WEIGHT-MONODROMY CONJECTURE FOR p-ADICALLY UNIFORMIZED VARIETIES

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Abstract. The aim of this paper is to prove the weight-monodromy conjecture (Deligne's conjecture on the purity of monodromy filtration) for varieties with p-adic uniformization by the Drinfeld upper half spaces of any dimension. The ingredients of the proof is to prove a special case of the Hodge standard conjecture, and apply an argument of Steenbrink, M. Saito to the weight spectral sequence of Rapoport-Zink. As an application, by combining our results with the results of Schneider-Stuhler, we compute the local zeta functions of p-adically uniformized varieties in terms of representation theoretic invariants. We also consider a p-adic analogue by using the weight spectral sequence of Mokrane.

1. Introduction

Let K be a complete discrete valuation field with finite residue field $\mathbb{F}_q$ of characteristic $p > 0$, $\mathcal{O}_K$ the ring of integers of K, and l a prime number different from p. Let X be a proper smooth variety over K, and $V := H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ the l-adic cohomology of $X_{\overline{K}} := X \otimes_K \overline{K}$ on which the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ acts. We define the inertia group I_K of K by the exact sequence :

$$1 \longrightarrow I_K \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1.$$

The monodromy filtration $M_\bullet$ on V is determined by the action of I_K on V, and the weight filtration $W_\bullet$ on V is determined by the action of a lift of the geometric Frobenius element $\mathrm{Fr}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, which is the inverse of the q-th power map on $\overline{\mathbb{F}}_q$ (for details, see Definition 5.1, Definition 5.3).

The weight-monodromy conjecture claims the coincidence of these two filtrations up to some shift ([De1], [De3], [Il1], [Il2], [RZ1], [Rap]). Sometimes, the weight-monodromy conjecture is also called *Deligne's conjecture on the purity of monodromy filtration* in the literature.

Conjecture 1.1 (Weight-monodromy conjecture)**.**

$$M_i V = W_{i+w} V \qquad \textit{for all } i.$$

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Let us briefly recall the theory of p-adic uniformization by Tate, Raynaud, Mumford, Mustafin, Kurihara, and others ([Mum], [Mus], [Ku]). Let $\widehat{\Omega}^d_K$ be the Drinfeld upper half space of dimension d, which is a rigid analytic space obtained by removing all K-rational hyperplanes from $\mathbb{P}^d_K$. We have a natural action of $\mathrm{PGL}_{d+1}(K)$ on $\widehat{\Omega}^d_K$. For a cocompact torsion free discrete subgroup $\Gamma \subset \mathrm{PGL}_{d+1}(K)$, the rigid analytic quotient $\Gamma\backslash\widehat{\Omega}^d_K$ has a structure of a projective smooth variety X_Γ over K (for details, see §6). The main theorem of this paper is as follows.

Theorem 1.2 (Weight-monodromy conjecture for X_Γ)**.** *Let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a cocompact torsion free discrete subgroup. Then, the weight-monodromy conjecture (Conjecture 1.1) holds for X_Γ.*

Note that Conjecture 1.1 was known to hold if X/K satisfies at least one of the following conditions.

(1) X has a proper smooth model over $\mathcal{O}_K$ ([SGA4-III], [De2], [De3]).
(2) X is a curve or an abelian variety over K ([SGA7-I], IX).
(3) X is a surface ([RZ1] for the semistable case, [dJ] for the general case).
(4) X is a certain threefold with strictly semistable reduction ([It2]).
(5) K is of characteristic $p > 0$ ([De3], [Te], [It1]).

Therefore, Theorem 1.2 is new only in mixed characteristic although our proof of Theorem 1.2 is valid even if K is of characteristic $p > 0$ (However, see also Remark 6.2). In mixed characteristic and in dimension ≥ 3, Conjecture 1.1 is still open up to now. Therefore, Theorem 1.2 gives us an interesting class of varieties of any dimension in mixed characteristic for which Conjecture 1.1 holds.

Our proof of Theorem 1.2 is based on a careful analysis of the weight spectral sequence of Rapoport-Zink ([RZ1]). We prove a special case of the Hodge standard conjecture and use it to follow an argument of Steenbrink, M. Saito in [St], [SaM], where they proved a Hodge analogue of Conjecture 1.1 by using polarized Hodge structures. Namely, in our proof of Theorem 1.2, the Hodge standard conjecture plays a role of polarized Hodge structures. Note that in [It2], by basically the same idea as in this paper, the author proved Conjecture 1.1 for certain threefolds by using the Hodge index theorem which is nothing but the Hodge standard conjecture for surfaces.

The outline of this paper is as follows. In §2, we fix notation and recall some basic facts about the Hodge standard conjecture and related theorems. Note that, in this paper, we mainly consider varieties whose l-adic cohomology groups are generated by algebraic cycles. We introduce appropriate notation in §2.1.

In §3, we recall some properties of blowing-up and prove the Hodge standard conjecture of blow-ups under certain assumptions by looking at the asymptotic behavior of the Lefschetz operator and the cup product pairings (Proposition 3.4). Such argument seems well-known at least in the simplest case of a blow-up at a point on a surface. However, since the multiplicative structure of a blow-up is not so simple in higher dimensions, we need tedious computations to prove it.

In §4, we consider a projective smooth variety B^n of dimension n over $\overline{\mathbb{F}}_q$ which is obtained by successive blowing-ups of $\mathbb{P}^n_{\overline{\mathbb{F}}_q}$ along linear subvarieties. The main result in §4 is a proof of the Hodge standard conjecture for such varieties for certain choice of an ample line bundle (Proposition 4.7). Although we can, in principle, compute everything combinatorially about the variety B^n, the proof of Proposition 4.7 is far from trivial. We look at the combinatorial structure of B^n very carefully, and the proof proceeds by induction on n. This proof is inspired by McMullen's proof of an analogue of the hard Lefschetz theorem for non-rational polytopes ([Mc]).

In §5, we recall some basic facts on the weight spectral sequence of Rapoport-Zink. By an argument of Steenbrink, M. Saito, we give a proof of a special case of Conjecture 1.1 assuming the existence of $\mathbb{Q}$-structures on l-adic cohomology and the Hodge standard conjecture (Proposition 5.7). Note that, although the class of varieties considered in Proposition 5.7 is extremely small, this class is large enough to include all p-adically uniformized varieties as we will see in §6.

In §6, we firstly give a proof of Theorem 1.2. The key point is that there is an explicitly constructed formal scheme $\widehat{\Omega}^d_{\mathcal{O}_K}$ locally of finite type over $\operatorname{Spf}\mathcal{O}_K$, whose associated rigid analytic space is $\widehat{\Omega}^d_K$. From the construction, we see that X_Γ has a proper semistable model $\mathfrak{X}_\Gamma$ over $\mathcal{O}_K$. Moreover, if $\mathfrak{X}_\Gamma$ is strictly semistable, we see that all irreducible components of the special fiber of $\mathfrak{X}_\Gamma$ are isomorphic to the variety B^d in §4. Hence we can prove the Hodge standard conjecture for them by Proposition 4.7. The same is true for intersections of them. By Proposition 5.7, the proof of Theorem 1.2 is completed.

In the rest of §6, we also give several immediate applications. By combining Theorem 1.2 with the results of Schneider-Stuhler on the l-adic cohomology of X_Γ ([SS]), we give a proof of Schneider-Stuhler's conjecture on the filtration $F^\bullet$ on $H^d_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ (Theorem 6.3). As a consequence, we prove the following formula of the local zeta function $\zeta(s, X_\Gamma)$ of X_Γ :

$$\begin{aligned}\zeta(s, X_\Gamma) &:= \prod_{k=0}^{2d} \det\left(1 - q^{-s}\cdot \mathrm{Fr}_q\,;\, H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K},\, \mathbb{Q}_l)^{I_K}\right)^{(-1)^{k+1}} \\ &= (1-q^{-s})^{\mu(\Gamma)\cdot(-1)^{d+1}} \cdot \prod_{k=0}^{d} \frac{1}{1-q^{k-s}},\end{aligned}$$

where $\mu(\Gamma)$ is the multiplicity of the Steinberg representation in the representation of $\mathrm{PGL}_{d+1}(K)$ induced from the trivial character on Γ (Theorem 6.4). In particular, $\zeta(s, X_\Gamma)$ is independent of l. In §6.5, we give an application to mod l Galois representations (Theorem 6.7). Finally, in §6.6, we consider a p-adic analogue by using the weight spectral sequence of Mokrane ([Mo]). After proving a p-adic analogue of Conjecture 1.1, we also prove that the local zeta function $\zeta_{p\text{-adic}}(s, X_\Gamma)$ defined by p-adic Hodge theory coincides with $\zeta(s, X_\Gamma)$ above (Theorem 6.11, Theorem 6.12). However, we note that the computation of $\zeta_{p\text{-adic}}(s, X_\Gamma)$ is accomplished by comparing the l-adic and the p-adic case (This argument is inspired by [SaT]). The reason why we have to make such a detour is that Schneider-Stuhler's computation doesn't work directly on the p-adic étale cohomology. In [SS], Schneider-Stuhler also conjectured that their filtration $F^\bullet$ on de Rham cohomology coincides with Hyodo's monodromy filtration ([Hy]), but we can't prove this here (see Remark 6.14, Remark 6.15).

Needless to say, there is a very important theory so called the theory of p-adic uniformization of Shimura varieties established by Čerednik, Drinfeld for Shimura curves, and Varshavsky, Rapoport-Zink in higher dimensions ([Č], [Dr], [Va], [RZ2]). We strongly expect that Theorem 1.2 has applications to the zeta functions of Shimura varieties with p-adic uniformization ([Rap], [RZ1]). We also expect that this will establish a special case of the compatibility of the global and local Langlands correspondences ([Harr], Problem 1). We plan to return to these problems in the future.

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2. Review of the Hodge standard conjecture

2.1. **Notation and assumptions.** Let F be an algebraically closed field of any characteristic, l a prime number different from the characteristic of F. Let X be a projective smooth variety over F of dimension n. Let $Z^k(X)$ be the group of algebraic cycles on X of codimension k, and $\mathrm{cl}^k \colon Z^k(X) \to H^{2k}_{\text{ét}}(X, \mathbb{Q}_l(k))$ the cycle map for l-adic cohomology. Let $Z^k_{\text{num}}(X) \subset Z^k(X)$ be the subgroup consisting of algebraic cycles which are numerically equivalent to zero. It is known that

$$N^k(X) := Z^k(X)/Z^k_{\text{num}}(X)$$

is a finitely generated free $\mathbb{Z}$-module ([Kl], Lemma 5-2, see also Lemma 2.3 below). Here we put the following assumption on X.

Assumption 2.1.

$$\dim_{\mathbb{Q}_l} H^k_{\text{ét}}(X,\mathbb{Q}_l) = \begin{cases} \operatorname{rank}_{\mathbb{Z}} N^{k/2}(X) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Remark 2.2. Assumption 2.1 is independent of l because the l-adic Betti numbers are independent of l (This is a consequence of the Weil conjecture, see, for example, [KM], Corollary 1).

Lemma 2.3. *Under Assumption 2.1,* cl^k *induces an isomorphism :*

$$N^k(X)\otimes_{\mathbb{Z}}\mathbb{Q}_l \cong H^{2k}_{\text{ét}}(X,\mathbb{Q}_l(k)).$$

Proof. Fix an integer k. Let $V \subset H^{2k}_{\text{ét}}(X,\mathbb{Q}_l(k))$ be a $\mathbb{Q}_l$-vector subspace generated by the image of the cycle map cl^k. Take elements $x_1,\dots,x_r \in Z^k(X)$ such that $\{\mathrm{cl}^k(x_1),\dots,\mathrm{cl}^k(x_r)\}$ is a basis of V over $\mathbb{Q}_l$. Then, it is easy to see that the image of the map

$$Z^{n-k}(X) \longrightarrow \mathbb{Z}^{\oplus r}, \qquad y \mapsto (y\cdot x_1,\dots,y\cdot x_r)$$

is isomorphic to $N^{n-k}(X)$, where $\cdot$ denotes the intersection product. Hence we have the following inequalities :

$$\operatorname{rank}_{\mathbb{Z}} N^{n-k}(X) \le r \le \dim_{\mathbb{Q}_l} H^{2k}_{\text{ét}}(X,\mathbb{Q}_l) = \dim_{\mathbb{Q}_l} H^{2n-2k}_{\text{ét}}(X,\mathbb{Q}_l).$$

Therefore, Assumption 2.1 implies that the above inequalities are equalities. Hence $V = H^{2k}_{\text{ét}}(X,\mathbb{Q}_l(k))$ for all k, and all cohomology classes are generated by algebraic cycles. In particular, the numerical equivalence coincides with the l-adic homological equivalence. We have a surjective map $N^k(X)\otimes_{\mathbb{Z}}\mathbb{Q}_l \to H^{2k}_{\text{ét}}(X,\mathbb{Q}_l(k))$, between $\mathbb{Q}_l$-vector spaces of the same dimension, hence it is an isomorphism. □

Namely, $N^k(X)$ defines l-independent $\mathbb{Q}$-structures on the l-adic cohomology of X. Similarly, by using the cycle map for crystalline cohomology ([GM], [Gr]), it is easy to see that $N^k(X)$ also defines $\mathbb{Q}$-structures on the crystalline cohomology of X.

We define

$$H^k(X) := \begin{cases} N^{k/2}(X)\otimes_{\mathbb{Z}}\mathbb{R} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

and $H^*(X) := \bigoplus_k H^k(X)$. We consider $H^*(X)$ virtually as cohomology with coefficients in $\mathbb{R}$. The cup product $\cup$ on $H^*(X)$ is induced by the intersection product. Most properties of l-adic cohomology such as pullback, Poincaré duality, Künneth formula, etc. are valid for $H^*(X)$. To avoid confusion, we always put the subscript "ét" for étale cohomology.

A $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*) is a formal sum of divisors with coefficients in $\mathbb{Q}$ (resp. $\mathbb{R}$). A $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*) L is called *ample* if L is of the form $L = a_1L_1+\cdots+a_rL_r$ with $r\ge 1$, $a_1,\dots,a_r \in \mathbb{Q}_{>0}$ (resp. $\mathbb{R}_{>0}$), and $L_1,\dots,L_r$

ample. If L is an ample $\mathbb{Q}$-divisor, some positive integral multiple of L is an ample divisor. But this doesn't hold for ample $\mathbb{R}$-divisors.

There is a natural map from the group of $\mathbb{R}$-divisors on X to $H^2(X)$. In this paper, we don't usually distinguish an $\mathbb{R}$-divisor, a formal sum of line bundles with coefficients in $\mathbb{R}$ (i.e. $\mathbb{R}$-line bundle), and its class in $H^2(X)$. We can naturally define pullbacks and restrictions for $\mathbb{R}$-divisors. To avoid confusion, for an $\mathbb{R}$-divisor H, we sometimes use the notation $\mathcal{O}(H)$ instead of H. For example, $f^*\mathcal{O}(H)$ (resp. $\mathcal{O}(H)|_X$) denotes the pullback (resp. restriction) as a formal sum of line bundles.

2.2. **Hard Lefschetz conjecture.** As in §2.1, let X be a projective smooth variety over F of dimension n satisfying Assumption 2.1, and L an ample $\mathbb{R}$-divisor on X. By taking cup product with L, we have an $\mathbb{R}$-linear map

$$L\colon H^k(X) \longrightarrow H^{k+2}(X)$$

called the *Lefschetz operator*.

Conjecture 2.4 (Hard Lefschetz conjecture)**.** *For all k, L^k induces an isomorphism :*

$$L^k\colon H^{n-k}(X) \xrightarrow{\cong} H^{n+k}(X).$$

If L is a $\mathbb{Q}$-divisor, Conjecture 2.4 is proved by Deligne in all characteristics ([De3]). If $F = \mathbb{C}$, Conjecture 2.4 holds by transcendental methods ([We]). The general characteristic 0 case follows from this by Lefschetz principle. However, in characteristic $p > 0$, Conjecture 2.4 seems open in general.

Assume that Conjecture 2.4 holds for (X, L). For an integer k with $0 \leq k \leq n$, we define the *primitive part* $P^k(X)$ by

$$P^k(X) = \operatorname{Ker}\bigl(L^{n-k+1}\colon H^k(X) \longrightarrow H^{2n-k+2}(X)\bigr).$$

We define $P^k(X) = 0$ for $k < 0$ or $k > n$. Then, by Conjecture 2.4, we have the *primitive decomposition* :

$$H^k(X) = \bigoplus_{i \geq 0} L^i P^{k-2i}(X) \cong \bigoplus_{i \geq 0} P^{k-2i}(X) \quad \text{for all } k. \tag{2.1}$$

For even k with $0 \leq k \leq n$, we define the pairing $\langle , \rangle_{H^k(X)}$ on $H^k(X)$ by

$$\langle , \rangle_{H^k(X)}\colon H^k(X) \times H^k(X) \to \mathbb{R}, \quad \langle x, y \rangle_{H^k(X)} = (-1)^{k/2} L^{n-k} x \cup y.$$

Otherwise, we define $\langle , \rangle_{H^k(X)}$ to be the zero pairing. The pairings $\langle , \rangle_{H^k(X)}$ are nondegenerate by Conjecture 2.4 and Poincaré duality.

We denote the restriction of $\langle , \rangle_{H^k(X)}$ to $P^k(X)$ by $\langle , \rangle_{P^k(X)}$. The decomposition (2.1) is an orthogonal decomposition with respect to $\langle , \rangle_{H^k(X)}$. Therefore, each

$\langle , \rangle_{P^k(X)}$ is nondegenerate. Moreover, for all k with $0 \leq k \leq n$, $\langle , \rangle_{H^k(X)}$ is isomorphic to the alternating sum of $\langle , \rangle_{P^{k-2i}(X)}$ for $i \geq 0$:

$$\langle , \rangle_{H^k(X)} = \sum_{i \geq 0} (-1)^i \langle , \rangle_{P^{k-2i}(X)} \quad \text{on} \quad H^k(X) \cong \bigoplus_{i \geq 0} P^{k-2i}(X). \tag{2.2}$$

Remark 2.5. The primitive decomposition may depend on the choice of L. However, the dimensions of $P^k(X)$ are independent of L because

$$\dim_{\mathbb{R}} P^k(X) = \dim_{\mathbb{R}} H^k(X) - \dim_{\mathbb{R}} H^{k-2}(X).$$

2.3. **Hodge standard conjecture.** As in §2.2, let X be a projective smooth variety over F of dimension n satisfying Assumption 2.1, and L an ample $\mathbb{R}$-divisor on X. Assume further that the hard Lefschetz conjecture (Conjecture 2.4) holds for (X, L).

Conjecture 2.6 (Hodge standard conjecture)**.** *For all k, the pairing $\langle , \rangle_{P^k(X)}$ is positive definite.*

Remark 2.7. Usually, Conjecture 2.6 is stated for all projective smooth varieties X without Assumption 2.1 ($Hdg(X)$ in [Kl]). It is a part of Grothendieck's standard conjectures, and known to hold in the following cases.

(1) If $F = \mathbb{C}$, it is a consequence of the existence of polarized Hodge structures on singular cohomology ([We]). The general characteristic 0 case follows by Lefschetz principle.
(2) If X is of dimension ≤ 2, it is known as the Hodge index theorem ([Kl], 5, [Fu], Example 15.2.4). As a corollary, we can prove the case $k \leq 1$ by taking hyperplane sections.

Remark 2.8. Note that Conjecture 2.6 is stated for a pair (X, L). It seems that Conjecture 2.6 for (X, L) doesn't automatically imply Conjecture 2.6 for (X, L') with another choice of L' (see also Corollary 2.12 below).

Example 2.9. For an ample $\mathbb{R}$-divisor L on $\mathbb{P}^n$, it is clear that Conjecture 2.4 holds for $(\mathbb{P}^n, L)$. The only nontrivial primitive cohomology is $P^0(\mathbb{P}^n) \cong \mathbb{R}$ and the pairing $\langle , \rangle_{P^0(\mathbb{P}^n)}$ is

$$P^0(\mathbb{P}^n) \times P^0(\mathbb{P}^n) \cong \mathbb{R} \times \mathbb{R} \ni (x, y) \mapsto L^n \cdot xy \in \mathbb{R} \cong H^{2n}(\mathbb{P}^n).$$

Since L^n is positive, this pairing is positive definite. Hence Conjecture 2.6 holds for $(\mathbb{P}^n, L)$ for all ample $\mathbb{R}$-divisors L on $\mathbb{P}^n$.

2.4. **Signatures.** Let notation be as in §2.3. Firstly, we recall the definition of signatures. Let V be a finite dimensional $\mathbb{R}$-vector space with a bilinear symmetric perfect pairing $\langle , \rangle$. Take a basis $\{e_1, \ldots, e_r\}$ of V and consider the matrix $M = (m_{ij})$ defined by $m_{ij} = \langle e_i, e_j \rangle$. By Sylvester's law of inertia, all eigenvalues of M are in $\mathbb{R}$, and the number of positive eigenvalues minus the number of negative eigenvalues is independent of the choice of $\{e_1, \ldots, e_r\}$. We denote this number by $\mathrm{sign}\big(V, \langle , \rangle\big)$ and call it the *signature* of the pairing $\langle , \rangle$ on V.

