ON THE DIMENSION OF SOME LINEAR SPACES OVER $\mathbb{Q}$ (Part 4)

L.A. Gutnik

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§0. Foreword.

For some problems studied in [40]-[58], the properties of the roots η of the polynomial

$$D^{\vee}(z,\eta) = (\eta+1)(\eta+\gamma_1) + 2r\exp(i\psi)(1+\gamma_1)\eta, \tag{1}$$

where

$$-1 < \delta_0 < 1, 0 < r, \gamma_1 = (1-\delta_0)/(1+\delta_0), \psi \in \mathbb{R}, \tag{2}$$

play an important role.

The zeros of the polynomial (1) are partially studied in the papers [55], [53], [58] and in more detail in [47]. In Section 1, I continue the study of the zeros of the polynomial (1). In Section 2, I make some corrections to [53].

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"There is one point that you haven't noticed,"
said the wizad,
"and that is a secret entrance..."

J.R.R. Tolkien

§1. Several properties of the roots of the polynomial (1).

Let $\eta = \theta(1+\gamma_1), \theta^\vee = -\exp(-i\psi)\theta$. Then

$$D^\vee(z,\eta) = (1+\gamma_1)^2 D_1^\vee(z,\theta), \tag{3}$$

where

$$D_1^\vee(z,\theta) = \theta^2 + 2\left(\frac{1}{2} + r\exp(i\psi)\right)\theta + \frac{\gamma_1}{(1+\gamma_1)^2} = \exp(2i\psi)D_2^\vee(z,\theta^\vee) \tag{4}$$

with

$$D_2^\vee(z,\theta^\vee) = (\theta^\vee)^2 - 2\left(\frac{\exp(-i\psi)}{2} + r\right)\theta^\vee + \frac{\gamma_1\exp(-\,2i\psi)}{(1+\gamma_1)^2}. \tag{5}$$

Let

$$D_0(r,\psi,\delta_0) = r^2 + r\exp(-i\psi) + (\delta_0/2)^2\exp(-\,2i\psi). \tag{6}$$

If we put $t = cos(\psi)$, then

$$Im(D_0(r,\psi,\delta_0)) = -((\delta_0)^2/2)t + r)sin(\psi), \tag{7}$$

$$Re(D_0(r,\psi,\delta_0)) = (\delta_0)^2t^2/2 + rt + r^2 - (\delta_0/2)^2, \tag{8}$$

where $t = \cos(\psi), \psi \in \mathbb{R}$,

$$Re(D_0(r,0,\delta_0)) = r^2 + r + (\delta_0/2)^2, \tag{9}$$

$$Re(D_0(r,\pi,\delta_0)) = r^2 - r + (\delta_0/2)^2, \tag{10}$$

$$Im(D_0(r,k\pi,\delta_0)) = 0, \tag{11}$$

where $k \in \mathbb{Z}$. Clearly, for $t = cos(\psi)$ holds the equality

$$\begin{gathered}|D_0(r,\psi,\delta_0)|^2 = r^4 + r^2 + (\delta_0/2)^4 + \\ 2r^2(\delta_0/2)^2(2t^2-1) + 2r(r^2+(\delta_0/2)^2)t = \\ r^2 + (r^2-(\delta_0/2)^2)^2 + 2r(r^2+(\delta_0/2)^2)t + r^2(\delta_0)^2t^2.\end{gathered} \tag{12}$$

Lemma 4.1. *The value $|D_0(r,\psi,\delta_0)|$ is increasing and concave (convex upwards) function from $t = cos(\psi)$,*

$$\min_{-1\le t\le 1}|D_0(r,\psi,\delta_0)|^2 = |D_0(r,\pi,\delta_0)|^2 = |r^2 - r + (\delta_0/2)^2|. \tag{13}$$

Proof. According (12),

$$|D_0(r,\psi,\delta_0)| = \sqrt{at^2+bt+c}, \tag{14}$$

where

$$a = r^2(\delta_0)^2, b = 2r(r^2+(\delta_0/2)^2), c = r^2+(r^2-(\delta_0/2)^2)^2. \tag{15}$$

Then

$$\frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)|^2 = 2at+b = 2r^2(\delta_0)^2t + 2r(r^2+(\delta_0/2)^2) \geq$$

$$2r^2(\delta_0)^2t + 2r|\delta_0|(r^2+(\delta_0/2)^2) \geq 2r^2 \geq 2r|\delta_0|(r-|\delta_0|/2)^2 \geq 0.$$

The inequality

$$\left(\frac{\partial}{\partial t}\right)^2 |D_0(r,\psi,\delta_0)| \leq 0$$

proved in [53], Lemma 3.1.1. ■

Lemma 4.2. *Let $-1 < \delta_0 < 1, 0 < r$. Then there exist a continuos function $R_0(\psi) = R_0(r,\psi,\delta_0)$, defined on $\mathbb{R}$, such that*

$$R_0(r,\psi,\delta_0)^2 = D_0(r,\psi,\delta_0), \tag{16}$$

$$R_0(r,0,\delta_0) > 0, \tag{17}$$

$$R_0(r,-\psi,\delta_0) = \overline{R_0(r,\psi,\delta_0)}, \tag{18}$$

$$R_0(r,\psi+2\pi k,\delta_0) = (-1)^k R_0(r,\psi,\delta_0), \tag{19}$$

when $2r \in \left[1-\sqrt{1-(\delta_0)^2}, 1+\sqrt{1-(\delta_0)^2}\right)$, $\psi \in \mathbb{R}$,

$$R_0(r,\psi+2\pi k,\delta_0) = R_0(r,\psi,\delta_0), sign(2r-1)R_0(r,\pi,\delta_0) > 0, \tag{20}$$

when $2r \notin \left[1-\sqrt{1-\delta_0^2}, 1+\sqrt{1-(\delta_0)^2}\right)$, $\psi \in \mathbb{R}$.

Proof. According to (13), if

$$2r \neq 1 \pm \sqrt{1-(\delta_0)^2}, \tag{21}$$

then $|D_0(r,\psi,\delta_0)| > 0$ for all $\psi \in \mathbb{R}$. Therefore, in view of (9), in this case for any fixed r and δ_0 there exist a single continuos function

$$\psi_1 = \psi_1(\psi) = \psi_1(r,\psi,\delta_0)$$

such that

$$D_0(r,\psi,\delta_0) = |D_0(r,\psi,\delta_0)| \exp(-i\psi_1(r,\psi,\delta_0)) \tag{22}$$

for each $\psi \in \mathbb{R}$ and

$$\psi_1(r,0,\delta_0) = 0 \tag{23}$$

(See, for example, [29], supplement to the chapter 9, section 2). Since

$$D_0(r, -\psi, \delta_0) = \overline{D_0(r, \psi, \delta_0)}, \tag{24}$$

it follows that there exists $m_1 \in \mathbb{Z}$ with property

$$\psi_1(r, -\psi, \delta_0) = -\psi_1(r, \psi, \delta_0) + 2m_1\pi.$$

In view of (23), $m = 0$ and

$$\psi_1(r, -\psi, \delta_0) = -\psi_1(r, \psi, \delta_0). \tag{25}$$

Since $D_0(r, \psi, \delta_0) = D_0(r, \psi+2\pi, \delta_0) = |D_0(r, \psi, \delta_0)| \exp(-i\psi_1(r, \psi+2\pi, \delta_0))$, it follows that

$$\psi_1(r, \psi + 2\pi, \delta_0) = \psi_1(r, \psi, \delta_0) + 2m_0\pi \tag{26}$$

with constante $m_0 \in \mathbb{Z}$. If $\{\psi^\vee, \psi^\wedge\} \subset \mathbb{R}$ and

$$|\psi_1(r, \psi^\vee, \delta_0) - \psi_1(r, \psi^\wedge, \delta_0)| \geq \pi,$$

then there exist ψ^* between $\psi^\wedge$ and $\psi^\vee$ such that $\psi_1(r, \psi^*, \delta_0)/\pi \in \mathbb{Z}$ and therefore

$$Im(D_0(r, \psi^*, \delta_0)) = 0. \tag{27}$$

But in view of (7), if $2r \geq (\delta_0)^2$, then equality (27) holds only for $\psi = k\pi$ with $k \in \mathbb{Z}$; therefore, if $\{\psi^\vee, \psi^\wedge\} \subset (k\pi, (k+1)\pi)$ for some $k \in \mathbb{Z}$, then

$$|\psi_1(r, \psi^\vee, \delta_0) - \psi_1(r, \psi^\wedge, \delta_0)| < \pi, \tag{28}$$

If

$$2r \geq 1 + \sqrt{1 - (\delta_0)^2}$$

then $2r > 1 > (\delta_0)^2$ and, according to (10), (25), (28) and (23),

$$\psi_1(r, \pm\pi, \delta_0) = 0, \tag{29}$$

if $2r > 1 + \sqrt{1 - (\delta_0)^2}$. Taking now in (26) $\psi = -\pi$, we see that, according to (29), $m_0 = 0$ and

$$\psi_1(r, \psi + 2\pi k, \delta_0) = \psi_1(r, \psi, \delta_0), \tag{30}$$

for any $k \in \mathbb{Z}$ and $2r > 1 + \sqrt{1 - (\delta_0)^2}$. If

$$(\delta_0)^2 \leq 2r < 1 + \sqrt{1 - (\delta_0)^2}, \tag{31}$$

then, according to (10), (23), (25), (28), (31) and to the inequality

$$1 - \sqrt{1 - (\delta_0)^2} < (\delta_0)^2 \tag{32}$$

we have

$$\psi_1(r, \varepsilon\pi, \delta_0) = \varepsilon\pi, \tag{33}$$

where $\varepsilon^2 = 1$. Substituting in (26) $\psi = -\pi$, we conclude, in view of (23), that $m_0 = 1$ and

$$\psi_1(r, \psi + 2\pi k, \delta_0) = \psi_1(r, \psi, \delta_0) + 2\pi k, \tag{34}$$

if $k \in \mathbb{Z}$ and $(\delta_0)^2 \leq 2r < 1 + \sqrt{1 - (\delta_0)^2}$. If

$$2r < (\delta_0)^2, \tag{35}$$

and

$$(\psi^\vee - \arccos(-2r/(\delta_0) - 2k\pi)(\psi^\wedge - \arccos(-2r/(\delta_0)^2) - 2k\pi) > 0,$$

where $\{\psi^\vee, \psi^\wedge\} \subset (2k\pi, 2k\pi + \pi)$ for some $k \in \mathbb{Z}$, then

$$|\psi_1(r, \psi^\vee, \delta_0) - \psi_1(r, \psi^\wedge, \delta_0)| < \pi. \tag{36}$$

According to (8),

$$Re(D_0(r, \arccos(-2r/(\delta_0)^2), \delta_0)) = r^2 - (\delta_0/2)^2 < \tag{37}$$
$$(\delta_0^2/2)^2 - (\delta_0/2)^2 < 0,$$

if (35) holds; therefore, in view of (23), (25), (28), (36), if (35) holds, then

$$\psi_1(r, \varepsilon \arccos(-2r/(\delta_0)^2, \delta_0) = \varepsilon\pi, \tag{38}$$

$$|\psi_1(r, \pi, \delta_0) - \psi_1(r, \arccos(-2r/(\delta_0)^2), \delta_0)| \leq \pi. \tag{39}$$

If

$$1 - \sqrt{1 - (\delta_0)^2} < 2r < (\delta_0)^2,$$

then $Re(D_0(r, \pi, \delta_0)) < 0$, in view of (39), (38),

$$\psi_1(r, \pi, \delta_0) = \psi_1(r, \arccos(-2r/(\delta_0)^2, \delta_0)$$

and (33) holds. Substituting in (26) $\psi = -\pi$, we conclude, in view of (33), that $m_0 = 1$ and (34) holds. If $2r < 1 - \sqrt{1 - (\delta_0)^2}$, then (39) holds again, but now $Re(D_0(r, \pi, \delta_0)) > 0$. In view of (8), the value $Re(D_0(r, \psi, \delta_0))$ decreases together with increasing $\psi \in (0, \arccos(-2r/(\delta_0)^2))$ and it increases with increasing $\psi \in (\arccos(-2r/(\delta_0)^2), \pi)$. Moreover, in view of (7),

$$\frac{\partial}{\partial \psi}(-Im(D_0(r, \psi, \delta_0)))\,|_{\psi = \arccos(-2r/(\delta_0)^2))} = 2(r^2 - (\delta_0/)^4/4)/(\delta_0)^2 < 0.$$

Therefore now

$$\psi_1(r, \varepsilon\pi, \delta_0) = 2\varepsilon\pi. \tag{40}$$

Taking in (26) $\psi = -\pi$, we see, that, in view of (40), $m_0 = 2$ and

$$\psi_1(r, \psi + 2\pi k, \delta_0) = \psi_1(r, \psi, \delta_0) + 4k\pi, \tag{41}$$

If holds (21), let

$$R_0(r, \psi, \delta_0) = \sqrt{|D_0(r, \psi, \delta_0)|}\exp(-i\psi_1(r, \psi, \delta_0)/2). \tag{42}$$

The previous considerations show, that in this case all the assertions of the Lemma are fulfilled for $R_0(r,\psi,\delta_0)$ from (42).

