \in X \times (\mathbb{R}^n \setminus \{0\})$ However,

$$\begin{aligned} Re\ v^*\sigma(\Delta)(x,\xi)v &= Re\ (\sigma(P)(x,\xi)v)^*(\sigma(P)(x,\xi)) \\ &= |\sigma(P)(x,\xi)v|^2, \end{aligned}$$

and so $v = 0$ because $\sigma(P)(x,\xi) : \mathbb{C}^k \to \mathbb{C}^l$ is injective. This contradicts our assumption.

□

We also need a slightly modified version of Theorem 4.2, a version which relates to *inhomogeneous* elliptic boundary value problems.

Lemma 4.4 *We keep the assumptions of Theorem 4.2. Let* $\{B_j\}_{j=0,1,\dots,p-1}$ *be a Dirichlet system of order* $p-1$ *with real analytic coefficients on* S. *If the Dirichlet data* $u_j = B_j u|_S$ $(j = 0,1,\dots,p-1)$ *of a solution* u *to* $\Delta u = 0$ *in* $\mathcal{D}$ *are real analytic on* S, *then for each point* $x_0 \in S$ *there exists a neighborhood* $\mathcal{N}(x_0)$ *on* X *depending only on* Δ, *the domain* $\mathcal{D}$ *near* x_0 *and* $\{u_j\}$, *such that* u *may be extended to a solution of* $\Delta u = 0$ *on* $\mathcal{N}(x_0)$.

Proof. For $j = p, p+1, \dots, 2p-1$, set $B_j = (\partial/\partial n)^j$, the j th derivative along the unit outward normal vector to S. This completes $\{B_j\}_{j=0,1,\dots,p-1}$ to a Dirichlet system of order $2p-1$ with real coefficients on S.

By the *Cauchy-Kovalevskaya Theorem*, there is a unique solution u' to the *Cauchy problem*

$$\begin{cases} \Delta u' = 0 & \text{in} \quad \mathcal{N}, \\ B_j u' = u_j & \text{on} \quad S \qquad (j = 0,1,\dots,p-1), \\ B_j u' = 0 & \text{on} \quad S \qquad (j = p, p+1,\dots,2p-1), \end{cases} \tag{4.2}$$

defined on some neighborhood $\mathcal{N}$ of S in X. (We observe at once that u' is real analytic in $\mathcal{N}$.)

Let $\mathcal{N}_{x_0}$ be the neighborhood of x_0 which is guaranteed by Theorem 4.2. We can certainly assume that u' is defined in $\mathcal{N}_{x_0}$, for if not, we replace $\mathcal{N}_{x_0}$ by $\mathcal{N}_{x_0} \cap \mathcal{N}$.

By (4.2), the difference $u'' = u - u'$ satisfies the equation $\Delta u'' = 0$ in $\mathcal{D} \cap \mathcal{N}$ and vanishes up to order $p-1$ on S.

Repeated application of Theorem 4.2 enables us to assert that there is a neighborhood of x_0 on X depending only on Δ and the domain $\mathcal{D} \cap \mathcal{N}$ near x_0, such that u'' may be extended to a solution of $\Delta u'' = 0$ in this neighborhood. To shorten notation, we continue to write $\mathcal{N}_{x_0}$ for this new neighborhood. Obviously, $u = u' + u''$ extends to $\mathcal{N}_{x_0}$, and the lemma follows.

□

We are now able to prove Corollary 4.1.

Proof. By Theorem 3.4, we shall have established the corollary if we prove that

$$sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta) \overset{top.}{\cong} sol(\overline{\mathcal{D}}, \Delta). \tag{4.3}$$

To this end, define a mapping $\mathcal{E}: \; sol(\overline{\mathcal{D}}, \Delta) \to sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ in the following way (cf. Tarkhanov [14, 10.2.3]).

Given any $u \in sol(\overline{\mathcal{D}}, \Delta)$, there exists a unique solution v to the Dirichlet problem

$$\begin{cases} \Delta v = 0 & in \quad \mathcal{O} \setminus \overline{\mathcal{D}}, \\ (\partial/\partial n)^j v = (\partial/\partial n)^j u & on \quad \partial\mathcal{D} \quad (j = 0, 1, \dots, p-1), \\ (\partial/\partial n)^j v = 0 & on \quad \partial\mathcal{O} \quad (j = 0, 1, \dots, p-1). \end{cases} \tag{4.4}$$

By the *regularity* of solutions to the Dirichlet problem, v is C^∞ up to the boundary of $\mathcal{O} \setminus \overline{\mathcal{D}}$ and so $v \in sol(\hat{\mathcal{O}} \setminus \overline{\mathcal{D}})$.

Let us denote by Ω the neighborhood of ∂D where the coefficients of P^*P are real analytic. By the *Petrovskii Theorem* there is a neighborhood Ω' of ∂D where u is real analytic. Since the Dirichlet data $\{(\partial/\partial n)^j u\}_{j=0,1,\dots,p-1}$ are real analytic on the real analytic open portion $\partial\mathcal{D}$ of the boundary of $\Omega' \setminus \overline{\mathcal{D}}$ and $\partial\mathcal{D}$ is compact, Lemma 4.4 shows that there is a neighborhood $\mathcal{N}_v$ of $\hat{\mathcal{O}} \setminus \overline{\mathcal{D}}$ such that v extends as a solution of $\Delta v = 0$ to $\mathcal{N}_v$. Moreover, $\mathcal{N}_v$ depends only on Δ, the domain $\Omega' \setminus \overline{\mathcal{D}}$ near $\partial\mathcal{D}$ and u.