Proposition 2.10. *The Hodge standard conjecture (Conjecture 2.6) holds for (X, L) if and only if*

$$\operatorname{sign}\bigl(H^k(X), \langle,\rangle_{H^k(X)}\bigr) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{R}} P^{k-2i}(X) \quad \textit{for all } k \textit{ with } 0 \leq k \leq n.$$

Remark 2.11. By Remark 2.5, the right hand side of Proposition 2.10 doesn't depend on a choice of L.

Proof. By definition, Conjecture 2.6 is equivalent to the following equality :

$$\dim_{\mathbb{R}} P^k(X) = \operatorname{sign}\bigl(P^k(X), \langle,\rangle_{P^k(X)}\bigr) \quad \text{for all } k \text{ with } 0 \leq k \leq n. \tag{2.3}$$

By (2.2), we see that, for all k with $0 \leq k \leq n$,

$$\operatorname{sign}\bigl(H^k(X), \langle,\rangle_{H^k(X)}\bigr) = \sum_{i\geq 0}(-1)^i \operatorname{sign}\bigl(P^{k-2i}(X), \langle,\rangle_{P^{k-2i}(X)}\bigr).$$

Hence (2.3) implies the conditions in Proposition 2.10.

Conversely, assume the conditions in Proposition 2.10. We shall prove (2.3) by induction on k. The case $k = 0$ is trivial, since $P^0(X) \cong \mathbb{R}$ and L^n is positive. If (2.3) is proved for $k < k_0$, the condition of Proposition 2.10 for $k = k_0$ implies

$$\begin{aligned}
&\operatorname{sign}\bigl(H^{k_0}(X), \langle,\rangle_{H^{k_0}(X)}\bigr) \\
&\quad = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{R}} P^{k_0-2i}(X) \\
&\quad = \dim_{\mathbb{R}} P^{k_0}(X) + \sum_{i\geq 1}(-1)^i \operatorname{sign}\bigl(P^{k_0-2i}(X), \langle,\rangle_{P^{k_0-2i}(X)}\bigr).
\end{aligned}$$

By (2.2), we have $\dim_{\mathbb{R}} P^{k_0}(X) = \operatorname{sign}\bigl(P^{k_0}(X), \langle,\rangle_{P^{k_0}(X)}\bigr)$. $\square$

As a corollary, we have the following implications about the Hodge standard conjecture (Conjecture 2.6) by a "homotopy argument".

Corollary 2.12.
(1) *Let L, L' be ample $\mathbb{R}$-divisors on X. If the hard Lefschetz conjecture (Conjecture 2.4) holds for $(X, tL + (1-t)L')$ for all $t \in \mathbb{R}$ with $0 \leq t \leq 1$, then the Hodge standard conjecture (Conjecture 2.6) for (X, L), (X, L') are equivalent to each other.*
(2) *In particular, if Conjecture 2.4 holds for (X, L) for all ample $\mathbb{R}$-divisors L on X, then Conjecture 2.6 for (X, L) for all ample $\mathbb{R}$-divisors L on X are equivalent to each other.*

Proof. It is enough to prove the first assertion. Let $\langle,\rangle_{H^k(X)}$ (resp. $\langle,\rangle'_{H^k(X)}$) be the pairing on $H^k(X)$ defined by L (resp. L'). By Remark 2.5 and Proposition 2.10, it is enough to show that, for all k, $\langle,\rangle_{H^k(X)}$ and $\langle,\rangle'_{H^k(X)}$ have the same signatures. For $t \in \mathbb{R}$ with $0 \leq t \leq 1$, let $\langle,\rangle_t$ be the pairing on $H^k(X)$ defined by $tL + (1-t)L'$, which is an ample $\mathbb{R}$-divisor on X satisfying Conjecture 2.4 by assumption. We have $\langle x, y\rangle_0 = \langle x, y\rangle'_{H^k(X)}$, $\langle x, y\rangle_1 = \langle x, y\rangle_{H^k(X)}$, and $\langle,\rangle_t$ is nondegenerate for $t \in \mathbb{R}$ with $0 \leq t \leq 1$. Therefore, $\{\langle , \rangle_t\}_t$ is a continuous family of nondegenerate pairings on the $\mathbb{R}$-vector space $H^k(X)$. Hence, the signature of $\langle , \rangle_t$ is independent of t. By comparing the case $t = 0$ and $t = 1$, we have the assertion. □

2.5. **Products of varieties.** As in §2.1, let X (resp. Y) be a projective smooth variety over F satisfying Assumption 2.1, and L_X (resp. L_Y) an ample $\mathbb{R}$-divisor on X (resp. Y). Then, by the Künneth formula, we have an isomorphism :

$$H^k(X \times Y) \cong \bigoplus_{i=0}^{k} H^i(X) \otimes_{\mathbb{R}} H^{k-i}(Y).$$

Hence $X \times Y$ also satisfies Assumption 2.1. The condition that L_X, L_Y are ample is equivalent to the condition that $\mathrm{pr}_1^*\mathcal{O}(L_X) + \mathrm{pr}_2^*\mathcal{O}(L_Y)$ is ample, where $\mathrm{pr}_1 \colon X \times Y \to X$ and $\mathrm{pr}_2 \colon X \times Y \to Y$ are projections ([Hart1], I, [Hart2], II).

Proposition 2.13. (1) *If the hard Lefschetz conjecture (Conjecture 2.4) holds for (X, L_X), (Y, L_Y), then it also holds for $\big(X \times Y,\ \mathrm{pr}_1^*\mathcal{O}(L_X) + \mathrm{pr}_2^*\mathcal{O}(L_Y)\big)$.*

(2) *Moreover, if the Hodge standard conjecture (Conjecture 2.6) holds for (X, L_X), (Y, L_Y), then it also holds for $\big(X \times Y,\ \mathrm{pr}_1^*\mathcal{O}(L_X) + \mathrm{pr}_2^*\mathcal{O}(L_Y)\big)$.*

Proof. Here we give a proof by using the representation theory of the Lie algebra $\mathfrak{sl}(2)$ (for details, [De3], 1.6, Bourbaki Lie, VIII, §1). It is a consequence of Conjecture 2.4 that we can construct $\mathfrak{sl}(2)$-representations on $H^*(X)$, $H^*(Y)$ by using the Lefschetz operator L_X, L_Y. Then, the tensor product $H^*(X) \otimes_{\mathbb{R}} H^*(Y)$ has a natural $\mathfrak{sl}(2)$-action which corresponds to $\mathrm{pr}_1^*\mathcal{O}(L_X) + \mathrm{pr}_2^*\mathcal{O}(L_Y)$. This proves the first assertion. The second assertion follows from a basic property of certain pairings defined on $\mathfrak{sl}(2)$-representations over $\mathbb{R}$ (Bourbaki Lie, VIII, §1). □

Remark 2.14. Since $H^2(X \times Y) \cong H^2(X) \oplus H^2(Y)$, all ample $\mathbb{R}$-divisors on $X \times Y$ is necessarily of the form $\mathrm{pr}_1^*\mathcal{O}(L_X) + \mathrm{pr}_2^*\mathcal{O}(L_Y)$ in $H^2(X \times Y)$. Therefore, if Conjecture 2.4 (resp. Conjecture 2.6) holds for (X, L_X), (Y, L_Y) for all ample $\mathbb{R}$-divisors L_X, L_Y on X, Y, then it also holds for $(X \times Y, L_{X \times Y})$ for all ample $\mathbb{R}$-divisors $L_{X \times Y}$ on $X \times Y$.

3. Cohomology of varieties obtained by blowing-ups

In this section, we study the Hodge standard conjecture (Conjecture 2.6) for varieties obtained by blowing-ups. After recalling basic facts about blowing-ups, we show that, under certain assumptions, Conjecture 2.6 holds for a blow-up for certain choice of an ample $\mathbb{R}$-divisor (for details of the computation, see [SGA5], VII, [Hart2], II, [Fu], 6.7). We use the same notation as in §2.1.

3.1. **Setup.** Let X be a projective smooth variety over F of dimension n, and $Y_1, \dots, Y_r \subset X$ mutually disjoint smooth closed subvarieties of codimension $d \geq 2$. Let $i\colon Y = \coprod_{k=1}^{r} Y_k \hookrightarrow X$ be the closed immersion. Let $f\colon X' \to X$ be the blow-up of X along Y, $Y' \subset X'$ the proper transform of Y, and $j\colon Y' \hookrightarrow X'$, $g\colon Y' \to Y$ natural morphisms. Assume that $X, Y_1, \dots, Y_r$ satisfy Assumption 2.1 in §2.1. Then, it is easy to see that X', Y' also satisfy Assumption 2.1 in §2.1. This can be checked from the following computations of the cohomology of X' and Y'.

We have the following cartesian diagram :

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{i} & X \end{array}$$

Since Y is a disjoint union of mutually disjoint smooth closed subvarieties, $g\colon Y' \to Y$ has a structure of $\mathbb{P}^{d-1}$-bundle. More precisely, $Y' = \mathbb{P}(\check{N}_{Y/X})$, where $\check{N}_{Y/X}$ is the conormal bundle of Y in X. Let

$$\xi = c_1\big(\mathcal{O}_{Y'}(1)\big) \ \in \ H^2(Y')$$

be the first Chern class of $\mathcal{O}_{Y'}(1)$ of the $\mathbb{P}^{d-1}$-bundle $g\colon Y' \to Y$. The restriction of the line bundle $\mathcal{O}_{X'}(Y')$ to Y' itself is isomorphic to $\mathcal{O}_{Y'}(-1)$.

We define a vector bundle F of rank $d-1$ on Y' by the kernel of a natural map $g^*(\check{N}_{Y/X}) \to \mathcal{O}_{Y'}(1)$. Then we have an exact sequence of vector bundles on Y' :

$$0 \longrightarrow F \longrightarrow g^*(\check{N}_{Y/X}) \longrightarrow \mathcal{O}_{Y'}(1) \longrightarrow 0.$$

3.2. **Cohomology of X' and Y'.** For all k, we have the following exact sequence of cohomology groups ([SGA5], VII, 8.5) :

$$(3.1) \quad 0 \longrightarrow H^{k-2d}(Y) \xrightarrow{\ \lambda\ } H^{k-2}(Y') \oplus H^k(X) \xrightarrow{\ \mu\ } H^k(X') \longrightarrow 0,$$

where λ and μ are defined as follows :

$$\lambda(y) = \big(g^*(y) \cup c_{d-1}(\check{F}),\ -i_*(y)\big), \qquad \mu(y,x) = j_*(y) + f^*(x).$$

Note that $\check{F}$ is the dual vector bundle of F and $c_{d-1}(\check{F}) \in H^{2d-2}(Y')$ is the $(d-1)$-th Chern class of $\check{F}$. Define $\lambda'\colon H^{k-2}(Y') \oplus H^k(X) \to H^{k-2d}(Y)$ by $\lambda'(y,x) = g_*(y)$, then we have

$$\lambda' \circ \lambda(y) = g_*\big(g^*(y) \cup c_{d-1}(\check{F})\big) = y \cup g_*\big(c_{d-1}(\check{F})\big) = y$$

since $g_*\big(c_{d-1}(\check{F})\big) = (-1)^{d-1} g_*\big(c_{d-1}(F)\big) = g_*(\xi^{d-1}) = 1$ ([SGA5], VII, 8.4.2). Therefore, we have the following isomorphism of cohomology groups :

$$(3.2) \qquad H^k(X') \overset{\mu}{\cong} \operatorname{Ker}\big(g_*\colon H^{k-2}(Y') \to H^{k-2d}(Y)\big) \oplus H^k(X)$$

We shall compute the right hand side explicitly. For an integer m with $0 \leq m \leq d-1$, the map $H^{k-2m}(Y) \to H^k(Y'),\ y \mapsto g^*(y) \cup \xi^m$ is injective because

$$g_*\big((g^*(y) \cup \xi^m) \cup \xi^{d-1-m}\big) = y \cup g_*(\xi^{d-1}) = y.$$

Let $H^{k-2m}(Y)\xi^m \subset H^k(Y')$ be the image of this map. For simplicity, we sometimes write $y\,\xi^m$ instead of $g^*(y) \cup \xi^m$. Note that $g_*\colon H^k(Y') \to H^{k-2(d-1)}(Y)$ satisfies

$$g_*(a\,\xi^k) = a \cup g_*(\xi^k) = \begin{cases} a & \text{if } k = d-1 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have an isomorphism ([SGA5], VII, 2.2.6):

$$\begin{aligned} H^k(Y') &\cong \bigoplus_{m=0}^{d-1} H^{k-2m}(Y)\xi^m \\ &= H^k(Y) \oplus H^{k-2}(Y)\xi \oplus \cdots \oplus H^{k-2(d-1)}(Y)\xi^{d-1}. \end{aligned}$$

Therefore, by (3.2), we have the following explicit description of the cohomology groups of X' :

$$H^k(X') \cong H^{k-2}(Y) \oplus H^{k-4}(Y)\xi \oplus \cdots \oplus H^{k-2-2(d-2)}(Y)\xi^{d-2} \oplus H^k(X). \tag{3.3}$$

The following property of the cohomology of X' will be used later.

Proposition 3.1. *Let $x' \in H^k(X')$ be a cohomology class on X' whose restriction to Y' is zero. Then x' is of the form $x' = f^*(x)$ for a unique $x \in H^k(X)$.*

Proof. The uniqueness is clear because $f_*(x') = f_*\big(f^*(x)\big) = x$. For the existence, we write x' of the form $x' = x_0 + x_1\,\xi + \cdots + x_{d-2}\,\xi^{d-2} + x$ via (3.3). This means that there is a relation

$$x' = j_*\big(x_0 + x_1\,\xi + \cdots + x_{d-2}\,\xi^{d-2}\big) + f^*(x).$$

It is enough to show $x_0 = x_1 = \cdots = x_{d-2} = 0$. Since the restriction of $\mathcal{O}_{X'}(Y')$ to Y' is isomorphic $\mathcal{O}_{Y'}(-1)$, we have $j^*\big(j_*(1_{Y'})\big) = -\xi$. Therefore,

$$\begin{aligned} j^*(x') &= j^*j_*\big(x_0 + x_1\,\xi + \cdots + x_{d-2}\,\xi^{d-2}\big) + j^*f^*(x) \\ &= -x_0\,\xi - x_1\,\xi^2 - \cdots - x_{d-2}\,\xi^{d-1} + g^*i^*(x). \end{aligned}$$

On the other hand, $j^*(x') = 0$ by assumption. Since $H^*(Y')$ is a free $H^*(Y)$-module with basis $1, \xi, \ldots, \xi^{d-1}$, we have $x_0 = x_1 = \cdots = x_{d-2} = i^*(x) = 0$. □

3.3. **Multiplicative structure of $H^*(X')$.** The multiplicative structure of $H^*(X')$ can be computed as follows. Define $\nabla^k(X')$ by

$$\nabla^k(X') = H^{k-2}(Y') \oplus H^k(X)$$

and $\cup'\colon \nabla^k(X') \times \nabla^m(X') \to \nabla^{k+m}(X')$ by

$$(y_1', x_1) \cup' (y_2', x_2) = \big(-y_1' \cup y_2' \cup \xi + y_1' \cup g^*i^*(x_2) + (-1)^{km} y_2' \cup g^*i^*(x_1),\ x_1 \cup x_2\big).$$

Then the following diagram is commutative ([SGA5], VII, 8.6.3) :

$$\begin{array}{ccc} \nabla^k(X') \times \nabla^m(X') & \xrightarrow{\mu\times\mu} & H^k(X') \times H^m(X') \\ {\scriptstyle \cup'}\downarrow & & {\scriptstyle \cup}\downarrow \\ \nabla^{k+m}(X') & \xrightarrow{\mu} & H^{k+m}(X') \end{array}$$

where $\cup$ denotes the cup product on $H^*(X')$. Since the horizontal maps $\mu, \mu \times \mu$ are surjective, we can compute the multiplicative structure of $H^*(X')$ in terms of the multiplicative structure of $\nabla^*(X') := \bigoplus_k \nabla^k(X')$.

Recall that the multiplicative structure of $H^*(Y')$ is computed by Chern classes ([SGA5], VII, 3). Namely, we have the following isomorphism of $H^*(Y)$-algebras

$$H^*(Y') \cong H^*(Y)[T]/\big(T^d + c_1(\check{N}_{Y/X})\,T^{d-1} + c_2(\check{N}_{Y/X})\,T^{d-2} + \cdots + c_d(\check{N}_{Y/X})\big).$$

which sends ξ to T.

There is another relation on $H^*(X')$ coming from the exact sequence (3.1). Namely, the image of $1_Y \in H^0(Y)$ by λ,

$$\lambda(1_Y) = \big(c_{d-1}(\check{F}),\ -i_*(1_Y)\big) \in H^{2d-2}(Y') \oplus H^{2d}(X),$$

maps to 0 in $H^{2d}(X')$. By definition, $i_*(1_Y) \in H^{2d}(X)$ is nothing but the cohomology class of Y. We denote it by $[Y]$. Since the coefficient of ξ^{d-1} in the expression of $c_{d-1}(\check{F})$ is 1 ([SGA5], VII, 8.4.2), we can replace ξ^{d-1} by $\big(\xi^{d-1} - c_{d-1}(\check{F}),\ [Y]\big)$.

In conclusion, we can compute the cup product on $H^*(X')$ in terms of the cohomology groups $H^*(X), H^*(Y)$, the Chern classes of $\check{N}_{Y/X}$, and the cohomology class $[Y] \in H^{2d}(X)$.

3.4. **Ample divisors on X'.**

Proposition 3.2. *Let notation be as in §3.1. Let L be an ample $\mathbb{R}$-divisor on X. There exists $\alpha \in \mathbb{R}_{>0}$ such that, for all $\varepsilon \in \mathbb{R}$, the $\mathbb{R}$-divisor L' on X' of the form*

$$L' = f^*\mathcal{O}(L) - \varepsilon Y'$$

is ample if and only if $0 < \varepsilon < \alpha$.

Proof. We use the following basic facts. Although some results quoted below are only stated for usual ample divisors in the literature (for example, [Hart1], [Hart2]), it is easy to prove them for ample $\mathbb{R}$-divisors.

(1) Take an effective curve C on X' which passes through a point of Y' and intersects with Y' transversally. By the Nakai-Moishezon criterion of ampleness ([Hart1], I, §5), we have $\big(f^*\mathcal{O}(L) - \varepsilon Y'\big) \cdot C > 0$. Since $Y' \cdot C > 0$, ε can't be arbitrary large.
(2) If $f^*\mathcal{O}(L) - \varepsilon Y'$ is ample, then $\varepsilon > 0$. To see this, take an effective curve C contained in a fiber of $g\colon Y' \to Y$. Since $f^*\mathcal{O}(L) \cdot C = L \cdot f_*C = 0$ and $Y' \cdot C < 0$, we have $\varepsilon > 0$ by the Nakai-Moishezon criterion of ampleness.

(3) We see that $f^*\mathcal{O}(L)-\varepsilon Y'$ is ample for sufficiently small $\varepsilon\in\mathbb{R}_{>0}$, because the restriction of $\mathcal{O}_{X'}(-Y')$ to Y' is isomorphic to $\mathcal{O}_{Y'}(1)$ ([Hart2], II, 7.10).
(4) The set $I=\big\{\varepsilon\in\mathbb{R}\mid f^*\mathcal{O}(L)-\varepsilon Y' \text{ is ample}\big\}$ is an interval, because if D,D' are ample $\mathbb{R}$-divisors, then $tD+(1-t)D'$ is also ample for all $t\in\mathbb{R}$ with $0\le t\le 1$.
(5) Moreover, I is an open interval because, for a $\mathbb{R}$-divisor D and an ample $\mathbb{R}$-divisor D', $tD+(1-t)D'$ is ample for sufficiently small $t\in\mathbb{R}_{>0}$.

By combining above facts, we conclude that $I=\{\varepsilon\in\mathbb{R}\mid 0<\varepsilon<\alpha\}$ for some $\alpha\in\mathbb{R}_{>0}$. □

Remark 3.3. Since all $\mathbb{R}$-divisors L' on X' is of the form $L'=f^*f_*\mathcal{O}(L')-\varepsilon Y'$ for some $\varepsilon\in\mathbb{R}$, Proposition 3.2 determines the set of all ample $\mathbb{R}$-divisors on X' in some sense. However, in practice, it seems difficult to compute the upper bound α in Proposition 3.2 explicitly.

3.5. Hodge standard conjecture for X'.

Proposition 3.4. *Let notation be as in §3.1 (Recall that X,Y,X',Y' satisfy Assumption 2.1 in §2.1). Let L be an ample $\mathbb{R}$-divisor on X. Assume that the hard Lefschetz conjecture (Conjecture 2.4) and the Hodge standard conjecture (Conjecture 2.6) hold for (X,L) and $\big(Y,\mathcal{O}(L)|_Y\big)$, where $\mathcal{O}(L)|_Y$ denotes the restriction of $\mathcal{O}(L)$ to Y which is also ample. Then, there exists $\beta\in\mathbb{R}_{>0}$ such that, for all $\varepsilon\in\mathbb{R}$ with $0<\varepsilon<\beta$, the $\mathbb{R}$-divisor L' on X' of the form*

$$L'=f^*\mathcal{O}(L)-\varepsilon Y'$$

is an ample $\mathbb{R}$-divisor on X' for which Conjecture 2.4 and Conjecture 2.6 hold.