We consider the case

$$|2r-1| = \sqrt{1-(\delta_0)^2}. \tag{43}$$

now. Then holds the equality

$$r^2 - r + (\delta_0/2)^2 = 0. \tag{44}$$

Substracting (44) from (6) we obtain the equality

$$D_0(r,\psi,\delta_0) = (1+\exp(-i\psi))(r+(\delta_0/2)^2(\exp(-i\psi)-1)) \tag{45}$$

Let

$$D_h(r,\psi,\delta_0) = r + (\delta_0/2)^2(\exp(-i\psi)-1). \tag{46}$$

Clearly, the value $Im(D_h(r,\psi,\delta_0)) = -(\delta_0/2)^2\sin(\psi)$ vanish only for $\psi = k\pi$ with $k \in \mathbb{Z}$; but then

$$Re(D_h(r,k\pi,\delta_0)) = r + (\delta_0/2)^2((-1)^k - 1) \neq 0, \tag{47}$$

because of

$$1-\sqrt{1-(\delta_0)^2} < (\delta_0)^2 < 1+\sqrt{1-(\delta_0)^2}. \tag{48}$$

Therefore $D_h(r,\psi,\delta_0)$ not vanish for $\psi \in \mathbb{R}$ and for any fixed r and δ_0 there exist a single continuos function $\psi_h = \psi_h(\psi) = \psi_h(r,\psi,\delta_0)$ such that

$$D_h(r,\psi,\delta_0) = |D_h(r,\psi,\delta_0)|\exp(-i\psi_h(r,\psi,\delta_0)) \tag{49}$$

for each $\psi \in \mathbb{R}$ and

$$\psi_h(r,0,\delta_0) = 0. \tag{50}$$

Reasoning, as before, and using (47), we see that

$$\psi_h(r,-\psi,\delta_0) = -\psi_h(r,\psi,\delta_0) \tag{51}$$

$$\psi_h\left(\frac{1+\sqrt{1-(\delta_0)^2}}{2}, \pi, \delta_0\right) = 0, \psi_h\left(\frac{1-\sqrt{1-(\delta_0)^2}}{2}, \varepsilon\pi, \delta_0\right) = \varepsilon\pi, \tag{52}$$

with $\varepsilon^2 = 1$,

$$\psi_h(r,\psi+2\pi,\delta_0) = \psi_h(r,\psi,\delta_0) + 2m_h\pi \tag{53}$$

where $m_h = 0$, if $2r = 1+\sqrt{1-(\delta_0)^2}$, and $m_h = 1$, if $2r = 1-\sqrt{1-(\delta_0)^2}$.
Let

$$R_h(r,\psi,\delta_0) = \sqrt{|D_h(r,\psi,\delta_0)|}\exp(i\psi_h(r,\psi,\delta_0)/2). \tag{54}$$

Then $R_h(r,\psi,\delta_0)$ as function of ψ is continuos,

(55) $$R_h(r,\psi,\delta_0)^2 = D_h(r,\psi,\delta_0),\ R_h(r,0,\delta_0) > 0,$$

(56) $$R_h(r,\psi + 2\pi k,\delta_0) = (-1)^k R_h(r,\psi,\delta_0).$$

if $2r = 1 - \sqrt{1-(\delta_0)^2}, \psi \in \mathbb{R}$ and

(57) $$R_h(r,\psi + 2\pi k,\delta_0) = R_h(r,\psi,\delta_0).$$

if $2r = 1 + \sqrt{1-(\delta_0)^2}, \psi \in \mathbb{R}$. Clearly, the function

(58) $$R^{\vee}(\psi) = sign(\cos(\psi/2))\sqrt{2|\cos(\psi/2)|}\exp(-i\psi/2)$$

is continuos $2\pi-$periodical function on $\mathbb{R}$ such that

$$R^{\vee}(\psi)^2 = 1 + \exp(-i\psi), R^{\vee}(-\psi) = \overline{R^{\vee}(\psi)}.$$

Let now

(59) $$R_0(r,\psi,\delta_0) = R^{\vee}(\psi,\delta_0)R_h(r,\psi,\delta_0),$$

where r satisfies (43). The previous considerations show that in this case all the assertions of the Lemma are fulfilled for $R_0(r,\psi,\delta_0)$ from (59). ■

We considered in (6) the value $D_0(r,\psi,\delta_0)$ only for $|\delta_0| < 1$, but it may be considered also for $\delta_0 = 1$ and in this case from (6) follows that

(60) $$D_0(r,\psi,1) = \left(r + \frac{\exp(-i\psi)}{2}\right)^2.$$

If we let

(61) $$R_0(r,\psi,1)) = r + \frac{\exp(-i\psi)}{2},$$

then we obtain 2π - periodical continuos function $R_0(r,\psi,1) = R_0(r,\psi)$ such that

(62) $$(R_0(r,\psi))^2 = D_0(r,\psi,1), R_0(r,0) > 0, R_0(r,-\psi) = \overline{R_0(r,\psi)}.$$

According to (61), for $r > 0, |\delta_0| < 1$ and $k = 0,1$ the values

(63) $$\theta_k^{\vee}(r,\psi,\delta_0) = R_0(r,\psi,1) + (-1)^k R_0(r,\psi,\delta_0)$$

compose the set of all the roots of the polynomial (5), the values

(64) $$\theta_k^{\wedge}(r,\psi,\delta_0) = -\exp(i\psi)(R_0(r,\psi,1) + (-1)^k R_0(r,\psi,\delta_0))$$

compose the set of all the roots of the polynomial (4) and the values

(65) $$\eta_k^{\wedge}(r,\psi,\delta_0) = -(1+\gamma_1)exp(i\psi)(R_0(r,\psi,1) + (-1)^k R_0(r,\psi,\delta_0))$$

compose the set of all the roots of the polynomial (1) .

It follows from (65), Lemma 4.2 and (62) that

$$\eta_k^\wedge(r, -\psi, \delta_0) = \overline{\eta_k^\wedge(r, \psi, \delta_0)} \tag{66}$$

where $\psi \in \mathbb{R}$, $r > 0$, $|\delta_0| < 1$, $k = 0, 1$.

Let $\varepsilon^3 = \varepsilon$,

$$R_\varepsilon^*(r, \psi, \delta_0) = r + (\exp(-i\psi))/2 - \varepsilon \exp(-i\psi)(1 + \delta_0)/2; \tag{67}$$

according to (65), (61),

$$\eta_k^\wedge(r, \psi, \delta_0) + \varepsilon = -\frac{2\exp(i\psi)}{1 + \delta_0}(R_\varepsilon^*(r, \psi, \delta_0) + (-1)^k R_0(r, \psi, \delta_0)), \tag{68}$$

where $\varepsilon^3 = \varepsilon, k = 0, 1$.

Lemma 4.3. *The values*

$$|\eta_k^\wedge(r, \psi, \delta_0) + \varepsilon|^2,$$

where $k = 0, 1$, $0 < r$, $\psi \in \mathbb{R}$, $\varepsilon^3 = \varepsilon$, *are nonnegative zeros of the polyinomial*

$$x^2 - xp_{1+\varepsilon}(r, \psi, \delta_0) + q_{1+\varepsilon}(r, \psi, \delta_0), \tag{69}$$

where $\varepsilon^3 = \varepsilon$, *and*

$$p_{1+\varepsilon}(r, \psi, \delta_0) = 8\frac{|R_\varepsilon^*(r, \psi, \delta_0)|^2 + |R_0(r, \psi, \delta_0)|^2}{(1 + \delta_0)^2}, \tag{70}$$

$$\prod_{k=0}^{1} |\eta_k^\wedge(r, \psi, \delta_0) + \varepsilon|^2 = q_{1+\varepsilon}(r, \psi, \delta_0) = |D^\vee(z, -\varepsilon)|^2. \tag{71}$$

Proof. The proof may be found in [53], Lemma 3.1.2, or in [47], Lemma 2.

Lemma 4.4. *Let* $\varepsilon^2 = \varepsilon$. *For* $k = 0, 1$ *with increasing* $\psi \in (0, \pi)$ *the value*

$$(-1)^k |\eta_k^\wedge(r, \psi, \delta_0)|$$

monotonically decreases and, if $2r \neq \delta_0$, *then the value*

$$(-1)^k |\eta_k^\wedge(r, \psi, \delta_0) + 1| sign(2r - \delta_0)$$

also monotonically decreases. Moreover

$$|\eta_1^\wedge(r, \psi, \delta_0)| < |\eta_0^\wedge(r, \psi, \delta_0)|\,, \tag{72}$$

if $\psi \in [0, \pi)$;

$$sign(2r - \delta_0)(|\eta_1^\wedge(r, \psi, \delta_0) + 1| - |\eta_0^\wedge(r, \psi, \delta_0) + 1|) < 0\,, \tag{73}$$

if $\psi \in [0, \pi)$ *and* $2r \neq \delta_0$;

$$|\eta_1^\wedge(r, \psi, \delta_0) + 1| = |\eta_0^\wedge(r, \psi, \delta_0) + 1| = (2\delta_0/(1 + \delta_0))^{1/2}\,, \tag{74}$$

if $\psi \in [0, \pi]$ *and* $2r = \delta_0$;

$$(2sign(2r - \delta_0) + 1)^\varepsilon(|\eta_1^\wedge(r, \pi, \delta_0) + \varepsilon| - |\eta_0^\wedge(r, \pi, \delta_0) + \varepsilon|) < 0\,, \tag{75}$$

if $|2r-1| > \sqrt{1-(\delta_0)^2}$;

$$|\eta_1^\wedge(r,\pi,\delta_0)+\varepsilon| = |\eta_0^\wedge(r,\pi,\delta_0)+\varepsilon| = \sqrt{q_{1+\varepsilon}}\,, \tag{76}$$

if

$$|2r-1| \le \sqrt{1-(\delta_0)^2}\,; \tag{77}$$

$$|\eta_1^\wedge(r,\pi,\delta_0)+\varepsilon| \le \left|\frac{1-\delta_0}{1+|\delta|}+\varepsilon\right| = \frac{1-|\delta_0|}{1+\delta_0}+\varepsilon \le \tag{78}$$

$$\frac{1+|\delta_0|}{1+\delta_0}+\varepsilon \le |\eta_0^\wedge(r,\pi,\delta_0)+\varepsilon|\ ,$$

if $r \ge 1$; *moreover the first and the last inequaliies (78) are strong, if* $r > 1$, *the middle of these inequalities is strong, if* $\delta_0 \ne 0$;

$$|\eta_1^\wedge(r,\pi,\delta_0)+\varepsilon| = |\eta_0^\wedge(r,\pi,\delta_0)+\varepsilon| = 1+\varepsilon\,, \tag{79}$$

if $r=1$ *and* $\delta=0$.

Proof. For $r \ge 1$ the proof coincides with the proof of the Lemma 3.1.3 in [53]; we must make only the supplements for the case $0<r<1$ to this proof. For $r=\delta_0/2$ the inequality (3.1.49) in [53] holds and the value (3.1.51) in [53] for $r=\delta_0/2$ is equal to 0; therefore (74) holds. It follows from (3.1.52) in [53] that

$$|\eta_0^\wedge(r,0,\delta_0)| = \frac{2r+1+2\sqrt{r^2+r+(\delta_0/2)^2}}{1+\delta_0} > \tag{80}$$

$$|\eta_1^\wedge(r,0,\delta_0)| = \frac{|2r+1-2\sqrt{r^2+r+(\delta_0/2)^2}|}{1+\delta_0}$$

and, if $2r \ne \delta_0$, then

$$sign(2r-\delta_0)|\eta_0^\wedge(r,0,\delta_0)+1| = \tag{81}$$

$$sign(2r-\delta_0)\frac{2r-\delta_0+2\sqrt{r^2+r+(\delta_0/2)^2}}{1+\delta_0} >$$

$$sign(2r-\delta_0)|\eta_1^\wedge(r,0,\delta_0)+1| =$$

$$sign(2r-\delta_0)\frac{|2r-\delta_0-2\sqrt{r^2+r+(\delta_0/2)^2}|}{1+\delta_0}.$$

Therefore are fulfilled (72) and (73). The equality (3.1.54) in [53] holds now for $\varepsilon=0$ but for $\varepsilon=1$ it must be replaced by

$$|\eta_k^\wedge(r,\psi,\delta_0)+1|^2 = \frac{p_2}{2} + sign(2r-\delta_0)(-1)^k\sqrt{\left(\frac{p_2}{2}\right)^2-q_2}. \tag{82}$$

The equality (3.1.54) in [53] holds now for $\varepsilon=0$; therefore, in view of (3.1.43) in [53], the value

$$(-1)^k|\eta_k^\wedge(r,\psi,\delta_0)|$$

monotonically increases with increasing $t=cos(\psi) \in (-1,1)$.

In view of (82), if $2r \neq \delta_0$, then with increasing $t = cos(\psi) \in (-1, 1)$ the value

$$|\eta^{\wedge}_{(1-sign(2r-\delta_0))/2}(r, \psi, \delta_0) + 1|$$

monotonically increases and, in view of (3.1.43) in [53], the value

$$|\eta^{\wedge}_{(1+sign(2r-\delta_0))/2}(r, \psi, \delta_0) + 1|$$

monotonically decreases with increasing $t = cos\psi \in (-1, 1)$.

The equalities (74) follow from (3.1.43) in [53] and (82). For $t = -1$ each of the inequalities (3.1.46), (3.1.49) in [53] is equivalent to inequality

$$|2r - 1| \leq \sqrt{1 - (\delta_0)^2}.$$

Therfore (76) holds.