For our case, we can derive a little bit more of information on $\mathcal{N}_v$ than that given by Lemma 4.4. Namely, $\mathcal{N}_v$ depends on the domain $\Omega' \cup D \supset \mathcal{N}_u \supset \overline{\mathcal{D}}$ of u rather than on u. Indeed, the difference $v - u$ satisfies $\Delta(v-u) = 0$ in the open set $\mathcal{N}_u \setminus \overline{\mathcal{D}}$ and vanishes up to order $p-1$ on the real analytic portion $\partial\mathcal{D}$ of its boundary. By Theorem 4.2, there is a neighborhood $\mathcal{N}$ of $\mathcal{N}_u \setminus \mathcal{D}$ depending only on Δ and $\mathcal{N}_u \setminus \overline{\mathcal{D}}$ near $\partial\mathcal{D}$, such that $v - u$ extends to a solution on $\mathcal{N}$. Then $v = u + (v-u)$ also extends to $\mathcal{N}$, and so we can add $\mathcal{N}$ to $\mathcal{N}_v$.

It follows that $v \in sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$. We set $\mathcal{E}(u) = v$, thus obtaining the mapping $\mathcal{E}: \; sol(\overline{\mathcal{D}}, \Delta) \to sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$.

Since the solution of the Dirichlet problem in $\mathcal{D}$ is unique, the mapping $\mathcal{E}$ is injective. On the other hand, since this problem is solvable for all Dirichlet data, the mapping $\mathcal{E}$ is surjective. In other words, $\mathcal{E}$ is an isomorphism of the vector spaces $sol(\overline{\mathcal{D}}, \Delta) \overset{\cong}{\to} sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$.

We now argue as at the end of the proof of Theorem 3.4 to conclude that this algebraic isomorphism is in fact a topological one. Since $sol(\overline{\mathcal{D}}, \Delta)$ and $sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ are both spaces of type *DFS*, we are reduced to proving that $\mathcal{E}$ is continuous.

To do this, pick a sequence $\{u_\nu\}$ in $sol(\overline{\mathcal{D}}, \Delta)$ converging to zero. By the definition of inductive limit topology, there is a neighborhood $\mathcal{N}_{\{u_\nu\}}$ of $\overline{\mathcal{D}}$ such that each u_ν is defined in $\mathcal{N}_{\{u_\nu\}}$ and $u_\nu \to 0$ uniformly on compact subsets of $\mathcal{N}_{\{u_\nu\}}$.

Set $v_\nu = \mathcal{E}(u_\nu)$. From what has already been proved it follows that there is a neighborhood $\mathcal{N}_{\{v_\nu\}}$ of $\hat{\mathcal{O}} \setminus \mathcal{D}$ such that all the v_ν are defined in $\mathcal{N}_{\{v_\nu\}}$.

As the Dirichlet problem in $\hat{\mathcal{O}} \setminus \overline{\mathcal{D}}$ is well-posed, we can assert that $v_\nu \to 0$ uniformly on $\hat{\mathcal{O}} \setminus \mathcal{D}$. The same holds also for the derivatives of $\{v_\nu\}$. We have however to show that $v_\nu \to 0$ uniformly on some neighborhood of $\hat{\mathcal{O}} \setminus \mathcal{D}$.

For this purpose, we find an $r > 0$ and a finite number of points $x_1, \dots, x_J$ on $\partial\mathcal{D}$ such that

- the balls $\{B(x_j, r)\}_{j=1,\dots,J}$ cover $\partial\mathcal{D}$; and

- for any ν and j, the Taylor series of v_ν at x_j converges in the ball $B(x_j, r)$.

(That such r and $\{x_j\}$ exist, follows from the comment on Theorem 4.2.)

□

Let $\mathcal{N} = \left(\hat{\mathcal{O}} \setminus \overline{\mathcal{D}}\right) \cup \left(\cup_{j=1}^J B(x_j, \frac{r}{2})\right)$. This is a neighborhood of $\hat{\mathcal{O}} \setminus \mathcal{D}$, and we have

$$\sup_{x \in \mathcal{N}} |v_\nu(x)| \le \sup_{x \in \hat{\mathcal{O}} \setminus \mathcal{D}} |v_\nu(x)| + \sum_{j=1}^{J} \sup_{x \in B(x_j, \frac{r}{2})} |v_\nu(x)| . \tag{4.5}$$

As mentioned, $\sup_{x \in \hat{\mathcal{O}} \setminus \mathcal{D}} |v_\nu(x)| \to 0$ when $\nu \to \infty$. It remains to estimate each term $\sup_{x \in B(x_j, \frac{r}{2})} |v_\nu(x)|$.

Since the Taylor series of v_ν at x_j converges in the ball of radius r, we obtain by the *Cauchy-Hadamard formula*

$$\left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \le const(\nu) \left(\frac{1}{r}\right)^{|\alpha|} \qquad \text{for all } \alpha \in \mathbb{Z}_+.$$

Therefore

$$\begin{aligned} \sup_{x \in B(x_j, \frac{r}{2})} |v_\nu(x)| &= \sup_{x \in B(x_j, \frac{r}{2})} \left| \sum_\alpha \frac{D^\alpha v_\nu(x_j)}{\alpha!} (x - x_j)^\alpha \right| \\ &\le \sum_\alpha \left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \left(\frac{r}{2}\right)^{|\alpha|} \\ &\le const(\nu) \sum_\alpha \left(\frac{1}{2}\right)^{|\alpha|}. \end{aligned}$$

We may now invoke the *Theorem on Dominated Convergence* to conclude that

$$\begin{aligned} \lim_{\nu \to \infty} \sup_{x \in B(x_j, \frac{r}{2})} |v_\nu(x)| &\le \sum_\alpha \left(\lim_{\nu \to \infty} \left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \right) \left(\frac{r}{2}\right)^{|\alpha|} \\ &= 0, \end{aligned}$$

the last equality being a consequence of the fact that the derivatives of $\{v_\nu\}$ converge to zero uniformly on $\partial \mathcal{D}$.

Thus, (4.5) shows that the sequence $\{v_\nu\}$ converges to zero uniformly on $\mathcal{N}$. It follows that $\{v_\nu\}$ converges to zero in the topology of $sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$, and so $\mathcal{E}$ is continuous. This completes the proof.