Proof. First of all, by the isomorphism (3.3), we observe that $H^*(X')$ is isomorphic to $H^*(Y\times\mathbb{P}^{d-2})\oplus H^*(X)$ as an $\mathbb{R}$-*vector space*, although the multiplicative structures are different. Nevertheless, if we choose appropriate bases of $H^*(X')$ and $H^*(Y\times\mathbb{P}^{d-2})\oplus H^*(X)$, and take the limit $\varepsilon\to 0$, the matrices representing the action of the Lefschetz operator and the cup product pairings on $H^*(X')$ converge to those on $H^*(Y\times\mathbb{P}^{d-2})\oplus H^*(X)$.

Let us make the above observation precise. Let $\{x_\gamma\}$ (resp. $\{y_\delta\}$) be the basis of $H^*(X)$ (resp. $H^*(Y)$) which is independent of ε. For $\varepsilon\in\mathbb{R}_{>0}$, we see that $\big\{\varepsilon^{-(d-2)/2+k}\,y_\delta\,\xi^k\big\}$ is a ε-dependent basis of $H^*(Y)\xi^k$ for $0\le k\le d-2$. By (3.3), we have the following isomorphism :

$$H^*(X')\cong H^*(Y)\oplus H^*(Y)\xi\oplus\cdots\oplus H^*(Y)\xi^{d-2}\oplus H^*(X).$$

such that the following set

$$\big\{\varepsilon^{-(d-2)/2}\,y_\delta\big\}\cup\big\{\varepsilon^{-(d-2)/2+1}\,y_\delta\,\xi\big\}\cup\cdots\cup\big\{\varepsilon^{-(d-2)/2+(d-2)}\,y_\delta\,\xi^{d-2}\big\}\cup\{x_\gamma\}\tag{3.4}$$

is a basis of $H^*(X')$. By using the basis (3.4), we have an isomorphism

$$H^k(X')\cong H^{k-2}(Y\times\mathbb{P}^{d-2})\oplus H^k(X)\tag{3.5}$$

as an $\mathbb{R}$-*vector space* which sends $\varepsilon^{-(d-2)/2+m}\,\xi^m$ to $c_1\big(\mathcal{O}_{\mathbb{P}^{d-2}}(1)\big)^m$. The reason we multiply ξ^k by $\varepsilon^{-(d-2)/2+k}$ in (3.4) is as follows. If we take the cup product of two elements in (3.4), the coefficient of the top degree term ξ^{d-2} is a constant function in ε. If ξ^k for $k > d-2$ appears in the cup product, we can replace ξ^{d-1} by $\big(\xi^{d-1} - c_{d-1}(\check{F}),\ [Y]\big)$ in $H^*(X')$ (see §3.3). However, since the order of ε in the coefficient of ξ^{d-1} is one, this term vanishes if we take the limit $\varepsilon \to 0$. Therefore, by using the basis (3.4), we can consider the asymptotic behavior of the Lefschetz operator and the cup product pairings on $H^*(X')$ as $\varepsilon \to 0$.

We put $\widetilde{H}^k(X') = H^{k-2}(Y \times \mathbb{P}^{d-2}) \oplus H^k(X)$ and define the map

$$\widetilde{L}\colon \widetilde{H}^k(X') \longrightarrow \widetilde{H}^{k+2}(X') \tag{3.6}$$

as follows. Let $\mathrm{pr}_1\colon Y \times \mathbb{P}^{d-2} \to Y$, $\mathrm{pr}_2\colon Y \times \mathbb{P}^{d-2} \to \mathbb{P}^{d-2}$ be projections. Then

$$L_{Y\times\mathbb{P}^{d-2}} = \mathrm{pr}_1^*\big(\mathcal{O}(L)|_Y\big) + \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{d-2}}(1)$$

is an ample $\mathbb{R}$-divisor on $Y \times \mathbb{P}^{d-2}$. We define the map $\widetilde{L}$ as a direct sum of the Lefschetz operators $L_{Y\times\mathbb{P}^{d-2}}$ and L. By Example 2.9, Proposition 2.13, the hard Lefschetz conjecture (Conjecture 2.4) and the Hodge standard conjecture (Conjecture 2.6) hold for $(Y \times \mathbb{P}^{d-2},\ L_{Y\times\mathbb{P}^{d-2}})$. Therefore, we see that $\widetilde{L}^k$ induces the following isomorphism :

$$\widetilde{L}^k\colon \widetilde{H}^{n-k}(X') \xrightarrow{\ \cong\ } \widetilde{H}^{n+k}(X').$$

Namely, $\widetilde{L}$ satisfies an analogue of the hard Lefschetz conjecture (Conjecture 2.4). Moreover, an analogue of the Hodge standard conjecture (Conjecture 2.6) holds for $\widetilde{L}$ in the following sense. Let $\widetilde{P}^k(X')$ be the kernel of $\widetilde{L}^{n-k+1}$ on $\widetilde{H}^k(X')$, which is isomorphic to $P^{k-2}(Y \times \mathbb{P}^{d-2}) \oplus P^k(X)$. Then the following pairing on $\widetilde{P}^k(X')$

$$\widetilde{P}^k(X') \times \widetilde{P}^k(X') \longrightarrow \mathbb{R}, \qquad (a,b) \mapsto (-1)^{k/2}\,\widetilde{L}^{n-k}\,a \,\widetilde{\cup}\, b$$

is positive definite, where $\widetilde{\cup}$ is a difference of the cup products on $H^*(Y \times \mathbb{P}^{d-2})$ and $H^*(X)$ defined as follows :

$$\widetilde{\cup}\colon \widetilde{H}^*(X') \times \widetilde{H}^*(X') \ni \big((y_0, x_0),\ (y_1, x_1)\big) \mapsto -y_0 \cup y_1 + x_0 \cup x_1.$$

Note that the minus sign is inevitable for the Hodge standard conjecture (Conjecture 2.6) because, for $(y, x) \in \widetilde{P}^k(X')$, the degree of y is smaller than the degree of x by 2.

Therefore, to prove Proposition 3.4, it is enough to prove the following claim.

Claim 3.5. If we take the limit $\varepsilon \to 0$, the matrices representing the action of $L' = f^*\mathcal{O}(L) - \varepsilon Y'$ with respect to the basis (3.4) converge to the matrices representing $\widetilde{L}$ via (3.5). Similarly, for all k, the matrices representing the cup product pairing between $H^k(X')$ and $H^{2n-k}(X')$ converge to the matrices representing the pairing $\widetilde{\cup}$ between $\widetilde{H}^k(X')$ and $\widetilde{H}^{2n-k}(X')$.

We shall compute the action of $L' = f^*\mathcal{O}(L) - \varepsilon Y'$. Since $[Y'] = j_*(1_{Y'}) \in H^2(X')$, we compute the cup product

$$\big(f^*\mathcal{O}(L) - \varepsilon j_*(1_{Y'})\big) \cup \big(f^*(x) + j_*(y)\big) \tag{3.7}$$

for $x \in H^k(X),\ y \in H^{k-2}(Y')$ with $g_*(y) = 0$ (see (3.2)). Since (3.7) is equal to

$$f^*\mathcal{O}(L) \cup f^*(x) + f^*\mathcal{O}(L) \cup j_*(y) - \varepsilon j_*(1_{Y'}) \cup f^*(x) - \varepsilon j_*(1_{Y'}) \cup j_*(y),$$

we consider these four terms separately. Firstly, since the pullback preserves the cup product, we have $f^*\mathcal{O}(L) \cup f^*(x) = f^*\big(\mathcal{O}(L) \cup x\big) = f^*(L \cup x)$. Secondly, we have $f^*\mathcal{O}(L) \cup j_*(y) = j_*\big(j^*f^*\mathcal{O}(L) \cup y\big) = j_*\big(g^*i^*\mathcal{O}(L) \cup y\big) = j_*\big(g^*(\mathcal{O}(L)|_Y) \cup y\big)$ because $f \circ j = i \circ g$. Similarly, we have $-\varepsilon j_*(1_{Y'}) \cup f^*(x) = -\varepsilon j_*\big(1_{Y'} \cup j^*f^*(x)\big) = -\varepsilon j_*\big(g^*(x|_Y)\big)$. Finally, by the self-intersection formula $j^*j_*(1_{Y'}) = -\xi$, we have $-\varepsilon j_*(1_{Y'}) \cup j_*(y) = -\varepsilon j_*\big(j^*j_*(1_{Y'}) \cup y\big) = j_*(\varepsilon\xi \cup y)$. Therefore, we conclude that (3.7) is equal to

$$f^*(L \cup x) + j_*\Big(\big(g^*(\mathcal{O}(L)|_Y) + \varepsilon\xi\big) \cup y\Big) - \varepsilon j_*\big(g^*(x|_Y)\big).$$

It is clear that the first term $f^*(L \cup x)$ corresponds to the action of L on $H^*(X)$ via (3.5). The second term $j_*\big(\big(g^*(\mathcal{O}(L)|_Y) + \varepsilon\xi\big) \cup y\big)$ almost corresponds to the action of $L_{Y \times \mathbb{P}^{d-2}}$ on $H^*(Y \times \mathbb{P}^{d-2})$ via (3.5). The only difference is that ξ^{d-1} may appear in the cup product $\big(g^*(\mathcal{O}(L)|_Y) + \varepsilon\xi\big) \cup y$. Since the order of ε in the coefficient of ξ^{d-1} is higher than others, if we replace ξ^{d-1} by $\big(\xi^{d-1} - c_{d-1}(\check{F}),\ [Y]\big)$, this contribution vanishes if we take the limit $\varepsilon \to 0$. The contribution from the third term $-\varepsilon j_*\big(g^*(x|_Y)\big)$ vanishes if we take the limit $\varepsilon \to 0$. Hence the first assertion of Claim 3.5 is proved.

Finally, we shall compute the cup product pairing on $H^*(X')$. According to (3.2), we compute the cup product

$$\big(f^*(x_0) + j_*(y_0)\big) \cup \big(f^*(x_1) + j_*(y_1)\big) \tag{3.8}$$

for $x_0 \in H^k(X)$, $y_0 \in H^{k-2}(Y')$, $x_1 \in H^{2n-k}(X)$, $y_1 \in H^{2n-k-2}(Y')$ with $g_*(y_0) = 0,\ g_*(y_1) = 0$. Since (3.8) is in $H^{2n}(X') = \mathbb{R}$, and $f_* \colon H^{2n}(X') \to H^{2n}(X)$ is an isomorphism, it is enough to compute

$$\begin{aligned} f_*\Big(\big(f^*(x_0) + j_*(y_0)\big) \cup \big(f^*(x_1) + j_*(y_1)\big)\Big) &= f_*\big(f^*(x_0) \cup f^*(x_1)\big) \\ &\quad + f_*\big(f^*(x_0) \cup j_*(y_1)\big) + f_*\big(j_*(y_0) \cup f^*(x_1)\big) + f_*\big(j_*(y_0) \cup j_*(y_1)\big). \end{aligned}$$

For the first term, we have $f_*\big(f^*(x_0) \cup f^*(x_1)\big) = f_*\big(f^*(x_0 \cup x_1)\big) = x_0 \cup x_1$ since the pullback preserves the cup product and $f_* \circ f^* = \mathrm{id}$. For the second term, we have $f_*\big(f^*(x_0) \cup j_*(y_1)\big) = x_0 \cup f_*\big(j_*(y_1)\big) = x_0 \cup i_*\big(g_*(y_1)\big) = 0$ because $f \circ j = i \circ g$ and $g_*(y_1) = 0$. Similarly, the third term $f_*\big(j_*(y_0) \cup f^*(x_1)\big)$ is zero because $g_*(y_0) = 0$. For the last term, by the self-intersection formula $j^*j_*(y_0) = -\xi \cup y_0$, we have $f_*\big(j_*(y_0) \cup j_*(y_1)\big) = f_*\Big(j_*\big(j^*j_*(y_0) \cup y_1\big)\Big) = i_*\big(g_*(-\xi \cup y_0 \cup y_1)\big)$. Since

$g_*(\xi^{d-1}) = 1$ and $g_*(\xi^m) = 0$ for $0 \leq m \leq d-2$, if we take the limit $\varepsilon \to 0$, the matrix representing the pairing

$$H^{k-2}(Y') \times H^{2n-k-2}(Y') \ni (y_0, y_1) \mapsto i_*\big(g_*(-\xi \cup y_0 \cup y_1)\big) \in H^{2n}(X) = \mathbb{R}$$

with respect to the basis (3.4) converges to the *minus* of the matrix representing the cup product pairing

$$H^{k-2}(Y \times \mathbb{P}^{d-2}) \times H^{2n-k-2}(Y \times \mathbb{P}^{d-2}) \longrightarrow H^{2n-4}(Y \times \mathbb{P}^{d-2}) = \mathbb{R}$$

via (3.5). Therefore, the second assertion of Claim 3.5 is proved. Now the proof of Proposition 3.4 is completed. □

Corollary 3.6. *Let notation and assumptions be as in Proposition 3.4.*

1. *If the hard Lefschetz conjecture (Conjecture 2.4) holds for (X', L') for all ample $\mathbb{R}$-divisors L' on X' of the form $L' = f^*\mathcal{O}(L) - \varepsilon Y'$ ($\varepsilon \in \mathbb{R}_{>0}$), then Hodge standard conjecture (Conjecture 2.6) holds for (X', L') for all ample $\mathbb{R}$-divisors L' on X' of the form $L' = f^*\mathcal{O}(L) - \varepsilon Y'$.*
2. *In particular, if the hard Lefschetz conjecture (Conjecture 2.4) holds for (X', L') for all ample $\mathbb{R}$-divisors L' on X', then the Hodge standard conjecture (Conjecture 2.6) holds for (X', L') for all ample $\mathbb{R}$-divisors L' on X'.*

Proof. By a "homotopy argument" (Corollary 2.12), the assertion easily follows from Proposition 3.4. □

Remark 3.7. Our proof of Proposition 3.4 is based on a consideration of the asymptotic behavior as $\varepsilon \to 0$. Therefore, by our method, we can't prove whether we can take $\beta = \alpha$. In other words, here we can't prove Conjecture 2.6 holds for *all* ample $\mathbb{R}$-divisors L' on X' of the form $L' = f^*\mathcal{O}(L) - \varepsilon Y'$.

4. Some blowing-ups of projective spaces

In this section, we consider a projective smooth variety B^n of dimension n over $\overline{\mathbb{F}}_q$ which is obtained by successive blowing-ups of $\mathbb{P}^n_{\overline{\mathbb{F}}_q}$ along linear subvarieties. All varieties appearing in this section are defined over $\overline{\mathbb{F}}_q$ and satisfy Assumption 2.1 in §2.1. Hence, for simplicity, we omit the subscript "$\overline{\mathbb{F}}_q$" and denote $\mathbb{P}^n_{\overline{\mathbb{F}}_q}$ by $\mathbb{P}^n$, etc. This doesn't cause any confusion. We use the same notation as in §2.1.

4.1. **Construction of B^n.** Let $n \geq 1$ be an integer. We have a natural map $\varphi\colon \mathbb{A}^{n+1} \backslash \{0\} \to \mathbb{P}^n$. A subvariety $V \subset \mathbb{P}^n$ is called a *linear subvariety* if $\varphi^{-1}(V) \cup \{0\} \subset \mathbb{A}^{n+1}$ is a linear subspace. Let $\mathrm{Gr}_d(\mathbb{P}^n)$ be the Grassmann variety of linear subvarieties of dimension d in $\mathbb{P}^n$. Let $\mathrm{Gr}_*(\mathbb{P}^n)$ be the disjoint union of $\mathrm{Gr}_d(\mathbb{P}^n)$ for all d. $\mathrm{Gr}_d(\mathbb{P}^n)(\mathbb{F}_q)$ is the set of linear subvarieties of dimension d in $\mathbb{P}^n$ defined over $\mathbb{F}_q$.

We construct a birational map $f\colon B^n \to \mathbb{P}^n$ as follows. Firstly, we put $Y_0 := \mathbb{P}^n$. Then, let Y_1 be the blow-up of Y_0 along the disjoint union of all $\mathbb{F}_q$-rational points on $\mathbb{P}^n$. Let Y_2 be the blow-up of Y_1 along disjoint union of all proper transforms of lines in $\mathbb{P}^n$. Similarly, we construct projective smooth varieties $Y_0, \ldots, Y_{n-1}$ inductively as follows. Assume that Y_k was already constructed. Let $Z_k \subset Y_k$ be the union of all proper transforms of linear subvarieties of dimension k in $\mathbb{P}^n$ defined over $\mathbb{F}_q$. These are disjoint because all intersections of them were already blown-up. Let $g_k\colon Y_{k+1} \to Y_k$ be the blow-up of Y_k along Z_k. Finally, we put $B^n := Y_{n-1}$ and $f := g_0 \circ \cdots \circ g_{n-2}\colon B^n \to \mathbb{P}^n$. We have the following sequence of blowing-ups.

$$B^n = Y_{n-1} \xrightarrow{g_{n-2}} Y_{n-2} \xrightarrow{g_{n-3}} \cdots \xrightarrow{g_1} Y_1 \xrightarrow{g_0} Y_0 = \mathbb{P}^n$$

Note that there is a natural action of $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$ on $\mathbb{P}^n$, and the above construction is equivariant with respect to $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-action. Hence we have a natural action of $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$ on B^n.

4.2. **Divisors on B^n.** For $V \in \mathrm{Gr}_k(\mathbb{P}^n)(\mathbb{F}_q)$, we define a smooth irreducible divisor D_V on B^n as follows. If $k = n-1$, let $D_V \subset B^n$ be the proper transform of V. If $k < n-1$, the proper transform $\widetilde{V} \subset Y_k$ of V is a connected component of Z_k. Therefore, $g_k^{-1}(\widetilde{V}) \subset Y_{k+1}$ is a $\mathbb{P}^{n-d-1}$-bundle over $\widetilde{V}$. Let $D_V \subset B^n$ be the proper transform of $g_k^{-1}(\widetilde{V})$.

By construction, an $\mathbb{R}$-divisor D on B^n is written *uniquely* as

$$D = f^*\mathcal{O}(H) + \sum_{\substack{V \in \mathrm{Gr}_k(\mathbb{P}^n)(\mathbb{F}_q), \\ 0 \leq k \leq n-2}} a_V D_V \qquad \text{in} \quad H^2(B^n),$$

where H is an $\mathbb{R}$-divisor on $\mathbb{P}^n$, and $a_V \in \mathbb{R}$. We also see that, all $\mathbb{R}$-divisors D' on B^n is written as

$$D' = \sum_{V \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)} a'_V D_V \qquad (a'_V \in \mathbb{R}) \qquad \text{in} \quad H^2(B^n).$$

Note that this second expression is *not unique*.

For $0 \leq k \leq n-1$, we define a divisor D_k on B^n by

$$D_k := \sum_{V \in \mathrm{Gr}_k(\mathbb{P}^n)(\mathbb{F}_q)} D_V.$$

By construction, we see that the support of D_k is a disjoint union of smooth divisors on B^n.

In this section, we use the following terminology for $\mathbb{R}$-divisors on B^n.

Definition 4.1. Let D be an $\mathbb{R}$-divisor on B^n.

(1) D is called *positive* if D can be written as

$$D = \sum_{V \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)} a_V D_V \quad (a_V \in \mathbb{R}_{>0}) \qquad \text{in} \quad H^2(B^n).$$

(2) D is called $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-*invariant* if the class of D in $H^2(B^n)$ is $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant.

It is easy to see that D is $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant if and only if D can be written as $D = f^*\mathcal{O}(H) + \sum_{k=0}^{n-1} b_k D_k$ in $H^2(B^n)$ for some H, b_k (for a similar criterion of positive divisors, see Proposition 4.4).

4.3. **Intersections of divisors D_V on B^n.** Here we study the combinatorial structure of intersections of divisors D_V on B^n (see also [Mus], proof of Theorem 4.1, I).

Proposition 4.2. (1) *For $V, W \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)$, $D_V \cap D_W \neq \emptyset$ if and only if $V \subset W$ or $W \subset V$.*

(2) *For $V \in \mathrm{Gr}_d(\mathbb{P}^n)(\mathbb{F}_q)$, we have a noncanonical isomorphism*

$$D_V \cong B^d \times B^{n-d-1},$$

where B^d (resp. B^{n-d-1}) is a variety constructed by successive blowing-ups of $\mathbb{P}^d$ (resp. $\mathbb{P}^{n-d-1}$) by the same way as B^n.

(3) *Let $x \in V$ be a point which doesn't lie in any linear subvariety defined over $\mathbb{F}_q$ strictly contained in V. Let $\widetilde{V} \subset Y_d$ be the proper transform of V. Then $g_d \colon g_d^{-1}(\widetilde{V}) \to \widetilde{V}$ is a trivial $\mathbb{P}^{n-d-1}$-bundle. Moreover, $\widetilde{V}$ is isomorphic to B^d, and an isomorphism $D_V \cong B^d \times B^{n-d-1}$ as in 2. is induced from a choice of $x \in V$ and a trivialization $g_d^{-1}(\widetilde{V}) \cong \widetilde{V} \times \mathbb{P}(\check{N}_{V/\mathbb{P}^n,x})$, where $\check{N}_{V/\mathbb{P}^n,x}$ is the fiber at x of the conormal bundle $\check{N}_{V/\mathbb{P}^n}$.*

(4) *For $W \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)$ strictly contained in V, D_W intersects transversally with D_V, and the intersection $D_W \cap D_V$ is of the form*

$$D_W \cap D_V = \widetilde{D}_W \times B^{n-d-1} \qquad \text{on} \quad D_V,$$

where $\widetilde{D}_W$ is a divisor on B^d corresponding to the inclusion $W \hookrightarrow V \cong \mathbb{P}^d$.