In view of (3.1.54) with $\varepsilon = 0$ in [53], also (75) with $\varepsilon = 0$ holds. Clearly,

$$(1 - \sqrt{1 - (\delta_0)^2})/2 \leq |\delta_0|/2 \leq (1 + \sqrt{1 - (\delta_0)^2})/2, \tag{83}$$

and the inequality $|2r - 1| > \sqrt{1 - (\delta_0)^2}$ implies the inequality $2r \neq \delta_0$; therefore (75) holds also for $\varepsilon = 1$. ■

Remark 1. The equality (3.1.55) in [53] for $\varepsilon^3 = \varepsilon$, $k^2 = k$, in view of (65), (67), (68) and (20), must be modified as follows:

if $|2r - 1| \geq \sqrt{1 - \delta_0^2}$, then

$$\eta^{\wedge}_k(r, \pi, \delta_0) + \varepsilon = \tag{84}$$

$$\frac{2\left(r - \frac{1}{2} + \varepsilon\frac{1+\delta_0}{2} + (-1)^k sign(2r - 1)\sqrt{r^2 - r + \left(\frac{\delta_0}{2}\right)^2}\right)}{1 + \delta_0}.$$

and if $|2r - 1| \geq \sqrt{1 - (\delta_0)^2}$, then

$$\eta^{\wedge}_k(r, \pi, \delta_0) + \varepsilon = \overline{\eta^{\wedge}_{1-k}(r, \pi, \delta_0) + \varepsilon}.$$

If $\varepsilon^2 = \varepsilon$ and $2r > 1$ then from (84) follows the relation

$$|\eta^{\wedge}_0(r, \pi, \delta_0) + \varepsilon| = \eta^{\wedge}_0(r, \pi, \delta_0) + \varepsilon = \frac{2r - 1 + 2\sqrt{r^2 - r + (\frac{\delta_0}{2})^2}}{1 + \delta_0} + \varepsilon. \tag{85}$$

Remark 2. Let us to check (75) directly for $r < (1 - \sqrt{1 - (\delta_0)^2})/2$. In view of (83),now $sign(2r + \delta_0) = sign(\delta_0)$ and $2r < 1$. Therefore (75) in this case follows directly from (65) and second of the inequalities (20). This understrike the importance of the Lemma 4.2.

Corollary 1. *The following inequalities hold*

$$|\eta^{\wedge}_1(r, \psi, \delta_0)| \leq |\eta^{\wedge}_1(r, \pi, \delta_0)| \leq \tag{86}$$

$$\sqrt{\frac{1 - \delta_0}{1 + \delta_0}} \leq \eta^{\wedge}_0(r, \pi, \delta_0) \leq \eta^{\wedge}_0(r, \pi, \delta_0),$$

$$|\eta^{\wedge}_1(r, \psi, \delta_0) + 1| \leq |\eta^{\wedge}_1(r, \psi, \delta_0)| + 1 \leq |\eta^{\wedge}_1(r, \pi, \delta_0)| + 1 \leq \tag{87}$$

$$1+\sqrt{\frac{1-\delta_0}{1+\delta_0}},$$

(88) $$4r/(1+\delta_0+\sqrt{(1-(\delta_0)^2}\le |\eta_0^\wedge(r,\psi,\delta_0)+1|$$

and equalities hold only if $\psi=\pi, |2r-1|\le\sqrt{1-(\delta_0)^2}$.

Proof. The first and the last inequalities in (86) follow directly from assertions of the Lemma, the second and the third inequalities in (86) follow from the first equality (3.1.43) in [53], from (72), from (75) for $\varepsilon=0$ and from (76) for $\varepsilon=0$. The first inequality (87) is obvious, the others directly follow from (86). The inequality (88) directly follow from (87) and second equality (3.1.43) in [53]. ■

Corollary 2. *If* $2r-1>\sqrt{1-(\delta_0)^2}, \psi\in(0,\pi)$, *then*

(89) $$|\eta_0^\wedge(r,\psi,\delta_0)|>\eta_0^\wedge(r,\pi,\delta_0)=$$
$$\frac{2(r-1/2+\sqrt{r^2-r+(\delta_0/2)^2})}{1+\delta_0}\ge$$
$$\frac{2(r-1/2-\sqrt{r^2-r+(\delta_0/2)^2})}{1+\delta_0}=$$
$$\eta_1^\wedge(r,\pi,\delta_0)>|\eta_1^\wedge(r,\psi,\delta_0)|,$$

(90) $$|\eta_1^\wedge(r,\psi,\delta_0)+1|\le\eta_1^\wedge(r,\psi,\delta_0)|+1<$$
$$|\eta_1^\wedge(r,\pi,\delta_0)|+1=\eta_1^\wedge(r,\pi,\delta_0)+1=$$
$$\frac{2(r+\delta_0/2-\sqrt{r^2-r+(\delta_0/2)^2})}{1+\delta_0}\le$$
$$\frac{2(r+\delta_0/2+\sqrt{r^2-r+(\delta_0/2)^2})}{1+\delta_0}=$$
$$\eta_0^\wedge(r,\pi,\delta_0)+1=4r(\eta_1^\wedge(r,\pi,\delta_0)+1)^{-1}/(1+\delta_0)<$$
$$4r|\eta_1^\wedge(r,\psi,\delta_0)+1|^{-1}/(1+\delta_0)=|\eta_0^\wedge(r,\psi,\delta_0)+1|.$$

Proof. The assertions of the corollary follow from (65), (3.1.43) in [53], the assertions of the Lemma 4.4, Lemma 4.2 and Lemma 4.3. ■

Lemma 4.5. *The following equalities hold:*

$$\eta_k^\wedge(r,\psi,\delta_0)=1/\eta_{1-k}^\wedge(r,\psi,-\delta_0),$$

where $k^2=k$, $|\delta_0|<1$, $\psi\in\mathbb{R}$.

Proof. In view of (6)

$$D_0(r,\psi,\delta_0)=D_0(r,\psi,-\delta_0).$$

Therefore, for $D_0(r,\psi,\delta_0)$ from the Lemma 4.2 we have the equality

$$R_0(r,\psi,\delta_0)=R_0(r,\psi,-\delta_0).$$

By substitution $\delta\to-\delta$ the value $1+\gamma_1=2/(1+\delta_0)$ turns into $2/(1-\delta_0)$. In view of (65),

$$\eta_k^\wedge(r,\psi,\delta_0/\eta_k^\wedge(r,\psi,-\delta_0=\gamma_1,$$

and γ_1 is the prodoct of both roots of the polynomial (1). ■

Corollary. *Let $\delta_0 \geq 0$. For $k = 0, 1$ the value*

$$(-1)^k |1/\eta_k^\wedge(r, \psi, \delta_0) + 1|$$

monotonically increases with increasing $\psi \in (0, \pi)$.

Proof. According to the Lemma 4.5, and,if $2r \neq \delta_0$, then the value

$$(-1)^k |1/\eta_k^\wedge(r, \psi, \delta_0) + 1| = -(-1)^{1-k} |\eta_{1-k}^\wedge(r, \psi, -\delta_0) + 1|,$$

because $2r > -\delta_0$. ■

Let

$$s = s_0(r, \psi) = |r \exp(i\psi) + 1|/2 = \sqrt{\frac{(r-1)^2}{4} + r\cos^2\left(\frac{\psi}{2}\right)}, \tag{91}$$

$$\psi_0(r, \psi) = \arg(r \exp(i\psi) + 1) \in (-\pi, \pi), \tag{92}$$

where $\psi \in (-\pi, \pi)$. Clearly, $\psi_0(r, -\psi) = -\psi_0(r, \psi)$, when $\psi \in (-\pi, \pi)$.

If $0 < r \leq 1$, then $\psi_0(r, \psi)$ increases in $(0, \arcsin(r))$ together with increasing ψ in $(0, \pi/2 + \arcsin(r))$ and defines bijection of $(0, \pi/2 + \arcsin(r))$ onto $(0, \arcsin(r))$. If $0 < r < 1$, then $\psi_0(r, \psi)$ decreases in $(0, \arcsin(r))$ with increasing ψ in $(\pi/2 + \arcsin(r), \pi)$ and defines bijection of $(\pi/2 + \arcsin(r), \pi)$ onto $(0, \arcsin(r))$. If $1 < r$, then the function $\psi_0(r, \psi)$ increases in $(0, \pi)$ with increasing ψ in $(0, \pi)$ and defines bijection of $(0, \pi)$ onto $(0, \pi)$. Clearly,

$$|r - 1|/2 \leq s \leq (r + 1)/2, \tag{93}$$

for s in (91). It follows from (91) that

$$t = \cos(\psi) = (4s^2 - r^2 - 1)/2r \tag{94}$$

Lemma 4.6. *If at least one of the following conditions holds:*

a)

$$2r > \max(1 + \sqrt{1 - (\delta_0)^2}, 2 + \delta_0) \tag{95}$$

and $|r - 1|/2 = s$,

b)

$$2r < 1 - \sqrt{1 - (\delta_0)^2} \tag{96}$$

and $|r - 1|/2 = s$,

c) $\max(|r - 1|/2, \delta_0/4) < s \leq (r + 1)/2$, *then*

$$|\eta_0^\wedge(r, \psi, \delta_0) - 1| > |\eta_1^\wedge(r, \psi, \delta_0) - 1|. \tag{97}$$

If at least one of the following conditions holds:

d) $2s = |r - 1|$, $2r = 2 + \delta_0$,

e) $2s = |r - 1|$, $|2r - 1| \leq \sqrt{1 - (\delta_0)^2}$,

f) $2|r - 1| \leq 4s = \delta_0 \leq 2|r + 1|$, *then*

(98) $$|\eta_0^\wedge(r,\psi,\delta_0)-1|=|\eta_1^\wedge(r,\psi,\delta_0)-1|,.$$

If at least one of the following conditions hold

g)

(99) $$1+\sqrt{1-(\delta_0)^2}<2r<2+\delta_0.$$

and $2s=|r-1|$,

h)

(100) $$2|r-1|<4s<\delta_0,$$

then

(101) $$|\eta_0^\wedge(r,\psi,\delta_0)-1|<|\eta_1^\wedge(r,\psi,\delta_0)-1|,$$

Proof. The left part of (3.1.73) in [53] for $s=(r+1)/2$ is equal to the positive number $r^2+r+(\delta_0/2)^2$; therefore $s=(r+1)/2$ is not a solution of the system (3.1.73) - (3.1.74) in [53]; only $s=\frac{1-r}{2}$ and $s=\frac{\delta_0}{4}$ may be a solutions of this system. In view of (91) and (93), the equality $s=\frac{1-r}{2}$ is possible only if the inequality $r\le 1$ and equality $\psi=\pi$ are fulfilled in the same time. In view of (84), if $|2r-1|\ge\sqrt{1-(\delta_0)^2}$, then

(102) $$\eta_k^\wedge(r,\pi,\delta_0)-1=$$

$$\frac{2\left(r-1-\frac{\delta_0}{2}+(-1)^k sign(2r-1)\sqrt{r^2-r+\left(\frac{\delta_0}{2}\right)^2}\right)}{1+\delta_0};$$

if $|2r-1|\ge\sqrt{1-(\delta_0)^2}$, then

(103) $$\eta_k^\wedge(r,\pi,\delta_0)=\overline{\eta_{1-k}^\wedge(r,\pi,\delta_0)}$$

where $k=0,1$. If (95) is fulfiiled, then $2r>1$, in view of (102),

(104) $$|\eta_0^\wedge(r,\pi,\delta_0)-1|>|\eta_1^\wedge(r,\pi,\delta_0)-1|$$

and therefore the inequality (97) holds. If (96) holds, then $2r<1$; making use (102) and the inequality $(1-\sqrt{1-(\delta_0)^2})/2<1+\delta_0/2$, we conclude that the inequality (104) holds and therefore in the case b) the inequality (97) holds . In the cases d) and e) the equality (98) directly follows from (103). In the case g) the inequalities (99) hold, therefore $2r>1$ and, according to the equality (102),

(105) $$|\eta_0^\wedge(r,\pi,\delta_0)-1|<|\eta_1^\wedge(r,\pi,\delta_0)-1|.$$

Consequently, (101) holds in the case g).

We consider now $s=\delta_0/4$ as possible solution of system, consisting of the inequality (3.1.73) in [53], inequalities (93) and equation (3.1.74) in [53]; according to (93), the equality

(106) $$s=\delta_0/4$$

is possible only if

$$|r-1| \le \delta_0/2; \tag{107}$$

If for $s=\delta_0/4$ in (107) an equality achieves, then $r=1\pm\delta_0/2$ with $\delta_0\ge 0$, the case $r=1+\delta_0/2$ coincides with case d) considered above, and if $r=1-\delta_0/2$ with $\delta_0>0$ then $|2r-1|=|1-\delta_0|<|1-\delta_0^2|^{1/2}<(1+\sqrt{1-(\delta_0)^2})/2$ and this case coincides with case e) considered above.

We must consider now the case

$$|r-1|/2 < s = \delta_0/4; \tag{108}$$

But (3.1.73) in [53] for $s=\delta_0/4$ turns into inequality

$$\delta_0(r^2-(\delta_0/2)^2)/2 > 0,$$

which follows from (108).

So, if (107) is fulfilled, then $s=\delta_0/4$ is the solution of system, consisting of inequality (3.1.73) - (3.1.74) in [53] and inequalities (93). Therefore in case f) (98) holds .

In view of (3.1.52) in [53], the inequality (97) holds in the case c).

It remains to consider the case h).

If $2r\ge 2+\delta_0$, then inequality $2|r-1|<\delta_0$ is impossible. If (99) holds, then (103) holds and, consequently, (101) holds.

If $2r\le 1+\sqrt{1-(\delta_0)^2}$, then $2-2r\ge 1-\sqrt{1-(\delta_0)^2}>0$. Since

$$1-\sqrt{1-(\delta_0)^2} < 2-\delta_0 < 1+\sqrt{1-(\delta_0)^2},$$

if $\delta_0>0$, it follows that we must consider the case

$$2-\delta_0 < 2r \le 1+\sqrt{1-(\delta_0)^2}.$$

Let us consider the function $|\eta_0^\wedge(r,\psi,\delta_0)-1|-|\eta_1^\wedge(r,\psi,\delta_0)-1|$ in the open triangle

$$2-\delta_0<2r<2+\delta_0,\ 2|r-1|<4s<\delta_0,$$

where, in view of (94), $t=\cos(\psi)=\arccos(4s^2-r^2-1)/2r$; in this triaqngle it does not vanish and continuos; (101) holds in the part of this triangle, defined by means (99). ■

Lemma 4.7. *Let*

$$f(\Delta)=\frac{\Delta+1}{\Delta}\left(\frac{\Delta+1}{\Delta-1}\right)^{\Delta-1}, \tag{109}$$

where $\Delta>1$. *Then* $f(2)=9/2$,

$$f(\Delta)\ge 16/3 \tag{110}$$

for $\Delta\in 2+\mathbb{N}$.