□

An advantage in describing duality by (4.1) is the fact that it also provides an explicit formula for the pairing.

Corollary 4.5 *Under the hypothesis of Corollary 4.1, let $\mathcal{F}_v$ be defined by (3.4). Then the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ induces the topological isomorphism (4.1).*

Proof. This follows from Theorem 3.4 and the proof of Corollary 4.1.

□

5 Miscellaneous

As follows, the analyticity of the boundary of $\mathcal{D}$ is essential to the validity of Corollary 4.5 (cf. Stout [12]).

Example 5.1 If P is the Cauchy operator in $X = \mathbb{R}^2$, then P^*P is the usual Laplace operator Δ in $\mathbb{R}^2$ up to the factor $-\frac{1}{4}$. Assume that $\mathcal{D}$ is a bounded domain in $\mathbb{R}^2$ with connected boundary $\partial\mathcal{D}$ of class C^2. According to the *Riemann Theorem*, $\mathcal{D}$ is holomorphically equivalent to the unit ball $B(0,1)$ in $\mathbb{R}^2$, i.e., there exists a conformal mapping $m : \mathcal{D} \to B(0,1)$. Moreover, it is known that m is of class C^1 up to the boundary of $\mathcal{D}$) and $m' \neq 0$ on $\overline{\mathcal{D}}$. We denote by x^0 the point of $\mathcal{D}$ such that $m(x^0) = 0$. Let $\mathcal{O} = B(x^0, R)$, where R a positive number, and $\mathcal{D} \Subset B(x^0, R)$. For $u(x) = \log\left|\frac{x-x^0}{R\,m(x)}\right|$, an easy verification shows that $\mathcal{E}(u)\,(x) = \log\frac{|x-x^0|}{R}$ belongs to $sol(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$. Clearly, u is real analytic near the closure of $\mathcal{D}$ if and only if $m(x)$ is. Thus, if the boundary of $\mathcal{D}$ is not real analytic, then u can fail to be real analytic near the closure of $\mathcal{D}$.

□

However, Theorem A is still true for *certain* domains $\mathcal{D}$ with non-analytic boundary.

Example 5.2 Under the hypothesis of Example 5.1, the mapping $m : \mathcal{D} \to B(0,1)$ induces a topological isomorphism of $sol(\mathcal{D}, \Delta) \overset{\cong}{\to} sol(B(0,1), \Delta)$. Arguing in a similar way, we see that the complement of $\overline{\mathcal{D}}$ is holomorphically equivalent to the complement of the closed unit ball in $\mathbb{R}^2$. And the corresponding conformal mapping induces a topological isomorphism of $sol(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta) \overset{\cong}{\to} sol(\widehat{\mathbb{R}^2} \setminus \overline{B(0,1)}, \Delta)$. Using the Grothendieck duality and the reflexivity of the spaces $sol(B(0,1), \Delta)$ and $sol(\widehat{\mathbb{R}^2} \setminus \overline{B(0,1)}, \Delta)$, we conclude that $sol(\widehat{\mathbb{R}^2} \setminus \overline{B(0,1)}, \Delta) \overset{top.}{\cong} sol(B(0,1), \Delta)$. Hence $sol(\mathcal{D}, \Delta) \overset{top.}{\cong} sol(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta)$. Finally, because of the Grothendieck duality, we have

$$\begin{aligned} sol(\mathcal{D}, \Delta)' &\overset{top.}{\cong} sol(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta)' \\ &\overset{top.}{\cong} sol(\overline{\mathcal{D}}, \Delta). \end{aligned}$$

What is still lacking is an explicit description of this duality (cf. Aizenberg and Gindikin [1]).

□

6 Duality for solutions of $Pu = 0$

For a domain $\mathcal{D} \Subset \mathcal{O}$ with real analytic boundary, pairing corresponding to the duality (5.1) is explicitly defined as follows.

Let $v \in sol(\overline{\mathcal{D}}, \Delta)$. Denote by $\mathcal{E}(v)$ the unique solution to the Dirichlet problem for the Laplacian in $\mathcal{O} \setminus \overline{\mathcal{D}}$, with Dirichlet data $\{(\partial/\partial n)^j v\}_{j=0,1,\dots,p-1}$ on $\partial\mathcal{D}$ and zero Dirichlet data on $\partial\mathcal{O}$ (cf. (4.4)). There exists an open set $\mathcal{N}_{\mathcal{E}(v)} \Subset \mathcal{D}$ with piecewise smooth boundary, such that $\mathcal{E}(v)$ still satisfies $\Delta\,\mathcal{E}(v) = 0$ in a neighborhood of $\mathcal{O} \setminus \mathcal{N}_{\mathcal{E}(v)}$. Set

$$\langle \mathcal{F}_{\mathcal{E}(v)}, u\rangle = \int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_\Delta(\star\mathcal{E}(v), u) \qquad (u \in sol(\mathcal{D}, \Delta)). \tag{6.1}$$

Then the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ induces, by Corollary 4.5, the *topological* isomorphism $sol(\overline{\mathcal{D}}, \Delta) \xrightarrow{\cong} sol(\mathcal{D}, \Delta)'$.

Since $\Delta = P^*P$, we have

$$\begin{array}{rcl} sol(\mathcal{D}, P) & \hookrightarrow & sol(\mathcal{D}, \Delta), \\ sol(\overline{\mathcal{D}}, P) & \hookrightarrow & sol(\overline{\mathcal{D}}, \Delta) \end{array}$$

(and both subspaces are closed).

Moreover, equality (3.3) shows that the restriction of functional (6.1) to the subspace $sol(\mathcal{D}, P)$ is given by

$$\langle \mathcal{F}_{\mathcal{E}(v)}, u\rangle = \int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P\,\mathcal{E}(v), u) \qquad (u \in sol(\mathcal{D}, P)). \tag{6.2}$$

Again it follows from *Stokes' formula* that the value $\langle \mathcal{F}_{\mathcal{E}(v)}, u\rangle$ is independent of the particular choice of $\mathcal{N}_v$ with the properties previously mentioned. By the above, $\mathcal{F}_{\mathcal{E}(v)}$ is a continuous linear functional on the space $sol(\mathcal{D}, P)$.