(5) *For $W' \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)$ strictly containing V, $D_{W'}$ intersects transversally with D_V, and the intersection $D_{W'} \cap D_V$ is of the form*

$$D_{W'} \cap D_V = B^d \times \widetilde{\widetilde{D}}_{W'} \qquad \text{on} \quad D_V,$$

where $\widetilde{\widetilde{D}}_{W'}$ is a divisor on B^{n-d-1} corresponding to the inclusion $\mathbb{P}(\check{N}_{V/W',x}) \hookrightarrow \mathbb{P}(\check{N}_{V/\mathbb{P}^n,x}) \cong \mathbb{P}^{n-d-1}$.

Proof. Firstly, we shall prove the first assertion. If $V \subset W$ or $W \subset V$, it is clear that $D_V \cap D_W \neq \emptyset$. Assume that $V \cap W$ is strictly smaller than V, W. Then, after blowing-up along the proper transform of $V \cap W$ in the construction of B^n, the proper transforms of V and W become disjoint. Therefore, D_V doesn't intersect with D_W.

Recall that, for a vector bundle E of rank k over a smooth variety X, $\mathbb{P}(E)$ is a trivial $\mathbb{P}^{k-1}$-bundle if and only if $E \otimes_{\mathcal{O}_X} \mathcal{L}$ is a trivial vector bundle for a line bundle $\mathcal{L}$ over X ([Hart2], II, Exercise 7.10). We call such E a *twist of a trivial vector bundle* over X. For $V \in \mathrm{Gr}_d(\mathbb{P}^n)(\mathbb{F}_q)$, the normal bundle $N_{V/\mathbb{P}^n} \cong \mathcal{O}_V(1)^{\oplus n-d}$ is a twist of a trivial vector bundle over V. Hence the normal bundle $N_{\widetilde{V}/Y_d}$ is also a twist of a trivial vector bundle over Y_d, where $\widetilde{V} \subset Y_d$ is the proper transform of V ([Fu], B.6.10). Therefore, $g_d^{-1}(\widetilde{V})$ is a trivial $\mathbb{P}^{n-d-1}$-bundle over $\widetilde{V}$.

Here $\widetilde{V}$ is isomorphic to B^d. This follows from the fact that, for a sequence of regular embeddings $Z \hookrightarrow Y \hookrightarrow X$, the proper transform of Y in the blow-up of X along Z is isomorphic to the blow-up of Y along Z ([Fu], B.6.9).

We fix a point $x \in V$ which doesn't lie in any linear subvariety defined over $\mathbb{F}_q$ strictly contained in V, and a trivialization $g_d^{-1}(\widetilde{V}) \cong \widetilde{V} \times \mathbb{P}(\check{N}_{V/\mathbb{P}^n,x})$, where $\check{N}_{V/\mathbb{P}^n,x}$ denotes the fiber at x of the conormal bundle $\check{N}_{V/\mathbb{P}^n}$. For $W \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)$ strictly containing V, a natural inclusion $V \subset W \subset \mathbb{P}^n$ induces $N_{V/W} \subset N_{V/\mathbb{P}^n}$ and hence $\mathbb{P}(\check{N}_{V/W}) \subset \mathbb{P}(\check{N}_{V/\mathbb{P}^n})$. Thus, $\mathbb{P}(\check{N}_{V/W,x})$ is a linear subvariety of $\mathbb{P}(\check{N}_{V/\mathbb{P}^n,x})$. From this, we see that the blowing-up of $g_d^{-1}(\widetilde{V})$ along the proper transform of a linear subvariety strictly containing V corresponds to the blowing-up along the proper transform of a linear subvariety of the second factor $\mathbb{P}(\check{N}_{V/\mathbb{P}^n,x})$ of $g_d^{-1}(\widetilde{V}) \cong \widetilde{V} \times \mathbb{P}(\check{N}_{V/\mathbb{P}^n,x})$. Therefore, we have an isomorphism $D_V \cong B^d \times B^{n-d-1}$. The remaining assertions follow from the construction of this isomorphism. $\square$

Corollary 4.3. *Let H be an $\mathbb{R}$-divisor on $\mathbb{P}^n$, $V \in \mathrm{Gr}_d(\mathbb{P}^n)(\mathbb{F}_q)$, and $f\colon B^n \to \mathbb{P}^n$ a natural map as in §4.1. Fix an isomorphism $D_V \cong B^d \times B^{n-d-1}$ as in Proposition 4.2. Let $\mathrm{pr}_1\colon D_V \to B^d$ be the projection to the first factor. Let $f'\colon B^d \to \mathbb{P}^d \cong V$ be a map defined similarly as f. Then we have*

$$\big(f^*\mathcal{O}(H)\big)\big|_{D_V} = \mathrm{pr}_1^*\Big(f'^*\big(\mathcal{O}(H)|_V\big)\Big) \qquad \text{in} \quad H^2(D_V).$$

Proof. This follows from the construction of an isomorphism $D_V \cong B^d \times B^{n-d-1}$ in Proposition 4.2. $\square$

4.4. $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$**-invariant** $\mathbb{R}$**-divisors on** B^n**.**

Proposition 4.4. (1) *For $V \in \mathrm{Gr}_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)$, we have*

$$f^*\mathcal{O}(V) = \sum_{\substack{W \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q),\\ W \subset V}} D_W \qquad \text{in} \quad H^2(B^n).$$

(2) *For an $\mathbb{R}$-divisor H on $\mathbb{P}^n$, $f^*\mathcal{O}(H)$ is $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant.*

(3) *Let D be a $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^n which is written as*

$$D = \alpha\, f^*\mathcal{O}_{\mathbb{P}^n}(1) + \sum_{d=0}^{n-1} a_d D_d \quad (\alpha, a_d \in \mathbb{R}) \qquad \text{in} \quad H^2(B^n).$$

Then, D is positive if and only if $\alpha > 0$ and

$$a_d + \alpha \frac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|} > 0 \qquad \text{for all} \quad 0 \leq d \leq n-1.$$

(4) *Let D be a $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^n. Take $V \in \mathrm{Gr}_d(\mathbb{P}^n)(\mathbb{F}_q)$, and fix an isomorphism $D_V \cong B^d \times B^{n-d-1}$ as in Proposition 4.2. Let $\mathrm{pr}_1 \colon D_V \to B^d$, $\mathrm{pr}_2 \colon D_V \to B^{n-d-1}$ be projections. Then, there exist a $\mathrm{PGL}_{d+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor D' on B^d and a $\mathrm{PGL}_{n-d}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor D'' on B^{n-d-1} such that*

$$\mathcal{O}(D)|_{D_V} = \mathrm{pr}_1^*\mathcal{O}(D') + \mathrm{pr}_2^*\mathcal{O}(D'') \qquad \text{in} \quad H^2(D_V).$$

Proof. The first assertion follows from the fact that the multiplicity of V along $W \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)$ with $W \subset V$ is equal to 1.

The second assertion is obvious because $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$ acts trivially on $H^2(\mathbb{P}^n)$ and f^* is $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-equivariant.

We shall compute $f^*\mathcal{O}(H)$ explicitly. Since $H^2(\mathbb{P}^n)$ is generated by the class of $\mathcal{O}_{\mathbb{P}^n}(1)$, we write $H = \alpha\,\mathcal{O}_{\mathbb{P}^n}(1)$ in $H^2(\mathbb{P}^n)$ for some $\alpha \in \mathbb{R}$. Since $|\mathrm{Gr}_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)| = |\mathbb{P}^n(\mathbb{F}_q)|$, and, for each $W \in \mathrm{Gr}_d(\mathbb{P}^n)(\mathbb{F}_q)$, there exist $|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|$ elements in $\mathrm{Gr}_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)$ containing W, we have the following equality in $H^2(B^n)$:

$$\begin{aligned} f^*\mathcal{O}(H) &= \frac{\alpha}{|\mathbb{P}^n(\mathbb{F}_q)|} \sum_{V \in \mathrm{Gr}_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)} \Bigg(\sum_{\substack{W \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q), \\ W \subset V}} D_W \Bigg) \\ &= \frac{\alpha}{|\mathbb{P}^n(\mathbb{F}_q)|} \sum_{d=0}^{n-1} |\mathbb{P}^{n-d-1}(\mathbb{F}_q)|\, D_d. \end{aligned}$$

Of course, from this expression, we can check that $f^*\mathcal{O}(H)$ is $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant.

The third assertion easily follows from the above computation because

$$D = \sum_{d=0}^{n-1} \left(a_d + \alpha \frac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|} \right) D_d \qquad \text{in} \quad H^2(B^n).$$

Finally, we shall prove the last assertion. To avoid self-intersection, we write D in the following form

$$D = \alpha\, f^*\mathcal{O}_{\mathbb{P}^n}(1) + \sum_{0 \leq k \leq n-1,\ k \neq d} a_k D_k \quad (\alpha, a_k \in \mathbb{R}) \qquad \text{in} \quad H^2(B^n).$$

It is enough to treat the case $D = \alpha\, f^*\mathcal{O}_{\mathbb{P}^n}(1)$ and $D = D_k$ ($k \neq d$) separately. The case $D = \alpha\, f^*\mathcal{O}_{\mathbb{P}^n}(1)$ follows from Corollary 4.3 and the second assertion. The case $D = D_k$ ($k \neq d$) follows from Proposition 4.2. $\square$

4.5. **Ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors on B^n.** Here we study ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors on B^n. First of all, we note that there exists at least one ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^n. It follows from the construction of B^n in §4.1 and Proposition 3.2.

Proposition 4.5. *Let D be an ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^n. Then, D is positive (see Definition 4.1).*

Proof. We prove the assertion by induction on n. The case $n = 1$ is obvious. Assume that Proposition 4.5 was already proved in dimension $< n$.

Let D be written uniquely as

$$D = \alpha\, f^*\mathcal{O}_{\mathbb{P}^n}(1) + \sum_{d=0}^{n-2} a_d D_d \quad (\alpha, a_d \in \mathbb{R}) \qquad \text{in} \quad H^2(B^n).$$

By Proposition 4.4, D is positive if and only if $\alpha > 0$ and

$$a_d + \alpha\, \frac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|} > 0 \qquad \text{for all} \quad 0 \leq d \leq n-2.$$

Fix $V \in \mathrm{Gr}_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)$. By Proposition 4.2, D_V is isomorphic to B^{n-1}. We consider the restriction of $\mathcal{O}(D)$ to D_V. Let $f' \colon D_V \cong B^{n-1} \to \mathbb{P}^{n-1}$ be a map defined similarly as f. Then, by Proposition 4.2 and Corollary 4.3, we have

$$\mathcal{O}(D)|_{D_V} = \alpha\, f'^*\mathcal{O}_{\mathbb{P}^{n-1}}(1) + \sum_{d=0}^{n-2} a_d D'_d \qquad \text{in} \quad H^2(B^n),$$

where D'_d is a divisor on $D_V \cong B^{n-1}$ defined similarly as D_d on B^n.

By induction, $\mathcal{O}(D)|_{D_V}$ is positive. Therefore, by Proposition 4.4, we have $\alpha > 0$ and

$$a_d + \alpha\, \frac{|\mathbb{P}^{n-d-2}(\mathbb{F}_q)|}{|\mathbb{P}^{n-1}(\mathbb{F}_q)|} > 0 \qquad \text{for all} \quad 0 \leq d \leq n-2.$$

Therefore, it is enough to show the following inequality

$$\frac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|} > \frac{|\mathbb{P}^{n-d-2}(\mathbb{F}_q)|}{|\mathbb{P}^{n-1}(\mathbb{F}_q)|}.$$

Since $|\mathbb{P}^k(\mathbb{F}_q)| = \frac{q^{k+1}-1}{q-1}$, the above inequality is equivalent to

$$\begin{aligned}
&\frac{q^{n-d}-1}{q^{n+1}-1} > \frac{q^{n-d-1}-1}{q^n-1} \\
&\quad \iff q^{2n-d} - q^n - q^{n-d} + 1 > q^{2n-d} - q^{n+1} - q^{n-d-1} + 1 \\
&\quad \iff q^{n+1} + q^{n-d-1} > q^n + q^{n-d} \\
&\quad \iff q^n(q-1) > q^{n-d-1}(q-1).
\end{aligned}$$

Hence we prove Proposition 4.5. $\square$

4.6. Hodge standard conjecture for B^n with ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors.

Lemma 4.6. *For $0 \leq k \leq n-1$ and $a \in H^k(B^n)$, if the restriction of a to D_V is zero for all $V \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)$, then $a = 0$.*

Proof. By applying Proposition 3.1 successively, we see that a is of the form $a = f^* a'$ for some $a' \in H^k(\mathbb{P}^n)$. For $V \in \mathrm{Gr}_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)$, the restriction of a' to V is zero because the restriction of a to D_V is zero. Since restriction map $H^k(\mathbb{P}^n) \to H^k(V)$ is an isomorphism, we conclude that $a = 0$. □

The following proposition is the main result of this section.

Proposition 4.7. *Let D be an ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^n. Then the hard Lefschetz conjecture (Conjecture 2.4) and the Hodge standard conjecture (Conjecture 2.6) hold for (B^n, D).*

Proof. We prove the assertion by induction on n. The case $n = 1$ is obvious. Assume that Proposition 4.7 was already proved in dimension $< n$.

By the construction of B^n in §4.1 and Proposition 3.4, we see that there exists at least one ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^n for which Proposition 4.7 holds. If D, D' are ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors on B^n, then $tD + (1-t)D'$ is also an ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on B^n for $t \in \mathbb{R}$ with $0 \leq t \leq 1$. Therefore, by a "homotopy argument" (Corollary 2.12), it is enough to show that the hard Lefschetz conjecture (Conjecture 2.4) holds for (B^n, D) for all ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors D on B^n.

Assume that there exists an ample $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor L on B^n and a nonzero cohomology class $a \in H^k(B^n)$ for $0 \leq k \leq n-1$ such that

$$L^{n-k} \cup a = 0.$$

For $0 \leq d \leq n-1$, take $V \in \mathrm{Gr}_d(\mathbb{P}^n)(\mathbb{F}_q)$ and fix an isomorphism $D_V \cong B^d \times B^{n-d-1}$ as in Proposition 4.2. Let $\mathrm{pr}_1 \colon D_V \to B^d$, $\mathrm{pr}_2 \colon D_V \to B^{n-d-1}$ be projections. Then, by Proposition 4.4, the restriction $\mathcal{O}(L)|_{D_V}$ can be written as

$$\mathcal{O}(L)|_{D_V} = \mathrm{pr}_1^* \mathcal{O}(D') + \mathrm{pr}_2^* \mathcal{O}(D'') \qquad \text{in} \quad H^2(D_V),$$

where D' (resp. D'') is an ample $\mathrm{PGL}_{d+1}(\mathbb{F}_q)$-invariant (resp. $\mathrm{PGL}_{n-d}(\mathbb{F}_q)$-invariant) $\mathbb{R}$-divisor on B^d (resp. B^{n-d-1}). Therefore, by induction hypothesis and Proposition 2.13, the hard Lefschetz conjecture (Conjecture 2.4) and the Hodge standard conjecture (Conjecture 2.6) hold for $\big(D_V, \mathcal{O}(L)|_{D_V}\big)$.

By restricting $L^{n-k} \cup a = 0$ to D_V, we have $\big(\mathcal{O}(L)|_{D_V}\big)^{n-k} \cup (a|_{D_V}) = 0$. Since $\dim D_V = n-1$, this implies $a|_{D_V} \in H^k(D_V)$ is in the primitive part $P^k(D_V)$. Therefore, by the Hodge standard conjecture (Conjecture 2.6) for $\big(D_V, \mathcal{O}(L)|_{D_V}\big)$, we have

$$(-1)^{k/2} \cdot \big(\mathcal{O}(L)|_{D_V}\big)^{n-k-1} \cup (a|_{D_V}) \cup (a|_{D_V}) \geq 0$$

and the equality holds if and only if $a|_{D_V} = 0$.

By Proposition 4.5, L can be written as

$$L = \sum_{V \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)} a_V D_V \quad (a_V \in \mathbb{R}_{>0}) \qquad \text{in} \quad H^2(B^n).$$

Therefore, we have

$$\begin{aligned}
&(-1)^{k/2} \cdot L^{n-k} \cup a \cup a \\
&= \sum_{V \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)} a_V \cdot D_V \cup \big((-1)^{k/2} \cdot L^{n-k-1} \cup a \cup a\big) \\
&= \sum_{V \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)} a_V \Big((-1)^{k/2} \cdot \big(\mathcal{O}(L)|_{D_V}\big)^{n-k-1} \cup (a|_{D_V}) \cup (a|_{D_V})\Big) \\
&= 0
\end{aligned}$$

because $L^{n-k} \cup a = 0$. Since $a_V > 0$ for all $V \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)$, we have $a|_{D_V} = 0$ for all $V \in \mathrm{Gr}_*(\mathbb{P}^n)(\mathbb{F}_q)$. By Lemma 4.6, we have $a = 0$. This is a contradiction. Hence the proof of Proposition 4.7 is completed. $\square$

Remark 4.8. Mustafin, Kurihara showed that B^d is isomorphic to all irreducible components of the special fiber of the formal scheme model $\widehat{\Omega}^d_{\mathcal{O}_K}$ over $\mathrm{Spf}\,\mathcal{O}_K$ of the Drinfeld upper half space $\widehat{\Omega}^d_K$ of dimension d (for details, see §6, see also [Mus], [Ku]). Moreover, they also showed that

$$D = -(d+1) f^* \mathcal{O}_{\mathbb{P}^d}(1) + \sum_{k=0}^{d-1} (d-k) D_k$$

is an ample divisor on B^d, and D coincides with the restriction of the "relative dualizing sheaf" $\omega_{\widehat{\Omega}^d_{\mathcal{O}_K}/\mathcal{O}_K}$ to an irreducible component of the special fiber. These facts are crucial in our application to p-adic uniformization in §6.

Remark 4.9. In our proof of Proposition 4.5, we heavily use the assumption that D is $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant. Therefore, our proof doesn't work for ample non-$\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors on B^n. It is easy to see that if all ample $\mathbb{R}$-divisors on B^n are positive, then Proposition 4.7 holds for them. However, the author doesn't even know whether it is natural to expect this. He has neither evidence nor counter-example for ample non-$\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors.

5. Review of the weight spectral sequence of Rapoport-Zink

In this section, we recall the definitions and basic properties of monodromy filtration, weight filtration, and the weight spectral sequence of Rapoport-Zink. Since we work over a local field in this section, we slightly change the notation. As in §1, let K be a complete discrete valuation field with finite residue field $\mathbb{F}_q$ of characteristic $p > 0$, l be a prime number different from p. Let X be a proper smooth variety of dimension n over K, and $V := H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$.

5.1. **Monodromy filtration.** Let I_K be the inertia group of K, which is a subgroup of $\mathrm{Gal}(\overline{K}/K)$ defined by the exact sequence :

$$1 \longrightarrow I_K \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1.$$

$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts on I_K by conjugation ($\tau \colon \sigma \mapsto \tau\sigma\tau^{-1}$, $\tau \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, $\sigma \in I_K$). The pro-l-part of I_K is isomorphic to $\mathbb{Z}_l(1)$ as a $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-module by

$$t_l \colon I_K \ni \sigma \mapsto \left(\frac{\sigma(\pi^{1/l^m})}{\pi^{1/l^m}} \right)_m \in \varprojlim \mu_{l^m} = \mathbb{Z}_l(1),$$

where π is a uniformizer of K, and μ_{l^m} is the group of l^m-th roots of unity. It is known that t_l is independent of the choice of π and its l^m-th root π^{1/l^m} ([Se1]).

By Grothendieck's monodromy theorem ([ST], Appendix), there exist $r, s \geq 1$ such that $\big(\rho(\sigma)^r - 1\big)^s = 0$ for all $\sigma \in I_K$. Therefore, by replacing K by its finite extension, I_K acts on V through $t_l \colon I_K \to \mathbb{Z}_l(1)$ and this action is unipotent. Then there is a unique nilpotent map of $\mathrm{Gal}(\overline{K}/K)$-representations called the *monodromy operator* $N \colon V(1) \to V$ such that

$$\rho(\sigma) = \exp\big(t_l(\sigma)N\big) \quad \text{for all} \quad \sigma \in I_K.$$

Here N is a nilpotent map means that $N^r \colon V(r) \to V$ is zero for some $r \geq 1$.

Definition 5.1. ([De3], I, 1.7.2) There exists a unique filtration $M_\bullet$ on V called the *monodromy filtration* characterized by the following properties.

(1) $M_\bullet$ is an increasing filtration $\cdots \subset M_{i-1}V \subset M_iV \subset M_{i+1}V \subset \cdots$ of $\mathrm{Gal}(\overline{K}/K)$-representations such that $M_iV = 0$ for sufficiently small i and $M_iV = V$ for sufficiently large i.
(2) $N\big(M_iV(1)\big) \subset M_{i-2}V$ for all i.
(3) By the second condition, we can define $N \colon \mathrm{Gr}_i^M V(1) \to \mathrm{Gr}_{i-2}^M V$, where $\mathrm{Gr}_i^M V := M_iV/M_{i-1}V$. Then, for each $r \geq 0$, $N^r \colon \mathrm{Gr}_r^M V(r) \to \mathrm{Gr}_{-r}^M V$ is an isomorphism.