Proof. For $\Delta=3$ the inequality (114) turns into equality, for $\Delta=4$ and $\Delta=5$ it may be obtained directly. Finally,

$$\left(\left(\frac{\partial}{\partial\Delta}\log(f)\right)\right)(\Delta)=\log(1+2/(\Delta-1))-\frac{1}{\Delta+1}-\frac{1}{\Delta}\ge$$

$$\frac{2}{\Delta-1}-\frac{2}{(\Delta-1)^2}-\frac{1}{\Delta+1}-\frac{1}{\Delta}=\frac{1}{\Delta^2-\Delta}-\frac{2}{(\Delta-1)^2}+\frac{2}{\Delta^2-1}=$$
$$\frac{1}{\Delta^2-\Delta}-\frac{4}{(\Delta^2-1)(\Delta-1)}>0,$$

when $\Delta>5.$ ■

Lemma 4.8. *Let $\mu>0$ and all roots of quadratic trinomial x^2-px+q are positive. Let further $x_{\max}$ and $x_{\min}$ denote respectively the maximal and minimal of these roots. Then the inequality*

$$\frac{x_{\min}}{x_{\max}}\geq\mu^2 \tag{111}$$

is equivalent to the inequality

$$\frac{p/2}{\sqrt{q}}\leq\frac{\mu^2+1}{2\mu}. \tag{112}$$

Proof. We have

$$\frac{x_{\min}}{x_{\max}}=\frac{(x_{\min})^2}{q}=$$
$$(\frac{x_{\min}}{\sqrt{q}})^2=\left(\frac{p/2}{\sqrt{q}}-\sqrt{\left(\frac{p/2}{\sqrt{q}}\right)^2-1}\right)^2.$$

Therefore (111) is equivalent to the inequality

$$\frac{p/2}{\sqrt{q}}-\mu>\sqrt{\left(\frac{p/2}{\sqrt{q}}\right)^2-1},$$

which is equivalent to (112) ■.

It follows from the Lemma 3.1.4 and its proof that

$$1\leq\frac{|\eta_0^\wedge(r,\pi,\delta_0)|}{|\eta_1^\wedge(r,\pi,\delta_0)|}\leq\frac{|\eta_0^\wedge(r,\psi,\delta_0)|}{|\eta_1^\wedge(r,\psi,\delta_0)|}, \tag{113}$$

and if $2r\geq\delta_0$, then

$$1\leq\frac{|\eta_0^\wedge(r,\pi,\delta_0)+1|}{|\eta_1^\wedge(r,\pi,\delta_0)+1|}\leq\frac{|\eta_0^\wedge(r,\psi,\delta_0)+1|}{|\eta_1^\wedge(r,\psi,\delta_0)+1|}, \tag{114}$$

where $\psi\in\mathbb{R}$; moreover, if $2r>1+\sqrt{1-\delta_0^2}$, then left inequality in (113) and left inequality in (114) are strong; on the other hand, when $r\geq\delta_0/2$ and $\psi\in[0,\pi)$, then right inequality in (113) and right inequality in (114) are strong.

Let $\delta_0>0, 2r\geq\delta_0$,

$$\mu_0(r,\delta_0)=\left(\frac{|\eta_1^\wedge(r,\pi,\delta_0)|}{|\eta_0^\wedge(r,\pi,\delta_0)|}\right)^{(1/\delta_0-1)}\frac{|\eta_1^\wedge(r,\pi,\delta_0)+1|}{|\eta_0^\wedge(r,\pi,\delta_0)+1|}, \tag{115}$$

$$c_0(r,\delta_0)=\frac{(\mu_0(r,\delta_0))^2+1}{2\mu_0(r,\delta_0)}. \tag{116}$$

According to (3.1.43) in [29] and (113) - (115), if $2r \geq \delta_0$, then

$$0 < \mu_0(r, \delta_0) \leq 1, \tag{117}$$

$$1/\mu_0(r, \delta_0)) \leq \left(\frac{|\eta_0^\wedge(r, \psi, \delta_0)|}{|\eta_1^\wedge(r, \psi, \delta_0)|}\right)^{1/\delta_0 - 1} \frac{|\eta_0^\wedge(r, \psi, \delta_0) + 1|}{|\eta_1^\wedge(r, \psi, \delta_0) + 1|}. \tag{118}$$

and if $2r > 1 + \sqrt{1 - (\delta_0)^2}$, then right inequality in (117) is strong.

Lemma 4.9. *Let $\delta_0 \geq 0$ and at least one of the following conditionsis is fulfilled:*

A) $2r > \max(1 + \sqrt{1 - (\delta_0)^2}, 2 + \delta_0)$, $|r - 1| \leq 2s \leq (r + 1)$;

B) $\max(2|r - 1|, \delta_0 < 4s \leq 2(r + 1)$, $\delta_0 \leq 2r$

C) $2|r - 1| < \delta_0 = 4s \leq (r + 1)$, $\delta_0 \leq 2r$,

D) $|r - 1|/2 = s \leq (r + 1)/2$, $1 + \sqrt{1 - (\delta_0)^2} < 2r$,

E) the case h) of the Lemma 4.6 holds and

$$s > \frac{\delta_0((1 + \delta_0/2)^2 - r^2)}{4(c_0(r, \delta_0)(1 + \delta_0) - \delta_0/2 - r)}, \tag{119}$$

then

$$|\eta_0^\wedge(r, \psi, \delta_0)|^{1-\delta_0}|(\eta_0^\wedge(r, \psi, \delta_0))^2 - 1|^{\delta_0} \geq \tag{120}$$

$$|\eta_1^\wedge(r, \psi, \delta_0)|^{1-\delta_0}|(\eta_1^\wedge(r, \psi, \delta_0))^2 - 1|^{\delta_0}.$$

Proof. According to Lemma 4.4 and Lemma 4.6, (120) holds in each of of cases A), B), C), D); we must consider the case E). The inequality (120) is equivalent to the inequality

$$\left(\frac{|\eta_0^\wedge(r, \psi, \delta_0)|}{|\eta_1^\wedge(r, \psi, \delta_0)|}\right)^{(1/\delta_0 - 1)} \times \tag{121}$$

$$\frac{|\eta_0^\wedge(r, \psi, \delta_0) + 1|}{|\eta_1^\wedge(r, \psi, \delta_0) + 1|}|\eta_0^\wedge(r, \psi, \delta_0) - 1| > |\eta_1^\wedge(r, \psi, \delta_0) - 1|,$$

which, in view of (118), will be fulfilled, if the inequality

$$|\eta_0^\wedge(r, \pi, \delta_0) - 1|^2 > |\eta_1^\wedge(r, \pi, \delta_0) - 1|^2(\mu_0(r, \delta_0))^2 \tag{122}$$

will be fulfilled. Since we are found in the case h) of the Lemma 4.6, it follows that

$$x_{\max} = |\eta_1^\wedge(r, \pi, \delta_0) - 1|^2, x_{\min} = |\eta_0^\wedge(r, \pi, \delta_0) - 1|^2$$

and (122) will be fulfilled, when (112) will be fulfilled with $p = p_0$, $q = q_0$, where p_0 and q_0 are taken from (70) and (71) respectively.

The equality (3.1.69) in [53] may be rewritten in the form

$$|R_0(r, \psi, \delta_0)|^4 = |D_0(r, \psi, \delta_0)|^2 = \tag{123}$$

$$r^2 + (r^2 - (\delta_0/2)^2)^2 + 2r(r^2 + (\delta_0/2)^2)t + r^2(\delta_0)^2 t^2 =$$

$$r^2(\delta_0)^2((4s^2 - r^2 - 1)/2r)^2 +$$

$$2r(r^2 + (\delta_0/2)^2)((4s^2 - r^2 - 1)/2r) + r^2 + (r^2 - (\delta_0/2)^2)^2 =$$

$$4(\delta_0)^2 s^4 + (4r^2 - (\delta_0)^2(2r^2+1))s^2 + (\delta_0/2)^2 r^2(r^2-1) + (\delta_0/2)^4 =$$
$$(2rs + (\delta_0/2)^2)^2 + (\delta_0)^2\alpha(s),$$

where

$$\alpha(s) = 4s^4 - (2r^2+1)s^2 + r^2(r^2-1)/4 - rs = \tag{124}$$
$$(2s^2 - r^2/2)^2 - (s + r/2)^2 =$$
$$(2s^2 + s + r/2 - r^2/2)(2s^2 - s - r/2 - r^2/2) =$$
$$4(s - (r-1)/2)(s + r/2)^2(s - (r+1)/2)) =$$
$$(2s + r)^2(s - (r-1)/2)(s - (r+1)/2)).$$

Since $\alpha(s) \le 0$, when $\frac{r-1}{2} \le s \le \frac{r+1}{2}$, it follows, in view of (3.1.70) - (3.1.71) in [53] and (123), that

$$\frac{p_0}{2\sqrt{q_0(r,\psi,\delta_0)}} \le (2(1+\delta_0)s)^{-1}\times$$
$$(2(2+\delta_0)s^2 - \delta_0(r^2-1) + (\delta_0/2)^2 + 2rs + (\delta_0/2)^2).$$

Therefore (112) will be fulfilled, if the inequaliti

$$c_0(r,\delta_0) > (2(1+\delta_0)s)^{-1}\times$$
$$(2(2+\delta_0)s^2 - \delta_0(r^2-1) + (\delta_0/2)^2 + 2rs + (\delta_0/2)^2),$$

will be fulfilled; according to (100),

$$2(2+\delta_0)s^2 \le (2+\delta_0)(\delta_0/2)s \le \delta_0 s + (\delta_0/2)^3.$$

Therefore (112) will be fulfilled, if the inequality

$$2(c_0(r,\delta_0)(1+\delta_0) - \delta_0/2 - r)s > \tag{125}$$
$$\delta_0(1 + \delta_0/2)^2 - r^2)/2,$$

will be fulfilled; the last inequality is equivalent to (119). ■

Corollary. *If $r = 1$, $1/\delta_0 \ge 3$, then (119) may be replaced by the inequality*

$$\frac{1}{51} < s.$$

Proof. Clearly,

$$\eta_0^\wedge(1,\pi,\delta_0) = 1,\ \eta_0^\wedge(1,\pi,\delta_0) + 1 = 2,$$

$$\eta_1^\wedge(1,\pi,\delta_0) = \frac{1-\delta_0}{1+\delta_0},\ \eta_1^\wedge(1,\pi,\delta_0) + 1 = \frac{2}{1+\delta_0}.$$

According to (115), for $\Delta = 1/\delta \in 2 + \mathbb{N}$ holds the equality

$$\mu_0(1,\delta_0) = \left(\frac{\Delta-1}{\Delta+1}\right)^{\Delta-1}\frac{\Delta}{\Delta+1}.$$

According to the Lemma 4.6, $\mu_0(1, \delta_0) < \frac{3}{16}$, and therefore $c_0(1, \delta_0) > \frac{265}{96}$,

$$4(c_0(r, \delta_0)(1 + \delta_0) - \delta_0/2 - r) >$$

$$4(265 - 16 - 96)/96 = 153/24 = 51/8,$$

$$\delta_0((1 + \delta_/2)^2 - r^2) \leq 13/108,$$

$$\frac{\delta_0((1 + \delta_0/2)^2 - r^2)/2)}{4(c_0(r, \delta_0)(1 + \delta_0) - \delta_0/2 - r)} < \frac{1}{51}\frac{26}{27}. \blacksquare$$

Remark. *If* $r = 1, \psi = -\pi + 2l\pi/m,$ *where*

$$m = 2, , \dots , 150, l = 1 , \dots , m - 1],$$

then the inequality $s = cos(\psi/2) > 1/51$ *will be fulfilled. In fact,*

$$\cos(-\pi/2 + l\pi/m) = sin(l\pi/m) > \sin(\pi/m) > \sin(\pi/150) >$$

$$\sin(1/50) > 1/50 - 1/750000 > 1/51.$$

Since

$$|\eta_k^\wedge(r, \psi, \delta_0) - 1|^2 + |\eta_k^\wedge(r, \psi, \delta_0) + 1|^2 = 2|\eta_k^\wedge(r, \psi, \delta_0)|^2 + 2, \tag{126}$$

it follows that

$$\sum_{k=0}^{1}\sum_{l=0}^{1}(-1)^k|\eta_k^\wedge(r, \psi, \delta_0) + (-1)^l|^2 = 2\sum_{k=0}^{1}(-1)^k|\eta_k^\wedge(r, \psi, \delta_0)|^2, \tag{127}$$

$$\frac{\partial}{\partial t}|\eta_k^\wedge(r, \psi, \delta_0) - 1|^2 = \sum_{\varepsilon=0}^{1}(-1)^\varepsilon 2^{1-\varepsilon}\frac{\partial}{\partial t}|\eta_k^\wedge(r, \psi, \delta_0) + \varepsilon|^2. \tag{128}$$

According to (3.1.54) in [53] with $\varepsilon = 0$ *and (82),*

$$|\eta_k^\wedge(r, \psi, \delta_0) - 1|^2 = 2 + \frac{2p_1 - p_2}{2} + \tag{129}$$

$$(-1)^k\left(2\sqrt{\left(\frac{p_1}{2}\right)^2 - q_1} - sign(2r - \delta_0)\sqrt{\left(\frac{p_2}{2}\right)^2 - q_2}\right).$$