Of course, it is no longer true that to different solutions v_1 and v_2 in $sol(\overline{\mathcal{D}}, \Delta)$ there correspond different functionals $\mathcal{F}_{\mathcal{E}(v_1)}$ and $\mathcal{F}_{\Theta(v_2)}$ on $sol(\mathcal{D}, P)$ by (6.2). However, this still holds if we vary v within $sol(\overline{\mathcal{D}}, P)$ only.

Lemma 6.1 *If $v \in sol(\overline{\mathcal{D}}, P)$ satisfies*

$$\int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P\,\mathcal{E}(v), u) = 0 \qquad \textit{for all } u \in sol(\mathcal{D}, P), \tag{6.3}$$

then $v = 0$.

Proof. Take $u = v$ in (6.3). By Stokes' formula,

$$\begin{array}{rcl} 0 & = & \displaystyle\int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P\,\Theta(v), v) \\ & = & \displaystyle\int_{\partial \mathcal{D}} G_P(\star P\,\mathcal{E}(v), v) \\ & = & \displaystyle\int_{\partial \mathcal{D}} G_P(\star P\,\mathcal{E}(v), \mathcal{E}(v)), \end{array}$$

the last equality being a consequence of the fact that $v = \mathcal{E}(v)$ up to order $p-1$ on $\partial\mathcal{D}$. As $\mathcal{E}(v)$ vanishes up to order $p-1$ on $\partial\mathcal{O}$ and $\Delta\mathcal{E}(v) = 0$ in $\mathcal{O} \setminus \overline{\mathcal{D}}$, we obtain by the definition of Green operators

$$\begin{aligned} 0 &= -\int_{\partial(\mathcal{O}\setminus\mathcal{D})} G_P(\star P\,\mathcal{E}(v),\mathcal{E}(v)) \\ &= -\int_{\mathcal{O}\setminus\mathcal{D}} |P\,\mathcal{E}(v)|^2\,dx. \end{aligned}$$

Hence it follows that $P\,\mathcal{E}(v)=0$ in $\mathcal{O}\setminus\overline{\mathcal{D}}$.

Consider the section

$$\tilde{v}=\begin{cases} v & \text{in } \mathcal{D}, \\ \mathcal{E}(v) & \text{in } \mathcal{O}\setminus\overline{\mathcal{D}}. \end{cases}$$

It is of class $C^{p-1}_{loc}(E|_{\mathcal{O}})$ and satisfies $P\tilde{v}=0$ away from the hypersurface $\partial\mathcal{O}$. A familiar argument on removable singularities (see for instance Tarkhanov [13, Theorem 3.2]) shows that $\tilde{v}$ is actually a solution to $P\tilde{v}=0$ on the whole domain $\mathcal{O}$.

Since $\tilde{v}$ vanishes up to order $p-1$ on $\partial\mathcal{O}$, it follows that $\tilde{v}=0$ in $\mathcal{O}$. Hence $v=0$ in $\mathcal{D}$, as desired.

□

Thus, the correspondence $v\mapsto\mathcal{F}_{\mathcal{E}(v)}$, provides us with an injective mapping $sol(\overline{\mathcal{D}},P)\to sol(\mathcal{D},P)'$. One may ask whether this mapping is surjective. We prove that this is the case if and only if the domain $\mathcal{D}$ possesses a convexity property with respect to the operator P.

Theorem 6.2 *Let $\mathcal{D}\Subset\mathcal{O}$ be a domain with real analytic boundary. Assume that the operator P^*P possesses the Unique Continuation Property $(U)_s$ and has real analytic coefficients in a neighborhood of ∂D. If, given any neighborhood U of $\overline{\mathcal{D}}$, there is a neighborhood $U'\subset U$ of $\overline{\mathcal{D}}$ such that $sol(U')$ is dense in $sol(\mathcal{D})$ then the correspondence $v\mapsto\mathcal{F}_{\mathcal{E}(v)}$, when restricted to $v\in sol(\overline{\mathcal{D}},P)$, induces the topological isomorphism*

$$sol(\mathcal{D},P)'\overset{top.}{\cong} sol(\overline{\mathcal{D}},P).$$

This result sharpens Theorem A announced in Introduction.

7 Proof of the main theorem

The main step in the proof consists of verifying the surjectivity of the mapping $v\mapsto\mathcal{F}_{\mathcal{E}(v)}$.

Let $\mathcal{F}$ be a continuous linear functional on $sol(\mathcal{D},P)$. Since $sol(\mathcal{D},P)$ is a subspace of $C_{loc}(E|_{\mathcal{D}})$, this functional can be extended, by the Hahn-Banach Theorem, to an E^*-valued measure m with compact support in $\mathcal{D}$. We set $K=supp\,m$.

As in the previous section, we denote by Ω the neighborhood of ∂D where the coefficients of P^*P are real analytic. Fix an open set $\mathcal{N}\Subset\mathcal{D}$ with piecewise smooth boundary, such that $K\subset\mathcal{N}$ and $\partial\mathcal{N}\subset\Omega$. We first argue formally.

Sketch of the proof of surjectivity. For any $u \in sol(\mathcal{D}, P)$, we have by Green's formula

$$\begin{aligned} u(x) &= -\int_{\partial\mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x,\cdot), u) \\ &= -\int_{\partial\mathcal{N}} G_P(\star P\, \mathcal{E}\left(\mathcal{E}^{-1}(\star^{-1}\mathcal{G}(x,\cdot))\right), u) \end{aligned}$$

whenever $x \in \mathcal{N}$.