Remark 5.2. In the above definition, we replace K by its finite extension. We can easily see that $M_\bullet$ is stable under the action of $\mathrm{Gal}(\overline{K}/K)$ for the original K. Therefore, we can define the monodromy filtration $M_\bullet$ as a filtration of $\mathrm{Gal}(\overline{K}/K)$-representations without replacing K by its finite extension.

5.2. **Weight filtration.** Let $\mathrm{Fr}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the inverse of the q-th power map on $\overline{\mathbb{F}}_q$ called the *geometric Frobenius element.* A $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representation is said to have *weight* k if all eigenvalues of the action of $\mathrm{Fr}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ are algebraic integers whose all complex conjugates have complex absolute value $q^{k/2}$.

Definition 5.3. ([De1], [De3], I, 1.7.5) There exists a unique filtration $W_\bullet$ called the *weight filtration* on V characterized by the following properties (for existence, see §5.3).

(1) $W_\bullet$ is an increasing filtration $\cdots \subset W_{i-1}V \subset W_iV \subset W_{i+1}V \subset \cdots$ of $\mathrm{Gal}(\overline{K}/K)$-representations such that $W_iV = 0$ for sufficiently small i and $W_iV = V$ for sufficiently large i.
(2) For a lift $\widetilde{\mathrm{Fr}}_q$ of Fr_q in $\mathrm{Gal}(\overline{K}/K)$, all eigenvalues of the action of $\widetilde{\mathrm{Fr}}_q$ on each $\mathrm{Gr}_i^W V := W_iV/W_{i-1}V$ are algebraic integers whose all complex conjugates have complex absolute value $q^{i/2}$.

Remark 5.4. By the second condition, the monodromy operator $N\colon \mathrm{Gr}_i^W V(1) \to \mathrm{Gr}_i^W V$ is zero for each i (see §5.1). Hence, by replacing K by its finite extension K' with residue field $\mathbb{F}_{q^r}$, $I_{K'}$ acts on $\mathrm{Gr}_i^W V$ trivially, and $\mathrm{Gal}(\overline{\mathbb{F}}_{q^r}/\mathbb{F}_{q^r})$-representation $\mathrm{Gr}_i^W V$ have weight i for each i.

5.3. Weight spectral sequence of Rapoport-Zink.

Definition 5.5. A proper scheme $\mathfrak{X}$ over $\mathcal{O}_K$ is called a *proper semistable model* of X over $\mathcal{O}_K$ if the following conditions are satisfied. $\mathfrak{X}$ has an open covering $\{U_i\}$ consisting of finite open sets of $\mathfrak{X}$ such that, for each i, there exists $r_i \geq 1$ and a commutative diagram

$$\begin{tikzcd} U_i \arrow[r, "f_i"] \arrow[d] & \operatorname{Spec} \mathcal{O}_K[T_0, \ldots, T_n]/(T_1 \cdots T_{r_i} - \pi) \arrow[dl] \\ \operatorname{Spec} \mathcal{O}_K & \end{tikzcd}$$

where f_i is an étale morphism. Moreover, if every irreducible component of the special fiber of $\mathfrak{X}$ is smooth over F, $\mathcal{O}_K$ is called a *proper strictly semistable model* of X over $\mathcal{O}_K$.

We recall the weight spectral sequence of Rapoport-Zink ([RZ1]). Assume that X has a proper strictly semistable model $\mathfrak{X}$ over $\mathcal{O}_K$. Let $X_1, \ldots, X_m$ be the irreducible components of the special fiber of $\mathfrak{X}$, and

$$X^{(k)} := \coprod_{1 \leq i_1 < \cdots < i_k \leq m} X_{i_1} \cap \cdots \cap X_{i_k}.$$

Then $X^{(k)}$ is a proper smooth variety of dimension $n-k+1$ over F. The *weight spectral sequence of Rapoport-Zink* is as follows :

$$E_1^{-r,w+r} = \bigoplus_{k \geq \max\{0,-r\}} H_{\text{ét}}^{w-r-2k}\big(X^{(2k+r+1)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(-r-k)\big) \Longrightarrow H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l).$$

This spectral sequence is $\mathrm{Gal}(\overline{K}/K)$-equivariant. The map $d_1^{i,j}\colon E_1^{i,j} \to E_1^{i+1,j}$ can be described by restriction morphisms and Gysin morphisms explicitly (see [RZ1], 2.10 for details).

The action of the monodromy operator N on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ in §5.1 is induced from a natural map $N\colon E_1^{i,j}(1) \to E_1^{i+2,j-2}$ satisfying

$$N^r\colon E_1^{-r,w+r}(r) \xrightarrow{\cong} E_1^{r,w-r}$$

for all r, w. We can describe $N\colon E_1^{i,j}(1) \to E_1^{i+2,j-2}$ explicitly ([RZ1], 2.10). Hence we can define N without replacing K by its finite extension (see Remark 5.2).

I_K acts on each $E_1^{i,j}$ trivially and $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts on them. By the Weil conjecture ([De2], [De3]), $H^{w-r-2k}_{\text{ét}}\big(X^{(2k+r+1)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(-r-k)\big)$ has weight $(w-r-2k)-2(-r-k) = w+r$ (see §5.2). Hence $E_1^{i,j}$ has weight j. Therefore, the filtration on $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ induced by the weight spectral sequence is the weight filtration $W_\bullet$ in Definition 5.3. This proves the existence of $W_\bullet$ in Definition 5.3 (for general X, we may use de Jong's alteration [dJ]). Moreover, since $d_r^{i,j}\colon E_r^{i,j} \to E_r^{i+r,j-r+1}$ is a map between $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representations with different weights for $r \geq 2$, $d_r^{i,j} = 0$ for $r \geq 2$. Therefore, the weight spectral sequence degenerates at E_2.

By combining above facts, we see that the weight-monodromy conjecture (Conjecture 1.1) is equivalent to the following conjecture on the weight spectral sequence of Rapoport-Zink.

Conjecture 5.6 ([RZ1], [Il1], [Il2]). *Let X be a proper smooth variety over K which has a proper strictly semistable model $\mathfrak{X}$ over $\mathcal{O}_K$. Let $E_1^{-r,w+r} \Rightarrow H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ be the weight spectral sequence of Rapoport-Zink. Then N^r induces an isomorphism*

$$N^r\colon E_2^{-r,w+r}(r) \xrightarrow{\cong} E_2^{r,w-r}$$

on E_2-terms for all r, w.

5.4. An argument of Steenbrink, M. Saito.

Proposition 5.7. *Let X be a proper smooth variety over K which has a proper strictly semistable model $\mathfrak{X}$ over $\mathcal{O}_K$. Let $X_1, \ldots, X_m$ be the irreducible components of the special fiber of $\mathfrak{X}$. Assume that $\mathfrak{X}$ is projective over $\mathcal{O}_K$ with an ample line bundle $\mathcal{L}$. Assume further that, for $1 \leq i_1 < \cdots < i_k \leq m$, each irreducible component Y of $X_{i_1} \cap \cdots \cap X_{i_k}$ satisfies Assumption 2.1 in §2.1, and the Hodge standard conjecture (Conjecture 2.6) holds for $(Y, \mathcal{L}|_Y)$. Then, Conjecture 5.6 holds for $\mathfrak{X}$, and Conjecture 1.1 holds for X.*

The idea of the proof here is essentially the same as in [SaM], 4.2.5, where M. Saito used polarized Hodge structures to prove a Hodge analogue of Conjecture 5.6. The only difference is that we use $\mathbb{Q}$-structures on l-adic cohomology and the Hodge standard conjecture instead of the polarized Hodge structures (A similar argument can be found in [It2]).

Since the presentation of M. Saito in [SaM] is too sophisticated, it isn't very clear to non-specialists that their argument can be applied to the situation in Proposition 5.7 even in positive or mixed characteristic. So we reproduce their argument here with slight modification suitable for Proposition 5.7 for reader's convenience. Therefore, those who are familiar with their argument may skip to the next section. In the followings, the proof of Claim 5.13 is the only part where we crucially use the assumptions of Proposition 5.7. In [SaM], M. Saito proved a corresponding statement by using polarized Hodge structures.

First of all, we recall the structure of the E_1-terms of the weight spectral sequence of Rapoport-Zink. Let

$$d_1^{-r,\,w+r} = \sum_{k\geq \max\{0,-r\}} (\rho+\tau)\colon E_1^{-r,\,w+r} \longrightarrow E_1^{-r+1,\,w+r}$$

denote the differential on E_1-terms, where

$$\rho\colon H^s_{\text{ét}}(X^{(t)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) \longrightarrow H^s_{\text{ét}}(X^{(t+1)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$$

is a linear combination of restriction morphisms for some s, t (ρ is $(-1)^{r+k}\theta$ in [RZ1], 2.10), and

$$\tau\colon H^s_{\text{ét}}(X^{(t)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) \longrightarrow H^{s+2}_{\text{ét}}(X^{(t-1)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(1))$$

is a linear combination of Gysin morphisms for some s, t (τ is $(-1)^k d_1'$ in [RZ1], 2.10). From the construction, ρ increases the index k by 1, and τ preserves the index k. Moreover, ρ, τ satisfy $\rho\circ\rho = 0,\ \tau\circ\tau = 0,\ \tau\circ\rho+\rho\circ\tau = 0$. Hence ρ, τ also satisfy

$$\rho\circ\tau\circ\rho = -\rho\circ\rho\circ\tau = 0, \qquad \tau\circ\rho\circ\tau = -\tau\circ\tau\circ\rho = 0.$$

Let k be an integer with $1\leq k\leq n$. To distinguish ρ, τ for different degree, we use the following notation.

$$\begin{aligned} \rho_i^{(k)}\colon H^i_{\text{ét}}(X^{(k)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) &\longrightarrow H^i_{\text{ét}}(X^{(k+1)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l), \\ \tau_i^{(k+1)}\colon H^i_{\text{ét}}(X^{(k+1)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) &\longrightarrow H^{i+2}_{\text{ét}}(X^{(k)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(1)). \end{aligned}$$

In the followings, to simply the notation, we fix an isomorphism $\mathbb{Q}_l \cong \mathbb{Q}_l(1)$ to ignore the Tate twists, and simply write $H^i_{\text{ét}}(X^{(k)}) := H^i_{\text{ét}}(X^{(k)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$, where we add a subscript "ét" not to confuse with the notation $H^i(X)$ in §2.1.

Let $L\colon H^i_{\text{ét}}(X^{(k)}) \to H^{i+2}_{\text{ét}}(X^{(k)})$ be the Lefschetz operator defined by $\mathcal{L}$. Since L commutes with ρ, τ, we use the same letter L for different i, k, which doesn't cause any confusion. Let

$$H^i_{\text{ét}}(X^{(k)}) \times H^{2(n-k+1)-i}_{\text{ét}}(X^{(k)}) \to \mathbb{Q}_l, \quad H^i_{\text{ét}}(X^{(k+1)}) \times H^{2(n-k)-i}_{\text{ét}}(X^{(k+1)}) \to \mathbb{Q}_l$$

be the sum of the cup product pairings. For simplicity, we denote them by $\cup$. Then, for $a\in H^i_{\text{ét}}(X^{(k)}),\ b\in H^{2(n-k)-i}_{\text{ét}}(X^{(k+1)})$, we have

$$a\cup \tau^{(k+1)}_{2(n-k)-i}(b) = \rho_i^{(k)}(a)\cup b. \tag{5.1}$$

Namely, $\rho_i^{(k)}$ and $\tau^{(k+1)}_{2(n-k)-i}$ are dual to each other with respect to $\cup$.

Claim 5.8. In the following sequence of ρ, τ

$$H^{i-2}_{\text{ét}}(X^{(k+1)}) \xrightarrow{\tau^{(k+1)}_{i-2}} H^{i}_{\text{ét}}(X^{(k)}) \xrightarrow{\rho^{(k)}_{i}} H^{i}_{\text{ét}}(X^{(k+1)}) \xrightarrow{\tau^{(k+1)}_{i}} H^{i+2}_{\text{ét}}(X^{(k)}),$$

we have $\left(\operatorname{Ker}\tau^{(k+1)}_{i}\right) \cap \left(\operatorname{Im}\rho^{(k)}_{i}\right) = \operatorname{Im}\left(\rho^{(k)}_{i} \circ \tau^{(k+1)}_{i-2}\right)$. Moreover, in the following sequence of ρ, τ

$$H^{i}_{\text{ét}}(X^{(k)}) \xrightarrow{\rho^{(k)}_{i}} H^{i}_{\text{ét}}(X^{(k+1)}) \xrightarrow{\tau^{(k+1)}_{i}} H^{i+2}_{\text{ét}}(X^{(k)}) \xrightarrow{\rho^{(k)}_{i+2}} H^{i+2}_{\text{ét}}(X^{(k+1)}),$$

we have $\left(\operatorname{Ker}\rho^{(k)}_{i+2}\right) \cap \left(\operatorname{Im}\tau^{(k+1)}_{i}\right) = \operatorname{Im}\left(\tau^{(k+1)}_{i} \circ \rho^{(k)}_{i}\right)$.

By assuming the above claim, it is easy to prove Conjecture 5.6.

Proof of Proposition 5.7 assuming Claim 5.8. For $r \geq 1$, we write $E^{-r,\,w+r}_1$, $E^{r,\,w-r}_1$ explicitly as follows (see §5.3)

$$E^{-r,\,w+r}_1 = \bigoplus_{k\geq 0} H^{w-r-2k}_{\text{ét}}(X^{(2k+r+1)}), \qquad E^{r,\,w-r}_1 = \bigoplus_{k\geq r} H^{w+r-2k}_{\text{ét}}(X^{(2k-r+1)}).$$

Put $k' = k - r$ and rewrite $E^{r,\,w-r}_1$ as

$$E^{r,\,w-r}_1 = \bigoplus_{k'\geq 0} H^{w-r-2k'}_{\text{ét}}(X^{(2k'+r+1)}).$$

From this expression, there is a natural isomorphism between $E^{-r,\,w+r}_1$ and $E^{r,\,w-r}_1$, which is nothing but an isomorphism induced by N^r in §5.3. Now, we look at the following part of the weight spectral sequence of Rapoport-Zink :

$$\begin{tikzcd} E^{-r-1,\,w+r}_1 \arrow[r] & E^{-r,\,w+r}_1 \arrow[r] \arrow[drr, "N^r"] & E^{-r+1,\,w+r}_1 & & \\ & & E^{r-1,\,w-r}_1 \arrow[r] & E^{r,\,w-r}_1 \arrow[r] & E^{r+1,\,w-r}_1. \end{tikzcd} \tag{5.2}$$

Since we have

$$E^{-r-1,\,w+r}_1 = \bigoplus_{k\geq 0} H^{w-r-2k-2}_{\text{ét}}(X^{(2k+r+2)}),$$

$$E^{-r+1,\,w+r}_1 = \bigoplus_{k\geq 0} H^{w-r-2k+2}_{\text{ét}}(X^{(2k+r)}),$$

$$E^{r-1,\,w-r}_1 = \bigoplus_{k\geq r-1} H^{w+r-2k-2}_{\text{ét}}(X^{(2k-r+2)}) = \bigoplus_{k'\geq -1} H^{w-r-2k'-2}_{\text{ét}}(X^{(2k'+r+2)}),$$

$$E^{r+1,\,w-r}_1 = \bigoplus_{k\geq r+1} H^{w+r-2k+2}_{\text{ét}}(X^{(2k-r)}) = \bigoplus_{k'\geq 1} H^{w-r-2k'+2}_{\text{ét}}(X^{(2k'+r)}),$$

we put

$$
\begin{aligned}
A &= \bigoplus_{k\geq 0} H^{w-r-2k-2}_{\text{ét}}(X^{(2k+r+2)}),\\
B &= \bigoplus_{k\geq 1} H^{w-r-2k}_{\text{ét}}(X^{(2k+r+1)}), \quad B' = H^{w-r}_{\text{ét}}(X^{(r+1)}),\\
C &= \bigoplus_{k\geq 1} H^{w-r-2k+2}_{\text{ét}}(X^{(2k+r)}), \quad D = H^{w-r+2}_{\text{ét}}(X^{(r)}), \quad E = H^{w-r}_{\text{ét}}(X^{(r)}).
\end{aligned}
$$

Then, (5.2) becomes

$$
\begin{tikzcd}
A \arrow[r, "d+d'"] & B\oplus B' \arrow[r, "{(e+0,\, f+f')}"] \arrow[dr] & C\oplus D & \\
& A\oplus E \arrow[r, "{(d+d',\, 0+g)}"'] & B\oplus B' \arrow[r, "{(e,f)}"'] & C
\end{tikzcd}
\tag{5.3}
$$

for some $d\colon A\to B,\ d'\colon A\to B',\ e\colon B\to C,\ f\colon B'\to C,\ f'\colon B'\to D,\ g\colon E\to B'$, where $B\oplus B'\to B\oplus B'$ is the identity map. Recall that ρ increases the index k by 1, and τ preserves the index k.

It is enough to show that the identity map $B\oplus B'\to B\oplus B'$ induces an isomorphism between the cohomologies the first and second rows of (5.3).

We shall prove the injectivity. Take an element $(b,b')\in B\oplus B'$ such that $e(b)+f(b')=0,\ f'(b')=0$. Assume that $(b+b')\in \operatorname{Im}(d+d',0+g)$. Namely, there exists $(x,y)\in A\oplus E$ such that $d(x)=b,\ d'(x)+g(y)=b'$. Then we have

$$
\begin{aligned}
e(b)+f(b') &= e\circ d(x)+f\circ d'(x)+f\circ g(y)=0\\
f'(b') &= f'\circ d'(x)+f'\circ g(y)=0.
\end{aligned}
$$

From the definition, we see that $d'=\tau^{(r+2)}_{w-r-2},\ f'=\tau^{(r+1)}_{w-r},\ g=\rho^{(r)}_{w-r}$. Hence $f'\circ d'=\tau^{(r+1)}_{w-r}\circ\tau^{(r+2)}_{w-r-2}=0$. Therefore, we have

$$f'\circ g(y)=\tau^{(r+1)}_{w-r}\circ\rho^{(r)}_{w-r}(y)=0.$$

By the first assertion of Claim 5.8, there exists $z\in H^{w-r-2}_{\text{ét}}(X^{(r+1)})$ such that

$$g(y)=\rho^{(r)}_{w-r}(y)=\rho^{(r)}_{w-r}\circ\tau^{(r+1)}_{w-r-2}(z).$$

Since $\rho^{(r)}_{w-r}\circ\tau^{(r+1)}_{w-r-2}(z)=-\tau^{(r+2)}_{w-r-2}\circ\rho^{(r+1)}_{w-r-2}(z)$, we have

$$b'=d'(x)+g(y)=\tau^{(r+2)}_{w-r-2}(x)-\tau^{(r+2)}_{w-r-2}\circ\rho^{(r+1)}_{w-r-2}(z)=\tau^{(r+2)}_{w-r-2}\big(x-\rho^{(r+1)}_{w-r-2}(z)\big).$$

On the other hand, the restriction of $d\colon A\to B$ to the $k=0$ part $H^{w-r-2}_{\text{ét}}(X^{(r+2)})$ is $\rho^{(r+2)}_{w-r-2}$. Then, we have

$$d\left(x-\rho^{(r+1)}_{w-r-2}(z)\right)=d(x)-\rho^{(r+2)}_{w-r-2}\circ\rho^{(r+1)}_{w-r-2}(z)=d(x)=b,$$

since $\rho^{(r+2)}_{w-r-2}\circ\rho^{(r+1)}_{w-r-2}=0$. Hence (b,b') is the image of $x-\rho^{(r+1)}_{w-r-2}(z)$ by $d+d'$, i.e. $(b,b')\in\operatorname{Im}(d+d')$. This proves the injectivity.