According to the Lemma 4.4, if

$$\varepsilon^2 = \varepsilon, t \in (-1, 1], 2r \neq \delta_0,$$

then $\Delta^*_{1+\varepsilon}(r, \psi, \delta_0) \neq 0$; *therefore, in view of (3.1.43) in [53],*

$$\frac{\partial}{\partial t}(|\eta_k^\wedge(r, \psi, \delta_0) + \varepsilon|^2) = \tag{130}$$

$$(-1)^k\left(\frac{\partial}{\partial t}\frac{p_{1+\varepsilon}}{2}\right)\frac{|\eta_k^\wedge(r, \psi, \delta_0) + \varepsilon|^2}{(sign(2r - \delta))^\varepsilon\sqrt{\left(\frac{p_{1+\varepsilon}}{2}\right)^2 - q_{1+\varepsilon}}} =$$

$$\left(\frac{\partial}{\partial t}p_{1+\varepsilon}\right)\frac{(-1)^k|\eta_k^\wedge(r,\psi,\delta_0)+\varepsilon|^2}{\sum\limits_{k=0}^{1}(-1)^k|\eta_k^\wedge(r,\psi,\delta_0),\varepsilon|^2},$$

where

$$\varepsilon^2=\varepsilon, k^2=k, |\delta_0|<1, |\psi|<\pi,\ 2r\neq\delta_0.$$

According to (128), (129) and (130), if

$$k^2=k, |\delta_0|<1, |\psi|<\pi, 2r\neq\delta_0,$$

then

(131)
$$\frac{\partial}{\partial t}|\eta_k^\wedge(r,\psi,\delta_0)-1|^2=$$

$$\sum_{\varepsilon=0}^{1}(-1)^\varepsilon 2^{1-\varepsilon}(-1)^k\left(\frac{\partial}{\partial t}\frac{p_{1+\varepsilon}}{2}\right)\frac{|\eta_k^\wedge(r,\psi,\delta_0)+\varepsilon|^2}{(sign(2r-\delta))^\varepsilon\sqrt{\left(\frac{p_{1+\varepsilon}}{2}\right)^2-q_{1+\varepsilon}}}=$$

$$\sum_{\varepsilon=0}^{1}(-1)^\varepsilon 2^{1-\varepsilon}(-1)^k\left(\frac{\partial}{\partial t}p_{1+\varepsilon}\right)\frac{|\eta_k^\wedge(r,\psi,\delta_0)+\varepsilon|^2}{\sum\limits_{k=0}^{1}(-1)^k|\eta_k^\wedge(r,\psi,\delta_0)+\varepsilon|^2}.$$

In view of (74), if $2r=\delta_0$, *then*

$$\frac{\partial}{\partial t}|\eta_k^\wedge(r,\psi,\delta_0)+1|^2=0$$

and, according to (128), if $2r=\delta_0$, $k=0,1$ *then*

$$\frac{\partial}{\partial t}|\eta_k^\wedge(r,\psi,\delta_0)-1|^2=2\frac{\partial}{\partial t}|\eta_k^\wedge(r,\psi,\delta_0)|^2;$$

therefore, if $2r=\delta_0$, $k=0,1$ *then the value* $(-1)^k|\eta_k^\wedge(r,\psi,\delta_0)-1|^2$ *monotonically decreases with increasing* $\psi\in(0,\pi)$.

According to the proof of the Lemma 3.1.3 in [53],

(132)
$$\frac{\partial}{\partial t}\frac{(1+\delta_0)p_1}{8}=r+\frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)|>$$

$$-(\delta_0)r+\frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)|=\frac{\partial}{\partial t}\frac{(1+\delta_0)p_2}{8}>0$$

Therefore, in view of (131), if $2r>\delta_0$, *then*

(133)
$$(-1)^k\frac{\partial}{\partial t}|\eta_k^\wedge(r,\psi,\delta_0)-1|^2>$$

$$-\left(\frac{\partial}{\partial t}p_j\right)\sum_{\varepsilon=0}^{1}(-2)^{1-\varepsilon}\frac{|\eta_k^\wedge(r,\psi,\delta_0)+\varepsilon|^2}{\sum\limits_{k=0}^{1}(-1)^k|\eta_k^\wedge(r,\psi,\delta_0)+\varepsilon|^2},$$

where $j=1,2$ *and* $k=0,1$.

Lemma 4.10. *If* $2\le 2r\le 2+\delta_0$, *then the value*

$$|\eta_0^\wedge(r,\psi,\delta_0)-1|$$

increases together with increasing $s \in ((r-1)/2, \delta_0/4)$.

Proof. Since $s \in ((r-1)/2, \delta_0/4)$, it follows that we have the case h) of the Lemma 4.6. Therefore according to (127),

$$\sum_{k=0}^{1}(-1)^k|\eta_k^{\wedge}(r,\psi,\delta_0)+1|^2 > \tag{134}$$

$$\sum_{k=0}^{1}(-1)^k\sum_{l=0}^{1}|\eta_k^{\wedge}(r,\psi,\delta_0)+(-1)^l|^2 = 2\sum_{k=0}^{1}(-1)^k|\eta_k^{\wedge}(r,\psi,\delta_0)|^2.$$

According to (132), (133) and (134),

$$(-1)^k\frac{\partial}{\partial t}|\eta_k^{\wedge}(r,\psi,\delta_0)-1|^2 > \tag{135}$$

$$-\left(\frac{\partial}{\partial t}p_2\right)\sum_{\varepsilon=0}^{1}(-4)^{1-\varepsilon}\frac{|\eta_k^{\wedge}(r,\psi,\delta_0)+\varepsilon|^2}{\sum\limits_{k=0}^{1}(-1)^k|\eta_k^{\wedge}(r,\psi,\delta_0)+1|^2},$$

In view of (3.1.57) in [53],

$$|\eta_0^{\wedge}(r,\psi,\delta_0)+1|^2 \le 4|\eta_0^{\wedge}(r,\psi,\delta_0)|^2. \blacksquare$$

In view of (1) ,

$$D^{\vee}(z,\eta) = (-\eta+1)(-\eta+\gamma_1)+ \tag{136}$$

$$2(1+\gamma_1)(1+r\exp(i\psi))\exp(-i\pi)(-\eta) =$$

$$(-\eta+1)(-\eta+\gamma_1)+4(1+\gamma_1)s_0(r,\psi)\exp(i\psi_0(r,\psi-\pi))(-\eta),$$

where $s_0(r,\psi)$ and $\psi_0(r,\psi)$ are defined in (91) and (92) respectively. Therefore, in view of (72)

$$\eta_k^{\wedge}(r,\psi,\delta_0) = -\eta_k^{\wedge}(2s_0(r,\psi),\psi_0(r,\psi)-\pi,\delta_0) = \tag{137}$$

$$-\overline{\eta_k^{\wedge}(2s_0(r,\psi),\pi-\psi_0(r,\psi)\pi,\delta_0)},$$

where $k=0,1,\ -1<\delta_0<1,\ 1\le r,\ -\pi<\psi<\pi$. Consequently,

$$1-\eta_k^{\wedge}(r,\psi,\delta_0) = 1+\eta_k^{\wedge}(2s_0(r,\psi),\psi_0(r,\psi)-\pi,\delta_0) = \tag{138}$$

Making use (138) and Lemma 4.4, we can give

Another proof of the Lemma 4.6. Let

$$r^{\wedge} = 2s_0(r,\psi),\ \psi^{\wedge} = \psi_0(r,\psi)-\pi.$$

Then

$$sign(2r^{\wedge}-\delta_0) = sign(4s_0(r,\psi)-\delta_0)$$

and, if $0<\psi<\pi$, then assertion of the Lemma 4.6 directly follow from (138) and Lemma 4.4. If $\psi=0$, then (138) implies the equalities

$$1-\eta_k^{\wedge}(r,0,\delta_0) = 1+\eta_k^{\wedge}(1+r,-\pi,\delta_0),\ r^{\wedge}=r+1$$

and the equality (76) is impossible. If $r > 1$ and $\psi = \pi$, then (138) implies the equalities

$$1 - \eta_k^\wedge(r, \pi, \delta_0) = 1 + \eta_k^\wedge(r-1, 0, \delta_0),\ r^\wedge = r - 1$$

and, according to the Lemma 4.4, all depends from the $sign(r - 1 - \delta_0/2)$ in correspondence with the Lemma 4.6. If $r < 1$ and $\psi = \pi$, then (138) implies the equalities

$$1 - \eta_k^\wedge(r, \pi, \delta_0) = \overline{1 + \eta_k^\wedge(1-r, \pi, \delta_0)},\ r^\wedge = 1 - r,$$

the condition of the case e) of the Lemma 4.6 coincides with (77) for $r^\wedge$ instead r, the condition of the case d) of the Lemma 4.6 fulfilled now only for $\delta_0 < 0$, it may be rewritten in the form

$$s = |r-1|/2,\ 1 - 2r^\wedge = 2r - 1 = 1 - |\delta_0| \leq \sqrt{1 - (\delta_0)^2}$$

and we have (77) for $r^\wedge$ intead r; if (95) holds, then

$$2r^\wedge - 1 = 1 - 2r < -\sqrt{1 - (\delta_0)^2}$$

and, in view of the inequality $r^\wedge > 0 > \delta_0/2$, we have the case of (75) for $\varepsilon = 1$ and $r^\wedge$ intead r; if (96) holds, then

$$2r^\wedge = 2 - 2r > 1 + \sqrt{1 - (\delta_0)^2} > \delta_0$$

and we again have the case of (75) for $\varepsilon = 1$ and $r^\wedge$ intead r; if (99) holds, then

$$\delta_0 > 0,\ 2r^\wedge = 2 - 2r < 1 - \sqrt{1 - (\delta_0)^2} < \delta_0$$

and we again have the case of (75) for $\varepsilon = 1$ and $r^\wedge$ intead r. In the case, when $r = 1, \psi = \pi$, we have case g) of the Lemma 4.6, if $\delta_0 > 0$, case d) of the Lemma 4.6 if $\delta_0 = 0$, and case a) of the Lemma 4.6, if $\delta_0 < 0$; the assertions of the Lemma 4.6 may be directly checcked now by means of (3.1.55) in [53]. ■

Let

$$r^\sim = r^\sim(r, \psi, \delta_0) = 4rs_0(r, \psi)(1 + \gamma_1)^2/(1 + \gamma_1^2), \tag{139}$$

$$\delta^\sim = \delta^\sim(\delta_0) = (1 - \gamma_1^2)/(1 + \gamma_1^2), \psi^\sim = \psi_0^\sim(r, \psi) = \psi_0(r, \psi) + \psi. \tag{140}$$

Multiplying the equality

$$(\eta_k^\wedge(r, \psi, \delta_0) + 1)(\eta_k^\wedge(r, \psi, \delta_0) + \gamma_1) = -2r\exp(i\psi)(1 + \gamma_1)\eta_k^\wedge(r, \psi, \delta_0)$$

with the equality

$$(-\eta_k^\wedge(r, \psi, \delta_0) + 1)(-\eta_k^\wedge(r, \psi, \delta_0) + \gamma_1) =$$

$$-4(1 + \gamma_1)s_0(r, \psi)\exp(\psi_0(r, \psi))(\eta_k^\wedge(r, \psi, \delta_0)),$$

we obtain the equality

$$(-(\eta_k^\wedge(r, \psi, \delta_0))^2 + 1)(-(\eta_k^\wedge(r, \psi, \delta_0))^2 + (\gamma_1)^2) = \tag{141}$$

$$-r^{\sim}(r,\psi,\delta_0)(1+(\gamma_1)^2)s_0(r,\psi)\exp(\psi^{\sim}(r,\psi))(-(\eta_k^{\wedge}(r,\psi,\delta_0))^2).$$

In view of (72)

$$-(\eta_k^{\wedge}(r,\psi,\delta_0))^2)=\eta_k^{\wedge}(r^{\sim},\psi^{\sim},\delta_0)). \tag{142}$$

where $k=0,1$ The equalities (139), (140), (142) and Lemma 4.4 let us to make more preciese assertions about the behavior of value $|(\eta_k^{\wedge}(r,\psi,\delta_0))^2-1|$.
Lemma 4.11. If

$$\frac{|r-1|}{2}\le s=\frac{\delta_0}{8r}\le\frac{r+1}{2}, \tag{143}$$

or

$$r\le 1,\ \delta_0^2\le 4r(1-r),\ s=(1-r)/2 \tag{144}$$

then

$$\sum_{k=0}^{1}(-1)^k|(\eta_k^{\wedge}(r,\psi,\delta_0))^2-1|=0. \tag{145}$$

If $(4r(r+1)-\delta_0)((4r|r-1|)-\delta_0)>0$, then

$$sign\left(\sum_{k=0}^{1}(-1)^k|(\eta_k^{\wedge}(r,\psi,\delta_0))^2-1|\right)(sign(4r\max(r,1)-\delta_0). \tag{146}$$

If $4r|r-1|)<\delta_0\le 4(r(r+1)$ and $|r-1|<2s\le(r+1)$, then

$$sign\left(\sum_{k=0}^{1}(-1)^k|(\eta_k^{\wedge}(r,\psi,\delta_0))^2-1|\right)=(sign(s-\delta_0/8r) \tag{147}$$

Proof. In view of (141), we apply the Lemma 4.4, taking $r^{\sim}$, $\delta_0^{\sim}$, and

$$-(\eta_k^{\wedge}(r,\psi,\delta_0))^2,\ k=0,1$$

respectively in the role of r,δ_0, and $\eta_k^{\wedge}(r,\psi,\delta_0)$, $k=0,1$. In view of the equalities (139) and (140),

$$sign(r^{\sim}(r,\psi,\delta_0)-\delta^{\sim}(\delta_0)/2)=sign(s_0(r,\psi)-\delta_0/8r) \tag{148}$$

According to the Lemma 4.4, the equality (145) may be fulfilled only if the equality

$$r^{\sim}(r,\psi,\delta_0)=\delta^{\sim}(\delta_0)/2 \tag{149}$$

is fulfilled or both conditions

$$\psi_0(r,\psi)+\psi\in\pi+2\pi\mathbb{Z}, \tag{150}$$

and

$$|2r^{\sim}-1|\le(1+\delta_0^{\sim})^2)^{1/2} \tag{151}$$

are fulfilled. In view of (148) and (93), the condition (149) is equivalent to the condition (143).