Suppose that outside of a larger open set $\mathcal{N}' \Subset \mathcal{D}$ with piesewise smooth boundary $K(x,\cdot) = \mathcal{E}^{-1} \star^{-1} \mathcal{G}(x,\cdot)$ can be decomposed into the sum $K(x,\cdot) = K_1(x,\cdot) + K_2(x,\cdot)$, where $K_1(x,\cdot) \in sol(\overline{\mathcal{D}}, P)$ is sufficiently smooth in $x \in \mathcal{N}$, and $K_2(x,\cdot)$ is orthogonal to u under the pairing $\int_{\partial\mathcal{N}'} G_P(\star P\, \mathcal{E}(K_2(x,\cdot)), u)$.

Then

$$u(x) = -\int_{\partial\mathcal{N}'} G_P(\star P\, \mathcal{E}(K_1(x,\cdot)), u), \qquad x \in \mathcal{N},$$

and so

$$\langle \mathcal{F}, u\rangle = \int_{\partial\mathcal{N}'} G_P(\star P\, \mathcal{E}(v), u)$$

with $v(y) = -\langle dm, K_1(\cdot, y)\rangle_{\mathcal{N}}$.

Hence it follows that $v \in sol(\overline{\mathcal{D}}, P)$ and $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$, as desired.

□

We now proceed to give a rigorous proof. By Theorem 2.3, the columns of the Green's function $\star_y^{-1}\mathcal{G}(x,y)$ belongs to $sol(\hat{\mathcal{O}} \setminus \mathcal{N}, \Delta)$ in the variable y, for each fixed $x \in \mathcal{N}$. In the following we will apply different operators and notations to matrices, understanding that they hold for each of their columns.

Given any fixed $x \in \mathcal{N}$, let $K(x,\cdot) = \mathcal{E}^{-1}(\star^{-1}\mathcal{G}(x,\cdot))$, i.e., $K(x,y)$ be the unique solution to the following Dirichlet problem:

$$\begin{cases} \Delta(y,D)K(x,y) = 0 & for \ \ y \in \overline{\mathcal{D}}, \\ (\partial/\partial n(y))^j K(x,y) = (\partial/\partial n(y))^j \left(\star_y^{-1}\mathcal{G}(x,y)\right) & for \ \ y \in \partial\mathcal{D}\ (j = 0, 1, \ldots, p-1). \end{cases}$$

Since $\overline{\mathcal{N}} \subset \mathcal{D}$, it follows from Lemma 4.4 that there is a neighborhood $U \Subset \Omega \cup D$ of $\overline{\mathcal{D}}$ independent of $x \in \mathcal{N}$, such that $K(x,\cdot)$ belongs to $sol(U, \Delta)$. (We use here the fact that Green's function is real analytic away from the diagonal in $\Omega \times \Omega$.)

Moreover, $K(x,\cdot)$ is real analytic in $x \in \mathcal{N} \cap \Omega$ because of the *Poisson formula* for solutions of the Dirichlet problem (cf. Tarkhanov [14, (9.3.12)].

As mentioned, $sol(U, P)$ is a closed subspace of $sol(U, \Delta)$. Our next goal is to extract a summand from $K(x,\cdot)$ which corresponds to this subspace, so that the rest is orthogonal to $sol(\mathcal{D}, P)$ in a suitable sense. To this end, we invoke Hilbert space techniques.

We first recall a result of Nacinovich and Shlapunov [9].

Lemma 7.1 *The Hermitian form*

$$h(u,v) = \int_{\mathcal{D}} (Pu, Pv)_x dx + \int_{\mathcal{O}\setminus\mathcal{D}} (P\, \mathcal{E}(u), P\, \mathcal{E}(v))_x dx \quad (u, v \in H^p(E|_{\mathcal{D}})), \qquad (7.1)$$

defines a scalar product on $H^p(E|_{\mathcal{D}})$, and the topologies induced in $H^p(E|_{\mathcal{D}})$ by $h(\cdot,\cdot)$ and by the standard scalar product are equivalent.

Proof. See *ibid* as well as in Tarkhanov [14, 10.2.3].

□

An easy calculation shows that if moreover v is sufficiently smooth up to the boundary of $\mathcal{D}$ (it suffices $v \in H^{2p}(E|_{\mathcal{D}})$) and u satisfies $Pu = 0$ in $\mathcal{D}$, then

$$h(u,v) = -\int_{\partial\mathcal{D}} G_P(\star P\,\mathcal{E}(v), \mathcal{E}(u)). \tag{7.2}$$

By assumption, there is a neighborhood $U' \Subset U$ of $\overline{\mathcal{D}}$ such that $sol(U'P)$ is dense in $sol(\mathcal{D}, P)$. We can certainly assume that $H^p(E_{|U'}) \cap sol(U', P)$ is dense in $sol(\mathcal{D}, P)$, for if not, we replace U' by a smaller neighborhood.

Denote by H_2 the closure of $H^p(E|_{U'}) \cap sol(U', P)$ in $H^p(E|_{\mathcal{D}})$; we endow H_2 with *scalar product* (7.1).

The following result is a particular case of a general theorem of Shlapunov and Tarkhanov [11] (see also [14, 12.1.2]).

Lemma 7.2 *There exists an orthonormal basis $\{e_\nu\}$ in $H^p(E|_{U'}) \cap sol(U', P)$ such that the restriction of $\{e_\nu\}$ to $\mathcal{D}$ is an orthogonal basis in H_2.*

Proof. Consider the mapping $R : H^p(E|_{U'}) \cap sol(U'; P) \to H_2$ given by restricting sections over U' to $\mathcal{D}$. (It will cause no confusion if we use the same symbol for a section $u \in H^p(E|_{U'}) \cap sol(U', P)$ and its restriction Ru to $\mathcal{D}$.)

By the Unique Continuation Property $(U)_s$, R is injective. Moreover, by *Stiltjes-Vitali Theorem* R is compact. It follows that R^*R is a compact selfadjoint operator of zero null-space in the Hilbert space $H^p(E|_{U'}) \cap sol(U', P)$. (Here R^* stands for the adjoint of R in the sense of Hilbert spaces.)