Finally, we shall prove the surjectivity. Take an element $(b, b') \in B \oplus B'$ with $e(b)+f(b') = 0$. From the construction, we see that $f = \rho^{(r+1)}_{w-r}$ and the restriction of $e\colon B \to C$ to the $k = 1$ part $H^{w-r-2}_{\text{ét}}(X^{(r+3)})$ is $\tau^{(r+3)}_{w-r-2}$. Therefore, the equality $e(b) + f(b') = 0$ implies $\tau^{(r+3)}_{w-r-2}(\tilde{b}) + \rho^{(r+1)}_{w-r}(b') = 0$ in the $k = 1$ part $H^{w-r}_{\text{ét}}(X^{(r+2)})$ of C, where $\tilde{b}$ is the $k = 1$ part of b. Hence we have

$$\rho^{(r)}_{w-r+2} \circ \tau^{(r+1)}_{w-r}(b') = -\tau^{(r+2)}_{w-r} \circ \rho^{(r+1)}_{w-r}(b') = \tau^{(r+2)}_{w-r} \circ \tau^{(r+3)}_{w-r-2}(\tilde{b}) = 0,$$

since $\rho^{(r)}_{w-r+2} \circ \tau^{(r+1)}_{w-r} = -\tau^{(r+2)}_{w-r} \circ \rho^{(r+1)}_{w-r}$ and $\tau^{(r+2)}_{w-r} \circ \tau^{(r+3)}_{w-r-2} = 0$. By the second assertion of Claim 5.8, there exists $a \in H^{w-r}_{\text{ét}}(X^{(r)}) = E$ such that

$$\tau^{(r+1)}_{w-r}(b') = \tau^{(r+1)}_{w-r} \circ \rho^{(r)}_{w-r}(a).$$

Consider the element $(b, b' - g(a)) \in B \oplus B'$ which is in the same class as (b, b') in the cohomology of the second row. Then, we have

$$f'(b' - g(a)) = \tau^{(r+1)}_{w-r}(b') - \tau^{(r+1)}_{w-r} \circ \rho^{(r)}_{w-r}(a) = 0,$$

hence $(b, b' - g(a)) \in \operatorname{Ker}(e + 0,\ f + f)$. This proves the surjectivity. □

Therefore, it is enough to prove Claim 5.8. First, recall the primitive decomposition of $H^i_{\text{ét}}(X^{(k)})$:

$$H^i_{\text{ét}}(X^{(k)}) = \bigoplus_{j \geq 0} L^j P^{i-2j}_{\text{ét}}(X^{(k)}),$$

where $P^{i-2j}_{\text{ét}}(X^{(k)})$ is the primitive part of $H^{i-2j}_{\text{ét}}(X^{(k)})$ as in §2.2. Recall that $\dim X^{(k)} = n-k+1$. The idea of our proof of Claim 5.8 is to analyze the primitive decomposition carefully. Since ρ, τ don't preserve the primitive decomposition, for $0 \leq i \leq \dim X^{(k+1)} = n - k$, we put

$$P^i \operatorname{Im}^0 \rho^{(k)}_i := \big(\operatorname{Im} \rho^{(k)}_i\big) \cap P^i_{\text{ét}}(X^{(k+1)}),$$

and $P^i \operatorname{Im}^0 \rho^{(k)}_i = 0$ otherwise. Then, we define

$$\operatorname{Im}^0 \rho^{(k)}_i := \bigoplus_{j \geq 0} L^j P^{i-2j} \operatorname{Im}^0 \rho_{i-2j}, \quad \operatorname{Im}^1 \rho^{(k)}_i := \operatorname{Im} \rho^{(k)}_i / \operatorname{Im}^0 \rho^{(k)}_i.$$

Roughly speaking, $\operatorname{Im}^0 \rho^{(k)}_i$ is a part which has a decomposition according to the primitive decomposition of $H^i_{\text{ét}}(X^{(k+1)})$, and $\operatorname{Im}^1 \rho^{(k)}_i$ is a part which doesn't have any decomposition compatible with the primitive decomposition of $H^i_{\text{ét}}(X^{(k+1)})$. Similarly for $\operatorname{Im} \tau^{(k+1)}_i$, for $0 \leq i \leq \dim X^{(k)} - 2 = n - k - 1$, we put

$$P^{i+2} \operatorname{Im}^0 \tau^{(k+1)}_i := \operatorname{Im} \tau^{(k+1)}_i \cap P^{i+2}_{\text{ét}}(X^{(k)})$$

and $P^{i+2} \operatorname{Im}^0 \tau^{(k+1)}_i = 0$ otherwise. Then, we define

$$\operatorname{Im}^0 \tau^{(k+1)}_i := \bigoplus_{j \geq 0} L^j P^{i+2-2j} \operatorname{Im}^0 \tau^{(k+1)}_{i-2j}, \quad \operatorname{Im}^1 \tau^{(k+1)}_i := \operatorname{Im} \tau^{(k+1)}_i / \operatorname{Im}^0 \tau^{(k+1)}_i.$$

Claim 5.9 (Hard Lefschetz theorem for Im^0)**.** We have the primitive decomposition for Im^0 in the following sense :

$$\begin{aligned}\mathrm{Im}^0\rho_i^{(k)} &= \bigoplus_{j\geq 0}\big(L^j P_{\text{ét}}^{i-2j}(X^{(k+1)})\cap \mathrm{Im}^0\rho_i^{(k)}\big),\\ \mathrm{Im}^0\tau_i^{(k+1)} &= \bigoplus_{j\geq 0}\big(L^j P_{\text{ét}}^{i-2j}(X^{(k)})\cap \mathrm{Im}^0\tau_i^{(k+1)}\big).\end{aligned}$$

Hence we have an analogue of the hard Lefschetz theorem for Im^0 in the following form :

$$\begin{aligned}L^{\dim X^{(k+1)}-i} = L^{(n-k)-i}\colon \mathrm{Im}^0\rho_i^{(k)} &\xrightarrow{\cong} \mathrm{Im}^0\rho_{2(n-k)-i}^{(k)},\\ L^{\dim X^{(k)}-(i+2)} = L^{(n-k+1)-(i+2)}\colon \mathrm{Im}^0\tau_i^{(k+1)} &\xrightarrow{\cong} \mathrm{Im}^0\tau_{2(n-k+1)-i-4}^{(k+1)}.\end{aligned}$$

Proof of Claim 5.9. These immediately follows from the definition of $\mathrm{Im}^0\rho_i^{(k)}$, $\mathrm{Im}^0\tau_i^{(k+1)}$. □

Claim 5.10 (Hard Lefschetz theorem for Im^1)**.** We also have an analogue of the hard Lefschetz theorem for Im^1 in the following sense :

$$\begin{aligned}L^{\dim X^{(k+1)}+1-i} = L^{(n-k)+1-i}\colon \mathrm{Im}^1\rho_i^{(k)} &\xrightarrow{\cong} \mathrm{Im}^1\rho_{2((n-k)+1)-i}^{(k)},\\ L^{\dim X^{(k)}+1-(i+2)} = L^{(n-k+1)+1-(i+2)}\colon \mathrm{Im}^1\tau_i^{(k+1)} &\xrightarrow{\cong} \mathrm{Im}^1\tau_{2((n-k+1)+1)-(i+4)}^{(k+1)}.\end{aligned}$$

Proof of Claim 5.10. The surjectivity follows from the hard Lefschetz theorem. The injectivity follows from the surjectivity of the maps

$$\begin{aligned}L^{(n-k)+1-i}\colon \mathrm{Im}^0\rho_i^{(k)} &\longrightarrow \mathrm{Im}^0\rho_{2((n-k)+1)-i}^{(k)}\\ L^{(n-k+1)+1-(i+2)}\colon \mathrm{Im}^0\tau_i^{(k+1)} &\longrightarrow \mathrm{Im}^0\tau_{2((n-k+1)+1)-(i+4)}^{(k+1)},\end{aligned}$$

which follows from Claim 5.9. □

Remark 5.11. The primitive decomposition of $\mathrm{Im}^0\rho_i^{(k)}$ (resp. $\mathrm{Im}^0\tau_i^{(k+1)}$) is the same as that of $H_{\text{ét}}^i(X^{(k+1)})$ (resp. $H_{\text{ét}}^{i+2}(X^{(k)})$). On the other hand, the primitive decomposition of $\mathrm{Im}^1\rho_i^{(k)}$ (resp. $\mathrm{Im}^1\tau_i^{(k+1)}$) is the same as that of $H_{\text{ét}}^i(X^{(k)})$ (resp. $H_{\text{ét}}^i(X^{(k+1)})$). Hence the center of symmetry of Im^1 differs by 1 from that of Im^0.

Claim 5.12. For all i, we have the following equality of dimensions :

$$\dim_{\mathbb{Q}_l}\mathrm{Im}^0\rho_i^{(k)} = \dim_{\mathbb{Q}_l}\mathrm{Im}^1\tau_i^{(k+1)}, \qquad \dim_{\mathbb{Q}_l}\mathrm{Im}^1\rho_{i+2}^{(k)} = \dim_{\mathbb{Q}_l}\mathrm{Im}^0\tau_i^{(k+1)}.$$

Proof of Claim 5.12. Since $\rho_i^{(k)}$ and $\tau_{2(n-k)-i}^{(k+1)}$ are dual to each other (see (5.1)), we have $\dim_{\mathbb{Q}_l}\mathrm{Im}\,\rho_i^{(k)} = \dim_{\mathbb{Q}_l}\mathrm{Im}\,\tau_{2(n-k)-i}^{(k+1)}$. Hence we have

$$\dim_{\mathbb{Q}_l}\mathrm{Im}^0\rho_i^{(k)} + \dim_{\mathbb{Q}_l}\mathrm{Im}^1\rho_i^{(k)} = \dim_{\mathbb{Q}_l}\mathrm{Im}^0\tau_{2(n-k)-i}^{(k+1)} + \dim_{\mathbb{Q}_l}\mathrm{Im}^1\tau_{2(n-k)-i}^{(k+1)}. \tag{5.4}$$

Since $\dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_{2(n-k)-i} = \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_{i-2}$ by Claim 5.9, and $\dim_{\mathbb{Q}_l} \mathrm{Im}^1 \tau^{(k+1)}_{2(n-k)-i} = \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \tau^{(k+1)}_{i}$ by Claim 5.10, we have

$$\dim_{\mathbb{Q}_l} \mathrm{Im}^0 \rho^{(k)}_i + \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_i = \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_{i-2} + \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \tau^{(k+1)}_{i}.$$

Equivalently, we have

$$\dim_{\mathbb{Q}_l} \mathrm{Im}^0 \rho^{(k)}_i - \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \tau^{(k+1)}_i = -\big(\dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_i - \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_{i-2} \big). \tag{5.5}$$

On the other hand, we have $\dim_{\mathbb{Q}_l} \mathrm{Im}^0 \rho^{(k)}_i = \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \rho^{(k)}_{2(n-k)-i}$ by Claim 5.9, and $\dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_i = \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_{2(n-k)-i+2}$ by Claim 5.10. Hence, by (5.4), we have

$$\begin{aligned} &\dim_{\mathbb{Q}_l} \mathrm{Im}^0 \rho^{(k)}_{2(n-k)-i} + \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_{2(n-k)-i+2} \\ &\qquad = \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_{2(n-k)-i} + \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \tau^{(k+1)}_{2(n-k)-i}. \end{aligned}$$

From this, by replacing $2(n-k)-i$ by i, we have

$$\dim_{\mathbb{Q}_l} \mathrm{Im}^0 \rho^{(k)}_i - \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \tau^{(k+1)}_i = -\big(\dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_{i+2} - \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_{i} \big). \tag{5.6}$$

By combining (5.5) and (5.6), we have

$$\dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_i - \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_{i-2} = \dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_{i+2} - \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_{i}.$$

Hence the difference $\dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_{i+2} - \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_i$ is independent of i. Since this is zero for sufficiently large (or small) i, we have

$$\dim_{\mathbb{Q}_l} \mathrm{Im}^1 \rho^{(k)}_{i+2} - \dim_{\mathbb{Q}_l} \mathrm{Im}^0 \tau^{(k+1)}_i = 0 \qquad \text{for all } i,$$

which proves the second assertion. Hence we also have the first assertion by (5.5) or (5.6). □

We call the pairing on $H^i_{\text{ét}}(X^{(k)})$ defined by

$$H^i_{\text{ét}}(X^{(k)}) \times H^i_{\text{ét}}(X^{(k)}) \to \mathbb{Q}_l, \qquad (a,b) \mapsto L^{n-k+1-i} a \cup b$$

the *sum of the Lefschetz pairings*. We denote it simply by $\langle , \rangle$. The pairing $\langle , \rangle$ is nondegenerate by the hard Lefschetz theorem (§2.2). However, the following claim is far from trivial, and we use the assumptions of Proposition 5.7 to prove it.

Claim 5.13. The restriction of $\langle , \rangle$ on $H^i_{\text{ét}}(X^{(k+1)})$ to $\mathrm{Im}^0 \rho^{(k)}_i$ is nondegenerate. Similarly, The restriction of $\langle , \rangle$ on $H^{i+2}_{\text{ét}}(X^{(k)})$ to $\mathrm{Im}^0 \tau^{(k+1)}_i$ is also nondegenerate.

Proof of Claim 5.13. Since the decomposition of $\mathrm{Im}^0 \rho^{(k)}_i$ in Claim 5.9 is orthogonal with respect to $\langle , \rangle$, it is enough to show that the restriction of $\langle , \rangle$ to

$L^j P^{i-2j} \operatorname{Im}^0 \rho_i^{(k)}$ is nondegenerate for each j. By the assumptions of Proposition 5.7, we observe that all cohomology groups of $X^{(k)}, X^{(k+1)}$, maps ρ, τ, L, and pairings $\langle , \rangle$ have natural $\mathbb{Q}$-structures (see §2.1). In particular, the inclusion

$$L^j P^{i-2j} \operatorname{Im}^0 \rho_i^{(k)} \subset L^j P_{\text{ét}}^{i-2j}(X^{(k+1)})$$

has a $\mathbb{Q}$-structure. Namely, let $V \subset L^j P^{i-2j} \operatorname{Im}^0 \rho_i^{(k)}$ and $W \subset L^j P_{\text{ét}}^{i-2j}(X^{(k+1)})$ be the $\mathbb{Q}$-vector subspaces generated by algebraic cycles, then we have

$$V \otimes_{\mathbb{Q}} \mathbb{Q}_l = L^j P^{i-2j} \operatorname{Im}^0 \rho_i^{(k)}, \qquad W \otimes_{\mathbb{Q}} \mathbb{Q}_l = L^j P_{\text{ét}}^{i-2j}(X^{(k+1)}).$$

It is enough to show that the restriction of $\langle , \rangle$ to V is nondegenerate. We may assume i is even because there is no odd degree cohomology. By assumption, the Hodge standard conjecture holds for all irreducible components of $X^{(k+1)}$. Hence the restriction of $\langle , \rangle$ to W is positive (resp. negative) definite if $(i-2j)/2$ is even (resp. odd). Therefore, the restriction of $\langle , \rangle$ to V is nondegenerate because it is definite. □

Claim 5.14. The composition of the following maps is an isomorphism :

$$\operatorname{Im}^0 \rho_i^{(k)} \xrightarrow{\tau_i^{(k+1)}} \operatorname{Im} \tau_i^{(k+1)} \longrightarrow \operatorname{Im}^1 \tau_i^{(k+1)} = \operatorname{Im} \tau_i^{(k+1)} / \operatorname{Im}^0 \tau_i^{(k+1)}.$$

Similarly, the composition of the following maps is also an isomorphism :

$$\operatorname{Im}^0 \tau_i^{(k+1)} \xrightarrow{\rho_{i+2}^{(k)}} \operatorname{Im} \rho_{i+2}^{(k)} \longrightarrow \operatorname{Im}^1 \rho_{i+2}^{(k)} = \operatorname{Im} \rho_{i+2}^{(k)} / \operatorname{Im}^0 \rho_{i+2}^{(k)}.$$

Proof of Claim 5.14. By Claim 5.12, we have only to show that the compositions are injective. Assume that the composition of the first row is not injective and take a nonzero $x \in \operatorname{Im}^0 \rho_i^{(k)}$ such that $\tau_i^{(k+1)}(x) \in \operatorname{Im}^0 \tau_i^{(k+1)}$. If $\tau_i^{(k+1)}(x) \neq 0$, by Claim 5.13, there exists $\tau_i^{(k+1)}(y) \in \operatorname{Im}^0 \tau_i^{(k+1)}$ such that

$$L^{2(n-k+1)-(i+2)}\, \tau_i^{(k+1)}(x) \cup \tau_i^{(k+1)}(y) \neq 0.$$

However, since $x = \rho_i^{(k)}(x')$ for some $x' \in H_{\text{ét}}^i(X^{(k)})$, by (5.1), we have

$$\begin{aligned} 0 &\neq L^{(n-k+1)-(i+2)}\, \tau_i^{(k+1)}(x) \cup \tau_i^{(k+1)}(y) \\ &= L^{(n-k+1)-(i+2)} \rho_{i+2}^{(k)} \circ \tau_i^{(k+1)} \circ \rho_i^{(k)}(x') \cup y \\ &= 0. \end{aligned}$$

Recall that $\rho_{i+2}^{(k)} \circ \tau_i^{(k+1)} \circ \rho_i^{(k)} = 0$. This is a contradiction. Hence we have $\tau_i^{(k+1)}(x) = 0$. By Claim 5.13 again, there exists $\rho_i^{(k)}(z) \in \operatorname{Im}^0 \rho_i^{(k)}$ such that

$$L^{n-k-i}\, x \cup \rho_i^{(k)}(z) \neq 0.$$

However, by (5.1), we have

$$0 \neq L^{n-k-i}\, x \cup \rho_i^{(k)}(z) = L^{n-k-i}\, \tau_i^{(k+1)}(x) \cup z = 0$$

since $\tau_i^{(k+1)}(x) = 0$. This is a contradiction. Hence the composition of the first row is injective. Similarly, we see that the composition of the second row is also injective. □

Claim 5.15. By Claim 5.14, the surjections $\operatorname{Im}\tau_i^{(k+1)} \to \operatorname{Im}^1\tau_i^{(k+1)}$ and $\operatorname{Im}\rho_{i+2}^{(k)} \to \operatorname{Im}^1\rho_{i+2}^{(k)}$ have canonical splittings. Therefore, we have the following decompositions of $\operatorname{Im}\tau_i^{(k+1)}$ and $\operatorname{Im}\rho_{i+2}^{(k)}$:

$$\operatorname{Im}\tau_i^{(k+1)} = \operatorname{Im}^0\tau_i^{(k+1)} \oplus \operatorname{Im}^1\tau_i^{(k+1)}, \quad \operatorname{Im}\rho_{i+2}^{(k)} = \operatorname{Im}^0\rho_{i+2}^{(k)} \oplus \operatorname{Im}^1\rho_{i+2}^{(k)}. \tag{5.7}$$

These are orthogonal decompositions with respect to $\langle , \rangle$.

Proof of Claim 5.15. The proof is easy and immediate. Take $(a, b) \in \operatorname{Im}^0\tau_i^{(k+1)} \oplus \operatorname{Im}^1\tau_i^{(k+1)}$. Then, by the first assertion of Claim 5.14, we can write $a = \tau_i^{(k+1)}(a')$ and $b = \tau_i^{(k+1)} \circ \rho_i^{(k)}(b')$ for some a', b'. Then, by (5.1), we have

$$\begin{aligned} L^{n-k+1-(i+2)}\, a \cup b &= L^{n-k+1-(i+2)}\, \tau_i^{(k+1)}(a') \cup \left(\tau_i^{(k+1)} \circ \rho_i^{(k)}(b')\right) \\ &= L^{n-k+1-(i+2)}\, a' \cup \left(\rho_{i+2}^{(k+1)} \circ \tau_i^{(k+1)} \circ \rho_i^{(k)}(b')\right) \\ &= 0, \end{aligned}$$

since $\rho_{i+2}^{(k+1)} \circ \tau_i^{(k+1)} \circ \rho_i^{(k)} = 0$. Similarly, we see that the decomposition $\operatorname{Im}\rho_{i+2}^{(k)} = \operatorname{Im}^0\rho_{i+2}^{(k)} \oplus \operatorname{Im}^1\rho_{i+2}^{(k)}$ is also orthogonal with respect to $\langle , \rangle$. $\square$

Finally, by combining above results, we shall prove Claim 5.8.

Proof of Claim 5.8. First, we shall prove the first assertion. Since $\tau_i^{(k+1)} \circ \rho_i^{(k)} \circ \tau_{i-2}^{(k+1)} = 0$, the inclusion

$$\left(\operatorname{Ker}\tau_i^{(k+1)}\right) \cap \left(\operatorname{Im}\rho_i^{(k)}\right) \supset \operatorname{Im}\left(\rho_i^{(k)} \circ \tau_{i-2}^{(k+1)}\right)$$

is obvious. To prove the opposite inclusion, we take $x \in \left(\operatorname{Ker}\tau_i^{(k+1)}\right) \cap \left(\operatorname{Im}\rho_i^{(k)}\right)$. Then, for all $y \in H^i_{\text{ét}}(X^{(k)})$, by (5.1), we have

$$L^{n-k-i}\, x \cup \rho_i^{(k)}(y) = L^{n-k-i}\, \tau_i^{(k+1)}(x) \cup y = 0$$

since $\tau_i^{(k+1)}(x) = 0$. By Claim 5.15, we write $x = x_0 + x_1$ for $x_0 \in \operatorname{Im}^0\rho_i^{(k)}$ and $x_1 \in \operatorname{Im}^1\rho_i^{(k)}$. If $x_0 \neq 0$, by Claim 5.13 and Claim 5.15, there exists $y' \in \operatorname{Im}^0\rho_i^{(k)}$ such that

$$\begin{aligned} L^{n-k-i}\, x \cup y' &= L^{n-k-i}\, x_0 \cup y' + L^{n-k-i}\, x_1 \cup y' \\ &= L^{n-k-i}\, x_0 \cup y' \\ &\neq 0, \end{aligned}$$

which is absurd. Hence $x_0 = 0$. Then, we have

$$x = x_1 \in \operatorname{Im}^1\rho_i^{(k)} = \rho_i^k(\operatorname{Im}\tau_{i-2}^{(k+1)}) \subset \operatorname{Im}\left(\rho_i^{(k)} \circ \tau_{i-2}^{(k+1)}\right).$$

This proves the first assertion. Similarly, we can prove the second assertion by using Claim 5.13 and Claim 5.15. Therefore, the proof of Claim 5.8 and hence Proposition 5.7 is completed. $\square$

6. Application to p-adically uniformized varieties

In this section, we give a proof of the main theorem (Theorem 1.2) of this paper. We also give some immediate applications.