In view of (91), (92) and (141), the condition (150) is equivalent to the condition

$$(152) \qquad (1 + r\exp(i\psi))\exp(i\psi) = -2s_0(r, \psi) = -(r^2 + 2rt + 1)^{1/2} \le 0,$$

which is equivalent to the system

$$(153) \quad (1 + 2rt))\sin(\psi)) = 0, t + r(2t^2 - 1) = t((1 + 2rt) - r \le 0, t = \cos(\psi);$$

the system (153) have only the following solutions:

$$(154) \qquad t = -1, r \le 1$$

and

$$(155) \qquad t = -1/2r, r \ge 1/2.$$

In view of (139) and (140), the condition (149) is equivalent to the condition

$$|8rs_0(r, \psi)(1 + \gamma_1)^2 - (1 + \gamma_1^2)| \le 2\gamma_1,$$

i.e to the condition

$$(156) \qquad \delta_0^2 \le 8rs_0(r, \psi) \le 1.$$

The system of conditions (154) - (156) implies the equality $2s = 1 - r$ and is equivalent to (144). In view of (152), if $2r \ge 1$, then

$$2s_0(r, \arccos(-1/2r)) = r$$

and therefore the system of conditions (155) - (156) takes the form

$$\delta_0^2 \le 4r^2 \le 1, 2r \ge 1$$

and for $r = 1/2$ coincides with (144). Other assertions of the Lemma directly follow from Lemma 4.4. ■

Lemma 4.12. *Let* $2r \ge 1$, $0 \le \psi \le \pi$. *Then*

$$(157) \qquad -sign(\arccos(-1/2r) - \psi)Re(\eta_0^\wedge(r, \psi, \delta_0)) \ge$$

.

$$-sign(\arccos(-1/2r) - \psi)Re(\eta_1^\wedge(r, \psi, \delta_0)) \ge 0$$

and

$$(158) \qquad Re(\eta_0^\wedge(r, \arccos(-1/2r), \delta_0)) = Re(\eta_1^\wedge(r, \arccos(-1/2r), \delta_0)) = 0.$$

.

Proof. In view of (42),

$$\exp(i\psi)R_0(r, \psi, \delta_0) = \sqrt{|D_0(r, \psi, \delta_0)|}\times$$

$$\exp(i(\psi - \psi_1(r, \psi, \delta_0)/2)).$$

In view of (6),

$$(159)\qquad (\exp(i\psi)R_0(r,\psi,\delta_0))^2 = r^2\exp(2i\psi) + r\exp(i\psi) + (\delta_0/2)^2 =$$

$$= |R_0(r,\psi,\delta_0)|^2(\exp(i(2\psi - \psi_1(r,\psi,\delta_0)),$$

$$(160)\qquad Re((\exp(i\psi)R_0(r,\psi,\delta_0))^2) = r^2\cos(2\psi) + r\cos(\psi) + \delta_0^2/4$$

$$(161)\qquad Im((\exp(i\psi)R_0(r,\psi,\delta_0))^2) =$$

$$r^2\sin(2\psi) + r\sin(\psi) = r\sin(\psi)(2r\cos(\psi)+1),$$

$$(162)\qquad sign(Im((\exp(i\psi)R_0(r,\psi,\delta_0))^2)) = sign(2rcos(\psi)+1),$$

if $0 < \psi < \pi$; In view of (8) and (9),

$$-Im(D_0(r,\psi,\delta_0))/Re(D_0(r,\psi,\delta_0)) =$$

$$\frac{((\delta_0)^2/2)t + r)sin(\psi)}{t((\delta_0)^2/2)t + r) + r^2 - (\delta_0/2)^2} \le \tan(\psi),$$

if $0 \le \psi < \pi/2$, $t = \cos(\psi)$ and $r \ge |\delta_0/2|$; consequently,

$$\psi_1(r,\psi,\delta_0) \le \psi$$

and therefore

$$(163)\qquad 0 \le \psi/2 \le \psi - \psi_1(r,\psi,\delta_0)/2 \le \pi,$$

if $0 \le \psi < \pi/2$ and $r \ge |\delta_0/2|$. In view of (33),

$$(164)\qquad \pi - \psi_1(r,\pi,\delta_0)/2 = \pi/2.$$

If

$$Im(\exp(i\psi)R_0(r,\psi,\delta_0)) = 0,$$

then also

$$Im((\exp(i\psi)R_0(r,\psi,\delta_0))^2) = 0$$

and $(2r\cos(\psi)+1)r\sin(\psi) = 0$. Therefore as continuous functions of ψ the values

$$Im(\exp(i\psi)R_0(r,\psi,\delta_0))$$

and

$$Im((\exp(i\psi)R_0(r,\psi,\delta_0))^2)$$

keep sign in each of the intervals $(0,\arccos(-1/2r), (\arccos(-1/2r),\pi)$; therefore $\psi - \psi_1(r,\psi,\delta_0)/2$ and $\pi - (\psi - \psi_1(r,\psi,\delta_0)/2)$ keep sign in each of the intervals $(0,\arccos(-1/2r), (\arccos(-1/2r),\pi)$; consequently, in view of (163),

$$(165)\qquad 0 \le \psi - \psi_1(r,\psi,\delta_0)/2 \le \pi,$$

for $\psi \in (0, \arccos(-1/2r)$ and in view of (164), the inequalities (165) hold also for $\psi \in (\arccos(-1/2r), \pi)$; so, the inequalities (165) hold for $\psi \in (0, \pi)$; moreover, in view of (162),

$$sign(2\psi - \psi_1(r, \psi, \delta_0) - \pi) = sign(\psi - \arccos(-1/2r)),$$

(166) $|\psi - \psi_1(r, \psi, \delta_0)/2 - \pi/2| \leq \pi/2$, $sign(\psi - \psi_1(r, \psi, \delta_0)/2 - \pi/2) =$

$$sign(\psi - \arccos(-1/2r)).$$

Let $A = Re(\exp(i\psi)R_0(r, \psi, \delta_0))^2), B = Im(\exp(i\psi)R_0(r, \psi, \delta_0))^2)$. In view of (166),

$$\exp(i\psi)R_0(r, \psi, \delta_0) = \tag{167}$$

$$sign\left(\arccos\left(\frac{-1}{2r}\right) - \psi\right)\sqrt{\frac{\sqrt{A^2+B^2}+A}{2}} +$$

$$i\sqrt{\frac{\sqrt{A^2+B^2}-A}{2}}.$$

Further we have $\frac{\partial}{\partial x}(\sqrt{x^2+B^2}+x) = (\sqrt{x^2+B^2}+x)/\sqrt{x^2+B^2} > 0$, where $x \in \mathbb{R}$, $B \in \mathbb{R}\diagdown\{0\}$. In view of (160),

$$Re(\exp(i\psi)R_0(r, \psi, 1))^2) - Re(\exp(i\psi)R_0(r, \psi, \delta_0))^2) > 0 \tag{168}$$

In view of (65),

$$-Re\eta_1^\wedge(r, \psi, \delta_0) = (1+\gamma_1)(Re(exp(i\psi)R_0(r, \psi, 1)) - Re(exp(i\psi)R_0(r, \psi, \delta_0)))$$

and, in view of (168), (167),

$$-signRe\eta_1^\wedge(r, \psi, \delta_0) = sign(\arccos(-1/2r) - \psi) \tag{169}$$

In view of (65),

$$-\sum_{k=0}^{1}(-1)^k Re(\eta_0^\wedge(r, \psi, \delta_0)) = 2(1+\gamma_1)Re(\exp(i\psi)R_0(r, \psi, \delta_0)) = \tag{170}$$

$$= 2(1+\gamma_1)sign(\arccos(-1/2r) - \psi)\sqrt{\frac{\sqrt{A^2+B^2}+A}{2}}.$$

Therefore

$$-sign(\arccos(-1/2r) - \psi)\sum_{k=0}^{1}(-1)^k Re(\eta_k^\wedge(r, \psi, \delta_0)) \geq 0. \tag{171}$$

The inequalities (157) follow from (169) and (171). In view of (65) and (164), the equalities (158) hold, if $\psi = \arccos(-1/2r)$. ■

Lemma 4.13. *If $1 \leq 2r$, then the value of*

$$(-1)^k|\eta_k^\wedge(r, \psi, \delta_0)^2 - 1|$$

decreases together with increasing $\psi \in (0, \arccos(-1/2r))$.

Proof. Clearly,

$$(-1)^k \frac{\partial}{\partial t} |\eta_k^{\wedge}(r,\psi,\delta_0)^2 - 1|^2 =$$

$$(-1)^k \sum_{l=0}^{1} |\eta_k^{\wedge}(r,\psi,\delta_0) - (-1)^l|^2 \frac{\partial}{\partial t} |\eta_k^{\wedge}(r,\psi,\delta_0) + (-1)^l|^2 =$$

$$|\eta_k^{\wedge}(r,\psi,\delta_0) - 1|^2 (-1)^k \frac{\partial}{\partial t} |\eta_k^{\wedge}(r,\psi,\delta_0) + 1|^2 + |\eta_k^{\wedge}(r,\psi,\delta_0) + 1|^2 \times$$

$$\sum_{\varepsilon=0}^{1} (-1)^{\varepsilon} 2^{1-\varepsilon} (-1)^k \frac{\partial}{\partial t} |\eta_k^{\wedge}(r,\psi,\delta_0) + \varepsilon|^2 =$$

$$(-1)^k \left(\frac{\partial}{\partial t} |\eta_k^{\wedge}(r,\psi,\delta_0) + 1|^2 \right) \times$$

$$\sum_{l=0}^{1} (-1)^l |\eta_k^{\wedge}(r,\psi,\delta_0) - (-1)^l|^2 +$$

$$2|\eta_k^{\wedge}(r,\psi,\delta_0) + 1|^2 (-1)^k \frac{\partial}{\partial t} |\eta_k^{\wedge}(r,\psi,\delta_0)|^2.$$

Assertion of the Lemma now directly follows from the Lemma 4.4. and the Lemma 4.12.■

Lemma 4.14. *If* $|r-1| < 2s < r+1$, *then*

$$(-1)^k sign(4s - \delta_0) = \tag{172}$$

$$sign(|\eta_0^{\wedge}(r,\psi,\delta_0) - 1|^2 - 8s/(1+\delta_0)),$$

where $k = 0, 1$.

Proof. In view of (3.1.34) in [53], (1) and (91),

$$q_0(r,\psi,\delta_0) = |D^{\vee}(z,1)|^2 = \tag{173}$$

$$4(1+\gamma_1)^2(2s)^2$$

and (172) follows now directly from the Lemma 4.6. ■

Lemma 4.15. *The value*

$$|\eta_0^{\wedge}(r,\psi,\delta_0) - 1|^2 \tag{174}$$

increases with increasing of $s \in ((1+\delta_0)/(4+2\delta_0), (r+1)/2)$. *If* $r = 1$, *then the value*

$$|\eta_1^{\wedge}(r,\psi,\delta_0) - 1|^2 \tag{175}$$

increases with increasing of $s \in (\max(\delta_0/4, 0), (1+\delta_0)/(6+2\delta_0)$.

If $r = 1$, $0 \le \delta_0$, *then the value (174) increases together with increasing of* $s \in (0, \delta_0/4)$.

Proof. According to (3.1.31) and (3.1.33) in [53],

$$|R_{\varepsilon}^{*}(r,\psi,\delta_0)|^2 = r^2 + r(2+\delta_0)t + (1+\delta_0/2)^2,$$

(176) $$p_0(r,\psi,\delta_0)=$$
$$8(r^2+r(2+\delta_0)t+(1+\delta_0/2)^2+|D_0(r,\psi,\delta_0)|)/(1+\delta_0)^2.$$

If $r=1$, then for a,b and c in (15), we have

$$0\le a<1, 0<b<5/2, c>25/16, 4c>b^2$$

and therefore, according to the Lemma 4.1,

$$0\le\frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)|,$$

where $t\in(-1,1)$,

$$\frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)|\big|_{t=1}=\sqrt{\frac{4a^2+4ab+b^2}{4a+4b+4c}}<1;$$

therefore

(177) $$\frac{\partial}{\partial t}p_0(r,\psi,\delta_0)/2\ge 4r(2+\delta_0)/(1+\delta_0)^2$$

and

$$\frac{\partial}{\partial t}p_0(1,\psi,\delta_0)/2\le 4(3+\delta_0)/(1+\delta_0)^2$$

According to the Lemma 4.6,

$$\Delta_0^*(r,\psi,\delta_0)\ne 0$$

if $s\in(|r-1|/2,(r+1)/2)$, $s\ne\delta_0/4$; therefore, in view of (173),

(178) $$\frac{\partial}{\partial t}(|\eta_k^\wedge(r,\psi,\delta_0)-1|^2)=$$
$$(-1)^k\left(\frac{\partial}{\partial t}\frac{p_0}{2}\right)\frac{|\eta_k^\wedge(r,\psi,\delta_0)-1|^2}{(sign(4s-\delta_0))\sqrt{\left(\frac{p_0}{2}\right)^2-q_0}}-$$
$$\frac{4r(1+\gamma_1)^2}{(sign(4s-\delta_0))\sqrt{\left(\frac{p_0}{2}\right)^2-q_0}}.$$

In view of (172), (178), (176), (173) and Lemma 4.1,

(179) $$\sqrt{\left(\frac{p_0}{2}\right)^2-q_0}\frac{\partial}{\partial t}(|\eta_0^\wedge(r,\psi,\delta_0)-1|^2)\ge$$
$$(4r(2+\delta_0)/(1+\delta_0)^2)4(1+\gamma_1)s-16r/(1+\delta_0)^2=$$
$$16r((4+2\delta_0)s/(1+\delta_0)-1)/(1+\delta_0)^2.$$