Let $\{e_\nu\}$ be a complete orthonormal system of eigenfunctions of the operator R^*R in $H^p(E|_{U'}) \cap sol(U', P)$ corresponding to eigenvalues $\{\lambda_\nu\}$. Since $H^p(E|_{U'}) \cap sol(U', P)$ is dense in H_2, we can assert that

- $\{e_\nu\}$ is an orthonormal basis in $H^p(E|_{U'}) \cap sol(U', P)$; and
- the system $\{Re_\nu\}$ is a basis in H_2 orthogonal with respect to the scalar product $h(\cdot, \cdot)$.

Thus, the system $\{e_\nu\}$ possesses the desired properties, and the lemma follows.

□

Note that the Fourier coefficients of a section $u \in H^p(E|_{U'}) \cap sol(U', P)$ with respect to the system $\{e_\nu\}$ are given by

$$\begin{aligned}
(u, e_\nu)_{H^p(E|_{U'})} &= \frac{1}{\lambda_\nu}\,(u, R^*R\,e_\nu)_{H^p(E|_{U'})} \\
&= \frac{1}{\lambda_\nu}\,h(Ru, Re_\nu) \\
&= \frac{1}{\lambda_\nu}\,h(u, e_\nu),
\end{aligned} \tag{7.3}$$

where $\lambda_\nu = h(e_\nu, e_\nu)$.

Our next objective is to treat the "projection" of the kernel $K(x,\cdot)$ on the space $H^p(E|_{U'}) \cap sol(U', P)$. To do this, we need the following technical lemma.

Lemma 7.3 *Let $\{e_\nu\}$ be an orthonormal system in a separable Hilbert space H, and $K(x)$ be a continuous function on a topological space T with values in H. Then the Fourier series $\sum_\nu (K(x), e_\nu)_H e_\nu$ converges in the norm of H uniformly in x on compact subsets of T.*

Proof. Denote by H_1 the closure of the linear span of $\{e_\nu\}$ in H. Pick a complete orthonormal system $\{b_\mu\}$ in the orthogonal complement of H_1 in H. Then $\{e_\nu\} \cup \{b_\mu\}$ is an orthonormal basis in H.

Given any $x \in T$, decompose $K(x)$ into the Fourier series with respect to this basis. Namely,

$$K(x) = \sum_{\nu=1}^{\infty} (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^{\infty} (K(x), b_\mu)_H b_\mu.$$

Hence it follows that

$$\begin{aligned} &\left\| K(x) - \left(\sum_{\nu=1}^{N} (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^{N} (K(x), b_\mu)_H b_\mu \right) \right\|_H^2 \\ &= \sum_{\nu=N+1}^{\infty} |(K(x), e_\nu)_H|^2 + \sum_{\mu=N+1}^{\infty} |(K(x), b_\mu)_H|^2, \end{aligned}$$

and so

$$\sum_{\nu=N+1}^{\infty} |(K(x), e_\nu)_H|^2 \le \left\| K(x) - \left(\sum_{\nu=1}^{N} (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^{N} (K(x), b_\mu)_H b_\mu \right) \right\|_H^2 \qquad (7.4)$$

for all $\nu = 1, 2, \ldots$.

Since the Fourier series converges in the norm H, for every $x^0 \in T$ and $\varepsilon > 0$ there is a number N^0 depending on x^0 and ε, such that

$$\left\| K(x^0) - \left(\sum_{\nu=1}^{N^0} (K(x^0), e_\nu)_H e_\nu + \sum_{\mu=1}^{N^0} (K(x^0), b_\mu)_H b_\mu \right) \right\|_H < \varepsilon.$$

Moreover, from the continuity of $K(x)$ at x^0 we deduce that the set

$$\mathcal{N}(x^0) = \left\{ x \in T : \left\| K(x) - \left(\sum_{\nu=1}^{N^0} (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^{N^0} (K(x), b_\mu)_H b_\mu \right) \right\|_H < \varepsilon \right\}$$

is an open neighborhood of x^0.

Applying (7.4) yields

$$\begin{aligned}\sum_{\nu=N+1}^{\infty} |(K(x), e_\nu)_H|^2 &\le \sum_{\nu=N^0+1}^{\infty} |(K(x), e_\nu)_H|^2 \\ &< \varepsilon^2,\end{aligned}$$

for all $N \geq N^0$ and $x \in \mathcal{N}(x^0)$. Therefore, the series $\sum_\nu (K(x), e_\nu)_H e_\nu$ converges in the norm of H uniformly in $x \in \mathcal{N}(x^0)$.

As each compact subset of T can be covered by a finite number of such neighborhoods, the lemma follows.

□

By the above, $K(x, \cdot)$ is a continuous function of $x \in \mathcal{N}$ with values in the Hilbert space $H^p(E|_{U'}) \cap sol(U', \Delta)$. Lemma 7.3 thus shows that the series

$$K_1(x, \cdot) = \sum_{\nu=1}^{\infty} (K(x, \cdot), e_\nu)_{H^p(E|_{U'})} e_\nu \tag{7.5}$$

converges in $H^p(E|_{U'}) \cap sol(U', \Delta)$ uniformly in x on compact subsets of $\mathcal{N}$. As the same holds for the derivatives of $K(x, \cdot)$ with respect to the x variables, we conclude that $K_1(x, \cdot)$ is of class C^∞ in $x \in \mathcal{N}$.