6.1. **Drinfeld upper half spaces and p-adic uniformization.** Let K be a complete discrete valuation field with finite residue field $\mathbb{F}_q$ of characteristic $p > 0$. Let $\widehat{\Omega}^d_K$ be the Drinfeld upper half space of dimension d over K, which is a rigid analytic space obtained by removing all K-rational hyperplanes from $\mathbb{P}^d_K$. We have a natural action of $\mathrm{PGL}_{d+1}(K)$ on $\widehat{\Omega}^d_K$.

As a formal scheme, $\widehat{\Omega}^d_K$ can be constructed as follows. Take the projective space $\mathbb{P}^d_{\mathcal{O}_K}$ over $\mathcal{O}_K$. Then, take successive blowing-ups of $\mathbb{P}^d_{\mathcal{O}_K}$ along all linear subvarieties in the special fiber $\mathbb{P}^d_{\mathbb{F}_q}$ as in §4.1. By continuing this process for *all* exceptional divisors appearing in the blowing-ups, we obtain a formal scheme $\widehat{\Omega}^d_{\mathcal{O}_K}$ locally of finite type over $\mathrm{Spf}\,\mathcal{O}_K$. By construction, the rigid analytic space associated with $\widehat{\Omega}^d_{\mathcal{O}_K}$ is isomorphic to $\widehat{\Omega}^d_K$. In other words, $\widehat{\Omega}^d_K$ is the "generic fiber" in the sense of Raynaud ([Ray]) of the formal scheme $\widehat{\Omega}^d_{\mathcal{O}_K}$ over $\mathrm{Spf}\,\mathcal{O}_K$.

Let Γ be a cocompact torsion free discrete subgroup of $\mathrm{PGL}_{d+1}(K)$. We have a natural action of Γ on $\widehat{\Omega}^d_{\mathcal{O}_K}$, and this action is discontinuous with respect to Zariski topology. Then, we take a quotient $\widehat{\mathfrak{X}}_\Gamma := \Gamma\backslash\widehat{\Omega}^d_{\mathcal{O}_K}$ as a formal scheme. Mustafin, Kurihara showed that the relative dualizing sheaf $\omega_{\widehat{\mathfrak{X}}_\Gamma/\mathcal{O}_K}$ is invertible and ample. Therefore, $\widehat{\mathfrak{X}}_\Gamma$ can be algebraized to a projective scheme $\mathfrak{X}_\Gamma$ over $\mathcal{O}_K$. The generic fiber $X_\Gamma := \mathfrak{X}_\Gamma \otimes_{\mathcal{O}_K} K$ is a projective smooth variety over K whose associated rigid analytic space is the rigid analytic quotient $\Gamma\backslash\widehat{\Omega}^d_K$. By construction, $\mathfrak{X}_\Gamma$ is a proper semistable model of X_Γ over $\mathcal{O}_K$ whose special fiber is described by the cell complex $\Gamma\backslash\mathfrak{T}$, where $\mathfrak{T}$ denotes the Bruhat-Tits building of $\mathrm{PGL}_{d+1}(K)$ (for details, see [Mus], Theorem 4.1, [Ku], Theorem 2.2.5).

6.2. **Proof of the main theorem (Theorem 1.2).** As in §6.1, let K be a complete discrete valuation field with finite residue field $\mathbb{F}_q$ of characteristic $p > 0$, $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ a cocompact torsion free discrete subgroup, and $\mathfrak{X}_\Gamma$ an algebraization of $\Gamma\backslash\widehat{\Omega}^d_{\mathcal{O}_K}$ with generic fiber X_Γ as in §6.1.

The following lemma seems well-known.

Lemma 6.1. *Let X, Y be proper smooth variety of dimension d over K, and $f\colon X \to Y$ a proper surjective generically finite morphism of degree* $\deg f$ *(f is called an* alteration *in* [dJ]*). If the weight-monodromy conjecture (Conjecture 1.1) holds for X, then Conjecture 1.1 holds also for Y.*

Proof. Since f is finite of degree $\deg f$ on some open dense subscheme $U \subset X$, $f^*\colon H^k_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l) \to H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ and $f_*\colon H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \to H^k_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)$ satisfy

$f_* \circ f^* =$ multiplication by $\deg f$. Since f_*, f^* commute with $\mathrm{Gal}(\overline{K}/K)$-action, $H^k_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)$ is a direct summand of $H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ as a $\mathrm{Gal}(\overline{K}/K)$-representation. Therefore, the restriction of the monodromy filtration (resp. weight filtration) on $H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ to $H^k_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)$ coincides with the monodromy filtration (resp. weight filtration) on $H^k_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)$ itself. Hence the assertion follows. □

Now, we shall prove the main theorem (Theorem 1.2) of this paper.

Proof of Theorem 1.2. First of all, it is known that the intersection of all finite index subgroups $\Gamma' \subset \Gamma$ is equal to the identity ([Ga], Theorem 2.7, Proof, (b)). From this, by looking at the action of Γ on the Bruhat-Tits building $\mathfrak{T}$ of $\mathrm{PGL}_{d+1}(K)$ ([Mus], [Ku]), it is easy to see that $\mathfrak{X}_{\Gamma'}$ is *strictly* semistable for some finite index subgroup $\Gamma' \subset \Gamma$. Since we have a finite étale covering $X_{\Gamma'} \to X_\Gamma$, by Lemma 6.1, we may assume that $\mathfrak{X}_\Gamma$ is a strictly semistable model of X_Γ over $\mathcal{O}_K$.

Under this assumption, from the construction of $\mathfrak{X}_\Gamma$ in §6.1, we see that all irreducible components $X_1, \ldots, X_m$ of the special fiber of $\mathfrak{X}_\Gamma$ are isomorphic to the variety B^d constructed in §4.1. By Proposition 4.2, 4, 5, for $i \neq j$, each irreducible component of $X_i \cap X_j$ is isomorphic to a divisor of the form D_V on B^d in §4.2. Moreover, by induction, we see that, for $1 \leq i_1 < \cdots < i_k \leq m$, each irreducible component Y of $X_{i_1} \cap \cdots \cap X_{i_k}$ is isomorphic to the product

$$Y \cong B^{n_1} \times \cdots \times B^{n_k}$$

for some $n_1, \ldots, n_k \geq 1$ with $n_1 + \cdots + n_k = d - k + 1$.

Let $\mathcal{L}$ be the relative dualizing sheaf $\omega_{\mathfrak{X}_\Gamma/\mathcal{O}_K}$, which is invertible and ample by Mustafin, Kurihara ([Mus], [Ku]). Take i with $1 \leq i \leq m$ and fix an isomorphism $X_i \cong B^d$. Then, by an explicit calculation in [Mus], [Ku], the restriction of $\mathcal{L}$ to each X_i is isomorphic to

$$-(n+1)f^*\mathcal{O}_{\mathbb{P}^d}(1) + \sum_{d=0}^{n-1}(n-d)D_d$$

(for notation, see §4.1, §4.2, see also Remark 4.8), which is an ample $\mathrm{PGL}_{d+1}(\mathbb{F}_q)$-invariant divisor on $X_i \cong B^d$. Therefore, by Proposition 4.7, the Hodge standard conjecture (Conjecture 2.6) holds for $(X_i, \mathcal{L}|_{X_i})$.

Moreover, for $1 \leq i_1 < \cdots < i_k \leq m$, take an irreducible component Y of $X_{i_1} \cap \cdots \cap X_{i_k}$. Then, Y is isomorphic to $B^{n_1} \times \cdots \times B^{n_k}$ for some $n_1, \ldots, n_k \geq 1$ with $n_1 + \cdots + n_k = d - k + 1$. By the construction of B^n in §4.1 and the Künneth formula, Y satisfies Assumption 2.1 in §2.1 (see also §2.5). Let $\mathrm{pr}_j \colon Y \to B^{n_j}$ be the projection to the i-th factor for $j = 1, \ldots, k$. Then, by applying Proposition 4.2, 4, 5 and Proposition 4.4, 4 inductively, we see that the restriction of $\mathcal{L}$ to Y is of the form

$$\mathcal{L}|_Y = \mathrm{pr}_1^*\mathcal{O}(L_1) + \cdots + \mathrm{pr}_k^*\mathcal{O}(L_k) \qquad \text{in} \quad H^2(Y)$$

where L_j is an ample $\mathrm{PGL}_{n_j+1}(\mathbb{F}_q)$-invariant divisor on B^{n_j} for $j = 1, \dots, k$. By Proposition 4.7, the Hodge standard conjecture (Conjecture 2.6) holds for (B^{n_j}, L_j) for $j = 1, \dots, k$. By applying Proposition 2.13 inductively, we conclude that the Hodge standard conjecture (Conjecture 2.6) also holds for $(Y, \mathcal{L}|_Y)$.

Therefore, $\mathfrak{X}_\Gamma$ satisfies all assumptions in Proposition 5.7. Hence the weight-monodromy conjecture (Conjecture 1.1) holds for X_Γ, and the proof of Theorem 1.2 is now completed. □

Remark 6.2. Theorem 1.2 is new only in mixed characteristic. However, we note that Theorem 1.2 in characteristic $p > 0$ doesn't automatically follows from Deligne's results in [De3] because $\mathfrak{X}_\Gamma$ doesn't automatically come from a family of varieties over a curve over a finite field (This was explained by Illusie in [Il2], 8.7). We need an approximation argument to reduce the general characteristic $p > 0$ case to Deligne's case ([Te], [It1]).

6.3. **Schneider-Stuhler's conjecture on the filtration $F^\bullet$.** In [SS], Schneider-Stuhler computed the cohomology of $\widehat{\Omega}^d_K$ and its quotient $\Gamma\backslash\widehat{\Omega}^d_K$ by methods from rigid analytic geometry and representation theory. Their results are valid for cohomology theories satisfying certain axioms, and they showed that de Rham cohomology and rigid analytic l-adic cohomology satisfy these axioms. Precisely speaking, at that time, they used unpublished results of Gabber to check the axioms for rigid analytic l-adic cohomology. After that, these axioms were established by Berkovich ([Be]), de Jong - van der Put ([dJvdP]). By comparison theorem between rigid analytic and usual l-adic cohomology, Schneider-Stuhler's computation is valid for l-adic cohomology of the variety X_Γ. Here we combine our results with the results of Schneider-Stuhler in [SS], and prove Schneider-Stuhler's conjecture on the filtration $F^\bullet$ for l-adic cohomology (However, for de Rham cohomology, see Remark 6.15).

First of all, we recall the results of Schneider-Stuhler (for details, see [SS], Theorem 4). As in §6.1, let K be a complete discrete valuation field with finite residue field $\mathbb{F}_q$ of characteristic $p > 0$. For a cocompact torsion free discrete subgroup $\Gamma \subset \mathrm{PGL}_{d+1}(K)$, let

$$\mathrm{Ind}_\Gamma := C^\infty\big(\mathrm{PGL}_{d+1}(K)/\Gamma, \mathbb{C}\big)$$

be the $\mathrm{PGL}_{d+1}(K)$-representation induced from the trivial character on Γ. Let $\mu(\Gamma)$ be the multiplicity of the Steinberg representation in Ind_Γ. In [SS], §5, Schneider-Stuhler explicitly computed the E_2-terms of a Hochschild-Serre type spectral sequence :

$$E_2^{r,s} = H^r\big(\Gamma, H^s_{\text{ét}}(\widehat{\Omega}^d_K \otimes_K \overline{K}, \mathbb{Q}_l)\big) \Longrightarrow H^{r+s}_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l) \tag{6.1}$$

([SS], §5, Proposition 2), and proved

$$H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l) \cong \begin{cases} \mathbb{Q}_l\big(-\frac{k}{2}\big) & \text{if } k \text{ is even, } 0 \le k \le 2d,\ k \ne d \\ 0 & \text{if } k \text{ is odd, } k \ne d. \end{cases} \tag{6.2}$$

For the middle degree $k = d$, they proved that the covering spectral sequence (6.1) defines a decreasing filtration $F^\bullet$ on $V := H^d_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)$ such that

$$V = F^0V \supset F^1V \supset \cdots \supset F^dV \supset F^{d+1}V = 0,$$

$$F^rV/F^{r+1}V \cong \begin{cases} \mathbb{Q}_l(r-d)^{\oplus\mu(\Gamma)} & \text{if } 0 \leq r \leq d,\ r \neq \frac{d}{2} \\ \mathbb{Q}_l\left(-\frac{d}{2}\right)^{\oplus(\mu(\Gamma)+1)} & \text{if } r = \frac{d}{2} \\ 0 & \text{otherwise,} \end{cases} \tag{6.3}$$

and conjectured that $F^\bullet$ essentially coincides with the monodromy filtration (see [SS], introduction and a remark following Theorem 5).

Theorem 6.3 (Schneider-Stuhler's conjecture for l-adic cohomology)**.** *Let $M_\bullet$ be the monodromy filtration on V (Definition 5.1). Define an increasing filtration $F'_\bullet$ on V by $F'_iV = F^{-\lfloor i/2 \rfloor}V$. Then, we have*

$$M_i V = F'_{i-d} V \qquad \textit{for all } i.$$

Proof. We compute the graded quotients of $F'_\bullet$ explicitly. If i is even, we have

$$\begin{aligned} F'_i V/F'_{i-1} V &= F^{-i/2}V/F^{-i/2+1}V \\ &= \begin{cases} \mathbb{Q}_l\left(-\frac{i}{2} - d\right)^{\oplus\mu(\Gamma)} & \text{if } -2d \leq i \leq 0,\ i \neq -d \\ \mathbb{Q}_l\left(-\frac{d}{2}\right)^{\oplus(\mu(\Gamma)+1)} & \text{if } i = -d \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If i is odd, we have $\lfloor i/2 \rfloor = \lfloor (i-1)/2 \rfloor$ hence $F'_i V/F'_{i-1} V = 0$. Therefore, each $F'_i V/F'_{i-1} V$ has weight $i + 2d$ as a $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representation. Now the assertion follows from Theorem 1.2. $\square$

6.4. **Application to the local zeta function of X_Γ.** Let notation be as in §6.3. Let us recall the definition of the local zeta functions (for details, see [Se2], [Rap]). For a continuous $\mathrm{Gal}(\overline{K}/K)$-representation V over $\mathbb{Q}_l$, the *local L-function* of V is defined by

$$L(s, V) := \det\left(1 - q^{-s} \cdot \mathrm{Fr}_q\,;\, V^{I_K}\right)^{-1}$$

(for notation, see §5.1, §5.2). For a variety X over K, the *local zeta function* of X is defined by

$$\zeta(s, X) := \prod_{k=0}^{2\dim X} L\big(s,\ H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)\big)^{(-1)^k}.$$

Let us compute the local zeta function of X_Γ. For an even integer k with $0 \leq k \leq 2d,\ k \neq d$, by (6.2), we easily see that

$$L\big(s, H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)\big) = \det\left(1 - q^{-s} \cdot \mathrm{Fr}_q\,;\, \mathbb{Q}_l\left(-\tfrac{k}{2}\right)\right)^{-1} = \frac{1}{1 - q^{k/2-s}}.$$

On the other hand, for the middle degree $k = d$, we can't compute the local L-function only from the results of Schneider-Stuhler (6.3) because (6.3) only gives us the semisimplification of the $\mathrm{Gal}(\overline{K}/K)$-representation $H^d_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)$. We need the weight-monodromy conjecture to recover the local L-function from the semisimplification (see [Rap], §2).

Now, we shall use Theorem 6.3. By Theorem 6.3, we see that the inertia fixed part is given by

$$H^d_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)^{I_K} = \begin{cases} \mathbb{Q}_l^{\oplus\mu(\Gamma)} \oplus \mathbb{Q}_l\left(-\frac{d}{2}\right) & \text{if } d \text{ is even} \\ \mathbb{Q}_l^{\oplus\mu(\Gamma)} & \text{if } d \text{ is odd} \end{cases}$$

(see also Definition 5.1, 3). Therefore, we have

$$L\big(s, H^d_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)\big) = \begin{cases} \dfrac{1}{(1-q^{-s})^{\mu(\Gamma)}} \cdot \dfrac{1}{1-q^{d/2-s}} & \text{if } d \text{ is even} \\ \dfrac{1}{(1-q^{-s})^{\mu(\Gamma)}} & \text{if } d \text{ is odd.} \end{cases}$$

By combining above results, we have the following theorem.

Theorem 6.4 (Local zeta function of X_Γ)**.** *Let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a cocompact torsion free discrete subgroup. Then, the local zeta function of X_Γ is given by*

$$\zeta(s, X_\Gamma) = (1-q^{-s})^{\mu(\Gamma)\cdot(-1)^{d+1}} \cdot \prod_{k=0}^{d} \frac{1}{1-q^{k-s}},$$

where $\mu(\Gamma)$ is the multiplicity of the Steinberg representation in Ind_Γ *as in §6.3.*

Remark 6.5. As a consequence, we also see that the local zeta function $\zeta(s, X_\Gamma)$ is independent of $l \neq p$, which is a nontrivial result. In §6.6, we consider the case $l = p$ by using p-adic cohomology.

6.5. **Application to mod l Galois representations.** Let notation be as in §6.3 and in §5.3. For $l \neq p$, we have the weight spectral sequence of Rapoport-Zink with coefficients in $\mathbb{F}_l$

$$'E_1^{-r,w+r} = \bigoplus_{k \geq \max\{0,-r\}} H^{w-r-2k}_{\text{ét}}(X^{(2k+r+1)}_{\overline{\mathbb{F}}_q}, \mathbb{F}_l(-r-k)) \qquad \Longrightarrow \quad H^w_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{F}_l),$$

similarly as in §5.3. We also have a natural monodromy operator $N\colon\ 'E_1^{i,j}(1) \to {}'E_1^{i+2,j-2}$. Here we study the above spectral sequence for all but finitely many l. Note that, in general, we don't even know the E_2-degeneracy because there doesn't exist good theory of weights for étale cohomology with coefficients in $\mathbb{F}_l$.

Fix an isomorphism $\mathbb{Z}_l \cong \mathbb{Z}_l(1)$, and let $E_1^{-r,w+r} \Rightarrow H^w_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)$ be the weight spectral sequence for l-adic cohomology as in §5.3. The crucial observation in the proof of Theorem 1.2 in §6.2 is the existence of $\mathbb{Q}$-structures on $E_1^{i,j}, d_1^{i,j}, N$. In fact, we can say little more. As we already saw in §6.2, all varieties appearing in the weight spectral sequence are products of the variety B^n in §4.1. Therefore, from the construction of B^n, we see that the cohomology of B^n has $\mathbb{Z}$-structures as well as $\mathbb{Q}$-structures. Namely, we have an isomorphism $N^k(B^n) \otimes_{\mathbb{Z}} \mathbb{F}_l \cong H^{2k}_{\text{ét}}(B^n, \mathbb{F}_l(k))$ for all k (see Lemma 2.3 in §2.1). Let $\widetilde{E}_1^{-r,\,w+r}$ be the free sub-$\mathbb{Z}$-module of $E_1^{-r,\,w+r}$ generated by algebraic cycles over $\mathbb{Z}$. Since $d_1^{i,\,j}, N$ are defined geometrically, there exist natural maps $\widetilde{d}_1^{i,\,j}, \widetilde{N}$ on $\widetilde{E}_1^{-r,\,w+r}$ which give $\mathbb{Z}$-structures on $d_1^{i,\,j}, N$. These are independent of l by Lemma 2.3. Hence we have the data $\{\widetilde{E}_1^{-r,\,w+r},\ \widetilde{d}_1^{i,\,j},\ \widetilde{N}\}$ such that, for all $l \neq p$ and for $\Lambda = \mathbb{Q}_l$ or $\mathbb{F}_l$, the data

$$\{\widetilde{E}_1^{-r,\,w+r} \otimes_{\mathbb{Z}} \Lambda,\ \widetilde{d}_1^{i,\,j} \otimes_{\mathbb{Z}} \Lambda,\ \widetilde{N} \otimes_{\mathbb{Z}} \Lambda\}$$

coincide with the E_1-terms of the weight spectral sequence with coefficients in Λ.

Lemma 6.6. *For all but finitely many l, the mod l weight spectral sequence ${}'E_1^{-r,\,w+r} \Rightarrow H^w_{\text{ét}}(X_\Gamma \otimes_K \overline{K},\, \mathbb{F}_l)$ degenerates at E_2, and N^r induces an isomorphism*

$$N^r :\ {}'E_2^{-r,\,w+r} \xrightarrow{\ \cong\ } {}'E_2^{r,\,w-r}$$

on E_2-terms for all r, w (Recall that we fix an isomorphism $\mathbb{Z}_l \cong \mathbb{Z}_l(1)$ here).

Proof. First of all, we have an inequality :

$$\sum_{i,\,j} \dim_{\mathbb{F}_l} {}'E_2^{i,\,j} \geq \sum_k \dim_{\mathbb{F}_l} H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K},\, \mathbb{F}_l),$$

and the equality holds if and only if the mod l weight spectral sequence degenerates at E_2. Since the E_2-degeneracy is known for l-adic cohomology (see §5.3), we have

$$\sum_{i,\,j} \dim_{\mathbb{Q}_l} E_2^{i,\,j} = \sum_k \dim_{\mathbb{Q}_l} H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K},\, \mathbb{Q}_l),$$

where $E_1^{-r,\,w+r} \Rightarrow H^w_{\text{ét}}(X_\Gamma \otimes_K \overline{K},\, \mathbb{Q}_l)$ is the l-adic weight spectral sequence. On the other hand, in general, we have the inequality :

$$\dim_{\mathbb{F}_l} H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K},\, \mathbb{F}_l) \geq \dim_{\mathbb{Q}_l} H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K},\, \mathbb{Q}_l),$$

and the equality holds if and only if the cohomology with coefficients in $\mathbb{Z}_l$ is torsion free. Hence it is enough to show the equality $\dim_{\mathbb{F}_l} {}'E_2^{i,\,j} = \dim_{\mathbb{Q}_l} E_2^{i,\,j}$ for all but finitely many l.