Since $\delta_0/4<(1+\delta_0)/(2+\delta_0)$ the first assertion of the Lemma is proved. Let $r=1$ and $4s>\delta_0$. Then

(180) $$-\frac{\partial}{\partial t}(|\eta_1^\wedge(r,\psi,\delta_0)-1|^2)\le$$

$$(4(3+\delta_0)/(1+\delta_0)^2)4(1+\gamma_1)s - 16/(1+\delta_0)^2 =$$
$$16((6+2\delta_0)s/(1+\delta_0) - 1)/(1+\delta_0)^2.$$

Since

(181) $$\delta_0/4 - (1+\delta_0)/(6+2\delta_0) = (\delta_0^2+\delta_0-2)/(12+4\delta_0) < 0,$$

it follows that the second assertion of the Lemma is proved. Let $r = 1$ and $4s < \delta_0$. Then

(182) $$-\frac{\partial}{\partial t}(|\eta_0^\wedge(r,\psi,\delta_0) - 1|^2) \leq$$
$$(4(3+\delta_0)/(1+\delta_0)^2)4(1+\gamma_1)s - 16/(1+\delta_0)^2 =$$
$$16((6+2\delta_0)s/(1+\delta_0) - 1)/(1+\delta_0)^2.$$

In view of (181), the third assertion of the Lemma is proved. ■

Remark. The third assertion of the Lemma corresponds to the Lemma 4.10

Lemma 4.16. *If* $0 \leq \delta_0$, *then*

$$(-1)^k\frac{\partial}{\partial t}\log(|\eta_k^\wedge(r,\psi,\delta_0)|) \geq (-1)^k\frac{\partial}{\partial t}\log(|\eta_0^\wedge(r,\psi,\delta_k)|+1)$$

Proof. Assertion of the Lemma follows from the Corollary to the Lemma 4.5. ■

Lemma 4.17. *Let* $a < b$. *Let the functions* $p = p(t)$ *and* $q = q(t)$ *are defined on* (a,b) *and belong to the class* C^1. *Let further*

(183) $$q(t) > 0,\ p(t) > 2\sqrt{q(t)}$$

(184) $$\frac{\partial}{\partial t}q(t) > 0$$

for any $t \in (a,b)$. *Let further*

$$x_k = x_k(t) = p(t)/2 + (-1)^k\sqrt{(p(t)/2)^2 - q(t)},$$

where $k^2 = k$.

Let $k^2 = k$, $t \in (a,b)$. *Then the condition*

(185) $$0 < (-1)^k\left(\frac{\frac{\partial}{\partial t}p(t)}{\frac{\partial}{\partial t}q(t)}q(t) - p(t)/2\right)$$

is equivalent to the condition

(186) $$0 < \frac{\partial}{\partial t}\left(x_k(t)/x_{1-k}(t)\right)$$

and imply the following relations:

(187) $$0 < \frac{\partial}{\partial t}x_k(t),$$

$$0 < \frac{\partial}{\partial t}(x_k(t) - x_{1-k}(t)) \tag{188}$$

$$-sign\left(\frac{\partial}{\partial t}x_{1-k}(t)\right) = \tag{189}$$

$$sign\left(\left(\frac{\frac{\partial}{\partial t}p(t)}{\frac{\partial}{\partial t}q(t)}q(t) - p(t)\right)\frac{\partial}{\partial t}p(t) + \frac{\partial}{\partial t}q(t)\right). \tag{190}$$

Proof Clearly,

$$(-1)^k\frac{\partial}{\partial t}x_k = \frac{x_k\frac{\partial}{\partial t}(p/2) - \frac{\partial}{\partial t}(q/2)}{\sqrt{(p/2)^2 - q}}. \tag{191}$$

Therefore

$$\sum_{k=0}^{1}(-1)^k x_{1-k}\frac{\partial}{\partial t}x_k = \frac{2q\frac{\partial}{\partial t}(p/2) - p\frac{\partial}{\partial t}(q/2)}{\sqrt{(p/2)^2 - q}}. \tag{192}$$

The condition (185) is equivalent to the the condition

$$0 < (-1)^k\frac{\partial}{\partial t}\left(\frac{p/2}{\sqrt{q}}\right), \tag{193}$$

and, in view of (192), to (186). According to (186), (183) and (184),

$$\frac{\partial}{\partial t}x_k = \frac{\partial}{\partial t}\sqrt{\frac{x_k}{x_{1-k}}q} > 0. \tag{194}$$

The condition (193) implies the inequality

$$0 < (-1)^k\frac{\partial}{\partial t}\sqrt{\left(\frac{p/2}{\sqrt{q}}\right)^2 - 1}, \tag{195}$$

which together with (183) and (184) implies the inequality

$$\frac{\partial}{\partial t}\left(x_k(t) - x_{1-k}(t)\right) = (-1)^k\frac{\partial}{\partial t}\left(\sqrt{\left(\frac{p/2}{\sqrt{q}}\right)^2 - 1}\,\sqrt{q}\right) > 0. \tag{196}$$

In view of (194) and (191),

$$0 < \frac{\partial}{\partial t}x_k = (-1)^k\frac{x_k\frac{\partial}{\partial t}p - \frac{\partial}{\partial t}q}{\sqrt{p^2 - 4q}} = \frac{x_k\frac{\partial}{\partial t}q}{q\sqrt{p^2 - 4q}}(-1)^k\left(\frac{q\frac{\partial}{\partial t}p}{\frac{\partial}{\partial t}q} - x_{1-k}\right), \tag{197}$$

$$-\frac{\partial}{\partial t}x_{1-k} = \frac{x_{1-k}\frac{\partial}{\partial t}q}{q\sqrt{p^2 - 4q}}(-1)^k\left(\frac{q\frac{\partial}{\partial t}p}{\frac{\partial}{\partial t}(q)} - x_k\right). \tag{198}$$

Multiplying (197) and (198) we obtain the equality

$$-sign\left(\frac{\partial}{\partial t}x_{1-k}\right) = sign\left(\frac{\left(\frac{\partial}{\partial t}p\right)^2 q^2}{\left(\frac{\partial}{\partial t}(q)\right)^2} - \frac{pq\frac{\partial}{\partial t}p}{\frac{\partial}{\partial t}q} + q\right) =$$

$$sign\left(\left(\frac{\frac{\partial}{\partial t}p}{\frac{\partial}{\partial t}q}q - p\right)\frac{\partial}{\partial t}p + \frac{\partial}{\partial t}q\right). \blacksquare$$

Remark. *The conditions and assertions of the Lemma are invariant relative to mulyiplying of both x_k on a positive constant γ and to substition $t = t(\tau)$ with continuous $\frac{\partial}{\partial \tau}t > 0$. Therefore one cane replace p and q by γp and $\gamma^2 q$.*

Lemma 4.18. *Let $r = 1$, $\delta_0 > 0$,*

(199) $$p = (1 + 1/\delta_0)^2 p_0/8,\ q = (1 + 1/\delta_0)^4 q_0/64.$$

Then

(200) $$sign\left(\frac{\frac{\partial}{\partial t}p(t)}{\frac{\partial}{\partial t}q(t)}q(t) - p(t)/2\right) = sign(4s - \delta_0),$$

where $s \in (0, 1)$.

Proof. Let

(201) $$\tau \in (0, 1/\delta_0),\ s = \sqrt{\delta_0\tau},\ \omega = 1/\delta_0,$$

(202) $$\mathfrak{R} = \sqrt{4\tau^2 + (4\omega^3 - 3\omega)\tau + 1/16},$$

(203) $$U = \frac{\frac{\partial}{\partial \tau}p}{\frac{\partial}{\partial \tau}q}q - p/2.$$

In view of (91), $t = 2s^2 - 1 = 2\delta_0\tau - 1$, and (200) is equivalent to the equality

(204) $$sign(U) = sign(16\omega\tau - 1).,$$

Since

$$r = 1,\ \delta_0 > 0,\ p = (1 + \omega)^2 p_0/8,\ q = (1 + \omega)^4 q_0/64,$$

it follows, according to (3.1.70) and (3.1.71) in [53], that

(205) $$p = (4\omega + 2)\tau + 1/4 + \mathfrak{R},\ q = (1 + \omega)^2\omega\tau.$$

$$\frac{\frac{\partial}{\partial t}p}{\frac{\partial}{\partial t}q}q - p/2 = (2\omega + 1)\tau - 1/8 + (2\tau^2 - 1/32)\mathfrak{R}^{-1}.$$

Therefore for $\tau \notin (1/(16\omega + 8), 1/8)$ the equality (200) is fulfilled. Since

$$(2\omega + 1)\tau - 1/8 - (2\tau^2 - 1/32)\mathfrak{R}^{-1} > 0,$$

if $\tau \in (1/(16\omega + 8), 1/8)$, it follows that

$$sign\left(\frac{\frac{\partial}{\partial t}p(t)}{\frac{\partial}{\partial t}q(t)}q(t) - p(t)/2\right) = sign(Q_0(\tau)),$$

where $\tau \in (1/(16\omega+8), 1/8)$, and

$$Q_0(\tau)) = ((2\omega+1)\tau - 1/8)^2\mathfrak{R}^2 - (2\tau^2 - 1/32)^2 =$$

$$(16\omega^2+16\omega)\tau^4 + (16\omega^5+16\omega^4-8\omega^3-12\omega^2-5\omega-1)\tau^3 +$$

$$(-2\omega^4-\omega3+7\omega^2/4+\omega+1/4)\tau^2+\omega^3/16-5\omega/64-1/64)\tau =$$

$$\tau(\tau-1/(16\omega)\omega(\omega+1)\times$$

$$16\tau^2+(16\omega^3-8\omega-4)\tau-\omega^2+\omega+1/4;$$

moreover, if $\tau(16\omega+8)>1$, then

$$(16\tau^2+(16\omega^3-8\omega-4)\tau-\omega^2+\omega+1/4)/\tau >$$

$$(16\omega^3-8\omega-4)-(\omega^2-\omega-1/4)(16\omega+8) =$$

$$8\omega^2+4\omega-2>0,$$

because $\omega > 1$. ■

Lemma 4.19. *Let are fulfilled conditions of the lemma 4.18. Then*

$$\left(\frac{\frac{\partial}{\partial t}p}{\frac{\partial}{\partial t}q}q - p\right)\frac{\partial}{\partial t}p + \frac{\partial}{\partial t}q \le 0, \tag{206}$$

and an equality reaches here only if $s = \delta_0/4$.

Proof. In view of (201) we must proof that

$$W = \left(\frac{\frac{\partial}{\partial\tau}p}{\frac{\partial}{\partial\tau}q}q - p\right)\frac{\partial}{\partial\tau}p + \frac{\partial}{\partial\tau}q \le 0, \tag{207}$$

and an equality reaches here only if $\tau = 1/(16\omega)$. First we prove that for $\tau = 1/(16\omega)$ equality reaches. Let

$$V = -\frac{p}{2}\frac{\partial}{\partial\tau}p + \frac{\partial}{\partial\tau}q.$$

Then $W = U\frac{\partial}{\partial\tau}p + V$ with U in (203). Since, accoding to the Lemma 18, $U\big|_{\tau=1/(16\omega)} = 0$ we must prove that $V\big|_{\tau=1/(16\omega)} = 0$. But in, view of (199),

$$-V = \frac{\partial}{\partial\tau}((p/2)^2 - q) = (1+1/\delta_0)^4\frac{\partial}{\partial\tau}\left(\frac{1}{2}\sum_{k=0}^{1}(-1)^k|\eta_k^{\wedge}(r,\psi,\delta_0)-1|^2\right)^2 =$$

$$(1+1/\delta_0)^4\left(\sum_{k=0}^{1}(-1)^k|\eta_k^{\wedge}(r,\psi,\delta_0)-1|^2\right)\times$$

$$\frac{\partial}{\partial\tau}\left(\frac{1}{2}\sum_{k=0}^{1}(-1)^k|\eta_k^{\wedge}(r,\psi,\delta_0)-1|^2\right),$$

according to the Lemma 4.4,

$$\frac{\partial}{\partial\tau}|\eta_k^{\wedge}(r,\psi,\delta_0)-1|^2 =$$

$$\frac{\partial}{\partial\tau}|\eta_k^\wedge(r,\psi,\delta_0)1|^2-\frac{\partial}{\partial\tau}\left(2|\eta_k^\wedge(r,\psi,\delta_0)1|\cos(\psi)\right),$$

exists at the point $\tau=1/(16\omega)$ and, according to the Lemma 4.6,

$$\left(\sum_{k=0}^{1}(-1)^k|\eta_k^\wedge(r,\psi,\delta_0)-1|^2\right)\Big|_{\tau=1/(16\omega)}=0.$$