We now apply the operator $\mathcal{E}$ to both sides of equality (7.5). Since $\mathcal{E}$ determines a topological isomorphism of $sol(\overline{\mathcal{D}}, \Delta) \to sol(\hat{\mathcal{O}} \backslash \mathcal{D}, \Delta)$, there is an open set $\mathcal{N}' \Subset \mathcal{D}$ with piesewise smooth boundary, such that every $\mathcal{E}(e_\nu)$ extends to a solution of $\Delta v = 0$ in a neighborhood of $\mathcal{O} \setminus \mathcal{N}'$, and the series

$$\mathcal{E}\left(K_1(x, \cdot)\right) = \sum_{\nu=1}^{\infty} \left(K(x, \cdot), e_\nu\right)_{H^p(E|_{U'})} \mathcal{E}(e_\nu)$$

converges in $sol(\hat{\mathcal{O}} \setminus \mathcal{N}', \Delta)$. (By construction, $\mathcal{N}'$ is larger than $\mathcal{N}$, since otherwise we obtain a gain in analyticity.)

Lemma 7.4 *For each $u \in sol(\mathcal{D}, P)$, it follows that*

$$u(x) = -\int_{\partial \mathcal{N}'} G_P(\star P\, \mathcal{E}(K_1(x, \cdot)), u), \qquad x \in \mathcal{N}. \tag{7.6}$$

Proof. Pick a system $\{b_\mu\}$ in $H^p(E|_{\mathcal{D}}) \cap sol(\mathcal{D}, \Delta)$ such that $\{e_\nu\} \cup \{b_\mu\}$ is a basis in this space orthogonal with respect to scalar product (7.1).

For a fixed $x \in \mathcal{N}$, we decompose $K(x, \cdot)$ into the Fourier series with respect to this basis, i.e.,

$$\begin{aligned} K(x, \cdot) &= \sum_{\nu=1}^{\infty} \frac{h(K(x, \cdot), e_\nu)}{h(e_\nu, e_\nu)} e_\nu + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu \\ &= \sum_{\nu=1}^{\infty} (K(x, \cdot), e_\nu)_{H^p(E|_{U'})} e_\nu + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu \\ &= K_1(x, \cdot) + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu, \end{aligned} \tag{7.7}$$

the second equality being a consequence of (7.3). (Note that the series on the right-hand here converges in the norm of $H^p(E|_{\mathcal{D}})$.)

As $\mathcal{E}$ is a topological isomorphism of

$$H^p(E|_{\mathcal{D}}) \cap sol(\mathcal{D}, \Delta) \to H^p(E|_{\mathcal{O}\setminus\overline{\mathcal{D}}}) \cap sol(\hat{\mathcal{O}} \setminus \overline{\mathcal{D}}, \Delta),$$

we may apply $\mathcal{E}$ to (7.7) termwise, thus obtaining

$$\star^{-1}\mathcal{G}(x,\cdot) = \mathcal{E}(K_1(x,\cdot)) + \sum_{\mu=1}^{\infty} \frac{h(K(x,\cdot), b_\mu)}{h(b_\mu, e_\mu)} \mathcal{E}(b_\mu), \qquad x \in \mathcal{N},$$

the series converging in the norm of $H^p(E|_{\mathcal{O}\setminus\overline{\mathcal{D}}})$.

Having disposed of this preliminary step, we can now return to representation (7.6). Let $u \in sol(\mathcal{D}, P)$. By assumption, there exists a sequence $\{u_j\}$ in $H^p(E|_{U'}) \cap sol(U', P)$ converging to u together with all derivatives uniformly on compact subsets of $\mathcal{D}$. Given any $x \in \mathcal{N}$, we have therefore by Green's formula

$$\begin{aligned} u(x) &= -\int_{\partial\mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x,\cdot), u) \\ &= -\lim_{j\to\infty} \int_{\partial\mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x,\cdot), u_j) \\ &= -\lim_{j\to\infty} \int_{\partial\mathcal{D}} G_P(\star P \star^{-1} \mathcal{G}(x,\cdot), u_j), \end{aligned}$$

the second equality being a consequence of Lemma 1.1, and the third equality being a consequence of Stokes' formula.

On the boundary of $\mathcal{D}$, we have $u_j = \mathcal{E}(u_j)$ up to order $p-1$. Therefore

$$\begin{aligned} u(x) &= -\lim_{j\to\infty} \int_{\partial\mathcal{D}} G_P(\star P \star^{-1} \mathcal{G}(x,\cdot), \mathcal{E}(u_j)) \\ &= \lim_{j\to\infty} h(u_j, K(x,\cdot)), \end{aligned}$$

which is due to (7.2).

On the other hand, since every u_j is in $H^p(E|_{U'}) \cap sol(U', P)$, we may write

$$u_j = \sum_{\nu=1}^{\infty} (u_j, e_\nu)_{H^p(E|_{U'})} e_\nu,$$

where the series converges in the norm of $H^p(E|_{U'})$. Combining this with (7.7) yields

$$h(u_j, K(x,\cdot)) = h(u_j, K_1(x,\cdot)),$$

for the systems of sections $\{e_\nu\}$ and $\{b_\mu\}$ are pairwise orthogonal with respect to $h(\cdot,\cdot)$.

Thus,

$$\begin{aligned} u(x) &= \lim_{j\to\infty} h(u_j, K(x,\cdot)) \\ &= -\lim_{j\to\infty} \int_{\partial\mathcal{D}} G_P(\star P\, \mathcal{E}(K_1(x,\cdot)), \mathcal{E}(u_j)) \\ &= -\int_{\partial\mathcal{N}'} G_P(\star P\, \mathcal{E}(K_1(x,\cdot)), u), \end{aligned}$$

for $x \in \mathcal{N}$. This is precisely the assertion of the lemma.

□

We are now in a position to finish the proof of Theorem 6.2.

Proof of Theorem 6.2. From Lemma 7.4 it follows that

$$\langle \mathcal{F}, u \rangle = \int_{\partial \mathcal{N}'} G_P(\star P\, \mathcal{E}(v), u) \qquad \text{for all } u \in sol(\mathcal{D}, P),$$

where $v(y) = -\langle dm, K_1(\cdot, y)\rangle_{\mathcal{N}}$.

One easily verifies that $Pv = 0$ in U'. Hence $v \in sol(\overline{\mathcal{D}}, P)$ and $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$, which proves the surjectivity of the mapping $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$.