To show this, we use the $\mathbb{Z}$-structures on the E_1-terms. We put

$$\widetilde{E}_2^{i,\,j} := \operatorname{Ker}(\widetilde{d}_1^{i,\,j})/\operatorname{Im}(\widetilde{d}_1^{i-1,\,j}).$$

Since $\mathbb{Q}_l$ is flat over $\mathbb{Z}$, we have $\widetilde{E}_2^{i,\,j} \otimes_{\mathbb{Z}} \mathbb{Q}_l = E_2^{i,\,j}$. In general, the natural map $\widetilde{E}_2^{i,\,j} \otimes_{\mathbb{Z}} \mathbb{F}_l \to {}'E_2^{i,\,j}$ isn't an isomorphism. However, since there are only finitely many nonzero terms at E_1 and each term is finitely generated over $\mathbb{Z}$, we have an isomorphism $\widetilde{E}_2^{i,\,j} \otimes_{\mathbb{Z}} \mathbb{F}_l \cong {}'E_2^{i,\,j}$ and the equality

$$\dim_{\mathbb{F}_l} \left(\widetilde{E}_2^{i,\,j} \otimes_{\mathbb{Z}} \mathbb{F}_l\right) = \dim_{\mathbb{Q}_l} \left(\widetilde{E}_2^{i,\,j} \otimes_{\mathbb{Z}} \mathbb{Q}_l\right)$$

for all but finitely many l. Hence we have the desired equality $\dim_{\mathbb{F}_l} {}'E_2^{i,\,j} = \dim_{\mathbb{Q}_l} E_2^{i,\,j}$ for all but finitely many l.

For the monodromy operator, by the same reason as above, it is enough to show that the map

$$\widetilde{N}^r \otimes_{\mathbb{Z}} \mathbb{F}_l : \widetilde{E}_2^{-r,w+r} \otimes_{\mathbb{Z}} \mathbb{F}_l \longrightarrow \widetilde{E}_2^{r,w-r} \otimes_{\mathbb{Z}} \mathbb{F}_l$$

is an isomorphism for all but finitely many l. By Theorem 1.2 (see also Conjecture 5.6, §6.2), $\widetilde{N}^r \otimes_{\mathbb{Z}} \mathbb{Q}_l$ is an isomorphism for all $l \neq p$. Since all $\widetilde{E}_2^{i,j}$ are finitely generated over $\mathbb{Z}$, we have the assertion. □

In conclusion, by Lemma 6.6, we have the following theorem for mod l Galois representations of X_Γ for all but finitely many l (see also an argument in §6.4). In the following statement, we don't ignore the Tate twists.

Theorem 6.7. *Let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a cocompact torsion free discrete subgroup. We put $V := H^d_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{F}_l)$. Then, for all but finitely many l, we have*

$$\dim_{\mathbb{F}_l} V = (d+1) \cdot \mu(\Gamma) + \varepsilon, \qquad \dim_{\mathbb{F}_l} V^{I_K} = \mu(\Gamma) + \varepsilon,$$

where $\varepsilon = 0$ (resp. $\varepsilon = 1$) if d is odd (resp. even), and $\mu(\Gamma)$ is the multiplicity of the Steinberg representation in Ind_Γ as in §6.3. More precisely, we have

$$V^{I_K} = \begin{cases} \mathbb{F}_l^{\oplus \mu(\Gamma)} \oplus \mathbb{F}_l\left(-\frac{d}{2}\right) & \text{if } d \text{ is even} \\ \mathbb{F}_l^{\oplus \mu(\Gamma)} & \text{if } d \text{ is odd.} \end{cases}$$

In particular, the characteristic polynomial of Fr_q acting on V^{I_K} is the reduction modulo l of the characteristic polynomial of Fr_q acting on $H^d_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)^{I_K}$. Moreover, if $\mu(\Gamma) \geq 1$, then the mod l $\mathrm{Gal}(\overline{K}/K)$-representation V is ramified.

Remark 6.8. The author doesn't know whether Theorem 6.7 holds for small l. This seems an interesting problem from the viewpoint of the theory of mod l automorphic forms and mod l Galois representations.

6.6. **A p-adic analogue.** We recall some results in p-adic Hodge theory to state an analogue of the weight-monodromy conjecture for p-adic étale cohomology.

Let K be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, $W(\mathbb{F}_q)$ the ring of Witt vectors with coefficients in $\mathbb{F}_q$, K_0 the field of fractions of $W(\mathbb{F}_q)$, and X a proper smooth variety over K. For simplicity, assume that there is a proper strictly semistable model $\mathfrak{X}$ of X over $\mathcal{O}_K$ (Definition 5.5).

For $V := H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, by the C_{st}-conjecture proved by Tsuji, there is a canonical isomorphism

$$(6.4) \qquad D_{\mathrm{st}}(V) := \left(V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}\right)^{\mathrm{Gal}(\overline{K}/K)} \cong H^w_{\text{log-crys}}\left(\mathfrak{X}^{\times}_{\mathbb{F}_q}/W(\mathbb{F}_q)^{\times}\right) \otimes_{W(\mathbb{F}_q)} K_0$$

where B_{st} is Fontaine's ring of p-adic periods, $\mathrm{Gal}(\overline{K}/K)$ acts on $V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$ diagonally, and $H^w_{\text{log-crys}}\left(\mathfrak{X}^{\times}_{\mathbb{F}_q}/W(\mathbb{F}_q)^{\times}\right)$ is the log crystalline cohomology of the special fiber $\mathfrak{X}_{\mathbb{F}_q} = \mathfrak{X} \otimes_{\mathcal{O}_K} \mathbb{F}_q$ endowed with a natural log structure ([Fo], [HK], [Ts]). It is known that $\dim_{K_0} D_{\mathrm{st}}(V) = \dim_{\mathbb{Q}_p} V$. Moreover, it is also known that and the monodromy operator N and the Frobenius φ act on both hand sides, and (6.4) is compatible with their actions. By using N, φ, we can define the monodromy filtration $M_\bullet$ and the weight filtration $W_\bullet$ on $D_{\mathrm{st}}(V)$ by the same way as in Definition 5.1, Definition 5.3. Then, we have the following p-adic analogue of the weight-monodromy conjecture.

Conjecture 6.9 (p-adic weight-monodromy conjecture).

$$M_i D_{\mathrm{st}}(V) = W_{i+w} D_{\mathrm{st}}(V) \qquad \textit{for all } i.$$

Let notation be as in §5.3. Mokrane constructed a p-adic analogue of the weight spectral sequence of Rapoport-Zink of the following form

$$\begin{aligned} E_1^{-r,\,w+r} = \bigoplus_{k \geq \max\{0,-r\}} & H^{w-r-2k}_{\mathrm{crys}}\big(X^{(2k+r+1)}/W(\mathbb{F}_q)\big)(-r-k) \\ & \Longrightarrow \quad H^w_{\text{log-crys}}\big(\mathfrak{X}^\times_{\mathbb{F}_q}/W(\mathbb{F}_q)^\times\big), \end{aligned}$$

where H^*_{crys} denotes the crystalline cohomology ([Mo], §3.23, Théorème 3.32). This spectral sequence has similar properties as the l-adic case (compare with §5.3). By the Weil conjecture for crystalline cohomology proved by Katz-Messing ([KM]), this spectral sequence degenerates at E_2 modulo torsion, and defines the weight filtration $W_\bullet$ on $H^w_{\text{log-crys}}\big(\mathfrak{X}^\times_{\mathbb{F}_q}/W(\mathbb{F}_q)^\times\big)$. Moreover, there is a monodromy operator N satisfying the same properties as the l-adic case (N is written as ν in [Mo], §3.33).

Therefore, by (6.4), Conjecture 6.9 is equivalent to the following conjecture on the weight spectral sequence of Mokrane (compare with Conjecture 5.6).

Conjecture 6.10 ([Mo], Conjecture 3.27, §3.33). *N^r induces an isomorphism*

$$N^r\colon E_2^{-r,\,w+r}(r) \otimes_{W(\mathbb{F}_q)} K_0 \xrightarrow{\ \cong\ } E_2^{r,\,w-r} \otimes_{W(\mathbb{F}_q)} K_0$$

for all r, w.

Theorem 6.11 (p-adic weight-monodromy conjecture for X_Γ). *Let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a cocompact torsion free discrete subgroup. Then, a p-adic analogue of the weight-monodromy conjecture (Conjecture 6.9, Conjecture 6.10) holds for X_Γ.*

Proof. The proof is basically the same as the l-adic case. Instead of l-adic cohomology, we use the cycle map for crystalline cohomology ([GM], [Gr]). □

Application to the p-adic local zeta function of X_Γ. Recall that a finite dimensional continuous $\mathrm{Gal}(\overline{K}/K)$-representation V over $\mathbb{Q}_p$ is called *semistable* if $\dim_{K_0} D_{\mathrm{st}}(V) = \dim_{\mathbb{Q}_p} V$ ([Ts]). For semistable V, the *local L-function* of V is defined by

$$L_{p\text{-adic}}(s, V) := \det\big(1 - q^{-s} \cdot \varphi\,;\, D_{\mathrm{st}}(V)^{N=0}\big)^{-1},$$

where $D_{\mathrm{st}}(V)^{N=0}$ denotes the kernel of N acting on $D_{\mathrm{st}}(V)$. In fact, we can define $L(s, V)$ for *potentially semistable* V, but we omit it here ([Fo]). For a variety X over K which has a proper strictly semistable model over $\mathcal{O}_K$, $H^k_{\mathrm{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is semistable for all k by [Ts]. We define the *p-adic local zeta function* $\zeta_{p\text{-adic}}(s,X)$ of X as follows :

$$\zeta_{p\text{-adic}}(s,X) := \prod_{k=0}^{2\dim X} L_{p\text{-adic}}\big(s,\, H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\big)^{(-1)^i}.$$

Here we put the subscript "p-adic" to distinguish it from the l-adic case in §6.4.

Theorem 6.12 (p-adic local zeta function of X_Γ). *Let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a cocompact torsion free discrete subgroup. Then, for all k, $L_{p\text{-adic}}\big(s, H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_p)\big)$ is the same as the usual local zeta function defined by l-adic cohomology for all $l \neq p$, which is computed explicitly in §6.4. In particular, we have the following equality of local zeta functions of X_Γ :*

$$\zeta_{p\text{-adic}}(s,X_\Gamma) = \zeta(s,X_\Gamma) = (1-q^{-s})^{\mu(\Gamma)\cdot(-1)^{d+1}} \cdot \prod_{k=0}^{d} \frac{1}{1-q^{k-s}},$$

where $\mu(\Gamma)$ is the multiplicity of the Steinberg representation in Ind_Γ as in §6.3.

Proof. We give a proof by comparing the l-adic and the p-adic local zeta functions because Schneider-Stuhler's computation doesn't work directly for p-adic étale cohomology (This argument is inspired by [SaT]). Fix a prime number $l \neq p$. Since the weight-monodromy conjecture holds for both l-adic and p-adic cohomology (Theorem 1.2, Theorem 6.11), it is enough to prove the equality :

$$\mathrm{Tr}\big(\mathrm{Fr}_q^m\,;\, H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)\big) = \mathrm{Tr}\big(\varphi^m\,;\, D_{\mathrm{st}}(H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_p))\big) \tag{6.5}$$

for all $k, m \geq 0$ (see [Rap], Lemma 2.12). For $k \neq d$, since the dimensions of l-adic and p-adic étale cohomology are the same, we see that, for odd $k \neq d$, $H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_p) = 0$, and for even $k \neq d$, $H^k_{\text{ét}}\big(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_p(\frac{k}{2})\big)$ is of dimension 1 and generated by algebraic cycles. Therefore, (6.5) holds for $k \neq d$. By Ochiai's result ([O], Theorem D), we know that the alternating sum of (6.5) for all k holds. Hence (6.5) also holds for $k = d$. □

Remark 6.13. For *strictly* semistable $\mathfrak{X}_\Gamma$, we can also prove Theorem 6.12 by comparing the l-adic and the p-adic weight spectral sequences as follows. As in the proof of Lemma 6.6 in §6.5, there are finitely generated $\mathbb{Z}$-modules $\widetilde{E}_2^{i,j}$ with operator $\widetilde{N}$ such that, up to Tate twists, $\{\widetilde{E}_2^{i,j} \otimes_{\mathbb{Z}} \mathbb{Q}_l,\ \widetilde{N} \otimes_{\mathbb{Z}} \mathbb{Q}_l\}$ coincide with the E_2-terms of the l-adic weight spectral sequence. By the same reason, $\{\widetilde{E}_2^{i,j} \otimes_{\mathbb{Z}} W(\mathbb{F}_q),\ \widetilde{N} \otimes_{\mathbb{Z}} W(\mathbb{F}_q)\}$ coincide with the E_2-terms of the p-adic weight spectral sequence up to torsion and Tate twists. Since the weight-monodromy conjecture holds for both l-adic and p-adic cohomology (Theorem 1.2, Theorem 6.11), we conclude that the characteristic polynomial of Fr_q, φ on each graded quotient of the monodromy filtration on $H^k_{\text{ét}}(X_\Gamma \otimes_K \overline{K}, \mathbb{Q}_l)$, $H^k_{\text{log-crys}}\big(\mathfrak{X}^\times_{\mathbb{F}_q}/W(\mathbb{F}_q)^\times\big)$ are the same for all k. Hence we have Theorem 6.12.

Remark 6.14. The proof above is somewhat indirect compared with the l-adic case in §6.3. The reason why we have to make such a detour is that Schneider-Stuhler's result on de Rham cohomology isn't enough to compute the p-adic local zeta functions because we don't have natural actions of N, φ on de Rham cohomology (see also Remark 6.15 below). It seems a very challenging problem to generalize Schneider-Stuhler's computation for appropriately modified p-adic cohomology theory with N, φ for rigid analytic spaces, and compute p-adic local zeta functions without using l-adic cohomology.

Remark 6.15. We say few words about Schneider-Stuhler's conjecture for de Rham cohomology (for l-adic cohomology, see Theorem 6.3). In [SS], §5, Schneider-Stuhler defined a decreasing filtration $F^\bullet$ on de Rham cohomology $H^d_{\mathrm{DR}}(X_\Gamma)$ having similar properties as in the l-adic case (§6.3, see also [SS], Theorem 5). They conjectured that $F^\bullet$ essentially coincides with the monodromy filtration defined by Hyodo ([Hy]). We can't prove this here because we don't have natural actions of N, φ on de Rham cohomology. The problem is that Hyodo's monodromy filtration is actually defined on the log crystalline cohomology, which is isomorphic to de Rham cohomology by *forgetting* N, φ ([HK]). Hence we have the same problem as in Remark 6.14 above. Schneider-Stuhler also conjectured that $F^\bullet$ on de Rham cohomology is opposite to the Hodge-de Rham filtration. Recently, this conjecture was proved by Iovita-Spiess in [IS].

References

[Be] V. G. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. No. 78 (1993), 5–161 (1994).

[Č] I. V. Čerednik, *Uniformization of algebraic curves by discrete arithmetic subgroups of* $\mathrm{PGL}_2(k_w)$ *with compact quotient spaces*, Mat. Sb. (N.S.) **100(142)** (1976), no. 1, 59–88, 165.

[De1] P. Deligne, *Théorie de Hodge I*, in *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, 425–430, Gauthier-Villars, Paris, 1971.

[De2] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. No. 43, (1974), 273–307.

[De3] P. Deligne, *La conjecture de Weil II*, Inst. Hautes Études Sci. Publ. Math. No. 52, (1980), 137–252.

[Dr] V. G. Drinfeld, *Coverings of p-adic symmetric domains*, Funkcional. Anal. i Priložen. **10** (1976), no. 2, 29–40.

[Fo] J.-M. Fontaine, *Représentations p-adiques semi-stables (with an appendix by Pierre Colmez)*, Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque No. 223 (1994), 113–184.

[Fu] W. Fulton, *Intersection theory*, Second edition, Springer, Berlin, 1998.

[Ga] H. Garland, *p-adic curvature and the cohomology of discrete subgroups of p-adic groups*, Ann. of Math. (2) **97** (1973), 375–423.

[GM] H. Gillet, W. Messing, *Cycle classes and Riemann-Roch for crystalline cohomology*, Duke Math. J. **55** (1987), no. 3, 501–538.

[Gr] M. Gros, *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique*, Mém. Soc. Math. France (N.S.) No. 21 (1985).

[Harr] M. Harris, *On the local Langlands correspondence*, Proceedings of the International Congress of Mathematics, Beijing, 2002.

[Hart1] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin-New York 1970.

[Hart2] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Hy] O. Hyodo, *On the de Rham-Witt complex attached to a semi-stable family*, Compositio Math. **78** (1991), no. 3, 241–260.

[HK] O. Hyodo, K. Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles*, Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque No. 223, (1994), 221–268.

[Il1] L. Illusie, *Autour du théorème de monodromie locale*, Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque No. 223, (1994), 9–57.

[Il2] L. Illusie, *An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic etale cohomology*, in *Cohomologies p-adiques et applications arithmétiques, II*, Astérisque No. 279 (2002), 271–322.

[IS] A. Iovita, M. Spiess, *Logarithmic differential forms on p-adic symmetric spaces*, Duke Math. J. **110** (2001), no. 2, 253–278.

[It1] T. Ito, *Weight-monodromy conjecture over equal characteristic local fields*, in preparation

[It2] T. Ito, *Weight-monodromy conjecture for certain threefolds in mixed characteristic*, math.NT/0212109, 2002.

[dJ] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. No. 83, (1996), 51–93.

[dJvdP] J. de Jong, M. van der Put, *Étale cohomology of rigid analytic spaces*, Doc. Math. **1** (1996), No. 01, 1–56.

[KM] N. M. Katz, W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.

[Kl] S. L. Kleiman, *The standard conjectures* in *Motives (Seattle, WA, 1991)*, 3–20, Proc. Sympos. Pure Math., Part 1, Amer. Math. Soc., Providence, RI, 1994.

[Ku] A. Kurihara, *Construction of p-adic unit balls and the Hirzebruch proportionality*, Amer. J. Math. **102** (1980), no. 3, 565–648.

[Mc] P. McMullen, *On simple polytopes*, Invent. Math. **113** (1993), no. 2, 419–444.

[Mo] A. Mokrane, *La suite spectrale des poids en cohomologie de Hyodo-Kato*, Duke Math. J. 72 (1993), no. 2, 301–337.

[Mum] D. Mumford, *An analytic construction of degenerating curves over complete local rings*, Compositio Math. **24** (1972), 129–174.

[Mus] G. A. Mustafin, *Non-archimedean uniformization*, Mat. Sb. (N.S.) **105(147)** (1978), no. 2, 207–237, 287.

[O] T. Ochiai, *l-independence of the trace of monodromy*, Math. Ann. **315** (1999), no. 2, 321–340.

[Rap] M. Rapoport, *On the bad reduction of Shimura varieties*, in *Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, 253–321, Academic Press, Boston, MA, 1990.

[RZ1] M. Rapoport, T. Zink, *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), no. 1, 21–101.

[RZ2] M. Rapoport, T. Zink, *Period spaces for p-divisible groups*, Ann. of Math. Stud., 141, Princeton Univ. Press, Princeton, NJ, 1996.

[Ray] M. Raynaud, *Géométrie analytique rigide d'après Tate, Kiehl,$\cdots$*, Bull. Soc. Math. France, Mém. **39**–**40**, 319–327, (1974).

[SaM] M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989).

[SaT] T. Saito, *Modular forms and p-adic Hodge theory*, Invent. Math. **129** (1997), no. 3, 607–620.

[SS] P. Schneider, U. Stuhler, *The cohomology of p-adic symmetric spaces*, Invent. Math., **105**, (1991), no. 1, 47–122.

[Se1] J.-P. Serre, *Corps locaux*, Deuxieme edition, Hermann, Paris, 1968.

[Se2] J.-P. Serre, *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*, Sem. Delange-Pisot-Poitou, 1970.

[ST] J.-P. Serre, J. Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.

[St] J. Steenbrink, *Limits of Hodge structures*, Invent. Math. **31** (1975/76), no. 3, 229–257.

[Te] T. Terasoma, *Monodromy weight filtration is independent of l*, math.AG/9802051, 1998.

[Ts] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), no. 2, 233–411.

[Va] Y. Varshavsky, *p-adic uniformization of unitary Shimura varieties*, Inst. Hautes Études Sci. Publ. Math. No. 87, (1998), 57–119.

[We] A. Weil, *Introduction à l'étude des variétés kählériennes*, Hermann, Paris, 1958.

[SGA4-III] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Math., 305, Springer, Berlin, 1973.

[SGA5] *Cohomologie l-adique et fonctions L*, Lecture Notes in Math., 589, Springer, Berlin, 1977.

[SGA7-I] *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Math., 288, Springer, Berlin, 1972.

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