So, $W\Big|_{\tau=1/(16\omega)}=0$. In view of (202) and (205)

$$\frac{\frac{\partial}{\partial t}p}{\frac{\partial}{\partial t}q}q-p=-1/4-\mathfrak{R}/2+(2\tau^2-1/32)\mathfrak{R}^{-1}=$$

$$-(\mathfrak{R}/4+(4\omega^3-3\omega)\tau/2+1/16)\mathfrak{R}^{-1},$$

$$\mathfrak{R}^2\left(\left(\frac{\frac{\partial}{\partial\tau}p}{\frac{\partial}{\partial\tau}q}q-p\right)+\frac{\partial}{\partial\tau}q\right)=$$

$$-(\mathfrak{R}/4+(4\omega^3-3\omega)\tau/2+1/16)\times$$

$$((4\omega+2)\mathfrak{R}+\tau+(4\omega^3-3\omega)/2)+$$

$$(1+\omega)^2\omega\mathfrak{R}^2.$$

Let

$$A_1=(4\omega^3-3\omega)\tau/2+1/16)\,,B_1=1/4,\ X_1=B_1\mathfrak{R},$$

$$A_2=(4\omega^3-3\omega)/2+4\tau,\ B_2=4\omega+2,\ X_2=B_2\mathfrak{R}.$$

Then

$$\mathfrak{R}^2\left(\left(\frac{\frac{\partial}{\partial\tau}p}{\frac{\partial}{\partial\tau}q}q-p\right)\frac{\partial}{\partial\tau}p+\frac{\partial}{\partial\tau}q\right)=$$

$$-(A_1+X_1)(A_2+X_2)+(1+\omega)^2\omega\mathfrak{R}^2=-(A_3+X_3)+(1+\omega)^2\omega\mathfrak{R}^2,$$

where $A_3=A_1A_2+X_1X_2$, $X_3=A_1X_2+A_2X_1$. In view of the inequality $X_3\geq 0$, it is sufficient to prove that

$$(X_3)^2\geq((1+\omega)^2\omega\mathfrak{R}^2-A_3)^2.$$

and an equality reaches here only if $\tau=1/(16\omega)$. Furter we have

$$A_1^2-X_1^2=(\omega^2-1)(4\omega^2-1)^2\tau^2/4,$$

$$A_2^2=(4\omega^3-3\omega)^2/4+4(4\omega^3-3\omega)\tau+16\tau^2=4\mathfrak{R}^2+(\omega^2-1)(4\omega^2-1)^2/4,$$

$$X_2^2-A_2^2=16(\omega+1)\omega\mathfrak{R}^2-(\omega^2-1)(4\omega^2-1)^2/4,$$

$$X_3^2-A_3^2=-(A_1^2-X_1^2)(A_2^2-X_2^2)=(\omega+1)^2(\omega-1)(4\omega^2-1)^2\times$$

$$(16\omega\mathfrak{R}^2-(\omega-1)(4\omega^2-1)^2/4)\tau^2/4,$$

$$2A_3-\omega(1+\omega)^2\mathfrak{R}^2=(\omega-1)(A_4(4\tau^2+1/16)++A_5\tau,$$

where $A_4=3\omega^2+\omega-1$, $A5=(8\omega^5-8\omega^4-18\omega^3+2\omega^2+5\omega-1)/2$,

$$(2A_3-\omega(1+\omega)^2\mathfrak{R}^2)\omega(1+\omega)^2\mathfrak{R}^2=\omega(1+\omega)^2(\omega-1)\times$$

$$(4\tau^2+1/16+(4\omega^3-3\omega)\tau)\times$$

$$(A_4(4\tau^2+1/16)+A_5\tau)=$$
$$\omega(1+\omega)^2(\omega-1)\times$$
$$(A_4(4\tau^2+1/16)^2+$$
$$(A_4(4\omega^3-3\omega)+A_5)(4\tau^2+1/16)\tau+$$
$$(4\omega^3-3\omega)A_5\tau^2)),$$
$$X_3^2-A_3^2+(2A_3-\omega(1+\omega)^2\mathfrak{R}^2)\omega(1+\omega)^2\mathfrak{R}^2=(\omega+1)^2(\omega-1)T_1,$$

where $T_1(\omega,\tau)=\sum\limits_{k=0}^{4}c_k\tau^{4-k}$ and

$$c_0=16A_4\omega+16\omega(4\omega^2-1)^2=16\omega((3\omega^2+\omega-1)+16\omega^4-8\omega^2+1)=$$
$$16(16\omega^5-5\omega^3+\omega^2),$$
$$c_1=4\omega A_4(4\omega^3-3\omega)+4\omega A_5+4\omega(4\omega^2-1)^2(4\omega^3-3\omega)=$$
$$(12\omega^3+4\omega^2-4\omega)(4\omega^3-3\omega)+(64\omega^5-32\omega^3+4\omega)(4\omega^3-3\omega)+4\omega A_5=$$
$$(64\omega^5-20\omega^3+4\omega^2)(4\omega^3-3\omega)+$$
$$16\omega^6-16\omega^5-36\omega^4+4\omega^3+10\omega^2-2\omega=$$
$$256\omega^8-256\omega^6+24\omega^4-8\omega^3+10\omega^2-2\omega,$$
$$c_2=(4\omega^2-1)^2\omega/4-(\omega-1)(4\omega^2-1)^4/16+$$
$$A_4\omega/2+A_5\omega(4\omega^3-3\omega)=$$
$$(4\omega^5-2\omega^3+\omega/4)-(\omega-1)(256\omega^8-256\omega^6+96\omega^4-16\omega^2+1)/16+$$
$$(3\omega^3+\omega^2-\omega)/2+(4\omega^4-3\omega^2)(8\omega^5-8\omega^4-18\omega^3+2\omega^2+5\omega-1)/2=$$
$$-32\omega^7+35\omega^5+\omega^4-7\omega^3+\omega^2-5\omega/16+1/16,$$
$$c_3=A_4(4\omega^3-3\omega)+A_5)\omega/16=$$
$$(3\omega^2+\omega-1)(8\omega^4-6\omega^2)/32+(8\omega^6-8\omega^5-18\omega^4+2\omega^3+5\omega^2-\omega)/32=$$
$$(32\omega^6-44\omega^4-4\omega^3+11\omega^2-\omega)/32,$$
$$c_4=A_4\omega/256=(3\omega^3+\omega^2-\omega)/256$$

We apply the Horner's method now twice and obtain the equality

$$T_1=(\tau-1/(16\omega))((256\omega^5-80\omega^3+16\omega^2)\tau^2+$$
$$(256\omega^8-256\omega^6+56\omega^4-8\omega^3-20\omega^2))\tau+$$
$$3\omega^5+\omega^4-\omega^3;$$

moreover, since $\omega\geq 1$, it follows that

$$256\omega^5-80\omega^3+16\omega^2\geq 192,$$
$$256\omega^8-256\omega^6+56\omega^4-8\omega^3-20\omega^2\geq 28,$$
$$3\omega^5+\omega^4-\omega^3\geq 3$$

and therefore $T_1\geq 3>0$ for $\tau\geq 0$ and $\omega\geq 1.\blacksquare$.

Lemma 4.20. *Let $r = 1$, $\delta_0 > 0$, $k^2 = k$. Then*

$$\frac{\partial}{\partial\psi}|\eta_k^\wedge(r,\psi,\delta_0) - 1|^2 \leq 0, \tag{208}$$

where $\psi \in (0, \pi)$,

$$|\eta_k^\wedge(r,\psi,\delta_0) - 1|^2 \geq 2\delta_0/(1+\delta_0), \tag{209}$$

if $s \geq \delta_0/4$.

Proof. If $s < \delta_0/4$, then, according to the Lemma 4.6, for $k = 0, 1$ the role of x_k in the Lemma 4.17 plais $|\eta_{1-k}^\wedge(r,\psi,\delta_0) - 1|^2$, according to the Lemma 4.18 and to lemma 4.19, all the conditions of the Lemma 4.17 are fulfilled and both $|\eta_0^\wedge(r,\psi,\delta_0) - 1|^2$ and $|\eta_1^\wedge(r,\psi,\delta_0) - 1|^2$ increases together with increasing of $s \in (0, \delta_0/4)$. If $s < \delta_0/4$, then, according to the Lemma 4.6, for $k = 0, 1$ the role of x_k in the Lemma 4.17 plais $|\eta_k^\wedge(r,\psi,\delta_0) - 1|^2$, according to the Lemma 4.18 and Lemma 4.19, both $|\eta_0^\wedge(r,\psi,\delta_0) - 1|^2$ and $|\eta_1^\wedge(r,\psi,\delta_0) - 1|^2$ increases together with increasing of $s \in (\delta_0/4, 0)$.

Moreover, $\frac{\partial}{\partial\psi}s < 0$. Therefore (208 is proved. It follows from (208 that

$$|\eta_k^\wedge(r,\psi,\delta_0) - 1|^2\Big|_{s\geq\delta_0/4} \geq |\eta_k^\wedge(r,\psi,\delta_0) - 1|^2\Big|_{s=\delta_0/4}, \blacksquare.$$

Corollary. *Let $r = 1$, $\delta_0 \geq 0$. Then the value*

$$|\eta_0^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|(\eta_0^\wedge(r,\psi,\delta_0))^2 - 1|^{\delta_0}, \tag{210}$$

decreases with increasing of psi $\in (0, \pi)$.

Proof. According to the Lemma 4.4 and Lemma 4.20, the value

$$|(\eta_0^\wedge(r,\psi,\delta_0))^2 + \varepsilon|^2,$$

where $\varepsilon^3 = \varepsilon$, decreases together with increasing of *psi* $\in (0, \pi)$. $\blacksquare$.

Lemma 4.21. *Let $r = 1$, $\delta_0 \geq 0$. Then the value*

$$-|\eta_1^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|\eta_1^\wedge(r,\psi,\delta_0)^2 - 1|^{\delta_0},$$

increases with increasing of $s \in (\delta_0/4, 1)$.

Proof. For $\delta_0 = 0$ assertion follows from the Lemma 4.4. Let $\delta_0 > 0$. According to the lemma 4.16 and (209),

$$-\frac{\partial}{\partial t}\left(\log\left(|\eta_1^\wedge(r,\psi,\delta_0)|^{2(1/\delta_0-1)}|\eta_1^\wedge(r,\psi,\delta_0)^2 - 1\right)\right) =$$

$$-(1/\delta_0 - 1)\frac{\partial}{\partial t}\log\left(|\eta_1^\wedge(r,\psi,\delta_0)|^2\right) -$$

$$\frac{\partial}{\partial t}\log\left(|\eta_1^\wedge(r,\psi,\delta_0) + 1|^2\right) -$$

$$-\partial\partial t\log\left(|\eta_1^\wedge(r,\psi,\delta_0) - 1|^2\right) =$$

$$-(1/\delta_0 - 1)\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)|^2}{|\eta_1^\wedge(r,\psi,\delta_0)|^2} - 2\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0) + 1|^2}{|\eta_1^\wedge(r,\psi,\delta_0) + 1|^2} +$$

$$\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0) + 1|^2}{|\eta_1^\wedge(r,\psi,\delta_0) + 1|^2} + \frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0) + 1|^2}{|\eta_1^\wedge(r,\psi,\delta_0) - 1|^2} -$$

$$2\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)|^2}{|\eta_1^\wedge(r,\psi,\delta_0)|^2}\frac{|\eta_1^\wedge(r,\psi,\delta_0)|^2}{|\eta_1^\wedge(r,\psi,\delta_0)-1|^2}\geq$$

$$-(1/\delta_0-1)\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}-2\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}+$$

$$\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}+\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)-1|^2}-$$

$$2\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}\frac{|\eta_1^\wedge(r,\psi,\delta_0)|^2}{|\eta_1^\wedge(r,\psi,\delta_0)-1|^2}=$$

$$-(1/\delta_0+1)\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}+$$

$$\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}\frac{2+2|\eta_1^\wedge(r,\psi,\delta_0)|^2}{|\eta_1^\wedge(r,\psi,\delta_0)-1|^2}-$$

$$2\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}\frac{|\eta_1^\wedge(r,\psi,\delta_0)|^2}{|\eta_1^\wedge(r,\psi,\delta_0)-1|^2}=$$

$$-\frac{\frac{\partial}{\partial t}|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}{|\eta_1^\wedge(r,\psi,\delta_0)+1|^2}\left(\frac{\delta_0+1}{\delta_0}-\frac{2}{|\eta_1^\wedge(r,\psi,\delta_0)-1|^2}\right)\geq 0.\blacksquare$$

§2. Some corrections to [53].

There are some misprints in the paper [53].

In the equality (3.1.41) instead of

$$p_1=8(|R_0^*(r,\psi,\delta_0)|^2+|R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2=$$

$$8\left(r^2+rt+\frac{1}{4}+|D_0(r,\psi,\delta_0)|)/(1+\delta_0\right)^2$$

should be

$$p_1=8(|R_0^*(r,\psi,\delta_0)|^2+|R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2=$$

$$8\left(r^2+rt+\frac{1}{4}+|D_0(r,\psi,\delta_0)|\right)/(1+\delta_0)^2.$$

In (3.1.51) instead of

$$2r(1+\delta_0(r-\delta/2))(1+t)$$

should be

$$2r(1+\delta_0(r-\delta_0/2))(1+t).$$

In (3.1.53) instead of

$$|\eta_0^\wedge(r,0,\delta_0)+\varepsilon|=\frac{2\left(r+\frac{1}{2}-\varepsilon\frac{1+\delta_0}{2}+\sqrt{r^2-r+\left(\frac{\delta_0}{2}\right)^2}\right)}{1+\delta_0}<$$

$$|\eta_1^\wedge(r,0,\delta_0)+\varepsilon| = \frac{2|\left|r+\frac{1}{2}-\varepsilon\frac{1+\delta_0}{2}-\sqrt{r^2-r+\left(\frac{\delta_0}{2}\right)^2}\right|}{1+\delta_0} <$$

should be

$$|\eta_0^\wedge(r,0,\delta_0)+\varepsilon| = \frac{2\left(r+\frac{1}{2}-\varepsilon\frac{1+\delta_0}{2}+\sqrt{r^2-r+\left(\frac{\delta_0}{2}\right)^2}\right)}{1+\delta_0} >$$

$$|\eta_1^\wedge(r,0,\delta_0)+\varepsilon| = \frac{2|\left|r+\frac{1}{2}-\varepsilon\frac{1+\delta_0}{2}-\sqrt{r^2-r+\left(\frac{\delta_0}{2}\right)^2}\right|}{1+\delta_0}.$$

On the page 9, line 7 from the bottom instead of

$$\frac{\partial}{\partial t}p_2 = -r\delta_0 + \frac{\partial}{\partial t}|D_0(r,\psi,\delta_0 >$$

should be

$$\frac{\partial}{\partial t}p_2 = -r\delta_0 + \frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)| > .$$

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