When combined with Lemma 6.1, this shows that the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ induces the isomorphism of vector spaces

$$sol(\overline{\mathcal{D}}, P) \stackrel{\cong}{\to} sol(\mathcal{D}, P)'.$$

We now argue as at the end of the proof of Theorem 3.4 to conclude that this algebraic isomorphism is in fact a topological one.

For this purpose, we note that the spaces $sol(\overline{\mathcal{D}}, P)$ and $sol(\mathcal{D}, P)'$ are both spaces of type *DFS*. (For $sol(\overline{\mathcal{D}}, P)$, see the proof of Theorem 1.5.5 in Morimoto [7, p.13]. For $sol(\mathcal{D}, P)'$, see Lemma 1.1 above.) As the *Closed Graph Theorem* is correct for linear maps between spaces of type *DFS* (see Corollary A.6.4 in Morimoto [7, p.254]), to see that $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ is a topological isomorphism, it suffices to show that it is continuous.

The latter conclusion is however a consequence of the following two facts already proved:

- the mapping $v \mapsto \mathcal{F}_v$ of $sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta) \to sol(\mathcal{D}, \Delta)$ is continuous (cf. Theorem 3.4); and
- the mapping $v \mapsto \mathcal{E}(v)$ of $sol(\overline{\mathcal{D}}, \Delta) \to sol(\hat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ is continuous (cf. Corollary 4.1).

This completes the proof.

□

Let us mention an important consequence of Theorem 6.2.

Corollary 7.5 *Under the hypotheses of Theorem 6.2, it follows that*

$$sol(\overline{\mathcal{D}}, P)' \stackrel{top.}{\cong} sol(\mathcal{D}, P).$$

Proof. By Lemma 1.1, $sol(\mathcal{D}, P)$ is a Fréchet-Schwartz space. Therefore, it is a *Montel space*. That $sol(\mathcal{D}, P)$ is a Montel space implies that it is reflexive, i.e., under the natural pairing, we have

$$(sol(\mathcal{D}, P)')' \stackrel{top.}{\cong} sol(\mathcal{D}, P),$$

where both $sol(\mathcal{D}, P)'$ and $(sol(\mathcal{D}, P)')'$ are provided with the strong topology. Thus, the desired statement follows immediately from Theorem 6.2.

□

8 The converse theorem

Assume that $\mathcal{D}$ is a relatively compact subdomain of $\mathcal{O}$ with real analytic boundary.

We have proved that if, for any neighborhood U of $\overline{\mathcal{D}}$, there exists a neighborhood $U' \subset U$ of $\overline{\mathcal{D}}$ such that $sol(U', P)$ is dense in $sol(\mathcal{D}, P)$, then the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ induces the topological isomorphism of $sol(\overline{\mathcal{D}}, P)$ onto the dual space to $sol(\mathcal{D}, P)$.

We now that this condition is almost necessary.

Theorem 8.1 *If the map $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ of $sol(\overline{\mathcal{D}}, P) \to sol(\mathcal{D}, P)'$ is surjective, then $sol(\overline{\mathcal{D}}, P)$ is dense in $sol(\mathcal{D}, P)$.*

Proof. Let $\mathcal{F}$ be a continuous linear functional on $sol(\mathcal{D}, P)$ vanishing on $sol(\overline{\mathcal{D}}, P)$. By the Hahn-Banach Theorem, our statement will be proved once we show that $\mathcal{F} \equiv 0$.

By assumption, there is a $v \in sol(\overline{\mathcal{D}}, P)$ such that $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$. It follows that

$$\begin{aligned} \langle \mathcal{F}_{\mathcal{E}(v)}, v \rangle &= \langle \mathcal{F}, v \rangle \\ &= 0, \end{aligned}$$

and so an argument similar to that in the proof of Lemma 6.1 shows that $v = 0$ in $\mathcal{D}$. Hence $\mathcal{F} \equiv 0$, as desired.

□

9 Duality in complex analysis

Aizenberg and Gindikin [1] obtained Theorem A, formulated in the Introduction, in the case where P is the *Cauchy-Riemann operator* in $\mathbb{C}^n$, and $n = 1, 2$ (for simply connected domains with real analytic boundary in $\mathbb{C}$, and for the so-called (p, q)-*circular domains* in $\mathbb{C}^2$).

Stout [12] proved Theorem A for the Cauchy-Riemann operator in $\mathbb{C}^n$ $(n \geq 1)$ and for domains $\mathcal{D}$ possessing the following property:

- the *Szegö kernel* $\mathcal{K}(\cdot, \zeta)$ of $\mathcal{D}$ has real analytic boundary values for each $\zeta \in \mathcal{D}$.

This condition is known to hold on some explicitly given domains. One supposes it to hold on *strictly pseudoconvex* domains with real analytic boundary. But, as far as Stout [12] has been able to determine, this result has not been written out anywhere.

However, the *approximation condition* in Theorem A holds true for strictly pseudoconvex domains in $\mathbb{C}^n$ (cf. Hörmander [4]). Thus, our viewpoint sheds some new light on the result of Stout [12].

Theorem 9.1 *Let $\mathcal{D} \Subset \mathbb{C}^n (n \geq 2)$ be a strictly pseudoconvex domain with real analytic boundary. Then the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$, when restricted to $v \in hol(\overline{\mathcal{D}})$, induces the topological isomorphism*

$$hol(\mathcal{D})' \overset{top.}{\cong} hol(\overline{\mathcal{D}}).$$

Here we use *hol* for the spaces of holomorphic functions.

Proof. This follows immediately by combining Theorem 6.2 with the *Runge theorem* as stated in Hörmander [4].

□

We note that, because the Cauchy-Riemann operator in $\mathbb{C}$ is determined elliptic, Theorem 9.1 holds true for spaces of holomorphic functions in every bounded domain in $\mathbb{C}$ with real analytic boundary.

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